Some Hierarchies of QCB_0-Spaces

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Abstract

We define and study hierarchies of topological spaces induced by the classical Borel and Luzin hierarchies of sets. Our hierarchies are divided into two classes: hierarchies of countably based spaces induced by their embeddings into $P\omega$, and hierarchies of spaces (not necessarily countably based) induced by their admissible representations. We concentrate on the non-collapse property of the hierarchies and on the relationships between hierarchies in the two classes.

1 Introduction

A basic notion of Computable Analysis (CA) [Wei00] is the notion of an *admissible representation* of a topological space X. This is a partial continuous surjection δ from the Baire space $\mathcal{N}$ onto X satisfying a certain universality property (see Subsection 2.6 for some more details). Such a representation of X induces a reasonable computability theory on X, and the class of admissibly represented spaces is wide enough to include most spaces of interest for Analysis or Numerical Mathematics. As shown by the first author [Sch03], this class coincides with the class of the so-called QCB_0-spaces, i.e. T_0-spaces which are quotients of countably based spaces, and it forms a cartesian closed category (with the continuous functions as morphisms). Thus, among QCB_0-spaces one meets many important function spaces including the Kleene-Kreisel continuous functionals [Kl959, Kr59] interesting for several branches of logic and computability theory.

Along with the mentioned nice properties of QCB_0-spaces, this class seems to be too broad to admit a deep understanding. Hence, it makes sense to search for natural subclasses of this class which still include "practically" important spaces but are (hopefully) easier to study. Interesting examples of such subclasses are obtained if we consider, for each level Γ of the classical Borel or Luzin (projective) hierarchies of Descriptive Set Theory (DST) [Ke95], the class of spaces which have an admissible representation of the complexity Γ (below we make this precise). The study of the resulting Borel and Luzin hierarchies of QCB_0-spaces is one of the aims of this paper.

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Along with the hierarchies of QCB_0-spaces, we will consider Borel and Luzin hierarchies of countably based T_0-spaces (CB_0-spaces for short) which are induced by the well-known fact that any CB_0-space may be embedded in the algebraic domain $P\omega$ of all subsets of ω. The precise definition depends crucially on the possibility to find the "right" extensions of the classical hierarchies from the class of Polish spaces to the arbitrary spaces (including $P\omega$). Such extensions were introduced by the second author in [Se04] and were studied in [Se05, Se05a, Se06, Se08, BY09, Br13]. In particular, it was shown that many properties of these hierarchies in ω-continuous domains resemble the properties of classical hierarchies in Polish spaces [Ke95].

Hierarchies of spaces obtained in this way turn out to be closely related to the corresponding hierarchies of QCB_0-spaces. Moreover, among the first levels of the Borel hierarchy of CB_0-spaces we meet some classes of spaces which attracted attention of several researches in the field of quasi-metric spaces, in particular the class of quasi-Polish spaces. The class of quasi-Polish spaces identified and studied in [Br13] is a good solution to the problem from [Se08] of finding a natural class of spaces that includes the Polish spaces and the ω-continuous domains and has a reasonable DST.

In this paper we establish some basic properties of the above-mentioned hierarchies of spaces. In particular, we show that the hierarchies of spaces do not collapse, and that any level of any hierarchy is closed under retracts. As main technical tools to prove these results we use suitable generalizations of some classical facts (e.g. of the injectivity property of $P\omega$ and of Lavrentyev's Theorem on extending partial homeomorphisms in Polish spaces). Some of those generalizations might be also interesting in their own right. We also show that the class of all spaces in our hierarchies forms in a sense the smallest cartesian closed category of QCB_0-spaces containing the discrete space ω of natural numbers, and establish the close relationship of the Luzin hierarchy of QCB_0-spaces to the continuous functionals of finite type. Hence the class of all spaces in the Luzin hierarchy of QCB_0-spaces seems to be a reasonable subclass of QCB_0-spaces that contains most of spaces of interest for CA, including the Kleene-Kreisel continuous functionals.

After recalling some notions and known facts in the next section, we establish the main technical facts in Section 3. In Sections 4 and 5 we introduce and study the mentioned hierarchies of CB_0-spaces and of QCB_0-spaces. In Section 6 we establish close relationships between hierarchies of CB_0-spaces to those of QCB_0-spaces (namely, any level of the CB_0-hierarchies coincides with the class of CB_0-spaces in the corresponding level of the corresponding QCB_0-hierarchy). In Section 7 we relate the Luzin hierarchy of QCB_0-spaces to the Kleene-Kreisel continuous functionals. In Section 8 we discuss the cartesian closedness of the corresponding categories, and we conclude in Section 9.

2 Notation and Preliminaries

2.1 Notation

We freely use the standard set-theoretic notation like $dom(f), rng(f)$ and $graph(f)$ for the domain, range and graph of a function f, respectively, $X \times Y$ for the Cartesian product, $X \sqcup Y$ for the disjoint union of sets X and Y, Y^X for the set of functions $f\colon X \to Y$ (but in the case when X, Y are QCB_0-spaces we use the same notation to denote the space of continuous functions from X to Y), and $P(X)$ for the set of all subsets of X. For $A \subseteq X$, $\overline{A}$ denotes the complement $X \setminus A$ of A in X. We identify the set of natural numbers with the first infinite ordinal ω. The first uncountable ordinal is denoted by ω_1. The notation $f : X \to Y$ (resp.

$f :\subseteq X \to Y$) means that f is a total (resp. a partial) function from a set X to a set Y.

2.2 Topological Spaces

We assume the reader to be familiar with the basic notions of topology. The collection of all open subsets of a topological space X (i.e. the topology of X) is denoted by τ_X; for the underlying set of X we will write X in abuse of notation. We will usually abbreviate "topological space" to "space". Remember that a space is *zero-dimensional* if it has a basis of clopen sets.

A space Y is called a *(continuous) retract* of a space X if there are continuous functions $s : Y \to X$ and $r : X \to Y$ such that composition rs coincides with the identity function id_Y on Y. Such a pair of functions (s, r) is called a *section-retraction* pair. Note that the section s is a homeomorphism between Y and the subspace $s(Y) = \{x \in X \mid sr(x) = x\}$ of X, and $s^{-1} = r|_{s(Y)}$.

Let ω be the space of non-negative integers with the discrete topology. Of course, the spaces $\omega \times \omega = \omega^2$, and $\omega \sqcup \omega$ are homeomorphic to ω, the first homeomorphism is realized by the Cantor pairing function $\langle \cdot, \cdot \rangle$.

Let $\mathcal{N} = \omega^\omega$ be the set of all infinite sequences of natural numbers (i.e., of all functions $\xi \colon \omega \to \omega$). Let ω^* be the set of finite sequences of elements of ω, including the empty sequence. For $\sigma \in \omega^*$ and $\xi \in \mathcal{N}$, we write $\sigma \sqsubseteq \xi$ to denote that σ is an initial segment of the sequence ξ. By $\sigma\xi = \sigma \cdot \xi$ we denote the concatenation of σ and ξ, and by $\sigma \cdot \mathcal{N}$ the set of all extensions of σ in $\mathcal{N}$. For $x \in \mathcal{N}$, we can write $x = x(0)x(1)\ldots$ where $x(i) \in \omega$ for each $i < \omega$. For $x \in \mathcal{N}$ and $n < \omega$, let $x^{<n} = x(0)\ldots x(n-1)$ denote the initial segment of x of length n. Notations in the style of regular expressions like 0^ω, 0^*1 or 0^m1^n have the obvious standard meaning.

By endowing $\mathcal{N}$ with the product of the discrete topologies on ω, we obtain the so-called *Baire space*. The product topology coincides with the topology generated by the collection of sets of the form $\sigma \cdot \mathcal{N}$ for $\sigma \in \omega^*$. The Baire space is of primary importance for DST and CA. The importance stems from the fact that many countable objects are coded straightforwardly by elements of $\mathcal{N}$, and it has very specific topological properties. In particular, it is a perfect zero-dimensional space and the spaces $\mathcal{N}^2$, $\mathcal{N}^\omega$, $\omega \times \mathcal{N} = \mathcal{N} \sqcup \mathcal{N} \sqcup \cdots$ (endowed with the product topology) are all homeomorphic to $\mathcal{N}$. Let $(x, y) \mapsto \langle x, y \rangle$ be a homeomorphism between $\mathcal{N}^2$ and $\mathcal{N}$. The Baire space $\mathcal{N}$ has the following universality property for zero-dimensional CB_0-spaces:

Proposition 2.1 *[Ke95, Theorems 1.1 and 7.8] A topological space X embeds into $\mathcal{N}$ iff X is a zero-dimensional CB_0-space.*

The subspace $\mathcal{C} = 2^\omega$ of $\mathcal{N}$ formed by the infinite binary strings (endowed with the relative topology inherited from $\mathcal{N}$) is known as the *Cantor space*. Along with $\mathcal{N}$ and $\mathcal{C}$, the space $P\omega$ is of principal importance for this paper. This is the space of subsets of the natural numbers with the Scott topology on $(P(\omega); \subseteq)$. The basic open sets of this topology are of the form $\{A \subseteq \omega \mid F \subseteq A\}$, where F ranges over the finite subsets of ω.

The importance of $P\omega$ for this paper is explained by its following well-known properties. First, $P\omega$ is universal for CB_0-spaces.

Proposition 2.2 *A topological space X embeds into $P\omega$ iff X is a CB_0-space.*

Proof. Since $P\omega$ is a CB_0-space, any space X homeomorphic to a subspace $P\omega$ is a CB_0-space as well. Conversely, if $\{\beta_i \mid i \in \omega\}$ is a base for a countably-based T_0-space X, then the function $e\colon X \to P\omega$ mapping x to $\{i \in \omega \mid x \in \beta_i\}$ is a homeomorphic embedding of X into $P\omega$. $\Box$

The second property shows that $P\omega$ is an injective object in the category of all topological spaces.

Proposition 2.3 *[G+80, Proposition 3.5] Let Y be topological space and X be a subspace of Y. Then any continuous function $f : X \to P\omega$ can be extended to a continuous function $g : Y \to P\omega$.*

Completely metrisable spaces satisfy the following extension theorem by Kuratowski.

Proposition 2.4 *[Ke95, Theorem 3.8] Let X be a metrisable space and Y be a completely metrisable space. Then any continuous function $f\colon A \to Y$ defined on a subset A of X can be extended to a continuous function $g : G \to Y$, where G is a G_δ-subset of Y with $A \subseteq G$.*

2.3 Families of Pointclasses

Here we recall a useful technical notion of a family of pointclasses introduced in [Se11].

A *pointclass* on X is simply a collection $\mathbf{\Gamma}(X)$ of subsets of X. We need the following "parameterized" version of the notion of pointclass. A *family of pointclasses* is a family $\mathbf{\Gamma} = \{\mathbf{\Gamma}(X)\}$ indexed by arbitrary topological spaces X such that each $\mathbf{\Gamma}(X)$ is a pointclass on X and $\mathbf{\Gamma}$ is closed under continuous preimages, i.e. $f^{-1}(A) \in \mathbf{\Gamma}(X)$ for every $A \in \mathbf{\Gamma}(Y)$ and every continuous function $f\colon X \to Y$. In particular, any pointclass $\mathbf{\Gamma}(X)$ in such a family is downward closed under the Wadge reducibility on X. Recall that $A \subseteq X$ is *Wadge reducible* to $B \subseteq Y$ if $A = f^{-1}(B)$ for some continuous function f on X.

Trivial examples of families of pointclasses are $\mathcal{E}, \mathcal{F}$, where $\mathcal{E}(X) = \{\emptyset\}$ and $\mathcal{F}(X) = \{X\}$ for any space X. A basic example of a family of pointclasses is given by the family $\mathcal{O} = \{\tau_X\}$ of the topologies of all the spaces X.

Finally, we define some operations on families of pointclasses which are relevant to hierarchy theory. First, the usual set-theoretic operations will be applied to the families of pointclasses pointwise: for example, the union $\bigcup_i \mathbf{\Gamma}_i$ of the families of pointclasses $\mathbf{\Gamma}_0, \mathbf{\Gamma}_1, \ldots$ is defined by $(\bigcup_i \mathbf{\Gamma}_i)(X) = \bigcup_i \mathbf{\Gamma}_i(X)$.

Second, a large class of such operations is induced by the set-theoretic operations of L.V. Kantorovich and E.M. Livenson (see e.g. [Se11] for the general definition). Among them are the operations $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_\sigma$ where $\mathbf{\Gamma}(X)_\sigma$ is the set of all countable unions of sets in $\mathbf{\Gamma}(X)$, the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_c$ where $\mathbf{\Gamma}(X)_c$ is the set of all complements of sets in $\mathbf{\Gamma}(X)$, the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_d$ where $\mathbf{\Gamma}(X)_d$ is the set of all differences of sets in $\mathbf{\Gamma}(X)$, and the operation $\Gamma \mapsto \Gamma_p$ defined by $\Gamma_p(X) = \{pr_X(A) \mid A \in \Gamma(\mathcal{N} \times X)\}$ where $pr_X(A) = \{x \in X \mid \exists p \in \mathcal{N}((p,x) \in A)\}$ is the projection of $A \subseteq \mathcal{N} \times X$ along the axis $\mathcal{N}$.

The next subsection contains some important examples of families of pointclasses from hierarchy theory.

2.4 Classical Hierarchies on Topological Spaces

Let us recall the definition of the Borel hierarchy on arbitrary spaces introduced in [Se04].

Definition 2.5 For $\alpha < \omega_1$, define the family of pointclasses $\mathbf{\Sigma}^0_\alpha = \{\mathbf{\Sigma}^0_\alpha(X)\}$ by induction on α as follows: $\mathbf{\Sigma}^0_0(X) = \{\emptyset\}$, $\mathbf{\Sigma}^0_1(X) = \tau_X$ is the collection of the open sets of X, $\mathbf{\Sigma}^0_2(X) = ((\mathbf{\Sigma}^0_1(X))_d)_\sigma$ is the collection of all countable unions of differences of open sets, and $\mathbf{\Sigma}^0_\alpha(X) = (\bigcup_{\beta<\alpha}(\mathbf{\Sigma}^0_\beta(X))_c)_\sigma$ (for $\alpha > 2$) is the class of countable unions of sets in $\bigcup_{\beta<\alpha}(\mathbf{\Sigma}^0_\beta(X))_c$.

The sequence $\{\mathbf{\Sigma}^0_\alpha(X)\}_{\alpha<\omega_1}$ is called *the Borel hierarchy* of X. We also let $\mathbf{\Pi}^0_\beta(X) = (\mathbf{\Sigma}^0_\beta(X))_c$ and $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$. The classes $\mathbf{\Sigma}^0_\alpha(X), \mathbf{\Pi}^0_\alpha(X), \mathbf{\Delta}^0_\alpha(X)$ are called the *levels* of the Borel hierarchy of X.

By the definition and remarks at the end of the previous subsection, any of $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha$ is a family of pointclasses. It is straightforward to check by induction on α, β that using Definition 2.5 one has the following result.

Proposition 2.6 *For every topological space X and for all $\alpha < \beta < \omega_1$, $\mathbf{\Sigma}^0_\alpha(X) \cup \mathbf{\Pi}^0_\alpha(X) \subseteq \mathbf{\Delta}^0_\beta(X)$.*

Remark. Definition 2.5 applies to all spaces X, and Proposition 2.6 holds true in the full generality. Note that Definition 2.5 differs from the classical definition for Polish spaces (see e.g. [Ke95, Section 11.B]) only for the level 2, and that for the case of Polish spaces our definition of the Borel hierarchy is equivalent to the classical one. Notice that the classical definition cannot be applied in general to non metrizable spaces (like e.g. the non discrete ω-algebraic domains) precisely because the inclusion $\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Sigma}^0_2$ may fail.

Let $\{\mathbf{\Sigma}^1_n(X)\}_{1\leq n<\omega}$ be Luzin's projective hierarchy in X (cf. [Br13]). Using the corresponding operation on families of pointclasses from the previous subsection we have $\mathbf{\Sigma}^1_1(X) = (\mathbf{\Pi}^0_2(X))_p$ and $\mathbf{\Sigma}^1_{n+1}(X) = (\mathbf{\Pi}^1_n(X))_p$ for any $n \geq 1$. Let also $\mathbf{\Sigma}^1_0 = \mathbf{\Pi}^1_0 = \mathbf{\Delta}^1_1$. The reason why the definition of the first level of the Luzin hierarchy is distinct from the classical definition $\mathbf{\Sigma}^1_1(X) = (\mathbf{\Pi}^0_1(X))_p$ for Polish spaces is that the inclusion $\mathbf{\Sigma}^0_1(X) \subseteq (\mathbf{\Pi}^0_1(X))_p$ may fail in general.

Any level $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n, \mathbf{\Delta}^1_n$ of the Luzin hierarchy is a family of pointclasses, and we have the natural inclusions among them similar to the inclusions for levels of the Borel hierarchy.

Levels of the introduced hierarchies in an arbitrary space have closure properties similar to those known for the classical hierarchies in Polish spaces, in particular:

Proposition 2.7 *Any non-zero $\mathbf{\Sigma}$-level of the Borel hierarchy on an arbitrary topological space is closed under finite intersections, countable unions and binary products (for the case of binary product this means that $A \in \mathbf{\Sigma}^0_\alpha(X)$ and $B \in \mathbf{\Sigma}^0_\alpha(Y)$ imply $A \times B \in \mathbf{\Sigma}^0_\alpha(X \times Y)$). Any non-zero $\mathbf{\Sigma}$-level of the Luzin hierarchy in an arbitrary space is closed under countable unions, countable intersections, binary products and the projection along $\mathcal{N}$-axis (for the case of projection this means that $A \in \mathbf{\Sigma}^1_n(\mathcal{N} \times X)$ implies $pr_X(A) \in \mathbf{\Sigma}^1_n(X)$).*

We will often use the following straightforward result which also extends the corresponding well-known facts for the classical hierarchies.

Proposition 2.8 *Let X be a subspace of a topological space Y, $A \subseteq X$, and let Γ be a $\mathbf{\Sigma}$- or a $\mathbf{\Pi}$-level of the introduced hierarchies. Then $A \in \Gamma(X)$ iff $A = X \cap B$ for some $B \in \Gamma(Y)$. If in addition $X \in \Gamma(Y)$, then $A \in \Gamma(X)$ iff $A \in \Gamma(Y)$.*

We will often cite the following topological complexity of the equality test $EQ_X := \{(x, x) \mid x \in X\}$ on a topological space X.

Proposition 2.9 *If X is a CB_0-space then $EQ_X \in \mathbf{\Pi}^0_2(X \times X)$. If X is a Hausdorff space, then $EQ_X \in \mathbf{\Pi}^0_1(X \times X)$.*

Proof. The first statement has been shown in [Br13]. The second (well-known) statement follows from the fact that in the Hausdorff case the complement of EQ_X is equal to $\bigcup\{U \times V \mid U, V$ open and disjoint$\}$ and therefore open. □

Let Γ be a family of pointclasses and X, Y be spaces. A function $f : X \to Y$ is called Γ-*measurable* if $f^{-1}(A) \in \Gamma(X)$ for each open set $A \subseteq Y$. Note that the continuous functions coincide with the $\mathbf{\Sigma}^0_1$-measurable functions.

We will also use the following basic property of Γ-measurable functions. Note that addition $+$ on ordinal numbers is associative, but not commutative.

Proposition 2.10 *[BY09, Lemma 1] Let X, Y be CB_0-spaces, let $\alpha, \beta < \omega_1$ be ordinals, let $f : X \to Y$ be a $\mathbf{\Sigma}^0_{1+\alpha}$-measurable function, and let $A \in \mathbf{\Sigma}^0_{1+\beta}(Y)$ and $B \in \mathbf{\Pi}^0_{1+\beta}(Y)$. Then $f^{-1}(A) \in \mathbf{\Sigma}^0_{1+\alpha+\beta}(X)$ and $f^{-1}(B) \in \mathbf{\Pi}^0_{1+\alpha+\beta}(X)$.*

Notice that the case $(\alpha, \beta) = (0, 0)$ denotes the fact that the preimage of an open (closed) set under a continuous function is open (resp. closed).

2.5 Polish and quasi-Polish spaces

Remember that a space X is *Polish* if it is countably based and metrisable with a metric d such that (X, d) is a complete metric space. Important examples of Polish spaces are the Baire space, the Cantor space, the space of reals $\mathbb{R}$ and its Cartesian powers $\mathbb{R}^n$ $(n < \omega)$, the closed unit interval $[0, 1]$, the Hilbert cube $[0, 1]^\omega$ and the Hilbert space $\mathbb{R}^\omega$.

Below we will cite the following result known as Lavrentyev's Theorem:

Proposition 2.11 *[Ke95, Theorem 3.9] Let X, Y be Polish spaces, $A \subseteq X$, $B \subseteq Y$, and let f be a homeomorphism of A onto B (equipped with the subspace topologies induced from X and Y). Then there exist $A^* \in \mathbf{\Pi}^0_2(X)$, $B^* \in \mathbf{\Pi}^0_2(Y)$ and a homeomorphism f^* of A^* onto B^* such that $A \subseteq A^*$, $B \subseteq B^*$, and $f^*|_A = f$.*

A natural variant of the class of Polish spaces has recently emerged. Given a set X, call a function d from $X \times X$ to the nonnegative reals *quasi-metric* whenever $x = y$ iff $d(x, y) = d(y, x) = 0$, and $d(x, y) \leq d(x, z) + d(z, y)$ (but we don't require d to be symmetric). In particular, every metric is a quasi-metric. Every quasi-metric on X canonically induces a topology on X which is denoted by τ_d, where τ_d is the topology generated by the open balls $B_d(x, \varepsilon) = \{y \in x \mid d(x, y) < \varepsilon\}$ for $x \in X$ and $0 < \varepsilon \in \mathbb{R}$. A space X is called *quasi-metrisable* if there is a quasi-metric on X which generates its topology. If d is a quasi-metric on X, let $\hat{d}$ be the metric on X defined by $\hat{d}(x, y) = \max\{d(x, y), d(y, x)\}$. A sequence $\{x_n\}_{n<\omega}$ in X is called *d-Cauchy sequence* if for every $\varepsilon > 0$ there is $n_0 \in \omega$ such that $d(x_n, x_m) < \varepsilon$ for all $n_0 \leq n \leq m$. We say that the quasi-metric d on X is *complete* if every d-Cauchy sequence converges with respect to $\hat{d}$ (notice that this definition is coherent with the notion of completeness for a metric d, as in this case $\hat{d} = d$).

A T_0-space X is called *quasi-Polish* if it is countably based and there is a complete quasi-metric on X which generates its topology. In particular, every Polish space is quasi-Polish, but by

[Br13, Corollary 45] also every ω-continuous domain is quasi-Polish. De Brecht [Br13] shows that there is a reasonable DST for the quasi-Polish spaces which extends the classical DST for Polish spaces [Ke95] and the DST for ω-continuous domains [Se06, Se08] in many directions.

An important example of a quasi-Polish space is the space $P\omega$ equipped with the following quasi-metric d: if $A \subseteq B$ then $d(A, B) = 0$, otherwise $d(A, B) = 1/2^a$ where a is the smallest number in $A \setminus B$. Note that Proposition 2.2 implies that any CB_0-space is quasi-metrisable.

De Brecht proved the following characterization of quasi-Polish spaces (cf. [Br13, Corollary 24]):

Proposition 2.12 *A topological space is quasi-Polish iff it is homeomorphic to a $\mathbf{\Pi}^0_2$-subset of $P\omega$ (endowed with the relative topology inherited from $P\omega$).*

2.6 Admissible Representations and QCB_0-spaces

A *representation* of a space X is a surjection of a subspace of the Baire space $\mathcal{N}$ onto X. Usually it is denoted as a partial function from $\mathcal{N}$ to X. The notion of admissible representation is basic in Computable Analysis (CA). A representation δ of X is *admissible*, if it is continuous and any continuous function $\nu : Z \to X$ from a zero-dimensional CB_0-space Z to X is continuously reducible to δ, i.e. $\nu = \delta g$ for some continuous function $g : Z \to \mathcal{N}$. A topological space is *admissibly representable* if it has an admissible representation.

The notion of admissibility was introduced in [KW85] for representations of countably based spaces (in a different but equivalent formulation) and was extensively studied by many authors. In [BH02] a close relation of admissible representations of countably based spaces to open continuous representations was established. In [Sch02, Sch03] the notion was extended to non-countably based spaces and a nice characterization of the admissibly represented spaces was achieved. Namely, the admissibly represented sequential topological spaces coincide with the QCB_0-spaces, i.e., T_0-spaces which are topological quotients of countably based spaces.

The category QCB_0 of QCB_0-spaces as objects and continuous functions as morphisms is known to be cartesian closed (cf. [ELS04, Sch03]). The exponential Y^X to QCB_0-spaces X, Y has the set of continuous functions from X to Y as the underlying set, and its topology is the sequentialization of the compact-open topology on Y^X. By the *sequentialization* of a topology τ we mean the family of all sequentially open sets pertaining to this topology. (Remember that *sequentially open* sets are defined to be the complements of the sets that are closed under forming limits of converging sequences.) The sequentialization of τ is finer than or equal to τ. The topology of the QCB_0-product to X and Y, which we denote by $X \times Y$, is the sequentialization of the classical Tychonoff topology on the cartesian product of the underlying sets of X and Y. So products and exponentials in QCB_0 are formed in the same way as in its supercategory Seq of sequential topological spaces.

From [Br13] it follows that admissible total representations are closely related to quasi-Polish spaces. The following assertion is contained among the results in [Br13].

Proposition 2.13 *For any CB_0-space X the following statements are equivalent:*

(1) *X is quasi-Polish.*

(2) *X has an open continuous total representation.*

(3) *X has an admissible total representation.*

(4) *X has an admissible representation whose domain is a Polish space.*

We will also cite the following facts from [Sch02, Sch03].

Proposition 2.14 *Let δ and γ be admissible representations of QCB_0-spaces X and Y, respectively. Then $f : X \to Y$ is continuous iff $f\delta = \gamma g$ for some partial continuous function g on $\mathcal{N}$.*

Note that if $u :\subseteq \mathcal{N}^2 \to \mathcal{N}$ is continuous and $p \in \mathcal{N}$ then $u_p = \lambda x.u(p, x)$ is a partial continuous function on $\mathcal{N}$.

Proposition 2.15 *There is a partial continuous function $u \colon \subseteq \mathcal{N}^2 \to \mathcal{N}$ such that $dom(u) \in \mathbf{\Pi}^0_2(\mathcal{N}^2)$ and for any partial continuous function g on $\mathcal{N}$ there is some $p \in \mathcal{N}$ such that u_p is an extension of g.*

The function u in Proposition 2.15 can be chosen as the application operator of the Second Kleene Algebra. We use it to describe the construction of admissible representations for function spaces formed in QCB_0 (cf. [Sch03, Wei00]).

Proposition 2.16 *Let δ and γ be admissible representations for QCB_0-spaces X and Y, respectively. Then admissible representations $[\delta \times \gamma]$ for the QCB_0-product $X \times Y$ and $[\delta \to \gamma]$ for the QCB_0-exponential Y^X can be defined by:*

$$[\delta \times \gamma](\langle p, q\rangle) = (x, y) \text{ iff } \delta(p) = x \wedge \gamma(q) = y \quad \text{and} \quad [\delta \to \gamma](p) = f \text{ iff } f\delta = \gamma u_p.$$

for $p, q \in \mathcal{N}$, $x \in X$, $y \in Y$, and $f \colon X \to Y$.

De Brecht and Yamamoto showed the following property of subsets of admissibly represented countably-based spaces.

Proposition 2.17 *[BY09, Corollary 3]. Let δ be an admissible representation of a countably-based T_0-space X, let $A \subseteq X$ and let $1 \leq \alpha < \omega_1$. Then $A \in \mathbf{\Sigma}^0_\alpha(X)$ iff $\delta^{-1}(A) \in \mathbf{\Sigma}^0_\alpha(dom(\delta))$.*

3 Main Technical Facts

In this section we prove a couple of facts that serve as main technical tools in the sequel, but some of these facts might be also of independent interest.

The first result generalizes the well-known fact that $P\omega$ is an injective space (see Proposition 2.2), because the continuous functions coincide with the $\mathbf{\Sigma}^0_1$-measurable functions.

Theorem 3.1 *Let Y be a topological space, let X be a subspace of Y, and let $\Gamma \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \mid 1 \leq \alpha < \omega_1, 1 \leq n < \omega\}$. Then any Γ-measurable function $f : X \to P\omega$ can be extended to a Γ-measurable function $g : Y \to P\omega$.*

Proof. For any $n < \omega$, the set $\uparrow\{n\} = \{A \subseteq \omega \mid n \in A\}$ is open in $P\omega$, hence $f^{-1}(\uparrow\{n\}) \in \Gamma(X)$, hence $f^{-1}(\uparrow\{n\}) = X \cap A_n$ for some $A_n \in \Gamma(Y)$ by Proposition 2.8. Define the function $g : Y \to P\omega$ by $g(y) = \{n \mid y \in A_n\}$. Then for any $y \in Y$ we have

$$y \in A_n \Leftrightarrow n \in g(y) \Leftrightarrow g(y) \in \uparrow\{n\} \Leftrightarrow y \in g^{-1}(\uparrow\{n\}),$$

hence $g^{-1}(\uparrow\{n\}) = A_n \in \Gamma(Y)$. Since $\{\uparrow\{n\} \mid n \in \omega\}$ is a subbasis in $P\omega$ and $\Gamma(Y)$ is closed under finite intersection and countable union by Proposition 2.7, g is Γ-measurable. If $y \in X$ then we have

$$n \in f(y) \Leftrightarrow f(y) \in \uparrow\{n\} \Leftrightarrow y \in f^{-1}(\uparrow\{n\}) \Leftrightarrow y \in A_n \Leftrightarrow n \in g(y),$$

hence $g(y) = f(y)$. $\square$

The second result is a remote relative of Lavrentyev's Theorem (cf. Proposition 2.11).

Theorem 3.2 *Let $A, B \subseteq P\omega$ be subspaces of $P\omega$, $\alpha, \beta < \omega_1$, and let $f : A \to B$ be a $\mathbf{\Sigma}^0_{1+\alpha}$-measurable bijection such that its inverse $g = f^{-1}$ is $\mathbf{\Sigma}^0_{1+\beta}$-measurable. Then there exist $A^* \in \mathbf{\Pi}^0_{1+\alpha+\mu+1}(P\omega)$, $B^* \in \mathbf{\Pi}^0_{1+\beta+\mu+1}(P\omega)$ (where $\mu = max\{\alpha, \beta\}$) and a $\mathbf{\Sigma}^0_{1+\alpha}$-measurable bijection $f^* : A^* \to B^*$ such that $g^* = f^{*-1}$ is $\mathbf{\Sigma}^0_{1+\beta}$-measurable, $A \subseteq A^*$, $B \subseteq B^*$, $f^*|_A = f$ and $g^*|_B = g$.*

Proof. By Theorem 3.1, there are a $\mathbf{\Sigma}^0_{1+\alpha}$-measurable extension $f_1 : P\omega \to P\omega$ of f and a $\mathbf{\Sigma}^0_{1+\beta}$-measurable extension $g_1 : P\omega \to P\omega$ of g. Consider the set

$$S = \{(x, y) \in P\omega \times P\omega \mid f_1(x) = y \wedge x = g_1(y)\}.$$

Clearly, $S = S_1 \cap S_2$ where $S_1 = \{(x, y) \mid f_1(x) = y\}$ and $S_2 = \{(x, y) \mid x = g_1(y)\}$. Since $EQ_{P\omega} \in \mathbf{\Pi}^0_2(P\omega \times P\omega)$ by Proposition 2.9 and S_1 is the preimage of $EQ_{P\omega}$ under the $\mathbf{\Sigma}^0_{1+\alpha}$-measurable function $(x, y) \mapsto (f_1(x), y)$, $S_1 \in \mathbf{\Pi}^0_{1+\alpha+1}(P\omega \times P\omega)$ by Proposition 2.10. Similarly, $S_2 \in \mathbf{\Pi}^0_{1+\beta+1}(P\omega \times P\omega)$ and therefore $S \in \mathbf{\Pi}^0_{1+\mu+1}(P\omega \times P\omega)$ by Propositions 2.6 and 2.7.

Now let $A^* = \{x \in P\omega \mid (x, f_1(x)) \in S\}$ and $B^* = \{y \in P\omega \mid (g_1(y), y) \in S\}$. Since the function $x \mapsto (x, f_1(x))$ is $\mathbf{\Sigma}^0_{1+\alpha}$-measurable and A^* is the preimage of S under this function, $A^* \in \mathbf{\Pi}^0_{1+\alpha+\mu+1}(P\omega)$ by Proposition 2.10. Similarly, $B^* \in \mathbf{\Pi}^0_{1+\beta+\mu+1}(P\omega)$. Then the sets A^*, B^* and the functions $f^* = f_1|_{A^*}$, $g^* = g_1|_{B^*}$ have the desired properties. $\square$

The next fact is a special case of the previous theorem.

Corollary 3.3 *If we take in the previous theorem $\alpha = 0$ and $\beta = 1$ then we obtain $A^* \in \mathbf{\Pi}^0_3(P\omega)$ and $B^* \in \mathbf{\Pi}^0_4(P\omega)$.*

Let us introduce one of the main notions of this paper.

Definition 3.4 Let Γ be a family of pointclasses. A topological space X is called a Γ-*space* if X is homeomorphic to a subspace $A \subseteq P\omega$ with $A \in \Gamma(P\omega)$. The class of all Γ-spaces is denoted $\mathsf{CB}_0(\Gamma)$.

The third result of this section extends Corollary 24 in [Br13] (which is obtained when $\Gamma = \mathbf{\Pi}^0_2$).

Proposition 3.5 *Let $\Gamma \in \{\mathbf{\Pi}^0_2, \mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \mid 3 \leq \alpha < \omega_1, 1 \leq n < \omega\}$ and $f :\subseteq X \to Y$ be a partial continuous function from a topological space X to a Γ-space Y. Then there is a continuous extension $g :\subseteq X \to Y$ of f with $dom(g) \in \Gamma(X)$.*

Proof. Without loss of generality we assume $Y \in \Gamma(P\omega)$, so in particular $f :\subseteq X \to P\omega$. By Proposition 2.3, there is a total continuous extension $h : X \to P\omega$ of f. Since $Y \in \Gamma(P\omega)$, $h^{-1}(Y) \in \Gamma(X)$. Since $dom(f) \subseteq h^{-1}(Y)$, we can take the restriction of h to $h^{-1}(Y)$ as the desired function g. $\square$

The fourth result of this section is the following extension of Lavrentyev's Theorem (see Proposition 2.11). For $\Gamma = \mathbf{\Pi}^0_2$ the result gives the extension of Lavrentyev's Theorem to quasi-Polish spaces (cf. [Br13]), and Lavrentyev's Theorem is obtained if we restrict the last fact to Polish spaces.

Theorem 3.6 *Let* $\Gamma \in \{\mathbf{\Pi}^0_2, \mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \mid 3 \leq \alpha < \omega_1, 1 \leq n < \omega\}$*,* X, Y *be* Γ*-spaces,* $A \subseteq X$*,* $B \subseteq Y$*, and let* f *be a homeomorphism of* A *onto* B*. Then there exist* $A^* \in \Gamma(X)$*,* $B^* \in \Gamma(Y)$ *and a homeomorphism* f^* *of* A^* *onto* B^* *such that* $A \subseteq A^*$*,* $B \subseteq B^*$*, and* $f^*|_A = f$*.*

Proof. Let $g = f^{-1}$. By Proposition 3.5 there exist $A_1 \in \Gamma(X)$, $B_1 \in \Gamma(Y)$ and continuous functions $f_1 : A_1 \to Y$, $g_1 : B_1 \to X$ such that $A \subseteq A_1$, $B \subseteq B_1$, $f_1|_A = f$ and $g_1|_B = g$. Let

$$S = \{(x, y) \in A_1 \times B_1 \mid f_1(x) = y \wedge x = g_1(y)\}.$$

Then *graph*$(f) \subseteq S$ and $S \in \Gamma(X \times Y)$, because $A_1 \times B_1 \in \Gamma(X \times Y)$, $EQ_X \in \mathbf{\Pi}^0_2(X \times X)$ and $EQ_Y \in \mathbf{\Pi}^0_2(Y \times Y)$ by Proposition 2.9, $(x, y) \mapsto (f_1(x), y))$ is a continuous function from $A_1 \times B_1$ to $Y \times Y$ and $(x, y) \mapsto (x, g_1(y))$ is a continuous function from $A_1 \times B_1$ to $X \times X$.

Let now

$$A^* = \{x \in A_1 \mid \exists y((x, y) \in A_1 \times B_1)\},\ B^* = \{y \in B_1 \mid \exists x((x, y) \in A_1 \times B_1)\},$$

$f^* = f_1|_A$ and $g^* = g_1|_B$. Then $A \subseteq A^*$, $B \subseteq B^*$, $f^*|_A = f$, $g^*|_B = g$ and $g^* = f^{*-1}$ (hence f^* is a homeomorphism of A^* onto B^*). Since

$$A^* = \{x \in A_1 \mid (x, f_1(x)) \in A_1 \times B_1\} \text{ and } B^* = \{y \in B_1 \mid (g_1(y), y) \in A_1 \times B_1\},$$

$A^* \in \Gamma(A_1)$ and $B^* \in \Gamma(B_1)$. By Proposition 2.8 we have $A^* \in \Gamma(X)$ and $B^* \in \Gamma(Y)$. □

Finally, we give a natural version of the previous theorem related to retracts, although this version is not used in the sequel.

Proposition 3.7 *Let* Γ, X, Y, A, B *be as in the previous theorem and let* $r : A \to B$ *and* $s : B \to A$ *be continuous functions such that* $rs = id_B$ *(hence,* B *is a retract of* A*). Then there exist* $A^* \in \Gamma(X)$*,* $B^* \in \Gamma(Y)$ *and continuous functions* $r^* : A^* \to B^*$*,* $s^* : B^* \to A^*$ *such that* $A \subseteq A^*$*,* $B \subseteq B^*$*,* $r^*|_A = r$*,* $s^*|_B = s$ *and* $r^* s^* = id_{B^*}$ *(hence,* B^* *is a retract of* A^**).*

Proof. By Proposition 3.5 there exist $A_1 \in \Gamma(X)$, $B_1 \in \Gamma(Y)$ and continuous functions $r_1 : A_1 \to Y$, $s_1 : B_1 \to X$ such that $A \subseteq A_1$, $B \subseteq B_1$, $r_1|_A = r$ and $s_1|_B = s$. Let

$$S = \{(x, y) \in A_1 \times B_1 \mid r_1(x) = y = r_1 s_1(y)\}$$

and

$$A^* = \{x \in A_1 \mid (x, r_1(x)) \in S\},\ B^* = \{y \in B_1 \mid (s_1(y), y) \in S\},$$

$r^* = r_1|_{A^*}$ and $s^* = s_1|_{B^*}$. Similarly to the previous proof one checks that these objects have the desired properties. □

4 Hierarchies of CB_0-Spaces

Here we introduce and study natural hierarchies of CB_0-spaces (see Definition 3.4) as well as the Borel hierarchy and the Luzin hierarchy on $P\omega$ discussed in Subsection 2.4.

Definition 4.1 (1) By the *Borel hierarchy* of CB_0-spaces we mean the sequence of classes $\{\mathsf{CB}_0(\mathbf{\Sigma}^0_\alpha)\}_{\alpha<\omega_1}$. By *levels* of this hierarchy we mean the classes $\mathsf{CB}_0(\mathbf{\Sigma}^0_\alpha)$ as well as the classes $\mathsf{CB}_0(\mathbf{\Pi}^0_\alpha)$ and $\mathsf{CB}_0(\mathbf{\Delta}^0_\alpha)$.

(2) By the *Luzin hierarchy* of CB_0-spaces we mean the sequence of classes $\{\mathsf{CB}_0(\mathbf{\Sigma}^1_n)\}_{n<\omega}$. By *levels* of this hierarchy we mean the classes $\mathsf{CB}_0(\mathbf{\Sigma}^1_n)$ as well as the classes $\mathsf{CB}_0(\mathbf{\Pi}^1_n)$ and $\mathsf{CB}_0(\mathbf{\Delta}^1_n)$.

Note that the Borel hierarchy of spaces includes some natural classes of spaces identified earlier. E.g., by Proposition 2.12 $\mathsf{CB}_0(\mathbf{\Pi}^0_2)$ coincides with the class of quasi-Polish spaces and, by Corollary 33 in [Br13], $\mathsf{CB}_0(\mathbf{\Pi}^0_3)$ coincides with the class of CB_0-spaces that admit a compatible bicomplete quasi-metric in the sense of [JK98].

Obviously, for any families of pointclasses Γ, Δ we have: if $\Gamma \subseteq \Delta$ (i.e. $\Gamma(X) \subseteq \Delta(X)$ for each space X) then $\mathsf{CB}_0(\Gamma) \subseteq \mathsf{CB}_0(\Delta)$. Therefore, we have the natural inclusions for levels of the introduced hierarchies, in particular $\mathsf{CB}_0(\mathbf{\Delta}^0_\alpha) \subseteq \mathsf{CB}_0(\mathbf{\Sigma}^0_\alpha) \cap \mathsf{CB}_0(\mathbf{\Pi}^0_\alpha)$ and $\mathsf{CB}_0(\mathbf{\Sigma}^0_\alpha) \cup \mathsf{CB}_0(\mathbf{\Pi}^0_\alpha) \subseteq \mathsf{CB}_0(\mathbf{\Delta}^0_\beta)$ for all $\alpha < \beta < \omega_1$. Below we establish some basic properties of the Borel and the Luzin hierarchy of spaces, but first we show that most of levels of the introduced hierarchies of spaces are closed under retracts (for $\mathbf{\Pi}^0_2$ the result is already known from [Br13]).

Proposition 4.2 *Let $\Gamma \in \{\mathbf{\Pi}^0_2, \mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \mid 3 \leq \alpha < \omega_1, 1 \leq n < \omega\}$. Then any retract of a Γ-space is a Γ-space.*

Proof. Let Y be a retract of a Γ-space X via a section-retraction pair (s, r) of continuous functions; we have to show that Y is a Γ-space. We may assume without loss of generality that $X \in \Gamma(P\omega)$. Since $s(Y) = \{x \in X \mid sr(x) = x\}$, $s(Y)$ is the preimage of EQ_X under the continuous function $x \mapsto (sr(x), x)$. Since $EQ_X \in \mathbf{\Pi}^0_2(X \times X)$ by Proposition 2.9, $s(Y) \in \mathbf{\Pi}^0_2(X)$ and thus $s(Y) \in \Gamma(X)$ by Proposition 2.6. Proposition 2.8 yields $s(Y) \in \Gamma(P\omega)$. Since Y is homeomorphic to $s(Y)$, Y is a Γ-space. $\square$

Now we establish an interesting property of the introduced hierarchies of spaces which implies the non-collapse property.

Proposition 4.3 *For any countable ordinal $\alpha \geq 2$, $\mathsf{CB}_0(\mathbf{\Sigma}^0_\alpha) \cap \mathsf{CB}_0(\mathbf{\Pi}^0_\alpha) = \mathsf{CB}_0(\mathbf{\Delta}^0_\alpha)$. For any positive integer n, $\mathsf{CB}_0(\mathbf{\Sigma}^1_n) \cap \mathsf{CB}_0(\mathbf{\Pi}^1_n) = \mathsf{CB}_0(\mathbf{\Delta}^1_n)$.*

This proposition is based on the following lemma.

Lemma 4.4 *Let $\Gamma \in \{\mathbf{\Pi}^0_2, \mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \mid 3 \leq \alpha < \omega_1, 1 \leq n < \omega\}$ and let X be a subspace of $P\omega$. Then $X \in \mathsf{CB}_0(\Gamma)$ iff $X \in \Gamma(P\omega)$.*

Proof. If the underlying set of X is in $\Gamma(P\omega)$, then X is a Γ-space by Definition 3.4. Conversely, let X be a Γ-space. Then there is some $B \in \Gamma(P\omega)$ such that B (endowed with the subspace topology inherited from $P\omega$) is homeomorphic to X. Let $h\colon X \to B$ be a homeomorphism. We apply Theorem 3.6 to extend h to a homeomorphism $h^*\colon X^* \to B^*$, where X^* and B^* are sets in $\Gamma(P\omega)$ with $X \subseteq X^*$ and $B \subseteq B^*$. By Proposition 2.8 we have $B \in \Gamma(B^*)$. Hence $X \in \Gamma(X^*)$, because $X = (h^*)^{-1}(B)$. Proposition 2.8 yields $X \in \Gamma(P\omega)$. $\square$

Now we are ready to give the proof of Proposition 4.3.

Proof. The inclusions from right to left are obvious. It remains to check $\mathsf{CB}_0(\Gamma) \cap \mathsf{CB}_0(\Gamma_c) \subseteq \mathsf{CB}_0(\Gamma \cap \Gamma_c)$ for each $\Gamma \in \{\mathbf{\Pi}^0_\alpha, \mathbf{\Pi}^1_n \mid 2 \leq \alpha < \omega_1, 1 \leq n < \omega\}$. Let $Z \in \mathsf{CB}_0(\Gamma) \cap \mathsf{CB}_0(\Gamma_c)$.

Then Z is homeomorphic to some subspaces A, B of $P\omega$ with $A \in \Gamma(P\omega)$ and $B \in \Gamma_c(P\omega)$. Hence B is a Γ-space by being homeomorphic to A. Lemma 4.4 yields $B \in \Gamma(P\omega)$. Therefore $B \in (\Gamma \cap \Gamma_c)(P\omega)$ and hence $Z \in \mathsf{CB}_0(\Gamma \cap \Gamma_c)$. $\square$

Corollary 4.5 *The Borel hierarchy and the Luzin hierarchy of CB_0-spaces do not collapse. More precisely $\mathsf{CB}_0(\mathbf{\Sigma}^0_\alpha) \not\subseteq \mathsf{CB}_0(\mathbf{\Pi}^0_\alpha)$ for each countable ordinal $\alpha \geq 2$, and $\mathsf{CB}_0(\mathbf{\Sigma}^1_n) \not\subseteq \mathsf{CB}_0(\mathbf{\Pi}^1_n)$ for each positive integer n.*

Proof. Proofs for both hierarchies are similar, so consider only the Borel hierarchy. According to [Se05a], there is a set A in $\mathbf{\Sigma}^0_\alpha(P\omega) \setminus \mathbf{\Pi}^0_\alpha(P\omega)$. The space A (with the topology induced from $P\omega$) is obviously a $\mathbf{\Sigma}^0_\alpha$-space. If it were a $\mathbf{\Pi}^0_\alpha$-space, then by Lemma 4.4 we would have $A \in \mathbf{\Pi}^0_\alpha(P\omega)$, a contradiction. $\square$

5 Hierarchies of QCB_0-Spaces

As shown in [Sch03], any QCB_0-space has an admissible representation. Here we introduce and study natural hierarchies of QCB_0-spaces induced by this fact. For any representation δ of a space X, let $EQ(\delta) := \{(p,q) \in \mathcal{N}^2 \mid p,q \in dom(\delta) \wedge \delta(p) = \delta(q)\}$.

Definition 5.1 (1) Let Γ be a family of pointclasses. A topological space X is called Γ-*representable* if X has an admissible representation δ with $EQ(\delta) \in \Gamma(\mathcal{N} \times \mathcal{N})$. The class of all Γ-representable spaces is denoted $\mathsf{QCB}_0(\Gamma)$.

(2) By the *Borel hierarchy* of QCB_0-spaces we mean the sequence $\{\mathsf{QCB}_0(\mathbf{\Sigma}^0_\alpha)\}_{\alpha<\omega_1}$. By *levels* of this hierarchy we mean the classes $\mathsf{QCB}_0(\mathbf{\Sigma}^0_\alpha)$ as well as the classes $\mathsf{QCB}_0(\mathbf{\Pi}^0_\alpha)$ and $\mathsf{QCB}_0(\mathbf{\Delta}^0_\alpha)$.

(3) By the *Luzin hierarchy* of QCB_0-spaces we mean the sequence $\{\mathsf{QCB}_0(\mathbf{\Sigma}^1_n)\}_{n<\omega}$. By *levels* of this hierarchy we mean the classes $\mathsf{QCB}_0(\mathbf{\Sigma}^1_n)$ as well as the classes $\mathsf{QCB}_0(\mathbf{\Pi}^1_n)$ and $\mathsf{QCB}_0(\mathbf{\Delta}^1_n)$.

The next assertion establishes an equivalent simpler definition of most levels for the case of CB_0-spaces.

Proposition 5.2 (1) *Let $\Gamma \in \{\mathbf{\Pi}^0_2, \mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \mid 3 \leq \alpha < \omega_1, 1 \leq n < \omega\}$ and let X be a CB_0-space. Then X is Γ-representable iff X has an admissible representation δ with $dom(\delta) \in \Gamma(\mathcal{N})$.*

(2) *Let $\Gamma \in \{\mathbf{\Pi}^0_1, \mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \mid 2 \leq \alpha < \omega_1, 1 \leq n < \omega\}$ and let X be a Hausdorff space. Then X is Γ-representable iff X has an admissible representation δ with $dom(\delta) \in \Gamma(\mathcal{N})$.*

Proof. The only-if-part of the first statement holds for any topological space X and any family of pointclasses Γ. Indeed, let δ be an admissible representation of X with $EQ(\delta) \in \Gamma(\mathcal{N} \times \mathcal{N})$. Since $dom(\delta)$ is the preimage of $EQ(\delta)$ under the continuous function $x \mapsto (x,x)$, $dom(\delta) \in \Gamma(\mathcal{N})$.

Conversely, let δ be an admissible representation of X with $dom(\delta) \in \Gamma(\mathcal{N})$. By Proposition 2.9, $EQ_X \in \mathbf{\Pi}^0_2(X \times X)$, so $EQ(\delta)$ is a $\mathbf{\Pi}^0_2$-set in $dom(\delta) \times dom(\delta)$ by the continuity of δ. The set $dom(\delta) \times dom(\delta)$ is a Γ-subset of $\mathcal{N} \times \mathcal{N}$ by being the intersection of the sets $\{(x,y) \in \mathcal{N} \times \mathcal{N} \mid$

$x \in dom(\delta)\} \in \Gamma(\mathcal{N} \times \mathcal{N})$ and $\{(x, y) \in \mathcal{N} \times \mathcal{N} \mid y \in dom(\delta)\} \in \Gamma(\mathcal{N} \times \mathcal{N})$. By Propositions 2.6 and 2.8 we obtain $EQ(\delta) \in \Gamma(\mathcal{N} \times \mathcal{N})$.

The second statement follows similarly by taking into account that EQ_X is closed in $X \times X$ by Proposition 2.9, if X is a Hausdorff space. □

The main advantage of the hierarchies of this section over the hierarchies from the previous section is that they include many natural non-countably based spaces, in particular the spaces of Kleene-Kreisel continuous functionals, as we will see later.

Obviously, for any families of pointclasses Γ, Δ we have: $\Gamma \subseteq \Delta$ implies $\mathsf{QCB}_0(\Gamma) \subseteq \mathsf{QCB}_0(\Delta)$, hence we have the natural inclusions for levels of the introduced hierarchies, in particular $\mathsf{QCB}_0(\mathbf{\Delta}^0_\alpha) \subseteq \mathsf{QCB}_0(\mathbf{\Sigma}^0_\alpha) \cap \mathsf{QCB}_0(\mathbf{\Pi}^0_\alpha)$ and $\mathsf{QCB}_0(\mathbf{\Sigma}^0_\alpha) \cup \mathsf{QCB}_0(\mathbf{\Pi}^0_\alpha) \subseteq \mathsf{QCB}_0(\mathbf{\Delta}^0_\beta)$ for all $\alpha < \beta < \omega_1$. We will show that these hierarchies do not collapse, but first we establish the closure of the levels under retracts.

Proposition 5.3 *Let $\Gamma \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \mid 1 \leq \alpha < \omega_1, 1 \leq n < \omega\}$. Then any retract of a Γ-representable space is a Γ-representable space.*

Proof. Let Y be a retract of a Γ-representable space X via a section-retraction pair (s, r) of continuous functions; we have to show that Y is a Γ-representable space. Let δ be an admissible representation for X with $EQ(\delta) \in \Gamma(\mathcal{N} \times \mathcal{N})$. Then $r\delta$ is clearly an admissible representation for Y, so it suffices to show that $EQ(r\delta) \in \Gamma(\mathcal{N} \times \mathcal{N})$. Since s is an injection, $EQ(r\delta)$ is the preimage of $EQ(\delta)$ under the continuous function $(x, y) \mapsto (sr(x), sr(y))$. Therefore, $EQ(r\delta) \in \Gamma(\mathcal{N} \times \mathcal{N})$. □

The following observation is well-known.

Lemma 5.4 *Let $r\colon D \to Y$ be a continuous function from a subspace D of $\mathcal{N}$ to a zero-dimensional CB_0-space Y. Then r (viewed as a partial function from $\mathcal{N}$ to Y) is an admissible representation for Y iff there is a continuous function $s\colon Y \to D$ satisfying $rs = id_Y$.*

Proof. Let Y be a retract of D via a section-retraction pair (s, r) of continuous functions. Then $r :\subseteq \mathcal{N} \to Y$ is admissible, because any continuous function $\nu : Z \to Y$ defined on a zero-dimensional CB_0-space Z is reducible to r via $s\nu$.

Conversely, assume that $r \subseteq: \mathcal{N} \to Y$ is an admissible representation of Y. Then the identity function id_Y is reducible to r via some continuous function $s : Y \to D$. Then $id_Y = rs$, hence Y is a retract of D via the continuous section s and the continuous retraction r. □

Finally, we establish the non-collapse property of the introduced hierarchies of spaces.

Theorem 5.5 *The Borel hierarchy and the Luzin hierarchy of QCB_0-spaces do not collapse. More precisely, $\mathsf{QCB}_0(\mathbf{\Sigma}^0_\alpha) \not\subseteq \mathsf{QCB}_0(\mathbf{\Pi}^0_\alpha)$ for each countable ordinal $\alpha \geq 2$, and $\mathsf{QCB}_0(\mathbf{\Sigma}^1_n) \not\subseteq \mathsf{QCB}_0(\mathbf{\Pi}^1_n)$ for each positive integer n.*

Proof. Proofs for both hierarchies are similar, so consider only the Borel hierarchy. As is well-known [Ke95], there is a set Y in $\mathbf{\Sigma}^0_\alpha(\mathcal{N}) \setminus \mathbf{\Pi}^0_\alpha(\mathcal{N})$. Consider Y as a subspace of $\mathcal{N}$. Since id_Y is an admissible representation for Y, $Y \in \mathsf{QCB}_0(\mathbf{\Sigma}^0_\alpha)$. Suppose for a contradiction that $Y \in \mathsf{QCB}_0(\mathbf{\Pi}^0_\alpha)$, so there is an admissible representation $\delta : D \to Y$ for Y with $D \in \mathbf{\Pi}^0_\alpha(\mathcal{N})$. By Lemma 5.4 there is a continuous function $s : Y \to D$ satisfying $\delta s = id_Y$. Since D is Hausdorff, $S = s(Y)$ is a closed subset of D, hence $S \in \mathbf{\Pi}^0_\alpha(D)$ by Proposition 2.7. By Proposition 2.8 we obtain $S \in \mathbf{\Pi}^0_\alpha(\mathcal{N})$.

Since $\mathcal{N}$ is a $\mathbf{\Pi}^0_2$-space, hence a Γ-space, we can use Theorem 3.6 (applied to $X' = Y' = \mathcal{N}$) to extend the homeomorphism $s : Y \to S$ to a homeomorphism s^* between larger sets $Y^*, S^* \in \mathbf{\Pi}^0_2(\mathcal{N})$. By Proposition 2.6 we have $Y^*, S^* \in \mathbf{\Pi}^0_\alpha(\mathcal{N})$. Since $S \in \mathbf{\Pi}^0_\alpha(S^*)$ by Proposition 2.8, $Y = s^{*-1}(S) \in \mathbf{\Pi}^0_\alpha(Y^*)$. Therefore $Y \in \mathbf{\Pi}^0_\alpha(\mathcal{N})$ by Proposition 2.8, a contradiction. □

Remark 5.6 The spaces Y witnessing the non-collapse property above are rather artificial. In Theorem 7.7 we will find very natural spaces witnessing the non-collapse of the Luzin hierarchy of QCB_0-spaces.

6 Relating the Hierarchies

In this section we establish close relationships of the hierarchies of CB_0-spaces with the corresponding hierarchies of QCB_0-spaces. The next result implies that any level of a hierarchy of QCB_0-spaces extends the corresponding level in the corresponding hierarchy of CB_0-spaces.

Proposition 6.1 *Let* $\Gamma \in \{\mathbf{\Pi}^0_2, \mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \mid 3 \leq \alpha < \omega_1, 1 \leq n < \omega\}$. *Then* $\mathsf{CB}_0(\Gamma) \subseteq \mathsf{QCB}_0(\Gamma)$.

Proof. It suffices to show that any space $X \in \Gamma(P\omega)$ is a Γ-representable space. Let $\rho : \mathcal{N} \to P\omega$ be the total admissible representation of $P\omega$ defined by $\rho(x) = \{n \mid \exists i(x(i) = n + 1)\}$ (see e.g. [Wei00, Br13] for details). Then the restriction of ρ to $\rho^{-1}(X) \in \Gamma(\mathcal{N})$ is an admissible representation of X. Since X is a CB_0-space, X is a Γ-representable space by Proposition 5.2. □

In particular, we have $\mathsf{CB}_0(\Gamma) \subseteq \mathsf{QCB}_0(\Gamma) \cap \mathsf{CB}_0$ for each level Γ of the Borel hierarchy or the Luzin hierarchy. The main question of this section is: for which levels Γ we have the equality $\mathsf{CB}_0(\Gamma) = \mathsf{QCB}_0(\Gamma) \cap \mathsf{CB}_0$? Proposition 2.12 implies that the equality holds for $\Gamma = \mathbf{\Pi}^0_2$.

The next result implies that the equality holds for all zero-dimensional CB_0-spaces.

Proposition 6.2 *Let* $\Gamma \in \{\mathbf{\Pi}^0_2, \mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \mid 3 \leq \alpha < \omega_1, 1 \leq n < \omega\}$ *and* X *be a zero-dimensional space in* $\mathsf{QCB}_0(\Gamma) \cap \mathsf{CB}_0$. *Then* $X \in \mathsf{CB}_0(\Gamma)$.

Proof. Let $r : D \to X$ be an admissible representation of X with $D \in \Gamma(\mathcal{N})$. By Lemma 5.4 there is a continuous function $s : X \to D$ satisfying $rs = id_X$. Then $s(X) = \{z \in D \mid sr(z) = z\} \in \mathbf{\Pi}^0_2(D)$ by Proposition 2.9, hence $s(X) \in \Gamma(\mathcal{N})$ by Proposition 2.8. Since $\mathcal{N}$ is a $\mathbf{\Pi}^0_2$-space, there is a homeomorphism f of $\mathcal{N}$ onto a subspace Y of $P\omega$ with $Y \in \mathbf{\Pi}^0_2(P\omega)$. As fs is a homeomorphism of X onto $fs(X)$, we have $fs(X) \in \Gamma(Y)$. Propositions 2.6 and 2.8 yield $fs(X) \in \Gamma(P\omega)$. Therefore $X \in \mathsf{CB}_0(\Gamma)$. □

By the next theorem the equality $\mathsf{CB}_0(\Gamma) = \mathsf{QCB}_0(\Gamma) \cap \mathsf{CB}_0$ holds for almost all levels. For finite levels, this was pointed out to us by Matthew de Brecht. We thank him for giving the permission to use his proof of Theorem 6.3(2). Note that his proof may be used to obtain also a proof of Theorem 6.3(1) which is slightly different from ours.

Theorem 6.3 (1) *For any level* $\Gamma \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \mid \omega \leq \alpha < \omega_1, 1 \leq n < \omega\}$, *we have*

$$\mathsf{QCB}_0(\Gamma) \cap \mathsf{CB}_0 = \mathsf{CB}_0(\Gamma)\,.$$

(2) *For all natural numbers $m \geq 2$ and $n \geq 3$, we have*

$$\mathsf{QCB_0}(\mathbf{\Pi}^0_m) \cap \mathsf{CB_0} = \mathsf{CB_0}(\mathbf{\Pi}^0_m) \quad \textit{and} \quad \mathsf{QCB_0}(\mathbf{\Sigma}^0_n) \cap \mathsf{CB_0} = \mathsf{CB_0}(\mathbf{\Sigma}^0_n)\,.$$

Proof.

(1) Let $\Gamma \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \mid \omega \leq \alpha < \omega_1, 1 \leq n < \omega\}$. We have already seen $\mathsf{CB_0}(\Gamma) \subseteq \mathsf{QCB_0}(\Gamma) \cap \mathsf{CB_0}$.

Let $X \in \mathsf{QCB_0}(\Gamma) \cap \mathsf{CB_0}$. Then there is an admissible representation $\delta : D \to X$ of X with $D \in \Gamma(\mathcal{N})$. By Proposition 2.1 we may assume w.l.o.g. X to be a subspace of $P\omega$. We have to show $X \in \mathsf{CB_0}(\Gamma)$.

Let $\chi : P\omega \to \mathcal{C}$ be the bijection between $P\omega$ and the Cantor space $\mathcal{C} = 2^\omega$ that sends any $A \subseteq \omega$ to its characteristic function χ_A. Obviously, χ is $\mathbf{\Sigma}^0_2$-measurable while its inverse χ^{-1} is continuous (i.e. $\mathbf{\Sigma}^0_1$-measurable). Let $\sigma = \chi|_X$ be the restriction of χ to $X \subseteq P\omega$. By Proposition 2.8, σ is a $\mathbf{\Sigma}^0_2$-measurable bijection between X and $\sigma(X) \subseteq \mathcal{C}$ such that σ^{-1} is $\mathbf{\Sigma}^0_1$-measurable.

Since σ^{-1} is continuous as a function from the zero-dimensional $\mathsf{CB_0}$-space $\sigma(X)$ to X and δ is admissible, there is a continuous function $g\colon \sigma(X) \to \mathcal{N}$ with $\sigma^{-1} = \delta g$. Then $id_X = \sigma^{-1}\sigma = \delta g \sigma$, hence the function $h = g\sigma$ satisfies $h(X) = \{y \in D \mid h\delta(y) = y\}$. Thus, h is a $\mathbf{\Sigma}^0_2$-measurable bijection between X and $H = h(X)$, whereas its inverse $h^{-1} = \delta|_{h(X)}$ is $\mathbf{\Sigma}^0_1$-measurable (i.e. continuous).

By Proposition 6.2 we may w.l.o.g. assume that D is a subspace of $P\omega$ and $D \in \Gamma(P\omega)$. Since H is the preimage of $EQ_D \in \mathbf{\Pi}^0_2(D \times D)$ under the $\mathbf{\Sigma}^0_2$-measurable function $y \mapsto (h\delta(y), y)$, we have $H \in \mathbf{\Pi}^0_3(D)$ by Proposition 2.10 and hence $H \in \Gamma(D)$ by Proposition 2.6. Proposition 2.8 yields $H \in \Gamma(P\omega)$.

By Theorem 3.2 and Corollary 3.3, there exist $H^* \in \mathbf{\Pi}^0_3(P\omega)$, $X^* \in \mathbf{\Pi}^0_4(P\omega)$, a continuous bijection $\delta^* : H^* \to X^*$ and a $\mathbf{\Sigma}^0_2$-measurable bijection $h^* : X^* \to H^*$ such that δ^* is the inverse of h^*, δ^* is an extension of h^{-1} and h^* is an extension of h. Since $H \in \Gamma(H^*)$ by Proposition 2.8 and $X = h^{*-1}(H)$, Proposition 2.10 yields us $X \in \Gamma(X^*)$. By Propositions 2.6 and 2.8, we obtain $X \in \Gamma(P\omega)$, hence $X \in \mathsf{CB_0}(\Gamma)$. This completes the proof.

(2) Let δ be an admissible representation of a subspace X of $P\omega$ such that $dom(\delta) \in \Gamma(\mathcal{N})$, where $\Gamma \in \{\mathbf{\Pi}^0_m, \mathbf{\Sigma}^0_n \mid m \geq 2,\, n \geq 3\}$. Let ρ be the total admissible representation of $P\omega$ defined in the proof of Proposition 6.1. By Proposition 2.17 it is enough to show that $\rho^{-1}(X)$ is in $\Gamma(\mathcal{N})$. Since the corestriction of ρ to X is continuous, there is a continuous function $h\colon \rho^{-1}(X) \to \mathcal{N}$ satisfying $\rho(p) = \delta h(p)$ for all $p \in \rho^{-1}(X)$. As $P\omega$ is an injective space and $\mathcal{N}$ is a Polish space, δ and h can be extended to continuous functions $\delta^*\colon \mathcal{N} \to P\omega$ and $h^*\colon G \to \mathcal{N}$, where G is a $\mathbf{\Pi}^0_2$-subset of $\mathcal{N}$, see Propositions 2.3 and 2.4. By Proposition 2.9, $A := \{p \in G \mid \rho(p) = \delta^* h^*(p)\}$ is a $\mathbf{\Pi}^0_2$-subset of G and thus of $\mathcal{N}$. It is easy to verify that $\rho^{-1}(X)$ is the intersection of A with $(h^*)^{-1}(dom(\delta))$. By Proposition 2.7 and 2.8 this implies that $\rho^{-1}(X)$ is Γ-subset of $\mathcal{N}$. Proposition 2.17 yields that X is a Γ-subset of $P\omega$, i.e., $X \in \mathsf{CB_0}(\Gamma)$.

□

7 The Luzin Hierarchy and Continuous Functionals

In this section we establish close relations of the Luzin hierarchy to the continuous functionals of finite types.

We start by relating the exponentiation operation on admissibly represented spaces (see Proposition 2.16) to the Luzin hierarchy.

Theorem 7.1 *Let $k \in \omega$, $X \in \mathsf{QCB}_0(\mathbf{\Pi}^1_k)$ and $Y \in \mathsf{QCB}_0(\mathbf{\Sigma}^1_k)$. Then $Y^X \in \mathsf{QCB}_0(\mathbf{\Pi}^1_{k+1})$.*

Proof. Let δ and γ be admissible representations of X and Y respectively such that $EQ(\delta) \in \mathbf{\Pi}^1_k(\mathcal{N}^2)$ and $EQ(\gamma) \in \mathbf{\Sigma}^1_k(\mathcal{N}^2)$. By Proposition 2.16 it suffices to show that $EQ([\delta \to \gamma]) \in \mathbf{\Pi}^1_{k+1}(\mathcal{N}^2)$.

From the definition of $[\delta \to \gamma]$ we obtain

$$(p,q) \in EQ([\delta \to \gamma]) \Longleftrightarrow \left\{ \begin{array}{l} \text{for all } (x,y) \in EQ(\delta) \\ \big(u_p(x), u_p(y)\big), \big(u_q(x), u_q(y)\big), \big(u_p(x), u_q(x)\big) \in EQ(\gamma). \end{array} \right.$$

So we have

$$EQ([\delta \to \gamma]) = \big\{(p,q) \in \mathcal{N}^2 \,\big|\, \forall (x,y) \in \mathcal{N}^2.(p,q,x,y) \in M\big\}\,,$$

where

$$\begin{aligned} M := \Big\{ &(p,q,x,y) \in \mathcal{N}^4 \,\Big|\, (x,y) \notin EQ(\delta) \text{ or there is some } (a,b,c,d) \in \mathcal{N}^4 \text{ such that} \\ &(p,x,a),(p,y,b),(q,x,c),(q,y,d) \in \mathit{graph}(u) \text{ and } (a,b),(c,d),(a,c) \in EQ(\gamma)\Big\}. \end{aligned}$$

Since the universal function $u :\subseteq \mathcal{N}^2 \to \mathcal{N}$ is continuous and has a $\mathbf{\Pi}^0_2$-set as domain, $\mathit{graph}(u)$ is a $\mathbf{\Pi}^0_2(\mathcal{N}^3)$-set. This implies that M is a $\mathbf{\Sigma}^1_n(\mathcal{N}^4)$-set by Propositions 2.6 and 2.7. We conclude that $EQ([\delta \to \gamma])$ is a $\mathbf{\Pi}^1_{n+1}(\mathcal{N}^2)$-set. $\square$

The next result (which is known from [Kr59] for the particular case $Y = \mathbb{N}\langle k\rangle)$ is the key technical fact of this section.

Theorem 7.2 *Let Y be a QCB_0-space and let $f \colon Y \to \mathcal{N}$ be a continuous function with $\mathit{rng}(f) \neq \mathcal{N}$. Then there exists a continuous function $g \colon \mathcal{N} \times \omega^Y \to \mathcal{N}$ with $\mathit{rng}(g) = \mathcal{N} \setminus \mathit{rng}(f)$.*

We remark that by $\mathcal{N} \times \omega^Y$ we mean the product formed in the category of QCB_0-spaces (see Subsection 2.6), the topology of which is finer than (or equal to) the Tychonoff topology. However, the function g constructed in the proof is even continuous w.r.t. the Tychonoff topology.

Proof. We abbreviate $M = \mathcal{N} \setminus \mathit{rng}(f)$. The set $NEQ = \big\{(x,y) \in \mathcal{N} \times Y \,\big|\, f(y) \neq x\big\}$ is the countable union of the clopen sets

$$\{x \in \mathcal{N} \,|\, x(j) = a\} \times f^{-1}\{z \in \mathcal{N} \,|\, z(j) = b\}\,,$$

where (j,a,b) varies over all triples of natural numbers with $a \neq b$. Let $\{D_i\}_i$ denote a sequence consisting of these clopen sets. So we have $NEQ = \bigcup_{i \in \omega} D_i$ and $x \in M \iff \forall y \in Y. \exists i \in \omega.(x,y) \in D_i$.

Motivated by the above equivalence, we call a continuous function $H \colon Y \to \omega$ *a witness for* an element $x \in M$, if $(x,y) \in D_{H(y)}$ holds for all $y \in Y$. The idea of the proof is to construct the required function $g\colon \mathcal{N} \times \omega^Y \to \mathcal{N}$ in such a way that g maps (x, H) to x, if H is a witness for x. Otherwise g assigns to (x, H) some element of a countable dense subset of M.

To construct an appropriate dense subset of M, we choose for every finite sequence w in the set $W := \{x^{<k} \mid x \in M,\, k \in \omega\}$ some $\alpha(w) \in M$ such that $\alpha(w)$ has w as an initial segment. Clearly, $\{\alpha(w) \mid w \in W\}$ is dense in M.

By being a QCB_0-space, Y has a countable dense subset $\{\beta_j \mid j \in \omega\}$ (see Proposition 3.3.1 in [Sch03]). We construct a sequence $\{C_k\}_k$ of subsets of the QCB_0-space $\mathcal{N}\times\omega^Y$ by $C_0 := \mathcal{N}\times\omega^Y$ and

$$C_k := \big\{(x, H) \in \mathcal{N} \times \omega^Y \,\big|\, x^{<k} \in W \text{ and } (x, \beta_j) \in D_{H(\beta_j)} \text{ for all } j \le k\big\}$$

for all $k > 0$. The set C_k is clopen in the Tychonoff topology on $\mathcal{N} \times \omega^Y$ and thus in the QCB_0-topology, because C_k is the union of the clopen sets

$$\big((w \cdot \mathcal{N}) \times \omega^Y\big) \cap \bigcap_{j=0}^{k} \big(\{x \in \mathcal{N} \mid (x, \beta_j) \in D_{a_j}\} \times \{H \in \omega^Y \mid \beta_j \in H^{-1}\{a_j\}\}\big),$$

where w ranges over the finite sequence of length k in W and $a_0, \ldots, a_k$ vary over the natural numbers. Note that $\{H \in \omega^Y \mid \beta_j \in H^{-1}\{a_j\}\}$ is clopen even in the compact-open topology on ω^Y and therefore in the QCB_0-topology on ω^Y, which is finer than the former.

Next we define a function $\ell\colon \mathcal{N} \times \omega^Y \to \omega \cup \{\infty\}$ by

$$\ell(x, H) := \begin{cases} \infty & \text{if } (x, H) \in \bigcap_{k\in\omega} C_k \\ \max\{k \in \omega \mid (x, H) \in C_k\} & \text{otherwise} \end{cases}$$

We use $\{C_k\}_k$ and ℓ to define our function $g : \mathcal{N} \times \omega^Y \to \mathcal{N}$ by

$$g(x, H) := \begin{cases} x & \text{if } (x, H) \in \bigcap_{k\in\omega} C_k \\ \alpha(x^{<\ell(x,H)}) & \text{otherwise} \end{cases}$$

Note that $\infty \neq i \le \ell(x, H)$ implies $x^{<i} \in W$ and $g(x, H)^{<i} = x^{<i}$.

First we show the continuity of g. Let O be an open set of the Baire space and let $(x, H) \in g^{-1}(O)$. Then there is some $m \in \omega$ such that $g(x, H)^{<m} \cdot \mathcal{N} \subseteq O$.
Case 1: Assume $\ell(x, H) = \infty$. Then $(x, H) \in \bigcap_{k\in\omega} C_k$. We define a set U by

$$U := (x^{<m} \cdot \mathcal{N} \times \omega^Y) \cap \bigcap_{i=0}^{m} C_i$$

Then U is open and all $(x', H') \in U$ satisfy $\ell(x', H') \ge m$ and $g(x', H')^{<m} = g(x, H)^{<m} = x^{<m}$. Hence $(x, H) \in U \subseteq g^{-1}(O)$.
Case 2: Assume $\ell(x, H) < \infty$. Let $k := \ell(x, H)$ and define a set U by

$$U := (x^{<k} \cdot \mathcal{N} \times \omega^Y) \cap \bigcap_{i=0}^{k} C_i \setminus C_{k+1}$$

Then U is open, $(x, H) \in U$ and all $(x', H') \in U$ satisfy even $g(x', H') = g(x, H) = \alpha(x^{<k}) \in U$.

We conclude that in both cases g is continuous in the point (x, H). Therefore g is continuous, even with respect to the Tychonoff product on the set $\mathcal{N} \times \omega^Y$, which is coarser than the QCB_0-topology.

Now we show $M \subseteq rng(g)$. Let $x \in M$. We define a function $H_x \colon Y \to \omega$ by

$$H_x(y) := \min\{k \in \omega \mid (x,y) \in D_k\}\,.$$

Note that H_x is total, because we have $f(y) \neq x$ and therefore $(x,y) \in \bigcup_{k\in\omega} D_k$ for every $y \in Y$. Furthermore, H is continuous in every point $y \in Y$, because the set

$$U := \Big\{b \in Y \,\Big|\, (x,b) \in D_{H_x(y)} \setminus \bigcup_{i=0}^{H_x(y)-1} D_i\Big\}$$

satisfies $y \in U \subseteq H_x^{-1}\{H_x(y)\}$ and is open by being the preimage of the open set $D_{H_x(y)} \setminus \bigcup_{i=0}^{H_x(y)-1} D_i$ under the continuous map $b \mapsto (x,b)$. Therefore H_x is an element of the function space ω^Y. Since $(x, H_x) \in \bigcap_{k\in\omega} C_k$, we have $g(x, H_x) = x$.

It remains to show $rng(g) \subseteq M$. So let $x \in rng(g)$. The only interesting case is $x \notin \{\alpha(w) \mid w \in W\}$. Then there is some continuous function $H \colon Y \to \omega$ such that $g(x,H) = x$ and thus $(x,H) \in \bigcap_{k\in\omega} C_k$. Suppose for a contradiction that there is some $y \in Y$ such that $(x,y) \notin D_{H(y)}$. Then the set

$$V := \{b \in Y \mid (x,b) \notin D_{H(y)}\} \cap H^{-1}\{H(y)\}$$

is non-empty and open in Y, because $D_{H(y)}$ is closed and H is continuous. So there exists some $k \in \omega$ with $\beta_k \in V$. The element β_k satisfies $(x, \beta_k) \notin D_{H(y)}$ and $H(\beta_k) = H(y)$. But since $(x,H) \in C_k$, we have $(x,\beta_k) \in D_{H(\beta_k)}$, a contradiction.
We conclude $(x,y) \in D_{H(y)}$ and thus $f(y) \neq x$ for every $y \in Y$. Therefore $x \notin rng(f)$ and $x \in M$.

So g satisfies the required properties. □

Using the cartesian closedness of QCB_0, we define a sequence of spaces $\{\mathbb{N}\langle k\rangle\}_{k<\omega}$ by induction on k as follows: $\mathbb{N}\langle 0\rangle := \omega$ and $\mathbb{N}\langle k+1\rangle := \omega^{\mathbb{N}\langle k\rangle}$, where ω denotes the space of natural numbers endowed with the discrete topology. Obviously $\mathbb{N}\langle 1\rangle$ is equal to the Baire space $\mathcal{N}$. The space $\mathbb{N}\langle k\rangle$ is referred to as *the sequential space of (Kleene-Kreisel) continuous functionals of type k*. For any $k \geq 2$, the sequential topology on $\mathbb{N}\langle k\rangle$ is strictly finer than the corresponding compact-open topology [Hy79]. Furthermore it is neither zero-dimensional nor regular [Sch09]. Any of these spaces has a natural admissible representation. From Theorem 7.1 we obtain:

Corollary 7.3 *For any integer $k \geq 1$, $\mathbb{N}\langle k\rangle \in \mathsf{QCB}_0(\mathbf{\Pi}^1_{k-1})$.*

Proof. Obviously, $\mathbb{N}\langle 0\rangle, \mathbb{N}\langle 1\rangle \in \mathsf{QCB}_0(\mathbf{\Pi}^0_0) \subseteq \mathsf{QCB}_0(\mathbf{\Delta}^1_1)$. By Theorem 7.1, $\mathbb{N}\langle 2\rangle \in \mathsf{QCB}_0(\mathbf{\Pi}^1_1)$. Assuming by induction that $\mathbb{N}\langle k\rangle \in \mathsf{QCB}_0(\mathbf{\Pi}^1_{k-1})$ for a given $k \geq 2$, we obtain $\mathbb{N}\langle k+1\rangle \in \mathsf{QCB}_0(\mathbf{\Pi}^1_k)$ by Theorem 7.1. □

We remark that Lemma 2.34 in [No80] implies that $\mathbb{N}\langle k\rangle$ has an admissible representation such that its domain is even in the effective version Π^1_{k-1} of $\mathbf{\Pi}^1_{k-1}(\mathcal{N})$. This follows from the fact that the function which maps "associates" for $\mathbb{N}\langle k\rangle$ to the encoded functionals forms an admissible representation.

Note that the continuous functionals (also known under the somewhat misleading name "countable functionals") were first defined in [Kl959, Kr59] independently by S. Kleene and G. Kreisel. Their definitions look different from each other, as well as from the definition above, although all three definitions are known to be equivalent. Another equivalent definition in terms of domains was given in [Er74]. Additional information may be found in [No80, No99].

The following observation follows from the cartesian closedness of QCB_0.

Lemma 7.4 *For $k \geq 1$, the QCB_0-spaces $\mathbb{N}\langle k-1\rangle$ and $\mathcal{N} \times \mathbb{N}\langle k\rangle$ are continuous retracts of $\mathbb{N}\langle k\rangle$.*

Proof. As QCB_0 is a cartesian closed category, for all QCB_0-spaces X, Y, Z such that X is a continuous retract of Y the function space Z^X is a continuous retract of Z^Y and the QCB_0-product $X \times Z$ is a continuous retract of $Y \times Z$. Clearly, ω is a continuous retract of $\mathcal{N} = \mathbb{N}\langle 1\rangle$. So the first statement follows by induction on k. A further induction establishes $\mathcal{N}$ as a continuous retract of $\mathbb{N}\langle k\rangle$ for $k \geq 1$. Hence $\mathcal{N} \times \mathbb{N}\langle k\rangle$ is a continuous retract of $\mathbb{N}\langle k\rangle \times \mathbb{N}\langle k\rangle$. The cartesian closedness of QCB_0 allows us to calculate

$$\mathbb{N}\langle k\rangle = \omega^{\mathbb{N}\langle k-1\rangle} \cong (\omega \times \omega)^{\mathbb{N}\langle k-1\rangle} \cong \omega^{\mathbb{N}\langle k-1\rangle} \times \omega^{\mathbb{N}\langle k-1\rangle} = \mathbb{N}\langle k\rangle \times \mathbb{N}\langle k\rangle,$$

hence $\mathbb{N}\langle k\rangle$ and $\mathbb{N}\langle k\rangle \times \mathbb{N}\langle k\rangle$ are isomorphic in QCB_0 and thus homeomorphic. We conclude that $\mathcal{N} \times \mathbb{N}\langle k\rangle$ is a continuous retract of $\mathbb{N}\langle k\rangle$. $\square$

From Theorem 7.2 and Lemma 7.4 we can easily infer the following nice result.

Proposition 7.5 *For any positive integer k and for any non-empty set $M \in \mathbf{\Sigma}^1_k(\mathcal{N})$ there is a continuous function $f : \mathbb{N}\langle k\rangle \to \mathcal{N}$ with $rng(f) = M$.*

Proof. We proceed by induction on $k \geq 1$. For $k = 1$ the claim is well-known (cf. Section 14 in [Ke95]).
Let $M \in \mathbf{\Sigma}^1_{k+1}(\mathcal{N})$. Then there is some set $A \in \mathbf{\Pi}^1_k(\mathcal{N})$ such that

$$M = \big\{x \in \mathcal{N} \,\big|\, \exists p \in \mathcal{N}. \langle p, x\rangle \in A\big\},$$

where $\langle \cdot, \cdot \rangle$ denotes a canonical homeomorphism from $\mathcal{N}^2$ to $\mathcal{N}$. We set $B := \mathcal{N} \setminus A$, hence $B \in \mathbf{\Sigma}^1_k(\mathcal{N})$. Since $M \in \{\emptyset, \mathcal{N}\}$ if $A \in \{\emptyset, \mathcal{N}\}$, we can assume $\emptyset \neq A, B \neq \mathcal{N}$. The induction hypothesis yields us a continuous function $f_B \colon \mathbb{N}\langle k\rangle \to \mathcal{N}$ such that $rng(f_B) = B$. By Theorem 7.2 there exists a continuous function $g \colon \mathcal{N} \times \mathbb{N}\langle k+1\rangle \to \mathcal{N}$ such that $rng(g) = A$. By Lemma 7.4 there is a continuous retraction $r \colon \mathbb{N}\langle k+1\rangle \to \mathcal{N} \times \mathbb{N}\langle k+1\rangle$. Since r is surjective, A is also the range of gr. Using the unique continuous function $\pi^{(2)}_2 \colon \mathcal{N} \to \mathcal{N}$ satisfying $\pi^{(2)}_2\langle p, x\rangle = x$, we define $f \colon \mathbb{N}\langle k+1\rangle \to \mathcal{N}$ by $f(z) := \pi^{(2)}_2(gr(z))$. Then f is continuous and satisfies

$$\begin{aligned} x \in M &\iff \exists p \in \mathcal{N}.\, \langle p, x\rangle \in A \iff \exists \phi \in \mathbb{N}\langle k+1\rangle. \exists p \in \mathcal{N}.\, gr(\phi) = \langle p, x\rangle \\ &\iff \exists \phi \in \mathbb{N}\langle k+1\rangle.\, f(\phi) = x\,. \end{aligned}$$

Hence M is the range of f. $\square$

The last result can be even improved to the following characterization of levels of the Luzin hierarchy in terms of the Kleene-Kreisel continuous functionals.

Theorem 7.6 *Let k be a positive integer and B a non-empty subset of $\mathcal{N}$. Then $B \in \mathbf{\Sigma}^1_k(\mathcal{N})$ iff there is a continuous function $f \colon \mathbb{N}\langle k\rangle \to \mathcal{N}$ with $rng(f) = B$.*

Proof. The only-if-part is given by Proposition 7.5. For the if-part let $f\colon \mathbb{N}\langle k\rangle \to \mathcal{N}$ be a continuous function. According to Corollary 7.3, $\mathbb{N}\langle k\rangle$ has an admissible representation δ such that its domain $dom(\delta)$ is a $\mathbf{\Pi}^1_{k-1}$-set. We define a partial function $g\colon \mathcal{N} \to \mathcal{N}$ by $g(p) := f\delta(p)$. Then g is continuous. So it can be extended to a continuous function $g^*\colon X \to \mathcal{N}$ such that X is a $\mathbf{\Pi}^0_2$-subset of $\mathcal{N}$ (cf. Proposition 2.15 or 3.5). The graph of g^* is a closed set in $X \times \mathcal{N}$ and therefore a $\mathbf{\Pi}^0_2$-set in $\mathcal{N}\times\mathcal{N}$. By Propositions 2.6 and 2.7, the set $A := graph(g^*)\cap(dom(\delta)\times\mathcal{N})$ is a $\mathbf{\Pi}^1_{k-1}$-subset of $\mathcal{N}\times\mathcal{N}$. Moreover, we have

$$M = \{y \in \mathcal{N} \mid \exists x \in \mathcal{N}.\, (x,y) \in A\} = pr_{\mathcal{N}}(A)\,.$$

Therefore $M \in \mathbf{\Sigma}^1_k(\mathcal{N})$. $\square$

A similar characterization for the effective version of $\mathbf{\Sigma}^1_k(\mathcal{N})$ can be found in [No80, Theorem 5.22].

Finally, we relate continuous functionals to the Luzin hierarchy of QCB_0-spaces (for a similar relationship see [No81]). The next result provides the exact estimation of the spaces of continuous functionals of finite types in the Luzin hierarchy of QCB_0-spaces. On the other hand, the result provides "natural" witnesses for the non-collapse property of this hierarchy.

Theorem 7.7 *For any positive integer k, $\mathbb{N}\langle k+1\rangle \in \mathsf{QCB}_0(\mathbf{\Pi}^1_k) \setminus \mathsf{QCB}_0(\mathbf{\Sigma}^1_k)$.*

Proof. We know $\mathbb{N}\langle k+1\rangle \in \mathsf{QCB}_0(\mathbf{\Pi}^1_k)$ from Corollary 7.3. It remains to show that $\mathbb{N}\langle k+1\rangle \notin \mathsf{QCB}_0(\mathbf{\Sigma}^1_k)$. We prove the stronger assertion that $\mathbb{N}\langle k+1\rangle$ has no continuous representation δ with $dom(\delta) \in \mathbf{\Sigma}^1_k(\mathcal{N})$.

Suppose for a contradiction that δ is a continuous representation of $\mathbb{N}\langle k+1\rangle$ with $dom(\delta) \in \mathbf{\Sigma}^1_k(\mathcal{N})$. Then there is a continuous function $f\colon \mathbb{N}\langle k\rangle \to \mathcal{N}$ with $rng(f) = dom(\delta)$ by Proposition 7.5. We define a function $g\colon \mathbb{N}\langle k\rangle \to \omega$ by

$$g(\phi) := 1 + \delta(f(\phi))(\phi) = 1 + eval\big(\delta(f(\phi)), \phi\big)\,.$$

Then g is continuous, because f, δ and the evaluation function $eval\colon \mathbb{N}\langle k+1\rangle \times \mathbb{N}\langle k\rangle \to \omega$ mapping (ψ,ϕ) to $\psi(\phi)$ are continuous (note that $\mathbb{N}\langle k+1\rangle \times \mathbb{N}\langle k\rangle$ carries the sequential QCB_0-topology). So there is some $p \in dom(\delta)$ and some $h \in \mathbb{N}\langle k\rangle$ such that $\delta(p) = g$ and $f(h) = p$. We obtain

$$g(h) = 1 + \delta(f(h))(h) = 1 + g(h)\,,$$

a contradiction.
We conclude $\mathbb{N}\langle k+1\rangle \notin \mathsf{QCB}_0(\mathbf{\Sigma}^1_k)$. $\square$

8 The category of projective QCB_0-spaces

In this section we show that the Luzin hierarchy of QCB_0-spaces gives rise to a nice cartesian closed category. This is the full subcategory of the category QCB_0 consisting of the $\bigcup_n \mathsf{QCB}_0(\mathbf{\Sigma}^1_n)$ as objects and all continuous function between them as morphisms. We denote this category by $\mathsf{QCB}_0(\mathbf{P})$ and call its objects *projective qcb-spaces*.

Theorem 8.1 *The category $\mathsf{QCB}_0(\mathbf{P})$ of projective qcb-spaces is cartesian closed.*

Proof. We only have to check that $\mathsf{QCB}_0(\mathbf{P})$ is closed under binary products and exponentiation formed in the supercategory QCB_0. For exponentiation this follows from Theorem 7.1. It is an easy exercise to show that the product representation $[\delta \times \gamma]$ constructed in Proposition 2.16 satisfies $EQ([\delta \times \gamma]) \in \mathbf{\Sigma}^1_n(\mathcal{N}^2)$, whenever $EQ(\delta)$ and $EQ(\gamma)$ are $\mathbf{\Sigma}^1_n$-subsets of $\mathcal{N}^2$. □

It turns out that $\mathsf{QCB}_0(\mathbf{P})$ is in a sense the smallest cartesian closed subcategory of QCB_0 containing ω.

Proposition 8.2 *There is no full cartesian closed subcategory* C *of* QCB_0 *such that* C *inherits binary products from* QCB_0*, contains the discrete space* ω *of natural numbers and is contained itself in* $\mathsf{QCB}_0(\mathbf{\Sigma}^1_n)$ *for some* $1 \leq n < \omega$.

Proof. We show at first that any cartesian closed subcategory D of QCB_0 that contains ω and inherits binary products from QCB_0 has the property that exponentials formed in D are homeomorphic to the corresponding QCB_0-exponentials.

Let E be an exponential formed in D to spaces $X, Y \in \mathsf{D}$. Remember that this means that there exists a continuous evaluation function $\mathit{eval}\colon E \times X \to Y$ such that for every space $Z \in \mathsf{D}$ and every continuous function $h\colon Z \times X \to Y$ there is a unique continuous function $\hat{h}\colon Z \to E$ such that

$$h(z,x) = \mathit{eval}\big(\hat{h}(z)(x)\big) \quad \text{for all } z \in Z \text{ and } x \in X.$$

For every continuous function $f\colon X \to Y$ it follows from the uniqueness condition (applied to $Z = \omega$ and the continuous function $(i,x) \mapsto f(x)$) that there exists a unique element $t(f) \in E$ satisfying $f(x) = \mathit{eval}(t(f),x)$ for all $x \in X$. Conversely, since Y^X is an exponential to X and Y in the supercategory QCB_0, there is a continuous function $Ev\colon E \to Y^X$ satisfying $Ev(e)(x) = \mathit{eval}(e,x)$ for all $e \in E$ and $x \in X$. Clearly, $Ev(t(f)) = f$ for every $f \in Y^X$. The uniqueness condition implies $t(Ev(e)) = e$ for every $e \in E$. Hence Ev is a continuous bijection with t as its inverse.

To show that t is a continuous function from Y^X to E, let $\{f_i\}_i$ be a sequence that converges in Y^X to some function $f_\infty \in Y^X$. This can be reformulated by stating that the function $F\colon \mathbb{N}_\infty \times X \to Y$ defined by $F(i,x) := f_i(x)$ for $i \in \omega \cup \{\infty\}$ and $x \in X$ is continuous, where $\mathbb{N}_\infty$ denotes the one-point compactification of the discrete natural number with ∞ as the infinity point. (This is a well-known property of exponentials in the category of sequential topological spaces and therefore in the category QCB_0, see e.g. [ELS04, Sch03]).

Below we will show that $\mathbb{N}_\infty$ is a continuous retract of the exponential $E_\mathcal{N}$ formed in D to $X = Y = \omega$. So let (s,r) be a continuous section-retraction pair. We define $G\colon E_\mathcal{N} \times X \to \omega$ by $G(p,x) := F(r(p),x)$. Clearly, G is continuous. Since E is an exponential in D, there is a continuous function $\widehat{G}\colon E_\mathcal{N} \to E$ satisfying $\mathit{eval}(\widehat{G}(p),x) = G(p,x)$ for all $p \in E_\mathcal{N}$ and $x \in X$. The uniqueness condition implies $\widehat{G}(p) = t(f_{r(p)})$. As $\{i\}_i$ converges to ∞ in $\mathbb{N}_\infty$, $\{\widehat{G}(s(i))\}_i$ converges to $\widehat{G}(s(\infty))$ in E. Clearly, we have $\widehat{G}(s(i)) = t(f_i)$ for all $i \in \omega \cup \{\infty\}$. Therefore $\{t(f_i)\}_i$ converges to $t(f_\infty)$ in E.

We conclude that t is sequentially continuous. Since Y^X is a sequential topological space, t is even continuous in the topological sense. This means that Y^X is homeomorphic to E.

Now we show that $\mathbb{N}_\infty$ is indeed a retract of $E_\mathcal{N}$. As in the general case, there is a continuous bijection $Ev_\mathcal{N}\colon E_\mathcal{N} \to \mathcal{N}$. Hence $E_\mathcal{N}$ is Hausdorff, but not discrete by being an uncountable space with a dense countable subset (see Proposition 3.3.1 in [Sch03]). So there exists an injective sequence $\{z_i\}_i$ converging in $E_\mathcal{N}$ to some point $z_\infty \in E_\mathcal{N} \setminus \{z_i \,|\, i \in \omega\}$. Since $\mathcal{N}$ is zero-dimensional and $Ev_{\mathcal{N}}$ is injective, there is a sequence $\{C_k\}_k$ of clopen sets in $\mathcal{N}$ satisfying

$$Ev_{\mathcal{N}}(z_k) \in C_k \quad \text{and} \quad \{Ev_{\mathcal{N}}(z_i) \,|\, i \leq \infty,\, i \neq k\} \subseteq \mathcal{N} \setminus C_k$$

for every $k \in \omega$. We define functions $s\colon \mathbb{N}_\infty \to E_{\mathcal{N}}$ and $r\colon E_{\mathcal{N}} \to \mathbb{N}_\infty$ by

$$s(i) := z_i \text{ and } r(z) := \min\left\{\infty, k \in \omega \,\middle|\, z \in Ev_{\mathcal{N}}^{-1}(C_k)\right\}.$$

Then both functions are continuous and satisfy $rs = id_{\mathbb{N}_\infty}$. Hence $\mathbb{N}_\infty$ is a continuous retract of $E_{\mathcal{N}}$.

Finally, suppose that C were a cartesian closed subcategory of $\mathsf{QCB_0}$ with the desired properties. By the above statement, C contains a space homeomorphic to $\mathbb{N}\langle k\rangle$ for all $k \in \omega$. In particular, there is a space $E \in \mathsf{C}$ homeomorphic to $\mathbb{N}\langle n+1\rangle$. Hence $\mathbb{N}\langle n+1\rangle$ has an admissible representation δ such that $EQ(\delta) \in \mathbf{\Sigma}^1_n(\mathcal{N}^2)$. This contradicts Theorem 7.7. $\square$

Remark 8.3 The assumption $\omega \in \mathsf{C}$ in the last proposition is essential, because several cartesian closed categories of ω-algebraic domains are known [Ju90], which are all $\mathsf{QCB_0}(\mathbf{\Pi}^0_2)$-spaces by being quasi-Polish.

9 Conclusion

Hopefully, the results of this paper show that the introduced hierarchies are natural and interesting for CA and DST. The study of these hierarchies is of course in the very beginning and many natural questions remain open. In particular, we would like to see more natural and important witnesses for the non-collapse property of the hierarchies. A systematic development of DST for the introduced classes of spaces (in particular, for the projective $\mathsf{QCB_0}$-spaces) seems also a natural direction of future research.

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