

# REALIZABILITY INTERPRETATION OF PA BY ITERATED LIMITING PCA

YOHJI AKAMA

**ABSTRACT.** For any partial combinatory algebra (PCA for short)  $\mathcal{A}$ , the class of  $\mathcal{A}$ -representable partial functions from  $\mathbb{N}$  to  $\mathcal{A}$  quotiented by the filter of cofinite sets of  $\mathbb{N}$ , is a PCA such that the representable partial functions are exactly the limiting partial functions of  $\mathcal{A}$ -representable partial functions (Akama, “Limiting partial combinatory algebras” Theoret. Comput. Sci. Vol. 311 2004). The  $n$ -times iteration of this construction results in a PCA that represents any  $n$ -iterated limiting partial recursive functions, and the inductive limit of the PCAs over all  $n$  is a PCA that represents any arithmetical, partial function. Kleene’s realizability interpretation over the former PCA interprets the logical principles of double negation elimination for  $\Sigma_n^0$ -formulas, and that over the latter PCA interprets Peano’s arithmetic (PA for short). A hierarchy of logical systems between Heyting’s arithmetic and PA is used to discuss the prenex normal form theorem, the relativized independence-of-premise schemes, and “PA is an unbounded extension of HA.”

## 1. INTRODUCTION

**1.1. Hierarchical of semi-classical arithmetical principles.** Following Section 1.3.2 of Troelstra (1973), by *Heyting’s arithmetic* HA, we mean an intuitionistic predicate calculus IQC with equality such that (1) the language of HA is a first-order language  $L_{\text{HA}}$ , with logical connectives  $\forall, \exists, \rightarrow, \wedge, \vee, \neg$ ; numeral variables  $l, m, n, \dots$ ; a constant symbol 0 (zero), a unary function symbol  $S$  (successor), constant function symbols for all primitive recursive functions, and a binary predicate symbol  $=$  (equality between numbers). *Bounded quantifications*  $\forall n < t. A$  and  $\exists n < t. A$  are abbreviations of  $\forall n(f(n, t) = 1 \rightarrow A)$  and  $\exists n(f(n, t) = 1 \wedge A)$ , where  $f(n, t)$  is a primitive recursive function such that  $f(n, t) = 1$  if and only if  $n < t$ ; and (2) besides the axioms for the equality, the axioms of HA are the defining equality of the primitive recursive functions and so-called Peano’s axiom  $\forall n(\neg S(n) = 0)$ ,  $\forall n \forall m(S(n) = S(m) \rightarrow n = m)$ , and an axiom scheme called *the induction scheme*:

$$B[0] \wedge \forall n(B[n] \rightarrow B[S(n)]) \rightarrow \forall n B[n] \quad (B \text{ is any formula.})$$

By *Peano’s arithmetic* PA, we mean the formal system obtained from HA by adjoining one of classical axiom scheme, such as *the law of excluded middle*  $A \vee \neg A$  ( $A$  is any  $L_{\text{HA}}$ -formula), and/or *the principle of double negation elimination*  $\neg \neg A \rightarrow A$  ( $A$  is any  $L_{\text{HA}}$ -formula). Kleene (1945) interpreted every theorem of HA by a recursive function/operation.

Kleene introduced *arithmetical hierarchy* of integer sets, over the class of recursive sets. The complexity of an integer set  $X$  in the arithmetical hierarchy is measured by the number of alternation of the quantifiers of the relation that defines the set  $X$ . The arithmetical hierarchy has a close relation to *oracle* computation,

such as the *complete sets* and the *jump hierarchy* (see Odifreddi (1989) for example).

According to Section 0.30 of Hájek and Pudlák (1998), a  $\Sigma_k^0$ -formula and a  $\Pi_k^0$ -formula are the following formulas preceded by  $k$  alternating quantifiers, respectively for  $k \geq 0$ :

- A  $\Sigma_k^0$ -formula is of the form  $\exists n_1 \forall n_2 \cdots Q n_{k-1} \overline{Q} n_k. P[n_1, \dots, n_k]$ .
- A  $\Pi_k^0$ -formula is of the form  $\forall m_1 \exists m_2 \cdots \overline{Q} m_{k-1} Q m_k. P[m_1, \dots, m_k]$ .

Here  $P[n_1, \dots, n_k]$  is an  $L_{\text{HA}}$ -formula with all the quantifiers being bounded, but may contain free variables other than its indicated variables. The  $L_{\text{HA}}$ -formula  $P[m_1, \dots, m_k]$  is understood similarly.

A formula in *prenex normal form* (PNF for short) is, by definition, a series of quantifiers followed by a quantifier-free formula. A formula

$$\exists n_1 \forall m_1 \exists n_2 \forall m_2 \cdots . P[n_1, m_1, n_2, m_2, \dots]$$

in PNF is true in classical logic, if and only if the formula represents a game between the quantifiers  $\exists$  and  $\forall$  where the player  $\exists$  has a winning strategy. Every formula is equivalent to a formula in PNF in classical logic, but it is not the case in HA. It may be interesting to think of an extension of HA from viewpoint of games which the formulas represent. We ask ourselves, “For which set  $\Gamma$  of  $L_{\text{HA}}$ -formulas, which extension  $T$  of HA admits the *prenex normal form theorem* for  $\Gamma$ ?” We will syntactically study the question.

For the study, we use *an arithmetical hierarchy of semi-classical principles*, introduced in Akama et al. (2004). In the hierarchy, the law of excluded middle and the principle of double negation elimination are relativized by various formula classes  $\Gamma = \Sigma_k^0, \Pi_k^0, \dots$  ( $k \geq 0$ ). The hierarchy has following axiom schemes:

$$\begin{array}{lll} (\Gamma\text{-LEM}) & A \vee \neg A & (A \text{ is any } \Gamma\text{-formula}). \\ (\Gamma\text{-DNE}) & \neg\neg A \rightarrow A & (A \text{ is any } \Gamma\text{-formula}). \end{array}$$

Any set  $X \subseteq \mathbb{N}$  in Kleene’s arithmetical hierarchy is identical to  $\mathbb{N} \setminus (\mathbb{N} \setminus X)$ . However, not every formula  $A$  is equivalent in HA to  $\neg\neg A$ . So we defined the *dual*  $A^\perp$  of  $A$  in a way similar to so-called *involution negation of classical logic*. We show that  $\text{HA} \vdash (A^\perp)^\perp \leftrightarrow A$  for any formula  $A$  in PNF, and consider an axiom scheme

$$(\Gamma\text{-LEM}') \quad A \vee A^\perp \quad (A \text{ is any } \Gamma\text{-formula}).$$

The axiom scheme  $\Sigma_k^0\text{-LEM}$  turns out to be equivalent in HA to  $\Sigma_k^0\text{-LEM}'$ . Motivated by  $\Delta_k^0$ -sets of Kleene’s arithmetical hierarchy, the hierarchy of semi-classical principles has the following axiom scheme

$$\Delta_k^0\text{-LEM} \quad (A \leftrightarrow B) \rightarrow (A \vee \neg A) \quad (A \in \Pi_k^0, B \in \Sigma_k^0).$$

According to Akama et al. (2004), it is weaker than the variant

$$\text{fp}\Delta_k^0\text{-LEM} \quad (A \leftrightarrow B) \rightarrow (B \vee A^\perp) \quad (A \in \Pi_k^0, B \in \Sigma_k^0).$$

Among these axiom schemes appearing in the arithmetical hierarchy of semi-classical principles, we answer, “Which axiom scheme is stronger than which axiom scheme?”

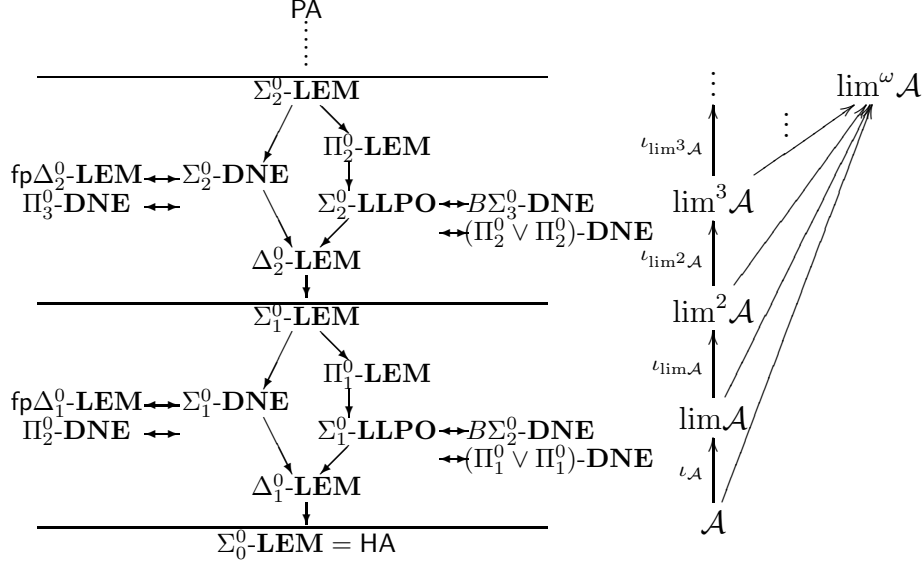


FIGURE 1. The left is the arithmetical hierarchy of semi-classical principles. The one-way arrows means implication which is not reversible. The non-reversibility, the the axiom schemes principle  $\Sigma_k^0$ -LLPO,  $B\Sigma_{k+1}^0$ -DNE and  $(\Pi_k^0 \vee \Pi_k^0)$ -DNE are not discussed in this paper, but in Akama et al. (2004). The right diagram consisting of PCAs and homomorphisms is a *colimit diagram*, in the category of PCAs and homomorphisms between them. The vertical arrows are canonical injections (see Section 3 for detail)

**Theorem 1.1.** *For any  $k \geq 0$ ,*

- (1)  $\Sigma_k^0$ -LEM proves  $\Pi_k^0$ -LEM in HA .
- (2)  $\Sigma_{k+1}^0$ -DNE proves  $\Sigma_k^0$ -LEM in HA .
- (3)  $\Sigma_k^0$ -LEM intuitionistically proves  $\Sigma_k^0$ -DNE .
- (4)  $\Pi_{k+1}^0$ -LEM intuitionistically proves  $\Sigma_k^0$ -LEM .
- (5)  $\text{fp}\Delta_k^0$ -LEM is equivalent in HA to  $\Sigma_k^0$ -DNE .
- (6)  $\Sigma_k^0$ -DNE proves  $\Delta_k^0$ -LEM in HA .

Let  $T$  be a consistent extension of HA. For a formula  $A$  of  $T$ , let a formula  $A'$  be obtained from  $A$  by moving a quantifier of  $A$  over a subformula  $D$  of  $A$ . If the subformula  $D$  is *decidable* in  $T$  (i.e.  $T$  proves  $D \vee \neg D$ ), then the formulas  $A$  and  $A'$  are equivalent in  $T$ . Based on this observation, by Theorem 1.1, we prove the following:

**Theorem 1.2** (Prenex Normal Form Theorem). *For every  $L_{\text{HA}}$ -formula  $A$  having at most  $k$  quantifiers, we can find an  $L_{\text{HA}}$ -formula  $\hat{A}$  in PNF which has  $k$  quantifiers and is equivalent in  $\text{HA} + \Sigma_k^0$ -LEM to  $A$ .*

Actually, for  $k$ , we can take an “essential” number of alternation of nested quantifiers. See Subsection 2.2 for detail.

**1.2. Iterated Limiting PCA and Realizability Interpretations.** Akama (2004) introduced a limit operation  $\lim(\bullet)$  for partial combinatory algebras (PCAs for short) such that from any PCA  $\mathcal{A}$ , the limit operation  $\lim(\bullet)$  builds hierarchies  $\{\lim^\alpha \mathcal{A}\}_{\alpha=0,1,\dots,\omega}$  of PCAs satisfying Figure 1 (right). The limit operation corresponds to the jump operation of the arithmetical hierarchies, as in Shoenfield’s limit lemma (see Odifreddi (1989) for instance). The introduction of the limit operation aimed to represent approximation algorithms needed in proof animation (Hayashi et al., 2002). Hayashi proposed proof animation in order to make interactive formal proof development easier.

In this paper, we provide a realizability interpretation of PA by a PCA  $\lim^\omega \mathcal{A}$  for every PCA  $\mathcal{A}$ .

**Theorem 1.3** (Iterated Limiting Realizability Interpretation). *For any PCA  $\mathcal{A}$  and for any nonnegative integer  $k$ , the system  $\text{HA} + \Sigma_{k+1}^0\text{-DNE}$  is sound by the realizability interpretation for the PCA  $\lim^k(\mathcal{A})$ . PA is sound by the realizability interpretation for the PCA  $\lim^\omega(\mathcal{A})$ .*

Let us call realizability interpretation by a PCA  $\lim^\alpha \mathcal{A}$  an *iterated limiting realizability interpretation* ( $\alpha = 0, 1, 2, \dots, \omega$ ). The feature of our realizability interpretation of PA are:

- if non-constructive objects are allowed to exist by the double negation elimination axioms, the realization of the non-constructive objects requires the jump of mathematical intuition. The jump is achieved by the limit.
- Our realizability interpretation of PA is simpler than those by Berardi et al. (1998) and Avigad (2000). They embedded classical logic to intuitionistic logic by the Gödel-Gentzen’s negative translation (see Section 81 of Kleene (1952) for example) or the Friedman-Dragalin translation, and then carried out the recursive realizability interpretation. However, they needed a special observation in interpreting the translation results of logical principles. Berardi (2005) developed a theory for “classical logic as limit.”

**1.3. Two Consequences of Our Prenex Normal Form Theorem and Our Iterated Limiting Realizability Interpretation of PA.** We derive a result for *independence-of-premise schemes* (see Section 1.11.6 of Troelstra (1973)), and that for *n-consistent extension of HA*.

**Definition 1.4** (Independence-of-premise scheme). *Let  $\Gamma$  be a set of  $L_{\text{HA}}$ -formulas.  $(\Gamma\text{-IP})$  is an axiom scheme*

$$(A \rightarrow \exists m. B) \rightarrow \exists m. (A \rightarrow B)$$

where  $m$  does not occur free in  $A$ ,  $A$  is any in  $\Gamma$ , and  $B$  is any  $L_{\text{HA}}$ -formula.

Let an  $F_n$ -formula be any  $L_{\text{HA}}$ -formulas having at most  $n$  quantifiers.

**Theorem 1.5** (Non-derivability between  $F_{k+1}\text{-IP}$  and  $\Sigma_{k+1}^0\text{-DNE}$ ).  *$\text{HA} + \Sigma_{k+1}^0\text{-DNE} + F_{k+1}\text{-IP}$  does not admit a realizability interpretation by the PCA  $\lim^k(\mathbb{N})$ , where  $\mathbb{N}$  is the PCA of all natural numbers such that the partial application operation  $\{n\}(m)$  is the application of the unary partial recursive function of Gödel number  $n$  applied to  $m$ . Hence  $\Sigma_{k+1}^0\text{-DNE} \not\vdash_{\text{HA}} F_{k+1}\text{-IP}$  and  $F_{k+1}\text{-IP} \not\vdash_{\text{HA}} \Sigma_{k+1}^0\text{-DNE}$ .*

No reasonable subsystem  $T$  of HA seems to admit prenex normal form theorem, because for all  $k$ ,  $T$  does not prove  $F_k\text{-IP}$ .

The next consequence of our prenex normal form theorem (Theorem 1.2) and our iterated limiting realizability interpretation (Theorem 1.3) of PA is about “PA is unbounded extension of HA.”

Before Akama et al. (2004), strict infinite hierarchies of formal arithmetics  $\text{HA} \subsetneq T_1 \subsetneq T_2 \subsetneq \dots \subsetneq \text{PA}$  was provided in a proof of a theorem “any set  $\Gamma$  of  $L_{\text{HA}}$ -sentences with bounded quantifier-complexity does not axiomatize PA over HA.” The proof was sketched in Section 3.2.32 of Troelstra (1973), and was based on C. Smoryński’s idea given in his unpublished note “*Peano’s arithmetic is unbounded extension of Heyting’s arithmetic.*” Troelstra (1973) used a *realizability interpretation* (Kleene (1945)) but the realizers are Gödel numbers of partial functions recursive in a complete  $\Pi_k^0$ -set of the Kleene’s arithmetical hierarchies.

We say an arithmetic  $T$  is  $n$ -consistent, provided every  $\Sigma_n^0$ -sentence provable in  $T$  is true in the standard model  $\omega$ . Note that HA is  $n$ -consistent for each positive integer  $n$ .

**Theorem 1.6** (PA as bounded extension of HA). *Let  $n \geq 2$  be a natural number, and  $\Gamma$  be a set of  $L_{\text{HA}}$ -sentences containing at most  $n$  quantifiers. If  $\text{HA} + \Gamma$  is  $n$ -consistent, then  $\text{HA} + \Gamma$  does not prove the axiom scheme  $\Sigma_{n+1}^0$ -LEM.*

The background and a possible research direction of the theorem is given in Section 4. The rest of the paper is organized as follows. In Section 2, the hierarchies of logical systems between HA and PA are introduced to discuss the prenex normal form theorem (Theorem 1.2). In Section 3, we introduce iterated autonomous limiting PCAs, In Section 4, by using the such PCAs, we introduce and study the iterated limiting realizability interpretation of arithmetics between HA and PA. In Subsection 4.1, we verify Theorem 1.5 and Theorem 1.6.

## 2. HIERARCHY OF SEMI-CLASSICAL PRINCIPLES

When we move quantifiers of a formula  $A$  outside the scope of propositional connectives, we ask ourselves when the resulting formula  $A'$  is equivalent in HA to the formula  $A$ .

**Lemma 2.1.** *If a variable  $n$  does not occur in a formula  $A$ , then intuitionistic predicate logic IQC proves: (1)  $A \vee \forall n B \rightarrow \forall n(A \vee B)$ ; (2)  $\exists n(A \circ B) \leftrightarrow A \circ \exists n B$  for  $\circ = \vee, \wedge$ ; and (3)  $\forall n(A \wedge B) \leftrightarrow A \wedge \forall n B$ .*

As usual, the symbol  $\vdash$  denotes the derivability.

**Fact 2.2.** *Suppose  $T$  is a formal system of arithmetic extending IQC. We say a formula  $D$  of  $T$  is decidable in  $T$ , if  $T \vdash D \vee \neg D$ .*

- (1) *If formulas  $D$  and  $D'$  are decidable in  $T$ , so are  $\neg D$  and  $D \circ D'$  for  $\circ = \wedge, \vee, \rightarrow$ .*
- (2) *If a formula  $D$  is decidable in HA, then bounded universal quantifications  $\forall n < t. D$  and  $\exists n < t. D$  are decidable in HA.*
- (3) *Every  $\Sigma_0^0$ -formulas is decidable in HA.*

**Fact 2.3.** *None of the following two formulas (7) and (8) are provable in IQC but both of two formulas  $(D \vee \neg D) \rightarrow (7)$  and  $(D' \vee \neg D') \rightarrow (8)$  are.*

$$(7) \quad (D \rightarrow B) \leftrightarrow (\neg D \vee B).$$

$$(8) \quad \forall n(D' \vee B) \rightarrow D' \vee \forall n B \quad (n \text{ does not occur free in } D').$$

IQC with the scheme (8) added is complete for the class of Kripke models of constant domains, and HA plus the schema is just PA, as explained in Section 1.11.3 of Troelstra (1973).

**2.1. Proof of Theorem 1.1.** For a formula  $A$ , we define a formula  $A^\perp$  classically equivalent to  $\neg A$ , as follows:

**Definition 2.4.** For any formula  $A$ , we define the dual  $A^\perp$  as follows:

- When  $A$  is prime,  $A^\perp$  is the negation  $\neg A$ .
- When  $A$  is a negated formula  $\neg B$ , then  $A^\perp$  is  $B$ .
- When  $A$  is  $B \vee C$ , then  $A^\perp$  is  $B^\perp \wedge C^\perp$ .
- When  $A$  is  $B \wedge C$ , then  $A^\perp$  is  $B^\perp \vee C^\perp$ .
- When  $A$  is  $B \rightarrow C$ , then  $A^\perp$  is  $B \wedge C^\perp$ .
- When  $A$  is  $\forall n. B$ , then  $A^\perp$  is  $\exists n. B^\perp$ .
- When  $A$  is  $\exists n. B$ , then  $A^\perp$  is  $\forall n. B^\perp$ .

The dual operation is more manageable than the propositional connective  $\neg$ .

**Fact 2.5.** (1)  $\text{HA} \vdash P^\perp \leftrightarrow \neg P$  ( $P$  is a  $\Sigma_0^0$ -formula.)  
 (2)  $\text{HA} \vdash (A^\perp)^\perp \leftrightarrow A$  ( $A$  is a  $\Sigma_k^0$ -formula or a  $\Pi_k^0$ -formula.)

*Proof.* (1) By induction on  $P$ . (2) First consider the case the formula  $A$  is a  $\Sigma_k^0$ -formula. Then  $A$  is written as  $\exists n_1 \forall n_2 \exists n_3 \cdots Q n_k. P$  for some  $\Sigma_0^0$ -formula  $P$ . Then  $(A^\perp)^\perp$  is  $\exists n_1 \forall n_2 \exists n_3 \cdots Q n_k. (P^\perp)^\perp$ . The Assertion (1) implies  $\vdash_{\text{HA}} (P^\perp)^\perp \leftrightarrow \neg \neg P$ . But Fact 2.2 (3), implies the decidability of  $P$ . So  $\vdash_{\text{HA}} \neg \neg P \leftrightarrow P$ . Hence  $\vdash_{\text{HA}} (P^\perp)^\perp \leftrightarrow P$ . Therefore  $\vdash_{\text{HA}} (A^\perp)^\perp \leftrightarrow A$ . When  $A$  is a  $\Pi_k^0$ -formula, the proof is similar.  $\square$

The axiom scheme  $\Sigma_k^0\text{-LEM}'$  is the axiom scheme consisting of the following form:

$$(9) \quad \exists n_1 \forall n_2 \cdots Q n_{k-1} \overline{Q} n_k P[n_1, \dots, n_k] \vee \forall m_1 \exists m_2 \cdots \overline{Q} m_{k-1} Q m_k (P[m_1, \dots, m_k])^\perp.$$

Here  $P[n_1, \dots, n_k]$  and  $P[m_1, \dots, m_k]$  are  $\Sigma_0^0$ -formulas possibly containing free variables other than indicated variables, and the quantifier  $Q$  is  $\forall$  for odd  $k$  and is  $\exists$  otherwise.  $\overline{Q}$  is  $\exists$  if  $Q$  is  $\forall$ , and is  $\forall$  otherwise.

$$(10) \quad \Sigma_k^0\text{-LEM}' \vdash_{\text{HA}} \Pi_k^0\text{-LEM}' \text{ and } \Pi_k^0\text{-LEM}' \vdash_{\text{HA}} \Sigma_k^0\text{-LEM}'$$

follows from Fact 2.5 (2), because the dual of a  $\Sigma_k^0$ -formula ( $\Pi_k^0$ -formula, resp.) is a  $\Pi_k^0$ -formula ( $\Sigma_k^0$ -formula, resp.).

**Fact 2.6.** For any formula  $A$ , IQC proves (1)  $\neg(A \wedge A^\perp)$  and (2)  $(A \vee A^\perp) \rightarrow (A^\perp \leftrightarrow \neg A)$ .

*Proof.* (1) The proof is by induction on the structure of  $A$ . When  $A$  is prime or negated, the assertion is trivial. When  $A$  is  $B \vee C$ , let us assume  $B \vee C$  and the dual  $A^\perp$ , that is,  $B^\perp \wedge C^\perp$ . The first conjunct contradicts by the induction hypothesis in case of  $B$ , and the second by the induction hypothesis in case of  $C$ . So,  $\neg(A \wedge A^\perp)$ . When  $A$  is a conjunction, the assertion is similarly verified. When  $A$  is  $B \rightarrow C$ , let us assume  $B \rightarrow C$  and the dual, that is  $B \wedge C^\perp$ . From the first conjunct  $B$  and  $B \rightarrow C$ , we infer  $C$ , which contradicts by the induction hypothesis against the second conjunct  $C^\perp$ . When  $A$  is  $\forall n. B[n]$ , let us assume  $\forall n. B[n]$  and the dual  $\exists n. (B[n])^\perp$ . For a fresh variable  $m$ , assume  $(B[m])^\perp$ . But we can

infer  $B[m]$  from  $A$ . This contradicts against the induction hypothesis. When  $A$  is existentially quantified, the assertion is similarly verified. (2) The Assertion (1) implies  $(A \vee A^\perp) \rightarrow (A^\perp \rightarrow \neg A)$ , while  $(A \vee A^\perp) \rightarrow (\neg A \rightarrow A^\perp)$  is immediate.  $\square$

The two axiom schemes  $\Sigma_k^0\text{-LEM}'$  and  $\Sigma_k^0\text{-LEM}$  are equivalent over HA, as we prove below:

**Lemma 2.7.** *For any  $k \geq 0$ , (1)  $\Sigma_k^0\text{-LEM}' \vdash_{\text{IQC}} \Sigma_k^0\text{-LEM}$ , and (2)  $\Sigma_k^0\text{-LEM} \vdash_{\text{HA}} \Sigma_k^0\text{-LEM}'$ .*

*Proof.* The first assertion follows from Fact 2.6 (2) in IQC. The second assertion is proved by induction on  $k$ . The assertion holds for  $k = 0$ , because  $\vdash_{\text{HA}} \Sigma_0^0\text{-LEM}'$  follows from Fact 2.2 (3) and Fact 2.5 (1). Let  $k > 0$ . Consider a  $\Sigma_k^0$ -formula  $\exists n. B$  with  $B$  being any  $\Pi_{k-1}^0$ -formula. By the induction hypothesis, we have  $\Sigma_k^0\text{-LEM} \vdash_{\text{HA}} \Sigma_{k-1}^0\text{-LEM}'$ . Because  $\Sigma_{k-1}^0\text{-LEM}'$  and  $\Pi_{k-1}^0\text{-LEM}'$  are equivalent over HA by (10), we have  $\Sigma_k^0\text{-LEM} \vdash_{\text{HA}} B^\perp \vee B$ . By this and Fact 2.6 (2), we have  $\Sigma_k^0\text{-LEM} \vdash_{\text{HA}} B^\perp \leftrightarrow \neg B$ . So  $\Sigma_k^0\text{-LEM} \vdash_{\text{IQC}} \exists n. B \vee \forall n. \neg B$  implies  $\Sigma_k^0\text{-LEM} \vdash_{\text{HA}} \exists n. B \vee \forall n. B^\perp$ . Therefore  $\Sigma_k^0\text{-LEM} \vdash_{\text{HA}} \Sigma_k^0\text{-LEM}'$ .  $\square$

We prepare the proof of Theorem 1.1 (2) below. An instance (9) of  $\Sigma_k^0\text{-LEM}'$  is equivalent in PA to the following  $\Sigma_{k+1}^0$ -formula:

$$(11) \quad \exists n_1 (\forall m_1 \forall n_2) (\exists m_2 \exists n_3) \cdots (Qm_{k-2} Qn_{k-1}) (\overline{Q}m_{k-1} \overline{Q}n_k) Qm_k \\ (P[n_1, \dots, n_k] \vee \neg P[m_1, \dots, m_k]).$$

Here  $P[n_1, \dots, n_k]$  and  $\neg P[m_1, \dots, m_k]$  are  $\Sigma_0^0$ -formulas possibly containing free variables other than indicated variables.

We apply  $\Sigma_{k+1}^0\text{-DNE}$  to the *Gödel-Gentzen translation* (Section 81 of Kleene (1952)) result of (11).

**Lemma 2.8.** *Let  $k \geq 1$ . The  $\Sigma_{k+1}^0$ -formula (11) is provable in  $\text{HA} + \Sigma_{k+1}^0\text{-DNE}$ .*

*Proof.* It is easy to see that the  $\Sigma_{k+1}^0$ -formula (11) is equivalent in a classical logic to an instance of  $\Sigma_k^0\text{-LEM}'$ . So, HA proves the Gödel-Gentzen translation of (11), which is obtained from (11)

- (1) by replacing each  $(\exists l)$  with  $(\neg \forall l \neg)$ ; and
- (2) by replacing the disjunction  $P[n_1, \dots, n_k] \vee \neg P[m_1, \dots, m_k]$  with a formula  $\neg(\neg P[n_1, \dots, n_k] \wedge \neg \neg P[m_1, \dots, m_k])$ .

However,

- (1) for each formula  $A$ ,  $\text{IQC} \vdash \neg \forall l \neg A \leftrightarrow \neg \neg \exists l. A$ ; and
- (2)  $\text{HA} \vdash P[n_1, \dots, n_k] \vee \neg P[m_1, \dots, m_k] \leftrightarrow \neg(\neg P[n_1, \dots, n_k] \wedge \neg \neg P[m_1, \dots, m_k])$ , by Fact 2.2 (3).

So, HA proves a formula obtained from (11) by only inserting  $\neg \neg$  just before each existential quantifier. The resulting formula is

$$\neg \neg \exists n_1 (\forall m_1 \forall n_2) (\neg \neg \exists m_2 \neg \neg \exists n_3) \cdots (P[n_1, \dots, n_k] \vee \neg P[m_1, \dots, m_k]), \quad (f_0)$$

and ends with

- ( $o_0$ ):  $\forall n_{k-1} \neg \neg \exists m_{k-1} \neg \neg \exists n_k \forall m_k (P[\vec{n}] \vee \neg P[\vec{m}])$  for odd  $k$ ; and
- ( $e_0$ ):  $\forall n_{k-1} \neg \neg \exists m_k (P[\vec{n}] \vee \neg P[\vec{m}])$  for even  $k$ .

In each case, the rightmost  $\neg\neg$  is just before a  $\Sigma_{1+(k \bmod 2)}^0$ -formula. So, if we can use  $\Sigma_{1+(k \bmod 2)}^0$ -DNE, then the rightmost  $\neg\neg$ (s) in the subformulas  $(o_0, e_0)$  can be safely eliminated from the formula  $(f_0)$ . But  $\Sigma_{1+(k \bmod 2)}^0$ -DNE follows from  $\Sigma_{k+1}^0$ -DNE. Thus  $\Sigma_{k+1}^0$ -DNE proves in HA the formula  $(f_0)$  with the rightmost  $\neg\neg$ (s) eliminated from the end-part  $(o_0, e_0)$ . The resulting formula  $(f_1)$  ends with

- ( $o_1$ ):  $\neg\neg\exists m_{k-3}\neg\neg\exists n_{k-2}(\forall m_{k-2}\forall n_{k-1})(\exists m_{k-1}\exists n_k)\forall m_k(P[\vec{n}] \vee \neg P[\vec{m}])$  for odd  $k$ ; and  
 ( $e_1$ ):  $\neg\neg\exists m_{k-2}\neg\neg\exists n_{k-2}(\forall m_{k-1}\forall n_k)(\exists m_k)(P[\vec{n}] \vee \neg P[\vec{m}])$  for even  $k$ .

In each case, the rightmost  $\neg\neg$  is just before a  $\Sigma_{3+(k \bmod 2)}^0$ -formula. So, if we can use  $\Sigma_{3+(k \bmod 2)}^0$ -DNE, then the rightmost  $\neg\neg$ 's in  $(o_1, e_1)$  can be safely eliminated from  $(f_1)$ . But  $\Sigma_{3+(k \bmod 2)}^0$ -DNE follows from  $\Sigma_{k+1}^0$ -DNE. Thus  $\Sigma_{k+1}^0$ -DNE proves in HA the formula  $(f_1)$  with the rightmost  $\neg\neg$ 's eliminated from the end-part  $(o_1, e_1)$ .

By iterating this argument, we can safely eliminate all  $\neg\neg$ 's from  $(f_0)$ . This establishes that  $\Sigma_{k+1}^0$ -DNE proves in HA the  $\Sigma_{k+1}^0$ -formula (11). This completes the proof of Lemma 2.8.  $\square$

We will present the proof of Theorem 1.1.

Assertion (1) " $\Sigma_k^0$ -LEM  $\vdash_{\text{HA}} \Pi_k^0$ -LEM" is verified as follows: By Lemma 2.7, we see that for every  $\Sigma_0^0$ -formula  $P[\vec{n}]$ , a disjunction of  $\exists n_1\forall n_2\cdots Qn_k\neg P[\vec{n}]$  and  $\forall n_1\exists n_2\cdots \overline{Q}n_kP[\vec{n}]$  is deducible in HA from  $\Sigma_k^0$ -LEM. When the first disjunct holds, then it contradicts against the dual of the first disjunct by Fact 2.6 (1), and thus we have the negation  $\neg\forall n_1\exists n_2\cdots \overline{Q}n_kP[\vec{n}]$  of the dual. In the other case, then we have the second disjunct  $\forall n_1\exists n_2\cdots \overline{Q}n_kP[\vec{n}]$ . In both cases, we have  $\forall n_1\exists n_2\cdots \overline{Q}n_kP[\vec{n}] \vee \neg\forall n_1\exists n_2\cdots \overline{Q}n_kP[\vec{n}]$ , which is an instance of  $\Pi_k^0$ -LEM.

Assertion (2) " $\Sigma_{k+1}^0$ -DNE  $\vdash_{\text{HA}} \Sigma_k^0$ -LEM" of Theorem 1.1 will be proved by induction on  $k$ . The case  $k = 0$  follows from Fact 2.2 (3). Next consider the case  $k > 0$ .

**Claim 2.9.** *Suppose that  $j \leq k$  is a positive odd number and that a variable  $m_j$  does not occur free in a  $\Pi_{k-j}^0$ -formula  $\forall n_{j+1}\exists n_{j+2}\cdots Qn_k.P[n_1, \dots, n_k]$ . Then  $\text{HA} + \Sigma_{k+1}^0$ -DNE proves the following equivalence formula:*

$$\begin{aligned} & \forall m_j (\forall n_{j+1}\exists n_{j+2}\cdots Qn_k.P[n_1, \dots, n_k] \vee \exists m_{j+1}\forall m_{j+2}\cdots \overline{Q}m_k.\neg P[m_1, \dots, m_k]) \\ \leftrightarrow & (\forall n_{j+1}\exists n_{j+2}\cdots Qn_k.P[n_1, \dots, n_k] \vee \forall m_j\exists m_{j+1}\forall m_{j+2}\cdots \overline{Q}m_k.\neg P[m_1, \dots, m_k]). \end{aligned}$$

*Proof.* In the left-hand side of the equivalence formula, we can easily see the first disjunct  $\forall n_{j+1}\exists n_{j+2}\cdots Qn_k.P[n_1, \dots, n_k]$  is a  $\Pi_{k-j}^0$ -formula. The system  $\text{HA} + \Sigma_{k+1}^0$ -DNE proves  $\Sigma_{k-j+1}^0$ -DNE which proves  $\Sigma_{k-j}^0$ -LEM by the induction hypothesis on Assertion (2) of Theorem 1.1. Hence the system  $\text{HA} + \Sigma_{k+1}^0$ -DNE proves  $\Pi_{k-j}^0$ -LEM by Assertion (1) of Theorem 1.1. Thus the  $\Pi_{k-j}^0$ -disjunct  $\forall n_{j+1}\exists n_{j+2}\cdots Qn_k.P[n_1, \dots, n_k]$  of the left-hand side is decidable in  $\text{HA} + \Sigma_{k+1}^0$ -DNE, where the variable  $m_j$  does not occur free. Because of Lemma 2.1 and Fact 2.3, the left-hand side and the right-hand side of the equivalence formula is indeed equivalent in the system  $\text{HA} + \Sigma_{k+1}^0$ -DNE.  $\square$



Next, we will consider when the universal quantifier can be safely moved over  $\Sigma_{k-i+1}^0$ -disjunct where  $i \geq 1$ .

**Claim 2.10.** *Suppose that  $i \leq k$  is a positive even number and that a variable  $n_i$  does not occur free in a  $\Sigma_{k-i+1}^0$ -disjunct  $\exists m_i \forall m_{i+1} \dots \overline{Q} m_k. \neg P[m_1, \dots, m_k]$  does not contain a free variable  $n_i$ . Then  $\text{HA} + \Sigma_{k+1}^0\text{-DNE}$  proves the following equivalence formula*

$$\begin{aligned} & \forall n_i (\exists n_{i+1} \forall n_{i+2} \dots Q n_k. P[n_1, \dots, n_k] \vee \exists m_i \forall m_{i+1} \dots \overline{Q} m_k. \neg P[m_1, \dots, m_k]) \\ \leftrightarrow & (\forall n_i \exists n_{i+1} \forall n_{i+2} \dots Q n_k. P[n_1, \dots, n_k] \vee \exists m_i \forall m_{i+1} \dots \overline{Q} m_k. \neg P[m_1, \dots, m_k]). \end{aligned}$$

*Proof.* In the left-hand side of the equivalence formula, we see that the second disjunct  $\exists m_i \forall m_{i+1} \dots \overline{Q} m_k. \neg P[m_1, \dots, m_k]$  is a  $\Sigma_{k-i+1}^0$ -formula. It is decidable in  $\text{HA} + \Sigma_{k+1}^0\text{-DNE}$ , because  $\Sigma_{k+1}^0\text{-DNE}$  proves  $\Sigma_{k-i+2}^0\text{-DNE}$  which proves  $\Sigma_{k-i+1}^0\text{-LEM}$  by the induction hypothesis of Assertion (2) of Theorem 1.1. The decidable  $\Sigma_{k-i+1}^0$ -disjunct  $\exists m_i \forall m_{i+1} \dots \overline{Q} m_k. \neg P[m_1, \dots, m_k]$  does not contain a free variable  $n_i$ . So move the universal quantifier  $\forall n_i$  over the decidable  $\Sigma_{k-i+1}^0$ -disjunct. The resulting formula is the right-hand side of the equivalence formula. It is equivalent in  $\text{HA} + \Sigma_{k+1}^0\text{-DNE}$  to the left-hand side of the equivalence formula, by Lemma 2.1 and Fact 2.3.  $\square$

We continue the proof of Assertion (2) “ $\Sigma_{k+1}^0\text{-DNE} \vdash_{\text{HA}} \Sigma_k^0\text{-LEM}$ ” of Theorem 1.1. To an instance (9) of  $\Sigma_k^0\text{-LEM}'$ , apply Lemma 2.1, Fact 2.3, Claim 2.9 with  $j = 1$ , and Claim 2.10 with  $i = 2$ . Next apply Lemma 2.1, Fact 2.3, Claim 2.9 with  $j = 3$ , and Claim 2.10 with  $i = 4$ . Then repeatedly apply them with  $(i, j) = (5, 6), (7, 8), \dots$  in this order. Then a formula (9) is equivalent in  $\text{HA} + \Sigma_{k+1}^0\text{-DNE}$  to the  $\Sigma_{k+1}^0$ -formula (11). But the formula (11) is provable in  $\text{HA} + \Sigma_{k+1}^0\text{-DNE}$  by Lemma 2.8. Hence every instance (9) of  $\Sigma_k^0\text{-LEM}'$  is provable in the system  $\text{HA} + \Sigma_{k+1}^0\text{-DNE}$ . Thus the system  $\text{HA} + \Sigma_{k+1}^0\text{-DNE}$  proves  $\Sigma_k^0\text{-LEM}'$  and thus  $\Sigma_k^0\text{-LEM}$  by Lemma 2.7. This completes the proof of Assertion (2).

To prove Assertion (3) “ $\Sigma_k^0\text{-LEM} \vdash_{\text{IQC}} \Sigma_k^0\text{-DNE}$ ,” let us assume  $\neg \neg A$  with  $A$  being a  $\Sigma_k^0$ -formula. By  $\Sigma_k^0\text{-LEM}$ , we have  $A \vee \neg A$ . In case of  $\neg A$ , by the assumption  $\neg \neg A$ , we have contradiction, from which  $A$  follows. Hence we concludes  $\neg \neg A \rightarrow A$ .

To prove Assertion (4) “ $\Pi_{k+1}^0\text{-LEM} \vdash_{\text{IQC}} \Sigma_k^0\text{-LEM}$ ,” note that any  $\Sigma_k^0$ -formula  $B$  is equivalent in IQC to a  $\Pi_{k+1}^0$ -formula  $\forall n. B$  where the variable  $n$  is fresh. Because  $\text{HA} + \Pi_{k+1}^0\text{-LEM}$  proves  $\forall n. B \vee \neg \forall n. B$ , so does  $B \vee \neg B$ , an instance of  $\Sigma_k^0\text{-LEM}$ .

We will prove Assertion (5) “ $\Sigma_k^0\text{-DNE}$  is equivalent in HA to  $\text{fp}\Delta_k^0\text{-LEM}$ ” of Theorem 1.1. First we will prove “ $\Sigma_k^0\text{-DNE} \vdash_{\text{HA}} \text{fp}\Delta_k^0\text{-LEM}$ .” Let us assume  $\Sigma_k^0\text{-DNE}$ . Let  $P[n_1, \dots, n_k]$  and  $R[m_1, \dots, m_k]$  be  $\Sigma_0^0$ -formulas possibly containing free variables other than indicated variables. Also assume the following equivalence formula between a  $\Sigma_k^0$ -formula and a  $\Pi_k^0$ -formula:

$$(12) \quad \exists n_1 \forall n_2 \dots Q n_{k-1} \overline{Q} n_k. P[n_1, \dots, n_k] \leftrightarrow \forall m_1 \exists m_2 \dots \overline{Q} m_{k-1} Q m_k. R[m_1, \dots, m_k],$$

We will derive the following disjunction of two  $\Sigma_k^0$ -formulas:

$$(13) \quad \exists n_1 \forall n_2 \cdots Q_{n_{k-1}} \overline{Q}_{n_k} . P[n_1, \dots, n_k] \vee \exists m_1 \forall m_2 \cdots Q_{m_{k-1}} \overline{Q}_{m_k} . (R[m_1, \dots, m_k])^\perp .$$

**Claim 2.11.** *The disjunction (13) is equivalent in  $\text{HA} + \Sigma_k^0\text{-DNE}$  to a  $\Sigma_k^0$ -formula:*

$$(14) \quad \exists n_1 \exists m_1 \forall n_2 \forall m_2 \cdots \overline{Q}_{n_k} \overline{Q}_{m_k} (P[n_1, \dots, n_k] \vee \neg R[m_1, \dots, m_k]).$$

*Proof.* The claim is proved by Lemma 2.1, in a similar argument as the Assertion (2) of Theorem 1.1 is. Since the  $\Sigma_k^0$ -formula (14) is obtained from the disjunction (13) by moving the quantifiers  $\exists n_{2i-1}, \exists m_{2i-1}, \forall n_{2i}, \forall m_{2i}$  ( $i = 1, 2, \dots$ ) out of the scope of the disjunction, the equivalence between (13) and (14) in  $\text{HA} + \Sigma_k^0\text{-DNE}$  is established by showing that the movement of the quantifiers are safe. The existential quantifiers  $\exists n_{2i-1}, \exists m_{2i-1}$  are safely moved by Lemma 2.1. Each quantifier  $\forall n_{2i}$  is moved over a  $\Pi_{k-2i+1}^0$ -disjunct  $\forall m_{2i} \exists m_{2i+1} \cdots \overline{Q}_{m_k} \neg R$ , and each quantifier  $\forall m_{2i}$  over a  $\Sigma_{k-2i}^0$ -disjunct  $\exists n_{2i+1} \forall n_{2i+2} \cdots \overline{Q}_{n_k} P$ . Here the  $\Pi_{k-2i+1}^0$ -disjunct and the  $\Sigma_{k-2i}^0$ -disjunct are both decidable by Theorem 1.1. So each  $\forall n_{2i}$  and  $\forall m_{2i}$  are safely moved. This completes the verification of the claim.  $\square$

To complete the verification of Assertion (5) “ $\Sigma_k^0\text{-DNE} \vdash_{\text{HA}} \text{fp}\Delta_k^0\text{-LEM}$ ,” it is sufficient to show that the  $\Sigma_k^0$ -formula (14) from the equivalence formula (12), by using  $\Sigma_k^0\text{-DNE}$ .

In view of  $\Sigma_k^0\text{-DNE}$ , we have only to derive the double negation of the  $\Sigma_k^0$ -formula (14). So assume the negation of the  $\Sigma_k^0$ -formula (14), that is,

$$\neg \exists n_1 \exists m_1 \forall n_2 \forall m_2 \cdots \overline{Q}_{n_k} \overline{Q}_{m_k} (P[n_1, \dots, n_k] \vee (R[m_1, \dots, m_k])^\perp) .$$

It is equivalent in  $\text{HA} + \Sigma_k^0\text{-DNE}$  to the dual

$$(15) \quad \forall n_1 \forall m_1 \exists n_2 \exists m_2 \cdots Q_{n_k} Q_{m_k} ((P[n_1, \dots, n_k])^\perp \wedge R[m_1, \dots, m_k]) ,$$

because  $\neg \exists n_1 \exists m_1$  is  $\forall n_1 \forall m_1 \neg$ , and because  $\Sigma_{k-1}^0\text{-LEM}$  is available in  $\text{HA} + \Sigma_k^0\text{-LEM}$ . By Lemma 2.1 (2) and (3), the  $\Pi_k^0$ -formula (15) implies a conjunction of two  $\Pi_k^0$ -formulas.

$$(\forall n_1 \exists n_2 \cdots Q_{n_k} . \neg P[n_1, \dots, n_k]) \quad \wedge \quad (\forall m_1 \exists m_2 \cdots Q_{m_k} . R[m_1, \dots, m_k])$$

By using assumption (12), the second  $\Pi_k^0$ -conjunct implies the dual of the first  $\Pi_k^0$ -conjunct. So the contradiction follows from Fact 2.6 (1). This establishes  $\Sigma_k^0\text{-DNE} \vdash_{\text{HA}} \text{fp}\Delta_k^0\text{-LEM}$ .

Next, we prove the converse  $\text{fp}\Delta_k^0\text{-LEM} \vdash_{\text{HA}} \Sigma_k^0\text{-DNE}$ . The axiom scheme  $\text{fp}\Delta_k^0\text{-LEM}$  has an instance (12)  $\rightarrow$  (13) with the  $\Sigma_0^0$ -formula  $P[n_1, \dots, n_k]$  being replaced by a false  $\Sigma_0^0$ -formula  $S(0) = 0$ . Hence  $\text{HA} + \text{fp}\Delta_k^0\text{-LEM}$  proves an implication formula  $\neg \forall m_1 \exists m_2 \cdots \overline{Q}_{m_k} . R \rightarrow \exists m_1 \forall m_2 \cdots Q_{m_k} . \neg R$ . So, we can derive  $\Sigma_k^0\text{-DNE}$  by using Modus Tolence if we can prove an implication formula

$$(16) \quad \neg \neg \exists m_1 \forall m_2 \cdots Q_{m_k} . \neg R \rightarrow \neg \forall m_1 \exists m_2 \cdots \overline{Q}_{m_k} . R .$$

To prove the formula (16), we use a *Gentzen-type sequent calculus G3* (see Section 81 of Kleene (1952)) for IQC. By the left- and the right-introduction rules of  $\neg$ , the G3-sequent (16) is inferred from a G3-sequent

$$\exists m_1 \forall m_2 \cdots Q_{m_k} . \neg R, \forall m_1 \exists m_2 \cdots \overline{Q}_{m_k} . R \rightarrow .$$

It does not contain the variable  $m_1$  free, so it is inferred by the left-introduction rule of  $\exists$  from a sequent

$$\forall m_2 \cdots Qm_k. \neg R, \forall m_1 \exists m_2 \cdots \overline{Q}m_k. R \rightarrow .$$

It is inferred by the left-introduction rule of  $\forall$  from a  $G3$ -sequent

$$\forall m_2 \exists m_3 \cdots Qm_k. \neg R, \exists m_2 \forall m_3 \cdots \overline{Q}m_k. R \rightarrow .$$

By repeating this argument, the  $G3$ -sequent (16) is inferred from a  $G3$ -sequent  $\neg R, R \rightarrow$ , which is inferred from an axiom sequent  $R \rightarrow R$  of  $G3$ . This establishes  $\text{fp}\Delta_k^0\text{-LEM} \vdash_{\text{HA}} \Sigma_k^0\text{-DNE}$ , and thus Assertion (5) of Theorem 1.1.

Assertion (6) “ $\Sigma_k^0\text{-DNE} \vdash_{\text{HA}} \Delta_k^0\text{-LEM}$ ” of Theorem 1.1 is proved as follows: By Assertion (5) of Theorem 1.1, we have  $\Sigma_k^0\text{-DNE} \vdash_{\text{HA}} (A \leftrightarrow B) \rightarrow (B \vee A^\perp)$  for any  $\Pi_k^0$ -formula  $A$  and any  $\Sigma_k^0$ -formula  $B$ . By Fact 2.6 (2), we have  $\Sigma_k^0\text{-DNE} \vdash_{\text{HA}} (A \leftrightarrow B) \rightarrow (B \vee \neg A)$ . Thus  $\Sigma_k^0\text{-DNE} \vdash_{\text{HA}} \Delta_k^0\text{-LEM}$ . This completes the proof of Theorem 1.1.

**Remark 2.12.** In HA, the axiom scheme  $\Delta_k^0\text{-LEM}$  is strictly weaker than the axiom scheme  $\Sigma_k^0\text{-DNE}$  for every positive integer  $k$ , according to Akama et al. (2004). Hence there is a  $\Pi_k^0$ -formula  $A$  such that  $\vdash_{\text{HA}} A^\perp \leftrightarrow \neg A$ . Otherwise, by Theorem 1.1 (5), axiom schemes  $\Delta_k^0\text{-LEM}$ ,  $\text{fp}\Delta_k^0\text{-LEM}$  and  $\Sigma_k^0\text{-DNE}$  are equivalent over HA.

The axiom scheme  $\Sigma_k^0\text{-DNE}$  has the following equivalent axiom schemes.

**Fact 2.13.** For  $k \geq 0$ ,  $\Sigma_k^0\text{-DNE}$  is equivalent in IQC to  $\Pi_{k+1}^0\text{-DNE}$ .

*Proof.* Let an  $L_{\text{HA}}$ -formula  $\forall n. A$  be a  $\Pi_{k+1}^0$ -formula with  $A$  being a  $\Sigma_k^0$ -formula. We can show  $\vdash_{\text{IQC}} \neg \neg \forall n. A \rightarrow \neg \neg A$ . We have  $\Sigma_k^0\text{-DNE} \vdash_{\text{IQC}} \neg \neg A \rightarrow A$ . By Modus Tolence, we have  $\Sigma_k^0\text{-DNE} \vdash_{\text{IQC}} \neg \neg \forall n. A \rightarrow A$ , and thus  $\Sigma_k^0\text{-DNE} \vdash_{\text{IQC}} \neg \neg \forall n. A \rightarrow \forall n. A$ . Hence  $\Sigma_k^0\text{-DNE} \vdash_{\text{IQC}} \Pi_{k+1}^0\text{-DNE}$ . To prove the converse  $\Pi_{k+1}^0\text{-DNE} \vdash_{\text{IQC}} \Sigma_k^0\text{-DNE}$ , let  $A$  be any  $\Sigma_k^0$ -formula. For any fresh variable  $l$ , the formula  $A$  is equivalent in IQC to a  $\Pi_{k+1}^0$ -formula  $\forall l. A$ . So an instance  $\neg \neg \forall l. A \rightarrow \forall l. A$  of the axiom scheme  $\Pi_{k+1}^0\text{-DNE}$  proves in IQC an instance  $\neg \neg A \rightarrow A$  of  $\Sigma_k^0\text{-DNE}$ .  $\square$

**2.2. Prenex Normal Form Theorem.** We will introduce three sets of  $L_{\text{HA}}$ -formulas such that the three correspond to  $\Sigma_k^0$ -,  $\Pi_k^0$ -, and  $\Delta_k^0$ -formulas of HA, respectively.

**Definition 2.14** ( $E_k, U_k, P_k$ ). For the language  $L_{\text{HA}}$ , we define  $E_k$ -,  $U_k$ -, and  $P_k$ -formulas.

- (1) Given an occurrence of a quantifier. If it is in a  $\Sigma_0^0$ -formula, then we do not assign the sign to it. Otherwise,
  - (a) The sign of an occurrence  $\exists$  in a formula  $A$  is the sign of the subformula  $\exists n. B$  starting with such  $\exists$ .
  - (b) The sign of an occurrence  $\forall$  in a formula  $A$  is the opposite of the sign of the subformula  $\forall n. B$  starting with such  $\forall$ .
- (2) The degree of a formula is the maximum number of nested quantifiers with alternating signs. Formulas of degree 0 are exactly  $\Sigma_0^0$ -formulas. Clearly the degree is less than or equal to the number of occurrences of the quantifiers.

- (3) By  $a(n)$   $U_k$ -( $E_k$ -)formula, we mean a formula of degree  $k$  such that all the outermost quantifiers are negative (positive). A  $P_{k+1}$ -formula is a propositional combination of  $U_k$ - and  $E_k$ -formulas.

The Heyting arithmetic  $\mathbf{HA}$  has the function symbols and the defining equations for a primitive recursive pairing  $p : \mathbb{N}^2 \rightarrow \mathbb{N}$  and primitive recursive, projection functions  $p_0 : \mathbb{N} \rightarrow \mathbb{N}$  and  $p_1 : \mathbb{N} \rightarrow \mathbb{N}$  such that  $p_0(p(l, m)) = l$ ,  $p_1(p(l, m)) = m$ , and  $p(p_0(n), p_1(n)) = n$ . It is fairly easy to verify the following fact:

**Fact 2.15.** An  $L_{\mathbf{HA}}$ -formula  $\dots(\dots QlQm\dots)(\dots l\dots m\dots)\dots$  is equivalent in  $\mathbf{HA}$  to an  $L_{\mathbf{HA}}$ -formula  $\dots(\dots Qn\dots)(\dots(p_0n)\dots(p_1n)\dots)\dots$  for all  $Q \in \{\forall, \exists\}$ .

**Theorem 2.16.** For any  $U_k^0$ -( $E_k^0$ -)formula  $A$ , we can find a  $\Pi_k^0$ -( $\Sigma_k^0$ -, resp.)formula  $\hat{A}$  which is equivalent in  $\mathbf{HA} + \Sigma_k^0$ -**LEM** to  $A$ .

*Proof.* The proof is by induction on the structure of  $A$ . When  $k = 0$ , we can take  $A$  as  $\hat{A}$  because  $A$  is a  $\Sigma_0^0$ -formula. Assume  $k > 0$ . Then  $A$  is not a prime formula. The rest of the proof proceeds by cases according to the form of the formula  $A$ .

Case 1.  $A$  is  $B_1 \circ B_2$  with  $\circ = \vee, \wedge, \rightarrow$

Subcase 1.1  $\circ = \vee, \wedge$ . Then  $B_1$  and  $B_2$  are both  $U_k^0$ -( $E_k^0$ -)formulas. We can use the induction hypotheses to find two  $\Pi_k^0$ -( $\Sigma_k^0$ -)formulas  $\hat{B}_1$  and  $\hat{B}_2$  which are equivalent in  $\mathbf{HA} + \Sigma_k^0$ -**LEM** to  $B_1$  and  $B_2$  respectively.

When  $A$  is a  $U_k^0$ -formula, then the  $\Pi_k^0$ -formulas  $\hat{B}_1$  and  $\hat{B}_2$  are  $\forall l. M_1l$  and  $\forall m. M_2m$  for some  $\Sigma_{k-1}^0$ -formulas  $M_1l$  and  $M_2m$ . Here  $M_1l$  and  $\hat{B}_2m$  are both decidable in  $\mathbf{HA} + \Sigma_k^0$ -**LEM** because the system  $\mathbf{HA} + \Sigma_k^0$ -**LEM** proves  $\Sigma_{k-1}^0$ -**LEM** and  $\Pi_k^0$ -**LEM** by Theorem 1.1. So by Lemma 2.1 and Fact 2.3 imply

$$\mathbf{HA} + \Sigma_k^0\text{-}\mathbf{LEM} \vdash A \leftrightarrow \hat{B}_1 \circ \hat{B}_2 \leftrightarrow \forall l(M_1l \circ \forall m M_2m) \leftrightarrow \forall l \forall m (M_1l \circ M_2m).$$

Here  $M_1l \circ M_2m$  is an  $E_{k-1}$ -formula. By  $\Sigma_k^0$ -**LEM**  $\vdash_{\mathbf{HA}} \Sigma_{k-1}^0$ -**LEM**, we can use the induction hypothesis to find a  $\Sigma_{k-1}^0$ -formula  $\hat{D}lm$  which is equivalent in  $\mathbf{HA} + \Sigma_k^0$ -**LEM** to the  $E_{k-1}$ -formula  $M_1l \circ M_2m$ . So, in  $\mathbf{HA} + \Sigma_k^0$ -**LEM**, the  $U_k^0$ -formula  $A$  is equivalent to  $\forall l \forall m. \hat{D}lm$  which is equivalent in  $\mathbf{HA} + \Sigma_k^0$ -**LEM** to a  $\Pi_k^0$ -formula.

When  $A$  is an  $E_k^0$ -formula, the proof proceeds as in the case  $A$  is a  $U_k^0$ -formula.

Subcase 1.2  $\circ = \rightarrow$ . Then  $B_1$  is an  $E_k^0$ -( $U_k^0$ -)formula, while  $B_2$  is an  $U_k^0$ -( $E_k^0$ -)formula. We can use the induction hypotheses to find a  $\Sigma_k^0$ -( $\Pi_k^0$ -)formula  $\hat{B}_1$  and a  $\Pi_k^0$ -( $\Sigma_k^0$ -)formula  $\hat{B}_2$  such that  $\mathbf{HA} + \Sigma_k^0$ -**LEM**  $\vdash (B_1 \leftrightarrow \hat{B}_1) \wedge (B_2 \leftrightarrow \hat{B}_2)$ . By Lemma 2.7 and Fact 2.6 (2),  $\mathbf{HA} + \Sigma_k^0$ -**LEM**  $\vdash \neg \hat{B}_1 \rightarrow (\hat{B}_1)^\perp$ . On the other hand, we can show  $\mathbf{IQC} \vdash (\hat{B}_1)^\perp \rightarrow \neg \hat{B}_1$  by using the sequent calculus  $G3$  for  $\mathbf{IQC}$ . Hence  $\mathbf{HA} + \Sigma_k^0$ -**LEM**  $\vdash (\hat{B}_1)^\perp \leftrightarrow \neg \hat{B}_1$ . In  $\mathbf{HA} + \Sigma_k^0$ -**LEM**, the  $\Sigma_k^0$ -( $\Pi_k^0$ -)formula  $\hat{B}_1$  is decidable, and thus  $(\hat{B}_1 \rightarrow \hat{B}_2) \xrightarrow{\text{Fact 2.3}} \neg \hat{B}_1 \vee \hat{B}_2 \leftrightarrow (\hat{B}_1)^\perp \vee \hat{B}_2$ . The two disjuncts  $(\hat{B}_1)^\perp$  and  $\hat{B}_2$  are both  $\Pi_k^0$ -( $\Sigma_k^0$ -)formulas decidable in  $\mathbf{HA} + \Sigma_k^0$ -**DNE**. Moreover, each subformula of  $(\hat{B}_1)^\perp$  and  $\hat{B}_2$  is so. Hence by Lemma 2.1, Fact 2.3 and Fact 2.15, the formula  $(\hat{B}_1)^\perp \vee \hat{B}_2$  is equivalent in  $\mathbf{HA} + \Sigma_k^0$ -**LEM** to a  $\Pi_k^0$ -( $\Sigma_k^0$ -)formula.

Case 2.  $A$  is  $\forall n. B[n]$  ( $\exists n. B[n]$ ).

Assume  $B[n]$  is a  $U_k^0$ -( $E_k^0$ -)formula. Then we can find by the induction hypothesis a  $\Pi_k^0$ -( $\Sigma_k^0$ -)formula  $\hat{B}[n]$  which is equivalent in  $\mathbf{HA} + \Sigma_k^0$ -**LEM** to  $B[n]$ . So, in  $\mathbf{HA} +$

$\Sigma_k^0$ -**LEM**, the formula  $A$  is equivalent to  $\forall n. \hat{B}[n]$  ( $\exists n. \hat{B}[n]$ ), which is equivalent to some  $\Pi_k^0$ -( $\Sigma_k^0$ )-formula by Fact 2.15.

Otherwise,  $B[n]$  is an  $E_{k-1}^0$ -( $U_{k-1}^0$ )-formula. By the induction hypothesis, we can find a  $\Sigma_{k-1}^0$ -( $\Pi_{k-1}^0$ )-formula  $\hat{B}[n]$  which is equivalent in  $\text{HA} + \Sigma_{k-1}^0$ -**LEM** to  $B[n]$ . So, in  $\text{HA} + \Sigma_k^0$ -**LEM**, the formula  $A$  is equivalent to  $\forall n. \hat{B}[n]$  ( $\exists n. \hat{B}[n]$ ).

Case 3.  $A$  is  $\neg B$ . The same argument as Subcase 1.2.  $\square$

Here we will prove a slightly stronger version of Theorem 1.2.

**Corollary 2.17.** *For any  $P_{k+1}^0$ -formula  $A$ , we can find a  $\Pi_{k+1}^0$ -formula  $\hat{B}$  and a  $\Sigma_{k+1}^0$ -formula  $\hat{C}$  such that  $\text{HA} + \Sigma_k^0$ -**LEM**  $\vdash A \leftrightarrow \hat{B} \leftrightarrow \hat{C}$ . Here the number of occurrences of quantifiers in  $\hat{B}$  and that of  $\hat{C}$  are less than or equal to that of  $A$ .*

*Proof.* By Theorem 2.16, the  $P_{k+1}^0$ -formula  $A$  is equivalent in  $\text{HA} + \Sigma_k^0$ -**LEM** to a propositional combination  $A^\circ$  of  $\Pi_k^0$ -formulas and  $\Sigma_k^0$ -formulas. In the formula  $A^\circ$ , move (0) all the outermost quantifiers of positive sign, out of all the propositional connectives, (1) all the outermost quantifiers of *negative* sign, out of all the propositional connectives, (2) all the outermost quantifiers of *positive* sign, out of all the propositional connectives, (3) all the outermost quantifiers of *negative* sign, out of all the propositional connectives,  $\dots$ . The resulting formula  $C$  is a block of quantifiers followed by a  $\Sigma_0^0$ -formula where the block has at most  $k+1$  alternations of quantifiers (e.g. If  $A$  is a  $P_2^0$ -formula  $\forall x Px \wedge (\exists y P'y \rightarrow \exists z P''z)$  with  $P, P', P''$  being  $\Sigma_0^0$ -formulas, then  $A^\circ$  is  $\exists z \forall xy (Px \wedge (P'y \rightarrow P''z))$  which has 2 alternations of quantifiers). All the  $\Pi_k^0$ - and all the  $\Sigma_k^0$ -formulas are  $(\text{HA} + \Sigma_k^0$ -**LEM**)-decidable. So, by Lemma 2.1 and Fact 2.3, the formula  $C$  is equivalent in  $\text{HA} + \Sigma_k^0$ -**LEM** to the  $P_{k+1}^0$ -formula  $A$ . By Fact 2.15, the resulting formula is equivalent in  $\text{HA} + \Sigma_k^0$ -**LEM** to a  $\Sigma_{k+1}^0$ -formula  $\hat{C}$ . In a similar way, the  $P_{k+1}^0$ -formula  $A$  is equivalent in  $\text{HA} + \Sigma_k^0$ -**LEM** to a  $\Pi_{k+1}^0$ -formula  $\hat{B}$ .  $\square$

### 3. ITERATED AUTONOMOUS LIMITING PCAS

We recall *autonomous limiting* PCAs (Akama, 2004). The construction was based on the Fréchet filter on  $\mathbb{N}$ , and is similar to but easier than the constructions of *recursive ultrapower* (Hirschfeld, 1975) and then semi-ring made from recursive functions modulo co- $r$ -maximal sets (Lerman, 1970).

We say a partial numeric function  $\varphi(n_1, \dots, n_k)$  is *guessed* by a partial numeric function  $\xi(t, n_1, \dots, n_k)$  as  $t$  goes to infinity, provided that  $\forall n_1, \dots, n_k \exists t_0 \forall t > t_0. \varphi(n_1, \dots, n_k) \simeq \xi(t, n_1, \dots, n_k)$ . Here, the relation  $\simeq$  means “if one side is defined, then the other side is defined with the same value.” In this case, we write  $\varphi(n_1, \dots, n_k) \simeq \lim_t \xi(t, n_1, \dots, n_k)$ . On the other hand, the symbol ‘=’ means both sides are defined with the same value. For every class  $\mathcal{F}$  of partial numeric functions,  $\lim(\mathcal{F})$  denotes the set of partial numeric functions guessed by a partial numeric function in  $\mathcal{F}$ .

A partial combinatory algebra (PCA for short) is a partial algebra  $\mathcal{A}$  equipped with two distinct constants  $\mathbf{k}, \mathbf{s}$  and a partial binary operation “application”  $(-)\cdot(\bullet)$  subject to  $(\mathbf{k} \cdot a) \cdot b = a$ ,  $((\mathbf{s} \cdot a) \cdot b) \cdot z \simeq (a \cdot c) \cdot (b \cdot c)$ , and  $(\mathbf{s} \cdot a) \cdot b$  is defined. We introduce the standard convention of associating the application to the left and writing  $ab$  instead of  $a \cdot b$ , omitting parentheses whenever no confusion occurs. If  $a \cdot b$  is defined then both of  $a$  and  $b$  are defined.

The 0-th Church numeral of  $\mathcal{A}$  is an element  $\mathbf{k} (\mathbf{s} \mathbf{k} \mathbf{k})$  of  $\mathcal{A}$ . The  $(n+1)$ -th Church numeral of  $\mathcal{A}$  is an element  $\mathbf{s} (\mathbf{s} (\mathbf{k} \mathbf{s}) \mathbf{k}) \bar{n}^{\mathcal{A}}$  of  $\mathcal{A}$ . By definition, for each natural number  $n$ , an element  $\bar{n}^{\mathcal{A}}$  of  $\mathcal{A}$  represents  $n$ , and an element  $a$  of  $\mathcal{A}$  represents itself. We say a partial function  $\varphi$  from  $M_1 \times M_2 \times \cdots \times M_k$  to  $M_0$  is represented by an element  $a$  of  $\mathcal{A}$ , whenever  $\varphi(x_1, \dots, x_k) = x_0$  if and only if for all representatives  $a_i \in \mathcal{A}$  of  $x_i$  ( $1 \leq i \leq k$ ),  $a a_1 \cdots a_{k-1} a_k$  is defined and is a representative of  $x_0$ . The set of  $\mathcal{A}$ -representable partial functions from  $M$  to  $M'$  is denoted by  $M \rightarrow_{\mathcal{A}} M'$ . Each partial recursive function is representable in any PCA.

Let  $\sim$  be the partial equivalence relation on  $\mathcal{A}$  such that  $a \sim b$  if and only if  $a \bar{t}^{\mathcal{A}} = b \bar{t}^{\mathcal{A}}$  for all but finitely many natural numbers  $t$ . A quotient structure  $(\mathbb{N} \rightarrow_{\mathcal{A}} \mathcal{A}) / \sim$  will be a PCA by the argument-wise application operation modulo  $\sim$ . More precisely, let  $[a]_{\sim}$  be  $\{b \in \mathcal{A} \mid b \sim a\}$ . Then the set  $\{[a]_{\sim} \mid a \in \mathcal{A} \text{ and } a \sim a\}$ ,  $\mathbf{k} := [\mathbf{k} \mathbf{k}]_{\sim}$ ,  $\mathbf{s} := [\mathbf{s} \mathbf{s}]_{\sim}$  and the following operation  $[a]_{\sim} * [b]_{\sim} \simeq [\mathbf{s} a b]_{\sim}$  defines a PCA. We denote it by  $\lim(\mathcal{A})$ .

By a *homomorphism* from a PCA  $\mathcal{A}$  to a PCA  $\mathcal{B}$ , we mean a function from  $\mathcal{A}$  to  $\mathcal{B}$  such that  $f(\mathbf{k}) = \mathbf{k}$ ,  $f(\mathbf{s}) = \mathbf{s}$ , and  $f(a) f(b) \simeq f(a b)$  for all  $a, b \in \mathcal{A}$ . A homomorphism fits in with a “strict, total homomorphism between PCAs” (see p. 23 of Hofstra and Cockett (2010)). A *canonical injection* of a PCA  $\mathcal{A}$  is, by definition, an injective homomorphism  $\iota_{\mathcal{A}} : \mathcal{A} \rightarrow \lim(\mathcal{A})$ ;  $x \mapsto [\mathbf{k} x]_{\sim}$ .

**Fact 3.1.**  $\iota_{\mathcal{A}}$  is indeed an injective homomorphism for every PCA  $\mathcal{A}$ .

*Proof.* We can see that  $\iota_{\mathcal{A}}$  is indeed a function from  $\mathcal{A}$  to  $\lim(\mathcal{A})$ . In other words,  $\iota_{\mathcal{A}}$  is “total” in a sense of Hofstra and Cockett (2010). It is proved as follows: For every  $x \in \mathcal{A}$ , we have  $\mathbf{k} x \bar{t} = \mathbf{k} x \bar{t}$  for every  $t \in \mathbb{N}$ . This implies  $\mathbf{k} x \sim \mathbf{k} x$ , from which  $\iota_{\mathcal{A}}(x) = [\mathbf{k} x]_{\sim}$  is in  $\lim(\mathcal{A})$ . The function  $\iota_{\mathcal{A}}$  is injective, because  $\iota_{\mathcal{A}}(x) = \iota_{\mathcal{A}}(y)$  implies  $\mathbf{k} x \bar{t}^{\mathcal{A}} = \mathbf{k} y \bar{t}^{\mathcal{A}}$  for all but finitely many natural numbers  $t$ , from which  $x = \mathbf{k} x \bar{t}^{\mathcal{A}} = \mathbf{k} y \bar{t}^{\mathcal{A}} = y$  holds for some natural number  $t$ .

It holds that (i) the injection  $\iota_{\mathcal{A}}$  maps the intrinsic constants  $\mathbf{k}, \mathbf{s}$  of the PCA  $\mathcal{A}$  to  $\mathbf{k}, \mathbf{s}$  of the PCA  $\lim(\mathcal{A})$ , and (ii)  $\iota_{\mathcal{A}}(a) \iota_{\mathcal{A}}(b) \simeq \iota_{\mathcal{A}}(a b)$ . In other words, the injection  $\iota_{\mathcal{A}}$  is “strict” in a sense of Hofstra and Cockett (2010). The Assertion (i) is clear by the definition. As for the Assertion (ii), we can prove that if  $\iota_{\mathcal{A}}(a b)$  is defined then  $\iota_{\mathcal{A}}(a) \iota_{\mathcal{A}}(b)$  is defined with the same value. The proof is as follows: By the premise,  $a b$  is defined. Because  $\mathbf{k} (a b) \bar{t} = (a b) = \mathbf{s} (\mathbf{k} a) (\mathbf{k} b) \bar{t}$  for all  $t \in \mathbb{N}$ , we have

$$(17) \quad \iota_{\mathcal{A}}(a) \iota_{\mathcal{A}}(b) \simeq [\mathbf{s} (\mathbf{k} a) (\mathbf{k} b)]_{\sim} \simeq [\mathbf{k} (a b)]_{\sim} \simeq \iota_{\mathcal{A}}(a b).$$

We can prove that if  $\iota_{\mathcal{A}}(a) \iota_{\mathcal{A}}(b)$  is defined then  $\iota_{\mathcal{A}}(a b)$  is defined with the same value. The proof is as follows: By the premise,  $[\mathbf{s} (\mathbf{k} a) (\mathbf{k} b)]_{\sim}$  is defined. So  $\mathbf{s} (\mathbf{k} a) (\mathbf{k} b) \sim \mathbf{s} (\mathbf{k} a) (\mathbf{k} b)$ . Hence for all but finitely many natural numbers  $t$ ,  $\mathbf{s} (\mathbf{k} a) (\mathbf{k} b) \bar{t} \simeq a b$  is defined. Thus  $(a b)$  is defined. By (17), the Assertion (ii) follows.  $\square$

Because  $\iota_{\mathcal{A}}$  is a homomorphism, we have  $\bar{n}^{\lim(\mathcal{A})} = \iota_{\mathcal{A}}(\bar{n}^{\mathcal{A}})$ . Hence the limit is the congruence class of the guessing function, as follows:

$$(18) \quad \lim_t (\xi \bar{t}) = \bar{n} \text{ in } \mathcal{A} \iff [\xi]_{\sim} = \bar{n} \text{ in } \lim(\mathcal{A}). \quad (\xi \in \mathcal{A})$$

The direct limit of  $\mathcal{A} \xrightarrow{\iota_{\mathcal{A}}} \lim(\mathcal{A}) \xrightarrow{\iota_{\lim(\mathcal{A})}} \lim^2(\mathcal{A}) \cdots$  is indeed a PCA, and will be denoted by  $\lim^{\omega}(\mathcal{A})$ . The application operator of a PCA and “limit procedure”

commute;

$$(\lim_t a \bar{t}) * (\lim_t b \bar{t}) = [a]_{\sim} * [b]_{\sim} = [s a b]_{\sim} = \lim_t s a b \bar{t} = \lim_t (a \bar{t}) (b \bar{t}).$$

The set of partial numeric functions represented by a PCA  $\mathcal{A}$  is denoted by  $\text{RpFn}(\mathcal{A})$ . By the bounded maximization of a function  $f(x, \vec{n})$ , we mean a function  $\max_{x < l} f(x, \vec{n})$ . The following fact is well-known.

**Fact 3.2.** *For every PCA  $\mathcal{B}$ , the set of functions represented by elements of  $\mathcal{B}$  is closed under the composition, the bounded maximization and under  $\mu$ -recursion.*

Then, we can prove  $\text{RpFn}(\lim^\alpha(\mathcal{A})) = \cup_{n < \max(1+\alpha, \omega)} \lim^n(\text{RpFn}(\mathcal{A}))$ . Shoenfield's limit lemma (see Odifreddi (1989) for instance) implies that the PCA  $\lim^\alpha(\mathcal{A})$  represents all  $\emptyset^{(\max(\alpha, \omega))}$ -recursive functions. So, the PCA  $\lim^\omega(\mathcal{A})$  can represent any arithmetical function.

#### 4. ITERATED LIMITING REALIZABILITY INTERPRETATION OF SEMI-CLASSICAL EONS

It is well-known that a form of Markov Principle over the language  $L_{\text{HA}}$ ,

$$\Sigma_1^0\text{-DNE} \quad \neg\neg\exists n\forall m < t. f(n, m, l) = 0 \rightarrow \exists n\forall m < t. f(n, m, l) = 0$$

is realized by an ordinary program  $r(t, l) = \mu n. \max_{m < t} f(n, m, l) = 0$  via recursive realizability interpretation of Kleene (1945). Here the program  $r(t, l)$  is representable by a PCA  $\mathcal{A}$ . A stronger principle of classical logic

$$\Sigma_2^0\text{-DNE} \quad \neg\neg\exists n\forall m. f(n, m, l) = 0 \rightarrow \exists n\forall m. f(n, m, l) = 0,$$

the “limit” with respect to  $t$  of a  $\Sigma_1^0\text{-DNE}$ , turns out to be realized by a limiting computation  $\lim_t r(t, l)$  which is representable by a limiting PCA  $\lim \mathcal{A}$ . This simple approach can be extended to an *iterated limiting realizability interpretation* of  $\Sigma_\alpha^0\text{-DNE}$  for  $\alpha \leq \omega$ , by  $\lim^\alpha \mathcal{A}$ .

For the convenience, we embed  $\text{HA} + \Sigma_{1+\alpha}^0\text{-DNE}$  in a corresponding extension of a constructive logic EON. It is EON plus a form of  $\Sigma_{1+\alpha}^0\text{-DNE}$ . The iterated limiting realizability interpretation is introduced by using an  $\alpha$ -iterated autonomous limiting PCAs  $\lim^\alpha(\mathcal{A})$ .

Here EON is a constructive logic of partial terms (see p. 98 of Beeson (1985)), and the language includes Curry's combinatory constants, and a partial application operator symbol. The language of EON is  $\{(-) \cdot (\bullet), \mathbf{s}, \mathbf{k}, \mathbf{d}, 0, \mathbf{s}_N, \mathbf{p}_N, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1; =, N, \downarrow\}$ . Here the constant symbols  $\mathbf{p}, \mathbf{p}_0, \mathbf{p}_1$  are intended to be the pairing function, the first projection, and the second projection, respectively. The predicate symbol  $=$  means “the both hand sides are defined and equal.” The 1-place predicate symbols  $N$  and  $\downarrow$  mean “is a natural number” and “is defined,” respectively. As before, we write  $a_0 a_1 a_2 \cdots a_{n-1} a_n$  for  $(\cdots ((a_0 \cdot a_1) \cdot a_2) \cdots a_{n-1}) \cdot a_n$ , whenever no confusion occurs.

In writing formulas of EON, variables  $n, m, l, i$  and  $j$  will be implicitly restricted to the predicate  $N$ , i.e. they are “natural number variables.” So,  $\forall n. An$  is the abbreviation for  $\forall x. (Nx \rightarrow Ax)$  and  $\exists m. Bm$  for  $\exists y. (Ny \wedge By)$ . We review the logical axioms of EON from p. 98 of Beeson (1985). The logical axioms and rules of EON are as follows: EON has the usual propositional axioms and rules. The quantifier axioms and rules are as follows: From  $B \rightarrow A$  infer  $B \rightarrow \forall x A$  ( $x$  not free in  $B$ ). From  $A \rightarrow B$  infer  $\exists x A \rightarrow B$  ( $x$  not free in  $B$ ).  $\forall x A[x] \wedge t \downarrow \rightarrow A[t]$ .  $A[t] \wedge t \downarrow \rightarrow \exists x A[x]$ .  $x = x$ .  $x = y \rightarrow y = x$ .  $t = s \rightarrow t \downarrow \wedge s \downarrow$ .  $R(t_1, \dots, t_n) \rightarrow t_1 \downarrow$

$\wedge \cdots \wedge t_n \downarrow$ . ( $R$  is any atomic formula).  $c \downarrow$  (every constant symbol  $c$ ).  $x \downarrow$  (every variable  $x$ ). Let us abbreviate  $t \simeq s$  for  $(t \downarrow \vee s \downarrow \rightarrow t = s)$ . EON has a logical axiom  $t \simeq s \rightarrow A[t] \rightarrow A[s]$ .

The non-logical axioms of EON consists of

$$\begin{aligned} & \mathbf{k}xy = x, \quad \mathbf{s}xyz \simeq xz(yz), \quad \mathbf{s}xy \downarrow, \quad \mathbf{k} \neq \mathbf{s}, \\ & \mathbf{p}xy \downarrow, \quad \mathbf{p}_0(\mathbf{p}xy) = x, \quad \mathbf{p}_1(\mathbf{p}xy) = y, \\ & N(0), \quad \forall x (Nx \rightarrow [N(\mathbf{s}Nx) \wedge \mathbf{p}_N(\mathbf{s}Nx) = x \wedge \mathbf{s}Nx \neq 0]), \\ & \forall x (Nx \wedge x \neq 0 \rightarrow N(\mathbf{p}_Nx) \wedge \mathbf{s}_N(\mathbf{p}_Nx) = x), \\ & Nx \wedge Ny \wedge x = y \rightarrow \mathbf{d}xyuv = u, \\ & Nx \wedge Ny \wedge x \neq y \rightarrow \mathbf{d}xyuv = v, \\ & A(0) \wedge \forall x (Nx \wedge A(x) \rightarrow A(\mathbf{s}Nx)) \rightarrow \forall x (Nx \rightarrow A(x)). \end{aligned}$$

We will interpret EON in a PCA, as we interpret classical logic in a model theory. The interpretations of the constant symbols  $\mathbf{s}, \mathbf{k}$  are the corresponding constants of the PCA  $\mathcal{A}$ . The interpretations of the constant symbols  $0, \mathbf{p}_N, \mathbf{s}_N, \mathbf{d}$  in  $\mathcal{A}$  are defined in a similar way that they are represented in Curry's combinatory logic by Church numerals. The interpretation of the pairing  $\mathbf{p}$  and projections  $\mathbf{p}_0, \mathbf{p}_1$  are as in Curry's combinatory logic. For detail, see Hindley and Seldin (1986). The application operator symbol  $(-) \cdot (\bullet)$  of EON is interpreted as the application of the PCA  $\mathcal{A}$ . The unary predicate symbols  $N$  and  $\downarrow$  are interpreted as the set of Church numerals of  $\mathcal{A}$  and  $\mathcal{A}$  itself, respectively. The binary predicate symbol  $=$  is interpreted as just the identity relation on  $\mathcal{A}$ . Given an assignment  $\rho : \{\text{EON-variables}\} \rightarrow \mathcal{A}$ . The interpretation of an EON-term  $t$  in  $\mathcal{A}$  and  $\rho$  is defined as an element of  $\mathcal{A}$  as usual. The interpretation of an EON-formula  $A$  in the PCA  $\mathcal{A}$  and  $\rho$  is defined as usual as one of the truth-value  $\top, \perp$ . We say an EON-formula  $A$  is true in a PCA  $\mathcal{A}$  and an assignment  $\rho : \{\text{EON-variables}\} \rightarrow \mathcal{A}$ , if the interpretation of  $A$  in  $\mathcal{A}$  and  $\rho$  is  $\top$ . In this case we write  $\mathcal{A}, \rho \models A$ . If  $\mathcal{A}, \rho \models A$  for every  $\rho$ , then we write  $\mathcal{A} \models A$ .

**Definition 4.1.** Let  $T$  be a formal system extending EON. The realizability interpretation of  $T$  is just an association to each formula  $A$  of  $T$  another formula  $\exists e. e \mathbf{r} A$  of  $T$  with a variable  $e$  being fresh. It is read “some  $e$  realizes  $A$ .” For an EON-term  $t$  and an EON-formula  $A$ , we define an EON-formula  $t \mathbf{r} A$  as follows:

- $t \mathbf{r} P$  is  $t \downarrow \wedge P$  for each atomic formula  $P$ .
- $t \mathbf{r} \neg A$  is  $t \downarrow \wedge \forall x (\neg x \mathbf{r} A)$ .
- $t \mathbf{r} A \rightarrow B$  is  $t \downarrow \wedge \forall x (x \mathbf{r} A \rightarrow tx \downarrow \wedge tx \mathbf{r} B)$ .
- $t \mathbf{r} \forall x. A$  is  $\forall x (tx \downarrow \wedge tx \mathbf{r} A)$ .
- $t \mathbf{r} \exists x. A[x]$  is  $\mathbf{p}_1 t \mathbf{r} A[\mathbf{p}_0 t]$ .
- $t \mathbf{r} A \wedge B$  is  $\mathbf{p}_0 t \mathbf{r} A \wedge \mathbf{p}_1 t \mathbf{r} B$ .
- $t \mathbf{r} A \vee B$  is  $N(\mathbf{p}_0 t) \wedge (\mathbf{p}_0 t = 0 \rightarrow \mathbf{p}_1 t \mathbf{r} A) \wedge (\neg \mathbf{p}_0 t = 0 \rightarrow \mathbf{p}_1 t \mathbf{r} B)$ .

**Definition 4.2.** A formal arithmetic  $T$  extending EON is said to be sound by the realizability interpretation for a PCA  $\mathcal{A}$ , provided that for every sentence  $B$  provable in  $T$ , a sentence  $\exists e. (e \mathbf{r} B)$  is true in  $\mathcal{A}$ .

(Realizability) interpretations and model theory of a (constructive) arithmetic  $T$  are often formalized within the system  $T$  plus reasonable axioms. For example, Troelstra (1973), Avigad (2000) and so on formalized realizability interpretations of constructive logics, while Smoryński (1978), Hájek and Pudlák (1998) and so on did



non-standard models of various arithmetic. However, as we defined in Definition 4.2, we will carry out our realizability interpretation within a naive set theory. This readily leads to the second assertion of the following Lemma.

**Lemma 4.3.** *Suppose  $B$  is an EON-formula in PNF with all the variables relativized by the predicate  $N$ .*

- (1) *For any EON-term  $t$ , we have  $\text{EON} \vdash t \mathbf{r} B \rightarrow B$ .*
- (2) *If  $B$  is an EON-sentence and  $\mathcal{A}$  is a PCA, then  $\mathcal{A} \models \neg\neg\exists x. x \mathbf{r} B$  implies  $\mathcal{A} \models B$ .*

*Proof.* (1) The proof is by induction on the structure of  $B$ . When  $B$  is prime, it is trivial. When  $B$  is  $\forall x(Nx \rightarrow Ax)$ , then  $t \mathbf{r} B$  is  $\forall x(t \cdot x \downarrow \wedge \forall y(Nx \rightarrow t \cdot x \cdot y \downarrow \wedge t \cdot x \cdot y \mathbf{r} Ax))$  where the variables  $x$  and  $y$  are fresh. So the induction hypothesis implies  $t \mathbf{r} B \rightarrow \forall x(t \cdot x \downarrow \wedge \forall y(Nx \rightarrow t \cdot x \cdot y \downarrow \wedge Ax))$ . Because  $y$  is fresh,  $t \mathbf{r} B \rightarrow \forall x(Nx \rightarrow Ax)$ . When  $B$  is  $\exists x(Nx \wedge Ax)$ , then  $t \mathbf{r} B$  is  $\mathbf{p}_0(\mathbf{p}_1 t) \downarrow \wedge N(\mathbf{p}_0 t) \wedge \mathbf{p}_1(\mathbf{p}_1 t) \mathbf{r} A(\mathbf{p}_0 t)$ . So the induction hypothesis implies  $t \mathbf{r} B \rightarrow \mathbf{p}_0(\mathbf{p}_1 t) \downarrow \wedge N(\mathbf{p}_0 t) \wedge A(\mathbf{p}_0 t)$ . Hence,  $t \mathbf{r} B \rightarrow N(\mathbf{p}_0 t) \wedge A(\mathbf{p}_0 t)$ . Thus  $t \mathbf{r} B \rightarrow \exists x(Nx \wedge Ax)$ .

(2) By Definition 4.1, the system EON proves a sentence  $\exists x. x \mathbf{r} \neg\neg B \rightarrow \neg\neg\exists x. x \mathbf{r} B$ . By the premise and the soundness of EON for any PCA,  $\neg\neg\exists x. x \mathbf{r} B$  is true in the PCA  $\mathcal{A}$ , and thus  $\exists x. x \mathbf{r} B$  is so. By the soundness of EON in any PCA and the Assertion (1) of this Lemma, the sentence  $B$  is true in the PCA.  $\square$

We will make the argument of the first paragraph of this section rigorous. It is instructive to consider the following Lemma.

**Lemma 4.4.** *For each closed EON-term  $t$  and for each PCA  $\mathcal{A}$ , whenever  $\mathcal{A} \models \forall m_1 \forall m_2. N(t \ m_1 \ m_2)$  holds, it holds*

$$\lim(\mathcal{A}) \models \exists x. [x \mathbf{r} (\neg\neg\exists m_1 \forall m_2. t \ m_1 \ m_2 = 0 \rightarrow \exists m_1 \forall m_2. t \ m_1 \ m_2 = 0)].$$

*Proof.* Let an EON-formula  $q \mathbf{r} \neg\neg\exists m_1 \forall m_2. t \ m_1 \ m_2 = 0$  be true in  $\lim(\mathcal{A})$ . By Lemma 4.3 (2), for some natural number  $n_1$ , the EON-sentence  $\forall m_2. t \ \overline{n_1} \ m_2 = 0$  is true in  $\lim(\mathcal{A})$ .

We can see that  $\mathcal{A}$  has an element  $\xi$  representing the following unary numeric function:

$$\text{minimal}(l) := \mu m_1. ((\max_{m_2 < l} t \ m_1 \ m_2) = 0).$$

Note that  $\text{minimal}(l) \leq \text{minimal}(l') \leq n_1$  if  $l \leq l'$ . So, some natural number  $m_1$  satisfies  $\lim_l \text{minimal}(l) = m_1$ . That is, for all natural numbers  $l$  but finitely many, we have  $\text{minimal}(l) = m_1$ . So, for all natural numbers  $l$  but finitely many, the formula  $\xi \ \bar{l} = \overline{m_1}$  is true in  $\mathcal{A}$ .

By the definition of  $\lim(\mathcal{A})$ , we have

$$[\xi]_{\sim} = \overline{m_1}$$

in  $\lim(\mathcal{A})$ . By the definition of  $\xi$ , for all natural numbers  $l$  but finitely many, an EON-sentence  $(\max_{m_2 < l} t \ \overline{m_1} \ \overline{m_2}) = 0$  is true in  $\lim(\mathcal{A})$ . Therefore, for all natural numbers  $m_2$ , an EON-sentence  $t \ \overline{m_1} \ \overline{m_2} = 0$  is true in  $\lim(\mathcal{A})$ . Hence,  $\forall m_2. t \ \overline{m_1} \ m_2 = 0$  is true in  $\lim(\mathcal{A})$ .

So, as a realizer  $x$  of  $\neg\neg\exists m_1 \forall m_2. t \ m_1 \ m_2 = 0 \rightarrow \exists m_1 \forall m_2. t \ m_1 \ m_2 = 0$ , take  $\mathbf{k}(\mathbf{p}(\mathbf{p} \ 0(\mathbf{k}(\mathbf{k} \ 0))) \ [\xi]_{\sim}) \in \lim(\mathcal{A})$ .  $\square$

**Definition 4.5.** For each PCA  $\mathcal{A}$  and each nonnegative integer  $k$ ,  $(\Sigma_k^0\text{-DNE}')$  is a rule

$$\frac{t \text{ is a closed term of EON} \quad \forall \vec{n} \forall m_1 \dots \forall m_k. N(t \vec{n} m_1 \dots m_k)}{\forall \vec{n} \left( \begin{array}{l} \neg \neg \exists m_1 \forall m_2 \exists m_3 \dots Q_k m_k. t \vec{n} m_1 m_2 \dots m_k = 0 \\ \rightarrow \exists m_1 \forall m_2 \exists m_3 \dots Q_k m_k. t \vec{n} m_1 m_2 \dots m_k = 0 \end{array} \right)}$$

Here  $Q_k$  is  $\exists$  for odd  $k$  and  $\forall$  for even  $k$ .

**Theorem 4.6.** For each nonnegative integer  $k$  and each PCA  $\mathcal{A}$ , if the system  $\text{EON} + (\Sigma_{k+1}^0\text{-DNE}')$  proves an EON-sentence  $A$ , then a sentence  $\exists e. e \text{ r } A$  is true in the PCA  $\lim^k(\mathcal{A})$ .

*Proof.* The verification is by induction on the length of the proof  $\pi$  of  $A$ . The axioms and rules other than  $(\Sigma_k^0\text{-DNE}')$  is manipulated as in the proof of Theorem 1.6 of Beeson (1985).

We will consider the case  $(\Sigma_k^0\text{-DNE}')$ . By the induction hypothesis on the proof  $\pi$ , an EON-sentence  $\exists e. e \text{ r } \forall \vec{n} \forall m_1 \dots \forall m_k. N(t \vec{n} m_1 \dots m_k)$  is true in the PCA  $\lim^k(\mathcal{A})$ . We will derive that an EON-sentence

$$\begin{aligned} \exists e. e \text{ r } \forall \vec{n} (\neg \neg \exists m_1 \forall m_2 \dots Q_k m_k. t \vec{n} m_1 m_2 \dots m_k = 0 \\ \rightarrow \exists m_1 \forall m_2 \dots Q_k m_k. t \vec{n} m_1 m_2 \dots m_k = 0) \end{aligned}$$

is true in  $\lim^k(\mathcal{A})$ . Let  $x$  be an element of  $\lim^k(\mathcal{A})$  and  $\vec{n}$  be nonnegative integers. Suppose

$$\lim^k(\mathcal{A}) \models x \text{ r } \neg \neg \exists m_1 \forall m_2 \dots Q_k m_k. t \vec{n} m_1 m_2 \dots m_k = 0.$$

By Lemma 4.3 (2), we have

$$\lim^k(\mathcal{A}) \models Q_1 m_1 Q_2 m_2 Q_3 m_3 \dots Q_k m_k. t \vec{n} m_1 \dots m_k = 0.$$

For every closed EON-term  $t'$ , the valuation of  $t'$  in  $\lim^k(\mathcal{A})$  is obtained from the valuation of  $t'$  in  $\mathcal{A}$  by the canonical injection  $\iota_{\lim^{k-1}(\mathcal{A})} \circ \dots \circ \iota_{\mathcal{A}}$ . Hence

$$(19) \quad \mathcal{A} \models Q_1 m_1 Q_2 m_2 Q_3 m_3 \dots Q_k m_k. t \vec{n} m_1 \dots m_k = 0$$

where  $Q_i = \exists$  ( $i$  : odd);  $\forall$  ( $i$  : even).

**Definition 4.7.** For each PCA  $\mathcal{A}$  and each  $j = 0, \dots, k-2$ , define a total function  $g_j(\vec{n}, \nu_1, \dots, \nu_{k-j}) : \vec{\mathbb{N}} \times \mathbb{N}^{k-j} \rightarrow \{0, 1\}$  such that

$$g_j(\vec{n}, \nu_1, \dots, \nu_{k-j}) = 0 \iff \mathcal{A} \models \left\{ \begin{array}{l} (Q_{k-j+1} m_{k-j+1}) (Q_{k-j+2} m_{k-j+2}) \dots (Q_k m_k). \\ t \vec{n} \overline{\nu_1} \dots \overline{\nu_{k-j}} m_{k-j+1} \dots m_k = 0. \end{array} \right.$$

**Claim 4.8.** For each  $j = 0, \dots, k-2$ , the total function  $g_j$  is represented by some element of a PCA  $\lim^j \mathcal{A}$ .

*Proof.* We can define  $g_j$  as a  $j$ -nested limiting function, as follows:

$$\begin{aligned} g_0(\vec{n}, \nu_1, \dots, \nu_k) &:= \min(1, l) \text{ such that } \mathcal{A} \models t \vec{n} \overline{\nu_1} \dots \overline{\nu_k} = \bar{l}. \\ g_j(\vec{n}, \nu_1, \dots, \nu_{k-j}) &:= \begin{cases} \lim_l \max_{\nu_{k-j+1} < l} g_{j-1}(\vec{n}, \nu_1, \dots, \nu_{k-j+1}), & (k-j \text{ is odd}); \\ \lim_l \min_{\nu_{k-j+1} < l} g_{j-1}(\vec{n}, \nu_1, \dots, \nu_{k-j+1}), & (k-j \text{ is even}). \end{cases} \end{aligned}$$

The claim is derived from (19) by induction on  $j$ , because  $g_j$  is the limit of a bounded monotone function which is either  $\max_{\dots < l}$  or  $\min_{\dots < l}$ . Each  $g_j$  is represented by some element of a PCA  $\lim^j \mathcal{A}$ , because of (18). This completes the proof of Claim 4.8.  $\square$

We continue the proof of Theorem 4.6. For an EON-formula

$$(20) \quad \exists m_1 \forall m_2 \exists m_3 \cdots Q_k m_k. t \bar{n} m_1 \cdots m_k = 0$$

appearing in (19), consider the “game” represented by (20) between the proponent  $\exists$  and the opponent  $\forall$ . From any moves  $\nu_2, \nu_4, \dots, \nu_{2p-2}$  ( $p = 1, 2, \dots, \lfloor (k+2)/2 \rfloor$ ) taken by the opponent  $\forall$ , the *minimum* move  $m_{2p-1}(\bar{n}, \nu_2, \nu_4, \dots, \nu_{2p-2})$  by the proponent  $\exists$  is given by the following limiting function

**Definition 4.9.** For  $p = 1, 2, \dots, \lfloor (k+2)/2 \rfloor$ , let

$$m_{2p-1}(\bar{n}, \nu_2, \nu_4, \dots, \nu_{2p-2}) := \lim_l \text{minimal}_{2p-1}(l, \bar{n}, \nu_2, \nu_4, \dots, \nu_{2p-2}).$$

Here the guessing function  $\text{minimal}_1(l, \bar{n}) = \mu m_1(\max_{\nu_2 < l} g_{k-2}(\bar{n}, m_1, \nu_2))$  is obtained from  $g_{k-2}$  by the bounded maximization  $\max_{\nu_2 < l}$  and the  $\mu$ -recursion. For  $p > 1$ , define the function  $\text{minimal}_{2p-1}$  by the composition, the bounded maximization  $\max_{\nu_{2p} < l}$  and the  $\mu$ -recursion  $\mu m_{2p-1}$ .

$$\text{minimal}_{2p-1}(l, \bar{n}, \nu_2, \nu_4, \dots, \nu_{2p-2})$$

$$:= \mu m_{2p-1} \cdot \left( \max_{\nu_{2p} < l} g_{k-2p}(\bar{n}, m_1(\bar{n}), \nu_2, m_3(\bar{n}, \nu_2), \nu_4, m_5(\bar{n}, \nu_2, \nu_4), \dots, m_{2p-3}(\bar{n}, \nu_2, \dots, \nu_{2p-4}), \nu_{2p-2}, m_{2p-1}, \nu_{2p}) = 0 \right).$$

For the function  $m_{2p-1}$  defined above, we have the following:

**Claim 4.10.** Assume  $p = 1, 2, 3, \dots, \lfloor (k+2)/2 \rfloor$ . Then the following assertions hold:

- (1)  $m_{2p-1}(\bar{n}, \nu_2, \nu_4, \dots, \nu_{2p-2})$  is indeed a total function of  $\bar{n}, \nu_2, \nu_4, \dots, \nu_{2p-2}$ .  
For the game the EON-formula (20) represents, consider the following alternating sequence  $\sigma$  of the proponent  $\exists$ 's moves and the opponent  $\forall$ 's moves of the game:

$$(m_1(\bar{n}), \nu_2, m_3(\bar{n}, \nu_2), \nu_4, m_5(\bar{n}, \nu_2, \nu_4), \dots, m_{2p-3}(\bar{n}, \nu_2, \dots, \nu_{2p-4}), \nu_{2p-2}) \in \mathbb{N}^{2p-2}$$

Suppose that  $n_{2p-1} \in \mathbb{N}$  is a proponent's move that immediately follows the sequence  $\sigma$ . Then  $n_{2p-1} \geq m_{2p-1}(\bar{n}, \nu_2, \nu_4, \dots, \nu_{2p-2})$ .

- (2) The limiting function  $m_{2p-1}$  is represented by an element of a PCA  $\lim^k(\mathcal{A})$ .

*Proof.* (1) The proof is by induction on  $p$ . The case where  $p = 1$  is essentially due to the proof of Lemma 4.4. Let  $p > 1$ . Assume (i) the opponent's  $2p$ -th move  $\nu_{2p}$  is bounded from above by  $l$ , (ii) the parameter  $\bar{n}$  of the game is supplied, and (iii) the opponent's moves  $\nu_2, \nu_4, \dots, \nu_{2p-2}$  so far are supplied. By the induction hypotheses, the functions  $m_1, m_3, \dots, m_{2p-3}$  are total. By this, Definition 4.7, and Definition 4.9, we see that  $\text{minimal}_{2p-1}(l, \bar{n}, \nu_2, \nu_4, \dots, \nu_{2p-2})$  is the *minimum*  $(2p-1)$ -th move of proponent  $\exists$  under the assumption (i). The guessing function  $\text{minimal}_{2p-1}$  is increasing with respect to the first argument  $l$ , because  $l$  is the bound of the maximization in the definition of  $\text{minimal}_{2p-1}$ . But there is  $n_{2p-1} \in \mathbb{N}$  such that for every  $l$ , we have  $\text{minimal}_{2p-1}(l, \bar{n}, \nu_2, \nu_4, \dots, \nu_{2p-2}) \leq n_{2p-1}$ , because of (19) and Claim 4.7. Therefore the limit  $m_{2p-1}(\bar{n}, \nu_2, \nu_4, \dots, \nu_{2p-2})$  of

$\text{minimal}_{2p-1}(l, \vec{n}, \nu_2, \nu_4, \dots, \nu_{2p-2})$  with respect to  $l$  is indeed a total function, and actually the limit from below. Therefore it is minimum among the possible winning moves. This completes the proof of Assertion (1).

(2) The proof is by induction on  $p$ . Consider the case where  $p = 1$ . Then  $m_1(\vec{n}) = \lim_l \text{minimal}_1(l, \vec{n}) = \lim_l \mu m_1(\max_{\nu_2 < l} g_{k-2}(\vec{n}, m_1, \nu_2)) = 0$ . By Claim 4.8, the total function  $g_{k-2}$  is represented by some element of  $\lim^{k-2}(\mathcal{A})$ . By Fact 3.2, the function  $\mu m_1(\max_{\nu_2 < l} g_{k-2}(\vec{n}, m_1, \nu_2)) = 0$  is represented by some element of  $\lim^{k-2}(\mathcal{A})$ . By (18), the function  $m_1(\vec{n})$  is represented by some element of  $\lim^{k-1}(\mathcal{A})$ . Fact 3.1 implies the function  $m_1(\vec{n})$  is represented by some element of  $\lim^k(\mathcal{A})$ .

Next consider the case where  $p > 1$ . By Claim 4.8, a (partial) function  $g_{k-2p}$  is indeed a total function represented by some element of the PCA  $\lim^{k-2p}(\mathcal{A})$ . By applying the bounded maximization and then  $\mu$ -recursion to  $g_{k-2p}$ , define a (partial) function of  $l, \vec{n}, x_1, \nu_2, x_3, \nu_4, \dots, x_{2p-3}, \nu_{2p-2}$ , as follows

$$(21) \quad \mu m_{2p-1} \left( \max_{\nu_{2p} < l} g_{k-2p}(\vec{n}, x_1, \nu_2, x_3, \nu_4, \dots, x_{2p-3}, \nu_{2p-2}, m_{2p-1}, \nu_{2p}) = 0 \right).$$

Then the (partial) function is also represented by some element of the PCA  $\lim^{k-2p}(\mathcal{A})$ , because of Fact 3.2. Let a (partial) function  $F$  of  $\vec{n}, x_1, \nu_2, x_3, \nu_4, \dots, x_{2p-3}, \nu_{2p-2}$  be guessed by a (partial) function (21) with respect to the variable  $l$ . Then  $F$  is represented by some element of a PCA  $\lim^{k-2p+1}(\mathcal{A})$  by (18). By Fact 3.1, the function  $F$  is represented by some element of a PCA  $\lim^k(\mathcal{A})$ .

By the induction hypothesis on  $p$ , all of  $(p-1)$  total functions  $m_1(\vec{n}), m_3(\vec{n}, \nu_2), m_5(\vec{n}, \nu_2, \nu_4), \dots, m_{p-1}(\vec{n}, \nu_2, \nu_4, \dots, \nu_{2p-4})$  are represented by some elements of the PCA  $\lim^k(\mathcal{A})$ . By composing the  $(p-1)$  total functions at the arguments  $x_1, x_3, \dots, x_{2p-3}$  of the (partial) function  $F(\vec{n}, x_1, \nu_2, x_3, \nu_4, \dots, x_{2p-3}, \nu_{2p-2})$ , we obtain the total function  $m_{2p-1}(\vec{n}, \nu_2, \nu_4, \dots, \nu_{2p-2})$ , according to Definition 4.9. Thus the total function  $m_{2p-1}$  is represented by some element of the PCA  $\lim^k(\mathcal{A})$  by Fact 3.2.  $\square$

The EON-formula (20) has a realizer  $q \in \lim^k(\mathcal{A})$ . Here  $q$  consists of the following elements of  $\lim^k(\mathcal{A})$ : the numerals  $\vec{n}^{\mathcal{A}}$ , and the representatives of the total functions  $m_1, m_3, \dots$ , in view of Claim 4.10. This completes the proof of Theorem 4.6.  $\square$

From Theorem 4.6, Theorem 1.3 follows, by embedding  $\text{HA} + \Sigma_k^0\text{-DNE}$  in a corresponding  $\text{EON} + (\Sigma_{k+1}^0\text{-DNE}')$  where  $\mathcal{A}$  is a PCA.

**4.1. Proofs of Theorem 1.5 and Theorem 1.6.** We prove the non-derivability between the axiom schemes  $F_{k+1}\text{-IP}$  and  $\Sigma_{k+1}^0\text{-DNE}$  (Theorem 1.5) by using iterated limiting realizability interpretation (Theorem 1.3). Let  $A \ n \ m$  be a  $\Pi_k^0$ -formula with all the variables indicated. The axiom scheme  $\Sigma_{k+1}^0\text{-DNE}$  proves a sentence

$$\forall n (\neg \neg \exists m. A \ n \ m \rightarrow \exists m. A \ n \ m).$$

By this and  $F_{k+1}\text{-IP}$ , we derive a sentence  $\forall n \exists m. (\neg \neg \exists m. A \ n \ m \rightarrow A \ n \ m)$ . If the system  $\text{HA} + \Sigma_{k+1}^0\text{-DNE} + F_{k+1}\text{-IP}$  is realizable by the PCA  $\lim^k(\mathbb{N})$ , then there exists  $e \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$  the following conditions hold:

- (1)  $f(n) := \lim_{t_1} \dots \lim_{t_k} \{e\}(t_1, \dots, t_k, n)$  is convergent (In this case,  $f$  is  $\emptyset^{(k)}$ -recursive and thus has a  $\Pi_{k+1}^0$ -graph); and
- (2) If  $A \ n \ m$  holds for some  $m \in \mathbb{N}$ , then  $A \ n \ f(n)$  holds.

Because  $A$  is a  $\Pi_k^0$ -formula and  $f$  has a  $\Pi_{k+1}^0$ -graph,  $A \wedge f(n)$  is a  $\Pi_{k+1}^0$ -relation for  $n$ . Note that  $\exists m. A \wedge m$  iff  $A \wedge f(n)$ . Because  $A$  is an arbitrary  $\Pi_k^0$ -formula, we can choose  $A$  such that  $\exists m. A(\bullet, m)$  is a complete  $\Sigma_{k+1}^0$ -relation. This contradicts against that  $A \wedge f(n)$  is a  $\Pi_{k+1}^0$ -relation. This completes the proof of Theorem 1.5.

Every arithmetical relation  $R$  satisfies the *uniformization property* (Odifreddi, 1989). That is, if for all natural numbers  $n$  there exists a natural number  $m$  such that  $R(n, m)$ , then there exists an arithmetical function  $f_R$  such that for all  $n$   $R(n, f(n))$ . In Section 3, we provide a PCA  $\lim^\omega(\mathbb{N})$  which represents all such  $f_R$ 's. In fact, the representative induces a realizer of  $\forall n \exists m. R(n, m)$ .

By our prenex normal form theorem (Theorem 1.2) and our iterated limiting realizability interpretations (Theorem 1.3), we will slightly refine Smoryński's result mentioned in Section 1 to Theorem 1.6.

*Proof of Theorem 1.6.* Assume otherwise. By Theorem 1.2, for every sentence  $A \in \Gamma$  there is a sentence  $\hat{A}$  in PNF such that  $\hat{A}$  contains at most  $n$  quantifiers and  $\text{HA} + \Gamma$  proves  $A \leftrightarrow \hat{A}$ .

Since  $\text{HA} + \Gamma$  is  $n$ -consistent, the sentence  $\hat{A}$  in PNF is true in the standard model  $\omega$ .

First consider the case  $\hat{A}$  is a  $\Pi_n^0$ -sentence. Then  $\hat{A}$  can be written as

$$\forall x_1 \exists x_2 \forall x_3 \cdots Q_n x_n. R x_1 x_2 x_3 \cdots x_n$$

for some  $\Sigma_0^0$ -formula  $R$ .

Here  $\forall x_3 \exists x_4 \cdots Q_n x_n. R x y x_3 \cdots x_n$  defines a  $\emptyset^{(n-2)}$ -recursive binary relation on  $\omega$ . By the relativization of the *uniformization property for recursive relations* (Odifreddi, 1989), there exists some  $\emptyset^{(n-2)}$ -recursive function

$$f_2(x) := \mu y. \forall x_3 \cdots Q_n x_n. R x y x_3 \cdots x_n$$

such that for each natural number  $x_1$  a formula  $\forall x_3 \exists x_4 \cdots Q_n x_n. R x_1 f_2(x_1) x_3 \cdots x_n$  is true on  $\omega$ .

In this way, there are  $\emptyset^{(n-2)}$ -functions  $f_2(x_1), f_4(x_1, x_3), \dots$  such that

$$\forall x_1 \forall x_3 \forall x_5 \cdots. R x_1 f_2(x_1) x_3 f_4(x_1, x_3) x_5 \cdots$$

holds on  $\omega$ .

If  $\hat{A}$  is not a  $\Pi_n^0$ -sentence, then  $\hat{A}$  is written as  $\exists x_1 \forall x_2 \exists x_3 \cdots Q_n x_n. R x_1 x_2 x_3 \cdots x_n$ . Then there are natural number  $n_1$  and  $\emptyset^{(n-3)}$ -recursive functions  $f_3(x_2), f_5(x_2, x_4), \dots$  such that a formula  $\forall x_2 \forall x_4 \forall x_6 \cdots R n_1 x_2 f_3(x_2) x_4 f_5(x_2, x_4) \cdots$  holds on  $\omega$ .

Because a PCA  $\lim^n(\mathbb{N})$  can represent all the  $\emptyset^{(n)}$ -functions  $f_i$ 's, we can find a realizer of  $\hat{A}$  in the PCA  $\lim^n(\mathbb{N})$ .

The PCA  $\lim^n(\mathbb{N})$  realizes  $\Sigma_n^0$ -**LEM**, and thus the formula  $A \in \Gamma$  by Theorem 1.2. Because the PCA  $\lim^n(\mathbb{N})$  does not realize  $\Sigma_{n+1}^0$ -**LEM**, we conclude  $\text{HA} + \Gamma \not\models \Sigma_{n+1}^0$ -**LEM**. This completes the proof of Theorem 1.6.

Our use of the complete set  $\emptyset^{(n)}$  contrasts against Kleene's use of *extended Church's thesis* on defining *effectively true* (*general recursively true*) prenex normal form (see Section 79 of Kleene (1952)).

Smoryński (1982) considered other versions  $HA$  and  $PA$  of Heyting's arithmetic and Peano's arithmetic, where  $HA$  and  $PA$  are formalized by the language

$$\{0, 1, 2, 3, \dots; Z(\cdot), S(\cdot), A(\cdot, \cdot), M(\cdot, \cdot), =\},$$

and then proved “Let  $\Gamma$  be a set of sentences of bounded quantifier-complexity, and suppose  $HA + \Gamma \vdash PA$ . Then  $HA + \Gamma$  is inconsistent.” For the proof, assuming otherwise, Smoryński constructed a model of  $PA$  by applying Orey’s compactness theorem to  $HA + \Gamma$ . For Orey’s compactness theorem, see Chapter 4 of Smoryński (1978), Orey (1961), Hájek and Pudlák (1998) and Theorem. III 2.39 (i)  $\iff$  (ii) of Hájek and Pudlák (1998). Then he constructed a Kripke model (see Section 5.2.3 of Troelstra (1973)) for  $HA$  to derive the contradiction. See Smoryński (1982) for a proof formalized within a formal system  $PA + 1\text{-Con}(PA)$ .

However, the referee wrote

*“As far as I can see Smoryński leaves open whether there can be a consistent, classically unsound, finite extension of HA that implies full sentential excluded third. I definitely do believe there isn’t. It is unknown whether the analogous result holds for all classically invalid constructive propositional schemes.”*

The author cannot help but suppose that the language of the HA referee meant consists of the function symbols for all the primitive recursive functions and the identity predicate. It may be important to construct Kripke models of such HA by employing model theory of arithmetic. The author thinks the referee’s last sentence suggests a possible research direction.

As in the proof of Theorem 4.6, we hope that the wording “game,” “strategy,” “move,” and so on are useful to explain realizability interpretation neatly, and that various realizability interpretations of logical principles over HA are related to circumstances where one or the other player of a various game have a winning strategy, and the consequences of the existence of such strategies.

#### ACKNOWLEDGEMENT

The author acknowledges Susumu Hayashi, Pieter Hofstra, Stefano Berardi, an anonymous referee and Craig Smoryński. The anonymous referee informed the author of Smoryński’s work and Smoryński let the author know his course notes.

#### REFERENCES

- Akama, Y. (2004). Limiting partial combinatory algebras. *Theoret. Comput. Sci.*, 311(1-3):199–220.
- Akama, Y., Berardi, S., Hayashi, S., and Kohlenbach, U. (2004). An arithmetical hierarchy of the laws of excluded middle and related principles. In *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science*, pages 192–201.
- Avigad, J. (2000). Realizability interpretation for classical arithmetic. In Buss, Hájek, and Pudlák, editors, *Logic Colloquium ’98*, number 13 in Lecture Notes in Logic, pages 57–90. AK Peters.
- Beeson, M. J. (1985). *Foundations of constructive mathematics*, volume 6 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin. Metamathematical studies.
- Berardi, S. (2005). Classical logic as limit completion. *Math. Structures Comput. Sci.*, 15(1):167–200.
- Berardi, S., Bezem, M., and Coquand, T. (1998). On the computational content of the axiom of choice. *J. Symbolic Logic*, 63(2):600–622.
- Hájek, P. and Pudlák, P. (1998). *Metamathematics of first-order arithmetic*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin. Second printing.

- Hayashi, S., Sumitomo, R., and Shii, K. (2002). Towards animation of proofs – testing proofs by examples –. *Theoret. Comput. Sci.*, 272(1–2):177–195.
- Hindley, J. and Seldin, J. (1986). *Introduction to Combinators and Lambda-calculus*. Cambridge University Press.
- Hirschfeld, J. (1975). Models of arithmetic and recursive functions. *Israel J. Math.*, 20(2):111–126.
- Hofstra, P. and Cockett, R. (2010). Unitary theories, Unitary categories. *Electronic Notes in Theoretical Computer Science*, 265:11–33.
- Kleene, S. C. (1945). On the interpretation of intuitionistic number theory. *J. Symbolic Logic*, 10:109–124.
- Kleene, S. C. (1952). *Introduction to metamathematics*. D. Van Nostrand Co., Inc., New York, N. Y.
- Lerman, M. (1970). Recursive functions modulo  $\text{CO}-r$ -maximal sets. *Trans. Amer. Math. Soc.*, 148:429–444.
- Odifreddi, P. (1989). *Classical recursion theory*, volume 125 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam. The theory of functions and sets of natural numbers, With a foreword by G. E. Sacks.
- Orey, S. (1961). Relative interpretations. *Z. Math. Logik Grundlagen Math.*, 7:146–153.
- Smoryński, C. (1978). Nonstandard models of arithmetic. Course notes, fall 1978, Utrecht University. Logic group preprint series.
- Smoryński, C. (1982). Nonstandard models and constructivity. In *The L. E. J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*, volume 110 of *Stud. Logic Found. Math.*, pages 459–464. North-Holland, Amsterdam.
- Troelstra, A. S., editor (1973). *Metamathematical investigation of intuitionistic arithmetic and analysis*. Springer-Verlag, Berlin. Lecture Notes in Mathematics, Vol. 344.