# On Recursive Operations Over Logic LTS ${ }^{\dagger}$ 

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Recently, in order to mix algebraic and logic styles of specification in a uniform framework, the notion of a logic labelled transition system (Logic LTS or LLTS for short) has been introduced and explored. A variety of constructors over LLTS, including usual process-algebraic operators, logic connectives (conjunction and disjunction) and standard temporal operators (always and unless), have been given. However, no attempt has made so far to develop general theory concerning (nested) recursive operations over LLTS and a few fundamental problems are still open. This paper intends to study this issue in pure process-algebraic style. A few fundamental properties, including precongruence and the uniqueness of consistent solutions of equations, will be established.

## 1. Introduction

Algebra and logic are two dominant approaches for the specification, verification and systematic development of reactive and concurrent systems. They take different standpoints for looking at specifications and verifications, and offer complementary advantages.

Logical approaches devote themselves to specifying and verifying abstract properties of systems. In such frameworks, the most common reasonable property of concurrent systems, such as safety, liveness, etc., can be formulated in terms of logic formulas without resorting to operational details and verification is a deductive or model-checking activity. However, due to their global perspective and abstract nature, logical approaches often give little support for modular designing and compositional reasoning.

Algebraic approaches put attention to behavioral aspects of systems, which have tended to use formalisms in algebraic style. These formalisms are referred to as process algebra or process calculus. In such a paradigm, a specification and its implementation usually are formulated by terms (expressions) of a formal language built from a number of operators,

[^0]and the underlying semantics is often assigned operationally. The verification amounts to comparing terms, which is often referred to as implementation verification or equivalence checking. Algebraic approaches often support compositional construction and reasoning, which bring us advantages in developing systems, such as, supporting modular design and verification, avoiding verifying the whole system from scratch when its parts are modified, allowing reusability of proofs and so on (Andersen et al. 1994). Thus such approaches offer significant support for rigorous systematic development of reactive and concurrent systems. However, since algebraic approaches specify a system by means of prescribing in detail how the system should behave, it is often difficult for them to describe abstract properties of systems, which is a major disadvantage of such approaches.

In order to take advantage of these two approaches when designing systems, so-called heterogeneous specifications have been proposed, which uniformly integrate these two specification styles. Amongst, based on Büchi automata and labelled transition system (LTS) augmented with a predicate, Cleaveland and Lüttgen provide a semantic framework for heterogenous system design (Cleaveland and Lüttgen 2000, 2002). In this framework, not only usual operational operators but also logic connectives are considered, and must-testing preorder presented in (Nicola and Hennessy 1983) is adopted to capture refinement relation. Unfortunately, this setting does not support compositional reasoning since must-testing preorder is not a precongruence in this situation. Moreover, the logic connective conjunction in this framework lacks the desired property that $r$ is an implementation of a given specification $p \wedge q$ if and only if $r$ implements both $p$ and $q$.

Recently, Lüttgen and Vogler have introduced the notion of a Logic LTS (LLTS), which combines operational and logic styles of specification in one unified framework (Lüttgen and Vogler 2007, 2010, 2011). In addition to usual operational constructors, e.g., CSPstyle parallel composition, hiding and so on, logic connectives (conjunction and disjunction) and standard modal operators (always and unless) are also integrated into this framework. Moreover, the drawbacks in (Cleaveland and Lüttgen 2000, 2002) mentioned above have been remedied by adopting ready-tree semantics (Lüttgen and Vogler 2007). In order to support compositional reasoning in the presence of the parallel constructor, a variant of the usual notion of ready simulation is employed to capture the refinement relation, which has been shown to be the largest precongruence satisfying some desired properties (Lüttgen and Vogler 2010).

Along the direction suggested by Lüttgen and Vogler in (Lüttgen and Vogler 2010), a process calculus called CLL is presented in (Zhang et al. 2011), which reconstructs their setting in pure process-algebraic style. Moreover, a sound and ground-complete proof system for CLL is provided. In effect, it gives an axiomatization of ready simulation in the presence of logic operators. However, CLL is lack of capability of describing infinite behaviour, that is important for representing reactive systems.

It is well known that recursive operations are fundamental mechanisms for representing objects with infinite behavior in terms of finite expressions (see, for instance (Bergstra et al. 2001)). We extend CLL with recursive operations and propose a process calculus named CLL $_{R}$. Since LLTS involves consideration of inconsistencies, it is far from straightforward to re-establish existent results concerning recursive operations in this framework. A solid effort is required, especially for handling inconsistencies. This
paper intends to explore recursive operations over LLTS in pure process-algebraic style. A behavioral theory of $\mathrm{CLL}_{R}$ will be established, especially, we prove that the behavioral relation (i.e., ready simulation mentioned above) is precongruent w.r.t all operators in $\mathrm{CLL}_{R}$. Moreover, the uniqueness of solution of equations will be obtained, provided conditions that, given an equation $X=t_{X}, X$ is strongly guarded and does not occur in the scope of any conjunction in $t_{X}$.

The remainder of this paper is organized as follows. The next section recalls some related notions. Section 3 introduces SOS rules of $\mathrm{CLL}_{R}$. In section 4, the existence and uniqueness of stable transition model for $\mathrm{CLL}_{R}$ is demonstrated, and a few of basic properties of the LTS associated with CLL $_{R}$ are given. More further properties are considered in Section 5 . In section 6 , we shall show that the variant of ready simulation presented by Lüttgen and Vogler is precongruent in the presence of (nested) recursive operations. In section 7, a theorem on the uniqueness of solution of equations is obtained. Finally, a brief conclusion and discussion are given in Section 8.

## 2. Preliminaries

### 2.1. Logic LTS and ready simulation

This subsection will set up notations and briefly recall the notions of Logic LTS and ready simulation presented by Lüttgen and Vogler. For motivation behind these notions we refer the reader to (Lüttgen and Vogler 2007, 2010, 2011).

Let $A c t$ be the set of visible actions ranged over by letters $a, b$, etc., and let $A c t_{\tau}$ denote $A c t \cup\{\tau\}$ ranged over by $\alpha$ and $\beta$, where $\tau$ represents invisible actions. A labelled transition system (LTS) with a predicate is a quadruple $\left(P, A c t_{\tau}, \longrightarrow, F\right)$, where $P$ is a set of processes (states), $\longrightarrow \subseteq P \times A c t_{\tau} \times P$ is the transition relation and $F \subseteq P$.

As usual, we write $p \xrightarrow{\alpha} q$ if $(p, \alpha, q) \in \longrightarrow . q$ is an $\alpha$-derivative of $p$ if $p \xrightarrow{\alpha} q$. We write $p \xrightarrow{\alpha}$ (or, $p \xrightarrow{\alpha}$ ) if $\exists q \in P . p \xrightarrow{\alpha} q(\nexists q \in P . p \xrightarrow{\alpha} q$ respectively). Given a process $p$, the ready set $\left\{\alpha \in A c t_{\tau} \mid p \xrightarrow{\alpha}\right\}$ of $p$ is denoted by $\mathcal{I}(p)$. A state $p$ is stable if it cannot engage in any $\tau$-transition, i.e., $p \xrightarrow{\tau}$. The list below contains some useful decorated transition relations:
$p \xrightarrow{\alpha}_{F} q$ iff $p \xrightarrow{\alpha} q$ and $p, q \notin F$.
$p \xrightarrow{\epsilon} q$ iff $p(\xrightarrow{\tau})^{*} q$, where $(\xrightarrow{\tau})^{*}$ is the transitive reflexive closure of $\xrightarrow{\tau}$.
$p \xrightarrow{\alpha} q$ iff $\exists r, s \in P . p \stackrel{\epsilon}{\longrightarrow} r \xrightarrow{\alpha} s \xrightarrow{\epsilon} q$.
$p \xrightarrow{\gamma} \mid q$ iff $p \xrightarrow{\gamma} q \not^{\tau}$ with $\gamma \in A c t_{\tau} \cup\{\epsilon\}$.
$p \stackrel{\epsilon}{\Longrightarrow}{ }_{F} q$ iff there exists a sequence of $\tau$-labelled transitions from $p$ to $q$ such that all states along this sequence, including $p$ and $q$, are not in $F$. The decorated transition $p{ }^{\alpha}{ }_{F} q$ may be defined similarly.
$p \stackrel{\epsilon}{\Longrightarrow}_{F} \mid q\left(\right.$ or, $\left.p \xlongequal{\alpha}_{F} \mid q\right)$ iff $p \stackrel{\epsilon}{\Longrightarrow}_{F} q\left(p{ }^{\alpha}{ }_{F} q\right.$ respectively $)$ and $q$ is stable.
Remark 2.1. Notice that some notations above are slightly different from ones adopted by Lüttgen and Vogler. In (Lüttgen and Vogler 2010, 2011) the notation $p \stackrel{\epsilon}{\Longrightarrow} \mid q$ (or, $p \xlongequal{\alpha} \mid q)$ has the same meaning as $p{ }^{\epsilon}{ }_{F} \mid q\left(p{ }^{\alpha}{\underset{F}{F}} \mid q\right.$ respectively) in this paper, while $p \xlongequal{\epsilon} \mid q$ in this paper does not involve any requirement on consistency.

Definition 2.1 (Lüttgen and Vogler 2010). An LTS $\left(P, A c t_{\tau}, \longrightarrow, F\right)$ is an LLTS if, for each $p \in P$,
(LTS1) $p \in F$ if $\exists \alpha \in \mathcal{I}(p) \forall q \in P(p \xrightarrow{\alpha} q$ implies $q \in F)$;
(LTS2) $p \in F$ if $\nexists q \in P . p{ }^{\epsilon}{ }_{F} \mid q$.

Here the predicate $F$ is used to denote the set of all inconsistent states. In the sequel, we shall use the phrase "inconsistency predicate" to refer to $F$. The condition (LTS1) formalizes the backward propagation of inconsistencies, and (LTS2) captures the intuition that divergence (i.e., infinite sequences of $\tau$-transitions) should be viewed as catastrophic.

Compared with usual LTSs, it is one distinguishing feature of LLTS that it involves consideration of inconsistencies. Roughly speaking, the motivation behind such consideration lies in dealing with inconsistencies caused by conjunctive composition. For example, consider a simple composition $a .0 \wedge b .0$, it cannot be interpreted as deadlock 0 because a run of a process cannot begin with both actions $a$ and $b$, that is $a .0$ and $b .0$ specify processes with different ready sets. It is proper to tag it as an inconsistent specification. Moreover, inconsistencies could propagate backward. a.b. $0 \wedge$ a.c. 0 specifies the absence of any alternative $a$-transition leading to a consistent state. It should also be tagged as an inconsistent one. For more intuitive ideas and motivation about inconsistency, the reader may refer to (Lüttgen and Vogler 2007, 2010).

Definition 2.2 (Lüttgen and Vogler 2010). An $\operatorname{LTS}\left(P, A c t_{\tau}, \longrightarrow, F\right)$ is $\tau$-pure if, for each $p \in P, p \xrightarrow{\tau}$ implies $\nexists a \in$ Act. $p \xrightarrow{a}$.

Hence, for any state $p$ in a $\tau$-pure LTS, either $\mathcal{I}(p)=\{\tau\}$ or $\mathcal{I}(p) \subseteq$ Act and intuitively, it represents either an external or internal (disjunctive) choice between its outgoing transitions. Following (Lüttgen and Vogler 2010), this paper will focus on $\tau$-pure LLTSs.

In (Lüttgen and Vogler 2010, 2011), the notion of ready simulation below is adopted to capture the refinement relation, which is a variant of the usual notion of weak ready simulation. Such kind of ready simulation cares only stable consistent states.

Definition 2.3 (Ready simulation on LLTS). Let $\left(P, A c t_{\tau}, \longrightarrow, F\right)$ be a LLTS. A relation $\mathcal{R} \subseteq P \times P$ is a stable ready simulation relation, if for any $(p, q) \in \mathcal{R}$ and $a \in$ Act
(RS1) both $p$ and $q$ are stable;
(RS2) $p \notin F$ implies $q \notin F$;
(RS3) $p \Longrightarrow_{F} \mid p^{\prime}$ implies $\exists q^{\prime} \cdot q{ }^{a}{ }_{F} \mid q^{\prime}$ and $\left(p^{\prime}, q^{\prime}\right) \in \mathcal{R}$;
(RS4) $p \notin F$ implies $\mathcal{I}(p)=\mathcal{I}(q)$.
We say that $p$ is stable ready simulated by $q$, in symbols $p{\underset{\sim}{\sim}}_{R S} q$, if there exists a stable ready simulation relation $\mathcal{R}$ with $(p, q) \in \mathcal{R}$. Further, $p$ is ready simulated by $q$, written $p \sqsubseteq_{R S} q$, if $\forall p^{\prime}\left(p{ }^{\epsilon}{ }_{F} \mid p^{\prime}\right.$ implies $\exists q^{\prime}\left(q \xlongequal{\epsilon}_{F} \mid q^{\prime}\right.$ and $\left.\left.p^{\prime}{\underset{\sim}{\sqsubset}}_{R S} q^{\prime}\right)\right)$. The kernels of $\sqsubset_{R S}$ and $\sqsubseteq_{R S}$ are denoted by $\approx_{R S}$ and $=_{R S}$ respectively. It is easy to see that $\sqsubset_{R S}$ is a stable ready simulation relation and both $\check{\sim}_{R S}$ and $\sqsubseteq_{R S}$ are pre-order (i.e., reflexive and transitive).

### 2.2. Transition system specification

Structural Operational Semantics (SOS) is proposed by G. Plotkin in (Plotkin 1981), which adopts a syntax oriented view on operational semantics, and gives operational semantics in logical style. Transition System Specifications (TSSs), as presented by Groote and Vaandrager in (Groote 1992), are formalizations of SOS. This subsection recalls basic concepts related to TSS. For further information on this issue we refer the reader to (Aceto 2001, Bol and Groote 1996, Groote 1992).

Let $V_{A R}$ be an infinite set of variables and $\Sigma$ a signature. The set of $\Sigma$-terms over $V_{A R}$, denoted by $T\left(\Sigma, V_{A R}\right)$, is the least set such that (I) $V_{A R} \subseteq T\left(\Sigma, V_{A R}\right)$ and (II) if $f \in \Sigma$ and $t_{1}, \ldots, t_{n} \in T\left(\Sigma, V_{A R}\right)$, then $f\left(t_{1}, \ldots, t_{n}\right) \in T\left(\Sigma, V_{A R}\right)$, where $n$ is the arity of $f$. $T(\Sigma, \emptyset)$ is abbreviated by $T(\Sigma)$, elements in $T(\Sigma)$ are called closed or ground terms.

A substitution $\sigma$ is a mapping from $V_{A R}$ to $T\left(\Sigma, V_{A R}\right)$. As usual, a substitution $\sigma$ may be lifted to a mapping $T\left(\Sigma, V_{A R}\right) \rightarrow T\left(\Sigma, V_{A R}\right)$ by $\sigma\left(f\left(t_{1}, \ldots, t_{n}\right)\right) \triangleq f\left(\sigma\left(t_{1}\right), \ldots, \sigma\left(t_{n}\right)\right)$ for any n-arity $f \in \Sigma$ and $t_{1}, \ldots, t_{n} \in T\left(\Sigma, V_{A R}\right)$. A substitution is closed if it maps all variables to ground terms.

A TSS is a quadruple $\mathcal{P}=(\Sigma, \mathbb{A}, \mathbb{P}, \mathbb{R})$, where $\Sigma$ is a signature, $\mathbb{A}$ is a set of labels, $\mathbb{P}$ is a set of predicate symbols and $\mathbb{R}$ is a set of rules. Positive literals are all expressions of the form $t \xrightarrow{a} t^{\prime}$ or $t P$, while negative literals are all expressions of the form $t \stackrel{a}{\longrightarrow}$ or $t \neg P$, where $t, t^{\prime} \in T\left(\Sigma, V_{A R}\right), a \in \mathbb{A}$ and $P \in \mathbb{P}$. A literal is a positive or negative literal, and $\varphi, \psi, \chi$ are used to range over literals. A literal is ground or closed if all terms occurring in it are ground. A rule $r \in \mathbb{R}$ has the form like $\frac{H}{C}$, where $H$, the premises of the rule $r$, denoted $\operatorname{prem}(r)$, is a set of literals, and $C$, the conclusion of the rule $r$, denoted $\operatorname{conc}(r)$, is a positive literal. Furthermore, we write $\operatorname{pprem}(r)$ for the set of positive premises of $r$ and $\operatorname{nprem}(r)$ for the set of negative premises of $r$. A rule $r$ is positive if nprem $(r)=\emptyset$. A TSS is positive if it has only positive rules. Given a substitution $\sigma$ and a rule $r \in \mathbb{R}$, $\sigma(r)$ is the rule obtained from $r$ by replacing each variable in $r$ by its $\sigma$-image, that is, $\sigma(r) \triangleq \frac{\{\sigma(\varphi) \mid \varphi \in \operatorname{prem}(r)\}}{\sigma(\operatorname{conc}(r))}$. Moreover, if $\sigma$ is closed then $\sigma(r)$ is a ground instance of $r$.

Definition 2.4 (Proof in positive TSS). Let $\mathcal{P}=(\Sigma, \mathbb{A}, \mathbb{P}, \mathbb{R})$ be a positive TSS. A proof of a closed positive literal $\psi$ from $\mathcal{P}$ is a well-founded, upwardly branching tree, whose nodes are labelled with closed positive literals, such that

- the root is labelled with $\psi$,
- if $\chi$ is the label of a node $q$ and $\left\{\chi_{i}: i \in I\right\}$ is the set of labels of the nodes directly above $q$, then there is a rule $\left\{\varphi_{i}: i \in I\right\} / \varphi$ in $\mathbb{R}$ and a closed substitution $\sigma$ such that $\chi=\sigma(\varphi)$ and $\chi_{i}=\sigma\left(\varphi_{i}\right)$ for each $i \in I$.

If a proof of $\psi$ from $\mathcal{P}$ exists, then $\psi$ is provable from $\mathcal{P}$, in symbols $\mathcal{P} \vdash \psi$.
A natural and simple method of describing the operational nature of closed terms is in terms of LTSs. Given a TSS, an important problem is how to associate LTS with any given closed terms. For positive TSS, the answer is straightforward. However, this problem is far from trivial for TSS containing negative premises. The notions of stable model and stratification of TSS play an important role in dealing with this issue. The remainder of this subsection intends to recall these notions briefly.

Given a $\operatorname{TSS} \mathcal{P}=(\Sigma, \mathbb{A}, \mathbb{P}, \mathbb{R})$, a transition model $M$ is a subset of $\operatorname{Tr}(\Sigma, \mathbb{A}) \cup$
$\operatorname{Pred}(\Sigma, \mathbb{P})$, where $\operatorname{Tr}(\Sigma, \mathbb{A})=T(\Sigma) \times \mathbb{A} \times T(\Sigma)$ and $\operatorname{Pred}(\Sigma, \mathbb{P})=T(\Sigma) \times \mathbb{P}$, elements $\left(t, a, t^{\prime}\right)$ and $(t, P)$ in $M$ are written as $t \xrightarrow{a} t^{\prime}$ and $t P$ respectively. A positive closed literal $\psi$ holds in $M$ or $\psi$ is valid in $M$, in symbols $M \models \psi$, if $\psi \in M$. A negative closed literal $t \stackrel{a}{\xrightarrow{\prime}}$ (or, $t \neg P)$ holds in $M$, in symbols $M \models t \stackrel{a}{\nmid}(M \models t \neg P$ respectively), if there is no $t^{\prime}$ such that $t \xrightarrow{a} t^{\prime} \in M(t P \notin M$ respectively). For a set of closed literals $\Psi$, we write $M \models \Psi$ iff $M \models \psi$ for each $\psi \in \Psi$. $M$ is a model of $\mathcal{P}$ if, for each $r \in \mathbb{R}$ and $\sigma: V_{A R} \longrightarrow T(\Sigma)$, we have $M \models \operatorname{conc}(\sigma(r))$ whenever $M \models \operatorname{prem}(\sigma(r))$. $M$ is supported by $\mathcal{P}$ if, for each $\psi \in M$, there exists $r \in \mathbb{R}$ and $\sigma: V_{A R} \longrightarrow T(\Sigma)$ such that $M \models \operatorname{prem}(\sigma(r))$ and $\operatorname{conc}(\sigma(r))=\psi . M$ is a supported model of $\mathcal{P}$ if $M$ is supported by $\mathcal{P}$ and $M$ is a model of $\mathcal{P}$.

Definition 2.5 (Aceto 2001; Bol and Groote 1996). Let $\mathcal{P}=(\Sigma, \mathbb{A}, \mathbb{P}, \mathbb{R})$ be a TSS and $\alpha$ an ordinal number. A function $S: \operatorname{Tr}(\Sigma, \mathbb{A}) \cup \operatorname{Pred}(\Sigma, \mathbb{P}) \longrightarrow \alpha$ is a stratification of $\mathcal{P}$ if, for every rule $r \in \mathbb{R}$ and every substitution $\sigma: V_{A R} \longrightarrow T(\Sigma)$, the following conditions hold.
(1) $S(\psi) \leq S(\operatorname{conc}(\sigma(r)))$ for each $\psi \in \operatorname{pprem}(\sigma(r))$,
(2) $S(t P)<S(\operatorname{conc}(\sigma(r)))$ for each $t \neg P \in \operatorname{nprem}(\sigma(r))$, and
(3) $S\left(t \xrightarrow{a} t^{\prime}\right)<S(\operatorname{conc}(\sigma(r)))$ for each $t^{\prime} \in T(\Sigma)$ and $t{ }^{a} \xrightarrow{a} \operatorname{nprem}(\sigma(r))$.

A TSS is stratified iff there exists a stratification function for it.
Definition 2.6 (Bol and Groote 1996; Gelfond and Lifchitz 1988). Let $\mathcal{P}=(\Sigma, \mathbb{A}, \mathbb{P}, \mathbb{R})$ be a TSS and $M$ a transition model. $M$ is a stable transition model for $\mathcal{P}$ if

$$
M=M_{\operatorname{Strip}(\mathcal{P}, M)}
$$

where $\operatorname{Strip}(\mathcal{P}, M)$ is the $\operatorname{TSS}(\Sigma, \mathbb{A}, \mathbb{P}, \operatorname{Strip}(\mathbb{R}, M))$ with

$$
\operatorname{Strip}(\mathbb{R}, M) \triangleq\left\{\left.\frac{\operatorname{pprem}(r)}{\operatorname{conc}(r)} \right\rvert\, \quad r \in \mathbb{R}_{\text {ground }} \text { and } M \models \operatorname{nprem}(r)\right\}
$$

where $\mathbb{R}_{\text {ground }}$ denotes the set of all ground instances of rules in $\mathbb{R}$, and $M_{\operatorname{Strip}(\mathcal{P}, M)}$ is the least transition model of the positive $\operatorname{TSS} \operatorname{Strip}(\mathcal{P}, M)$.

As is well known, stable models are supported models and each stratified TSS $\mathcal{P}$ has a unique stable model (Bol and Groote 1996); moreover, such stable model does not depend on particular stratification function (Groote 1993).

## 3. Syntax and SOS rules of $\mathrm{CLL}_{R}$

The calculus CLL $_{R}$ is obtained from CLL by enriching it with recursive operations. Following (Baeten and Bravetti 2008), this paper adopts the notation $\langle X \mid E\rangle$ to denote recursive operations, which encompasses both the CCS operator recX.t and standard way of expressing recursion in ACP. Formally, the terms in $\mathrm{CLL}_{R}$ are defined by BNF:

$$
t::=0|\perp|(\alpha . t)|(t \square t)|(t \wedge t)|(t \vee t)|\left(t \|_{A} t\right)|X|\langle X \mid E\rangle
$$

where $X \in V_{A R}, \alpha \in A c t_{\tau}, A \subseteq A c t$ and recursive specification $E=E(V)$ is a nonempty finite set of equations $E=\{X=t \mid X \in V\}$. As usual, 0 encodes deadlock. The prefix $\alpha . t$ has a single capability, expressed by $\alpha$; the process $t$ cannot proceed until $\alpha$ has been
exercised. $\square$ is an external choice operator. $\|_{A}$ is a CSP-style parallel operator, $t_{1} \|_{A} t_{2}$ represents a process that behaves as $t_{1}$ in parallel with $t_{2}$ under the synchronization set $A . \perp$ represents an inconsistent process with empty behavior. $\vee$ and $\wedge$ are logical operators, which are intended for describing logical combinations of processes.

In the sequel, we often denote $\left\langle X \mid\left\{X=t_{X}\right\}\right\rangle$ briefly by $\left\langle X \mid X=t_{X}\right\rangle$. Given a term $\langle X \mid E\rangle$ and variable $Y$, the phrase " $Y$ occurs in $\langle X \mid E\rangle$ " means that $Y$ occurs in $t_{Z}$ for some $Z=t_{Z} \in E$. Moreover, the scope of a recursive operation $\langle X \mid E\rangle$ exactly consists of all $t_{Z}$ with $Z=t_{Z} \in E$. An occurrence of a variable $X$ in a given $t$ is free if it does not occur in the scope of any recursive operation $\langle Y \mid E\rangle$ with $E=E(V)$ and $X \in V$. A variable $X$ in term $t$ is a free variable if all occurrences of $X$ in $t$ are free, otherwise $X$ is a recursive variable in $t$.

Convention 3.1. Throughout this paper, as usual, we make the assumption that recursive variables are distinct from each other. That is, for any two recursive specifications $E\left(V_{1}\right)$ and $E^{\prime}\left(V_{2}\right)$ we have $V_{1} \cap V_{2}=\emptyset$. Moreover, we will tacitly restrict our attention to terms where no recursive variable has free occurrences. For example we will not consider terms such as $X \square\langle X \mid X=a . X\rangle$ because this term could be replaced by the clear term $X \square\langle Y \mid Y=a . Y\rangle$ with the same meaning.

On account of the above convention, given a term $t$, the set $F V(t)$ of all free variables of $t$ may be defined recursively as:
$-F V(X)=\{X\} ; F V(0)=F V(\perp)=\emptyset ; F V(\alpha . t)=F V(t) ;$
$-F V\left(t_{1} \odot t_{2}\right)=F V\left(t_{1}\right) \cup F V\left(t_{2}\right)$ with $\odot \in\left\{\vee, \wedge, \square, \|_{A}\right\} ;$
$-F V(\langle Y \mid E\rangle)=\bigcup_{Z=t_{Z} \in E} F V\left(t_{Z}\right)-V$ where $E=E(V)$.
As usual, a term $t$ is closed if $F V(t)=\emptyset$. The set of all closed terms of $\mathrm{CLL}_{R}$ is denoted $T\left(\Sigma_{\mathrm{CLL}_{R}}\right)$. In the following, a term is a process iff it is closed. Unless noted otherwise we use $p, q, r$ to represent processes. We shall always use $t_{1} \equiv t_{2}$ to mean that expressions $t_{1}$ and $t_{2}$ are syntactically identical. In particular, $\langle Y \mid E\rangle \equiv\left\langle Y^{\prime} \mid E^{\prime}\right\rangle$ means that $Y \equiv Y^{\prime}$ and for any $Z$ and $t_{Z}, Z=t_{Z} \in E$ iff $Z=t_{Z} \in E^{\prime}$.

Definition 3.1. For any recursive specification $E(V)$ and term $t$, we define $\langle t \mid E\rangle$ to be $t\{\langle X \mid E\rangle / X: X \in V\}$, that is, $\langle t \mid E\rangle$ is obtained from $t$ by simultaneously replacing all free occurrences of each $X(\in V)$ by $\langle X \mid E\rangle$.

For example, consider $t \equiv X \square a .\langle Y \mid Y=X \square Y\rangle$ and $E(\{X\})=\left\{X=t_{X}\right\}$ then $\langle t \mid E\rangle \equiv\left\langle X \mid X=t_{X}\right\rangle \square a .\left\langle Y \mid Y=\left\langle X \mid X=t_{X}\right\rangle \square Y\right\rangle$. In particular, for any recursive specification $E(V)$ and $t \equiv X,\langle t \mid E\rangle \equiv\langle X \mid E\rangle$ whenever $X \in V$ and $\langle t \mid E\rangle \equiv X$ if $X \notin V$.

As usual, an occurrence of $X$ in $t$ is strongly (or, weakly) guarded if such occurrence is within some subexpression $a . t_{1}$ with $a \in \operatorname{Act}$ ( $\tau . t_{1}$ or $t_{1} \vee t_{2}$ respectively). A variable $X$ is strongly (or, weakly) guarded in $t$ if each occurrence of $X$ is strongly (weakly respectively) guarded. Notice that, since the first move of $r \vee s$ is a $\tau$-labelled transition (see Table 11), which is independent of $r$ and $s$, any occurrence of $X$ in $r \vee s$ is treated as being weakly guarded. A recursive specification $E(V)$ is guarded if for each $X \in V$ and $Z=t_{Z} \in E$, each occurrence of $X$ in $t_{Z}$ is (weakly or strongly) guarded.

Convention 3.2. It is well known that unguarded processes cause many problems in

$$
\begin{aligned}
& \left(R a_{1}\right) \frac{-}{\alpha \cdot x_{1} \xrightarrow{\alpha} x_{1}} \\
& \left(R a_{2}\right) \frac{x_{1} \xrightarrow{a} y_{1}, x_{2} \not 一 ⿱^{\top}}{x_{1} \square x_{2} \xrightarrow{a} y_{1}} \\
& \left(R a_{3}\right) \xrightarrow[{x_{1}{ }^{\tau}{ }^{\tau}, x_{2} \xrightarrow{a} y_{2}}]{x_{1}} y_{2} \\
& \left(R a_{4}\right) \xrightarrow[{x_{1} \xrightarrow{x_{1}} y_{1} \xrightarrow{\tau} y_{1} \square x_{2}}]{x_{1}} \\
& \left(R a_{5}\right) \frac{x_{2} \xrightarrow{\tau} y_{2}}{x_{1} \square x_{2} \xrightarrow{\tau} x_{1} \square y_{2}} \\
& \left(R a_{7}\right) \frac{x_{1} \xrightarrow{\tau} y_{1}}{x_{1} \wedge x_{2} \xrightarrow{\tau} y_{1} \wedge x_{2}} \\
& \left(R a_{6}\right) \frac{x_{1} \xrightarrow{a} y_{1}, x_{2} \xrightarrow{a} y_{2}}{x_{1} \wedge x_{2} \xrightarrow{a} y_{1} \wedge y_{2}} \\
& \left(R a_{8}\right) \frac{x_{2} \xrightarrow{\tau} y_{2}}{x_{1} \wedge x_{2} \xrightarrow{\tau} x_{1} \wedge y_{2}} \\
& \left(R a_{9}\right) \frac{-}{x_{1} \vee x_{2} \xrightarrow{\tau} x_{1}} \\
& \left(R a_{11}\right) \frac{x_{1} \xrightarrow{\tau} y_{1}}{x_{1}\left\|_{A} x_{2} \xrightarrow{\tau} y_{1}\right\|_{A} x_{2}} \\
& \left(R a_{13}\right) \frac{x_{1} \xrightarrow{a} y_{1}, x_{2} f^{\top}}{x_{1}\left\|_{A} x_{2} \xrightarrow{a} y_{1}\right\|_{A} x_{2}}(a \notin A) \\
& \left(R a_{15}\right) \frac{x_{1} \xrightarrow{a} y_{1}, x_{2} \xrightarrow{a} y_{2}}{x_{1}\left\|_{A} x_{2} \xrightarrow{a} y_{1}\right\|_{A} y_{2}}(a \in A) \\
& \left(R a_{10}\right) \frac{-}{x_{1} \vee x_{2} \xrightarrow{\tau} x_{2}} \\
& \left(R a_{12}\right) \frac{x_{2} \xrightarrow{\tau} y_{2}}{x_{1}\left\|_{A} x_{2} \xrightarrow{\tau} x_{1}\right\|_{A} y_{2}} \\
& \left(R a_{14}\right) \frac{x_{1} f^{\tau}, x_{2} \xrightarrow{a} y_{2}}{x_{1}\left\|_{A} x_{2} \xrightarrow{a} x_{1}\right\|_{A} y_{2}}(a \notin A) \\
& \left(R a_{16}\right) \frac{\left\langle t_{X} \mid E\right\rangle \xrightarrow{\alpha} y}{\langle X \mid E\rangle \xrightarrow{\alpha} y}\left(X=t_{X} \in E\right)
\end{aligned}
$$

Table 1．Operational rules
many aspects of the theory（Milner 1983）and unguarded recursion is incompatible with negative rules（Bloom 1994）．As usual，this paper will focus on guarded recursive specifi－ cations．That is，we assume that all recursive specifications considered in the remainder of this paper are guarded．

We now provide SOS rules to specify the behavior of processes（i．e．，closed terms） formally．All SOS rules are divided into two parts：operational and predicate rules．

Operational rules $R a_{i}(1 \leq i \leq 16)$ are listed in Table 1 where $a \in A c t, \alpha \in A c t_{\tau}$ and $A \subseteq A c t$ ．Negative premises in Rules $R a_{2}, R a_{3}, R a_{13}$ and $R a_{14}$ give $\tau$－transition prece－ dence over transitions labelled with visible actions，which guarantees that the transition model of $\mathrm{CLL}_{R}$ is $\tau$－pure．Rules $R a_{9}$ and $R a_{10}$ illustrate that the operational aspect of $t_{1} \vee t_{2}$ is same as internal choice in usual process calculus．Rule $R a_{6}$ reflects that con－ junction operator is a synchronous product for visible transitions．The operational rules of the other operators are as usual．

Predicate rules in Table 2 specify the inconsistency predicate $F .0$ and $\perp$ represent different processes．Rule $R p_{1}$ says that $\perp$ is inconsistent．Thus $\perp$ cannot be implemented． While 0 is consistent，which is an implementable process．Rule $R p_{3}$ reflects that if both two disjunctive parts are inconsistent then so is the disjunction．Rules $R p_{4}-R p_{9}$ de－ scribe the system design strategy that if one part is inconsistent，then so is the whole composition．Rules $R p_{10}$ and $R p_{11}$ reveal that a stable conjunction is inconsistent if its conjuncts have distinct ready sets．

$$
\left.\begin{array}{ll}
\left(R p_{1}\right) \frac{-}{\perp F} & \left(R p_{2}\right) \frac{x_{1} F}{\alpha \cdot x_{1} F} \\
\left(R p_{3}\right) \frac{x_{1} F, x_{2} F}{x_{1} \vee x_{2} F} & \left(R p_{4}\right) \frac{x_{1} F}{x_{1} \square x_{2} F} \\
\left(R p_{5}\right) \frac{x_{2} F}{x_{1} \square x_{2} F} & \left(R p_{6}\right) \frac{x_{1} F}{x_{1} \|_{A} x_{2} F} \\
\left(R p_{7}\right) \frac{x_{2} F}{x_{1} \|_{A} x_{2} F} & \left(R p_{8}\right) \frac{x_{1} F}{x_{1} \wedge x_{2} F} \\
\left(R p_{9}\right) \frac{x_{2} F}{x_{1} \wedge x_{2} F} & \left(R p_{10}\right) \xrightarrow{x_{1} \xrightarrow{a} y_{1}, x_{2} \not{ }^{a} \rightarrow, x_{1} \wedge x_{2} \AA^{\tau}} \\
x_{1} \wedge x_{2} F
\end{array}\right)
$$

Table 2. Predicate rules

Rules $R p_{13}$ and $R p_{15}$ are used to capture (LTS2) in Def. 2.1] which are the abbreviation of the rules with the format

$$
\frac{\left\{y F: \exists y_{0}, y_{1}, \ldots, y_{n}\left(z \equiv y_{0} \xrightarrow{\tau} y_{1} \xrightarrow{\tau} \cdots \xrightarrow{\tau} y_{n} \equiv y \text { and } y f^{\tau}\right)\right\}}{z F}
$$

with $z \equiv x_{1} \wedge x_{2}$ or $\langle X \mid E\rangle$. Intuitively, these two rules say that if all stable $\tau$-descendants of $z$ are inconsistent, then $z$ itself is inconsistent. Notice that, especially for readers who are familiar with notations used in (Lüttgen and Vogler 2010), the transition relation $\xlongequal{\epsilon} \mid$ occurring in these two rules does not involve any requirement on consistency (see Remark 2.1 and notations above it).

Since the behavior of any process in CLL is finite, each process can reach a stable state, and Rules $R p_{1}-R p_{12}$ suffice to capture the inconsistency predicate $F$. In particular, these rules guarantee that the LTS associated with CLL satisfies (LTS1) and (LTS2) in Def. 2.1 (Zhang et al. 2011). However, for $\mathrm{CLL}_{R}$, Rules $R p_{1}-R p_{12}$ are insufficient even if the usual rule for recursive operations (i.e. $R p_{14}$ ) is added. For instance, consider processes $q \equiv\langle X \mid X=\tau . X\rangle$ and $p \equiv\langle X \mid X=X \vee 0\rangle \wedge a .0$, it is not difficult to see that neither $q F$ nor $p F$ can be inferred by using only Rules $R p_{1}-R p_{12}$ and $R p_{14}$, however, both $p$ and $q$ should be inconsistent due to (LTS2). Fortunately, an inference of $p F$ (or, $q F$ ) is at hand by admitting Rule $R p_{13}$ ( $R p_{15}$ respectively).

Summarizing, the TSS for $\mathrm{CLL}_{R}$ is $\mathcal{P}_{\mathrm{CLL}_{R}}=\left(\Sigma_{\mathrm{CLL}_{R}}, A c t_{\tau}, \mathbb{P}_{\mathrm{CLL}_{R}}, \mathbb{R}_{\mathrm{CLL}_{R}}\right)$, where
$-\Sigma_{\mathrm{CLL}_{R}}=\{\square, \wedge, \vee, 0, \perp\} \cup\left\{\alpha .(): \alpha \in A_{\tau} t_{\tau}\right\} \cup\left\{\|_{A}: A \subseteq A c t\right\} \cup\{\langle X \mid E\rangle: E=$ $E(V)$ is a guarded recursive specification with $X \in V\}$,
$-\mathbb{P}_{\mathrm{CLL}_{R}}=\{F\}$, and
$-\mathbb{R}_{\mathrm{CLL}_{R}}=\left\{R a_{1}, \ldots, R a_{16}\right\} \cup\left\{R p_{1}, \ldots, R p_{15}\right\}$.

## 4. Stable transition model of $\mathcal{P}_{\mathrm{CLL}_{R}}$

This section will consider the well-definedness of the $\operatorname{TSS} \mathcal{P}_{\mathrm{CLL}_{R}}$ (i.e., the existence and uniqueness of the stable model of $\mathcal{P}_{\mathrm{CLL}_{R}}$ ) and provide a few basic properties of the LTS associated with $\mathcal{P}_{\mathrm{CLL}_{R}}$.

As we know, it is not trivial that a TSS with rules with negative premises and recursion has a unique stable model. In order to demonstrate that $\mathcal{P}_{\mathrm{CLL}_{R}}$ has one, it is sufficient to give a stratification function of $\mathcal{P}_{\mathrm{CLL}_{R}}$. To this end, a few preliminary notations are introduced. Given a term $t$, the degree of $t$, denoted by $|t|$, is inductively defined as:
$-|0|=|\perp|=|\langle X \mid E\rangle| \triangleq 1 ;$
$-\left|t_{1} \odot t_{2}\right| \triangleq\left|t_{1}\right|+\left|t_{2}\right|+1$ for each $\odot \in\left\{\wedge, \square, \vee, \|_{A}\right\}$;
$-|\alpha . t| \triangleq|t|+1$ with $\alpha \in A c t_{\tau}$.
The function $G: T\left(\Sigma_{\mathrm{CLL}_{R}}\right) \longrightarrow \mathbb{N}$ is defined by:
$-G(\langle X \mid E\rangle) \triangleq 1$;
$-G(0)=G(\perp)=G(\alpha . t)=G\left(t_{1} \vee t_{2}\right) \triangleq 0$ with $\alpha \in A c t_{\tau}$;
$-G\left(t_{1} \odot t_{2}\right) \triangleq G\left(t_{1}\right)+G\left(t_{2}\right)$ for each $\odot \in\left\{\wedge, \square, \|_{A}\right\}$.
Clearly, given a term $t, G(t)$ is the number of unguarded recursive operations occurring in $t$. Further, the function $S_{\mathcal{P}_{\mathrm{CLL}_{R}}}$ from $\operatorname{Tr}\left(\Sigma_{\mathrm{CLL}_{R}}, \operatorname{Act} t_{\tau}\right) \cup \operatorname{Pred}\left(\Sigma_{\mathrm{CLL}_{R}}, \mathbb{P}_{\mathrm{CLL}_{R}}\right)$ to $\omega \times 2+1$ is given below, where $\omega$ is the initial limit ordinal,
$-S_{\mathcal{P}_{\mathrm{CLL}_{R}}}\left(t \xrightarrow{\alpha} t^{\prime}\right) \triangleq G(t) \times \omega+|t|$;
$-S_{\mathcal{P}_{\mathrm{CLL}_{R}}}(t F) \triangleq \omega \times 2$.
Since each recursive specification is assumed to be guarded (see, Convention 3.2), it is not difficult to check that this function $S_{\mathcal{P}_{\mathrm{CLL}_{R}}}$ is a stratification of $\mathcal{P}_{\mathrm{CLL}_{R}}$. Moreover, since each stratified TSS has a unique stable model (Bol and Groote 1996), $\mathcal{P}_{\mathrm{CLL}_{R}}$ has a unique stable transition model. From now on, we use $M_{\mathrm{CLL}_{R}}$ to denote such stable model.

Definition 4.1. The LTS associated with $\mathrm{CLL}_{R}$, in symbols $\operatorname{LTS}\left(\mathrm{CLL}_{R}\right)$, is the quadruple $\left(T\left(\Sigma_{\mathrm{CLL}_{R}}\right), A c t_{\tau}, \longrightarrow \mathrm{CLL}_{R}, F_{\mathrm{CLL}_{R}}\right)$, where
$-p \xrightarrow{\alpha} \mathrm{CLL}_{R} p^{\prime}$ iff $p \xrightarrow{\alpha} p^{\prime} \in M_{\mathrm{CLL}_{R}}$;
$-p \in F_{\mathrm{CLL}_{R}}$ iff $p F \in M_{\mathrm{CLL}_{R}}$.
Therefore, $p \xrightarrow{\alpha} \operatorname{CLL}_{R} p^{\prime}$ (or, $p \in F_{\mathrm{CLL}_{R}}$ ) if and only if $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash p \xrightarrow{\alpha}$ $p^{\prime}$ ( $p F$ respectively) for any processes $p, p^{\prime}$ and $\alpha \in A c t_{\tau}$. This allows us to proceed by induction on the depth of inferences when demonstrating propositions concerning $\longrightarrow \mathrm{CLL}_{R}$ and $F_{\mathrm{CLL}_{R}}$.

Convention 4.1. For the sake of convenience, in the remainder of this paper, we shall omit the subscript in labelled transition relations $\xrightarrow{\alpha} \mathrm{CLL}_{R}$, that is, we shall use $\xrightarrow{\alpha}$ to denote transition relation in $\operatorname{LTS}\left(\mathrm{CLL}_{R}\right)$. Thus, the notation $\xrightarrow{\alpha}$ has double utility: predicate symbols in the TSS $\mathcal{P}_{\mathrm{CLL}_{R}}$ and labelled transition relations on processes in $\operatorname{LTS}\left(\mathrm{CLL}_{R}\right)$. However, it usually does not lead to confusion in a given context. Similarly,
the notation $F_{\mathrm{CLL}_{R}}$ will be abbreviated by $F$. Hence the symbol $F$ is overloaded, predicate symbol in the TSS $\mathcal{P}_{\mathrm{CLL}_{R}}$ and the set of all inconsistent processes within $\operatorname{LTS}\left(\mathrm{CLL}_{R}\right)$, in each case the context of use will allow us to make the distinction.

In the following, we intend to provide a number of simple properties of $\operatorname{LTS}\left(\mathrm{CLL}_{R}\right)$. In particular, we will show that $L T S\left(\mathrm{CLL}_{R}\right)$ is a $\tau$-pure LLTS.

Lemma 4.1. Let $p$ and $q$ be any two processes.
(1) $p \vee q \in F$ iff $p, q \in F$.
(2) $\alpha . p \in F$ iff $p \in F$ for each $\alpha \in A c t_{\tau}$.
(3) $p \odot q \in F$ iff either $p \in F$ or $q \in F$ with $\odot \in\left\{\square, \|_{A}\right\}$.
(4) Either $p \in F$ or $q \in F$ implies $p \wedge q \in F$.
(5) $0 \notin F$ and $\perp \in F$.
(6) $\langle X \mid X=\tau . X\rangle \in F$.
(7) If $\forall q(p \xlongequal{\epsilon} \mid q$ implies $q \in F)$ then $p \in F$.
(8) $\langle X \mid E\rangle \in F$ iff $\left\langle t_{X} \mid E\right\rangle \in F$ for each $X$ with $X=t_{X} \in E$.

Proof. Items (1) - (6) are straightforward. For item (7), it proceeds by induction on $p$, in particular, for the case where $p$ is of the format $p_{1} \wedge p_{2}$ (or $\langle X \mid E\rangle$ ), the conclusion immediately follows due to Rule $R p_{13}$ ( $R p_{15}$ respectively).

For item (8), the implication from right to left is straightforward. The argument of the converse implication splits into two cases based on the last rule applied in the proof tree of $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash\langle X \mid E\rangle F$. If Rule $R p_{14}$ is the last rule then the proof is trivial. For the other case where Rule $R p_{15}$ is used, it is also straightforward by applying item (7) in this lemma and the fact that $\langle X \mid E\rangle \xrightarrow{\tau} r$ iff $\left\langle t_{X} \mid E\right\rangle \xrightarrow{\tau} r$ for any $r$.

The notion of $\tau$-purity is a technical constraint for LLTSs (Lüttgen and Vogler 2007, 2010). The result below shows that $\operatorname{LTS}\left(\mathrm{CLL}_{R}\right)$ is indeed $\tau$-pure.

Theorem 4.1. $L T S\left(\mathrm{CLL}_{R}\right)$ is $\tau$-pure.
Proof. Suppose $p \xrightarrow{\tau}$. Hence $p \xrightarrow{\tau} q$ for some $q$. Then the lemma would be established by proving that $p \xrightarrow{a}$ for any $a \in A c t$. It is straightforward by induction on the depth of the proof tree of $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash p \xrightarrow{\tau} q$.

In order to prove that $L T S\left(\mathrm{CLL}_{R}\right)$ is a LLTS, the result below is needed. Its converse is an instance of (LTS1) with $\alpha=\tau$, and hence also holds by Theorem4.2,

Lemma 4.2. For any process $p$ with $\tau \in \mathcal{I}(p)$, if $p \in F$ then $\forall q(p \xrightarrow{\tau} q$ implies $q \in F)$.
Proof. Suppose $p \xrightarrow{\tau} q$. We may prove $q \in F$ by induction on the depth of the proof tree $\mathcal{T}$ of $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash p \xrightarrow{\tau} q$. It proceeds by distinguishing different cases based on the form of $p$. Here we handle only three cases as examples.

Case $1 p \equiv p_{1} \square p_{2}$.
W.l.o.g, assume the last rule applied in $\mathcal{T}$ is $\frac{p_{1} \xrightarrow{\tau} p_{1}^{\prime}}{p_{1} \square p_{2} \xrightarrow{\tau} p_{1}^{\prime} \square p_{2}}$. Hence $q \equiv p_{1}^{\prime} \square p_{2}$. Since $p \in F$, by Lemma 4.1(3), $p_{1} \in F$ or $p_{2} \in F$. If $p_{2} \in F$ then it immediately follows from

Lemma 4.1 (3) that $q \equiv p_{1}^{\prime} \square p_{2} \in F$. If $p_{1} \in F$ then $p_{1}^{\prime} \in F$ by induction hypothesis (IH, for short). Hence $p_{1}^{\prime} \square p_{2} \in F$, as desired.

Case $2 p \equiv\langle X \mid E\rangle$.
The last rule applied in $\mathcal{T}$ is $\frac{\left\langle t_{X} \mid E\right\rangle \xrightarrow{\tau} q}{\langle X \mid E\rangle \xrightarrow{\tau} q}$ with $X=t_{X} \in E$. Since $p \in F$, by Lemma 4.1(8), we have $\left\langle t_{X} \mid E\right\rangle \in F$. Then $q \in F$ by applying IH.

Case $3 p \equiv p_{1} \wedge p_{2}$.
W.l.o.g, assume the last rule applied in $\mathcal{T}$ is $\frac{p_{1} \xrightarrow{\tau} p_{1}^{\prime}}{p_{1} \wedge p_{2} \xrightarrow{\tau} p_{1}^{\prime} \wedge p_{2}}$. Hence $q \equiv p_{1}^{\prime} \wedge p_{2}$. In the following, we intend to show $q \in F$ by distinguishing four cases based on the last rule applied in the inference of $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash p_{1} \wedge p_{2} F$.

Case $3.1 \frac{p_{1} F}{p_{1} \wedge p_{2} F}$ or $\frac{p_{2} F}{p_{1} \wedge_{2} F}$.
Similar to Case 1, omitted.
Case $3.2 \xrightarrow{\stackrel{p_{1} \xrightarrow{a} r, p_{2} f^{a} \rightarrow, p_{1} \wedge p_{2} f^{\top}}{p_{1} \wedge p_{2} F}}$ or $\frac{p_{1} \not^{a} \xrightarrow{2} p_{2} \xrightarrow{a} r, p_{1} \wedge p_{2} f^{\top}}{p_{1} \wedge p_{2} F}$.
This case is impossible because of $\tau \in \mathcal{I}\left(p_{1} \wedge p_{2}\right)$.
Case $3.3 \xrightarrow{p_{1} \wedge p_{2} \xrightarrow{\alpha} r,\left\{r^{\prime} F: p_{1} \wedge p_{2} \xrightarrow{\alpha} r^{\prime}\right\}} p_{1} \wedge p_{2} F$.
Since $\operatorname{LTS}\left(\mathrm{CLL}_{R}\right)$ is $\tau$-pure and $p_{1} \wedge p_{2} \xrightarrow{\tau}$, we have $\alpha=\tau$. Hence $q \in F$ immediately.
Case 3.4 $\frac{\left\{r F: p_{1} \wedge p_{2} \xlongequal{\epsilon} \mid r\right\}}{p_{1} \wedge p_{2} F}$.
Assume $q \equiv p_{1}^{\prime} \wedge p_{2} \xlongequal{\epsilon} \mid r^{\prime}$. Thus $r^{\prime} \in F$ due to $p \xrightarrow{\tau} p_{1}^{\prime} \wedge p_{2} \xlongequal{\epsilon} \mid r^{\prime}$. Hence $p_{1}^{\prime} \wedge p_{2} \in F$ by applying Rule $R p_{13}$.

Now we are ready to show that $L T S\left(\mathrm{CLL}_{R}\right)$ is a LLTS.
Theorem 4.2. $L T S\left(\mathrm{CLL}_{R}\right)$ is a LLTS.
Proof. (LTS1) Suppose $\alpha \in \mathcal{I}(p)$ and $\forall r(p \xrightarrow{\alpha} r$ implies $r \in F)$. Then $p \xrightarrow{\alpha} q$ for some $q$. To complete the proof, we intend to show $p \in F$. It proceeds by induction on the depth of the proof tree $\mathcal{T}$ of $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash p \xrightarrow{\alpha} q$. We distinguish different cases based on the form of $p$. In particular, the proof for the case $p \equiv p_{1} \wedge p_{2}$ is immediate by Rule $R p_{12}$. In the following, we give the proof for the case $p \equiv p_{1} \|_{A} p_{2}$, the other cases are left to the reader. The argument splits into two cases depending on $\alpha$.

Case $1 \alpha=\tau$.
W.l.o.g, assume the last rule applied in $\mathcal{T}$ is $\frac{p_{1} \xrightarrow{\tau} p_{1}^{\prime}}{p_{1}\left\|_{A} p_{2} \xrightarrow[\longrightarrow]{\tau} p_{1}^{\prime}\right\|_{A} p_{2}}$. Thus $q \equiv p_{1}^{\prime} \|_{A} p_{2}$. If $p_{2} \in F$ then $p_{1} \|_{A} p_{2} \in F$ follows from Lemma 4.1(3) at once. For the other case $p_{2} \notin F$, it is not difficult to see that each $\tau$-derivative of $p_{1}$ is inconsistent, that is $\forall p_{1}^{\prime \prime}\left(p_{1} \xrightarrow{\tau} p_{1}^{\prime \prime}\right.$ implies $\left.p_{1}^{\prime \prime} \in F\right)$. Hence $p_{1} \in F$ by IH. Therefore it follows from Lemma 4.1(3) that $p_{1} \|_{A} p_{2} \in F$, as desired.

Case $2 \alpha \in$ Act.

In this situation, the last rule applied in $\mathcal{T}$ has one of the following three formats: (1) $\frac{p_{1} \xrightarrow{\alpha} p_{1}^{\prime}, p_{2} f^{\top}}{p_{1}\left\|_{A} p_{2} \xrightarrow{\alpha} p_{1}^{\prime}\right\|_{A} p_{2}}(\alpha \notin A)$; (2) $\frac{p_{2} \xrightarrow{\alpha} p_{2}^{\prime}, p_{1} \digamma^{\top}}{p_{1}\left\|_{A} p_{2} \xrightarrow{\longrightarrow} p_{1}\right\|_{A} p_{2}^{\prime}}(\alpha \notin A)$; (3) $\frac{p_{1} \xrightarrow{\alpha} p_{1}^{\prime}, p_{2} \xrightarrow{\alpha} p_{2}^{\prime}}{p_{1}\left\|_{A} p_{2} \xrightarrow{\hookrightarrow} p_{1}^{\prime}\right\|_{A} p_{2}^{\prime}}(\alpha \in A)$.

We consider only (3), the other two may be handled in a similar manner as the case $\alpha=\tau$. Since $\forall r\left(p_{1} \|_{A} p_{2} \xrightarrow{\alpha} r\right.$ implies $\left.r \in F\right)$, by Lemma 4.1(3), it is easy to see that either $\forall r\left(p_{1} \xrightarrow{\alpha} r\right.$ implies $\left.r \in F\right)$ or $\forall r\left(p_{2} \xrightarrow{\alpha} r\right.$ implies $\left.r \in F\right)$. Furthermore, due to $\alpha \in \mathcal{I}\left(p_{1}\right)$ and $\alpha \in \mathcal{I}\left(p_{2}\right)$, by IH, we have $p_{1} \in F$ or $p_{2} \in F$, which implies $p_{1} \|_{A} p_{2} \in F$.
(LTS2) It suffices to show that, for each $p$, if $p \notin F$ then $p{ }^{\epsilon}{ }_{F} \mid q$ for some $q$. Suppose $p \notin F$. By Lemma4.1(7), there exists $q$ such that $p \xlongequal{\epsilon} \mid q$ and $q \notin F$. Then it immediately follows from Lemma 4.2 that $p{ }^{\epsilon}{ }_{F} \mid q$, as desired.

Remark 4.1. It is worth pointing out that Lemma 4.2 does not always hold for LLTS. In fact, the property " $p \in F$ implies $q \in F$ for each $\tau$-derivative $q$ of $p$ " is logically independent of Def. [2.1, It is SOS rules adopted in this paper that bring such additional property. Hence this paper restricts itself to specific LLTSs, which makes reasoning about inconsistency a bit easier than in the general LLTS setting.

A simple observation on proof trees for $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash p \wedge q F$ is given below, which will be used in establishing a fundamental property of conjunctive compositions.

Lemma 4.3. For any finite sequence $p_{0} \wedge q_{0} \xrightarrow{\tau}, . ., \xrightarrow{\tau} p_{i} \wedge q_{i} \xrightarrow{\tau}, . ., \xrightarrow{\tau} \mid p_{n} \wedge q_{n}(n \geq 0)$, if $p_{i} \wedge q_{i} \in F$ and $p_{i}, q_{i} \notin F$ for each $i \leq n$, then the inference of $p_{0} \wedge q_{0} F$ essentially depends on $p_{n} \wedge q_{n} F$, that is, each proof tree for $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash p_{0} \wedge q_{0} F$ has a subtree with root $p_{n} \wedge q_{n} F$, in particular, such subtree is proper if $n \geq 1$.

Proof. We prove the statement by induction on $n$. For the inductive basis $n=0$, it holds trivially due to $p_{0} \wedge q_{0} \equiv p_{n} \wedge q_{n}$. For the inductive step, assume that $p_{0} \wedge q_{0} \xrightarrow{\tau}$ $p_{1} \wedge q_{1}(\xrightarrow{\tau})^{k} \mid p_{k+1} \wedge q_{k+1}$. Let $\mathcal{T}$ be any proof tree for $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash p_{0} \wedge q_{0} F$. Since $p_{0}, q_{0} \notin F$ and $p_{0} \wedge q_{0} \xrightarrow{\tau}$, the last rule applied in $\mathcal{T}$ is

$$
\text { either } \frac{p_{0} \wedge q_{0} \xrightarrow{\alpha} r^{\prime},\left\{r F: p_{0} \wedge q_{0} \xrightarrow{\alpha} r\right\}}{p_{0} \wedge q_{0} F} \text { or } \frac{\left\{r F: p_{0} \wedge q_{0} \stackrel{\epsilon}{\Longrightarrow} \mid r\right\}}{p_{0} \wedge q_{0} F} .
$$

For the first alternative, since $\operatorname{LTS}\left(\mathrm{CLL}_{R}\right)$ is $\tau$-pure, we have $\alpha=\tau$. Then it follows from $p_{0} \wedge q_{0} \xrightarrow{\tau} p_{1} \wedge q_{1}$ that, in the proof tree $\mathcal{T}$, one of nodes directly above the root is labelled with $p_{1} \wedge q_{1} F$. Thus, by IH, $\mathcal{T}$ has a proper subtree with root $p_{k+1} \wedge q_{k+1} F$.

For the second alternative, since $p_{0} \wedge q_{0} \xlongequal{\epsilon} \mid p_{k+1} \wedge q_{k+1}$, one of nodes directly above the root of $\mathcal{T}$ is labelled with $p_{k+1} \wedge q_{k+1} F$, as desired.

The next three results has been obtained for CLL in pure process-algebraic style in (Zhang et al. 2011), where the proof essentially depends on the fact that, for any process $p$ within CLL and $\alpha \in A c t_{\tau}, p$ is of more complex structure than its $\alpha$-derivatives. Unfortunately, such property does not always hold for CLL $_{R}$. For instance, consider the process $\left\langle X \mid X=a . X \|_{\emptyset} a . b . X\right\rangle$. Here we give another proof along lines presented in (Zhu et al. 2013).

Lemma 4.4. If $p_{1}{\underset{\sim}{\sim}}_{R S} p_{2}, p_{1}{\underset{\sim}{\sim}}_{R S} p_{3}$ and $p_{1} \notin F$ then $p_{2} \wedge p_{3} \notin F$.

Proof. Let $\Omega=\left\{q \wedge r: p{\underset{\sim}{~}}_{R S} q, p{\underset{\sim}{\square}}_{R S} r\right.$ and $\left.p \notin F\right\}$. Clearly, it suffices to prove that $F \cap \Omega=\emptyset$. Conversely, suppose that $F \cap \Omega \neq \emptyset$. In the following, we intend to prove that, for each $t \in \Omega$, any proof tree of $t F$ is not well-founded. Then a contradiction arises at this point due to Def. 2.4. Thus, to complete the proof, it suffices to show the claim below.

Claim For any $s \in \Omega$, each proof tree of $s F$ has a proper subtree with root $s^{\prime} F$ for some $s^{\prime} \in \Omega$.

Suppose $q \wedge r \in \Omega$. Then $p{\underset{\sim}{\sim}}_{R S} q, p{\underset{\sim}{\sim}}_{R S} r$ and $p \notin F$ for some $p$. Thus it follows that

$$
\begin{equation*}
q \notin F, r \notin F \text { and } \mathcal{I}(p)=\mathcal{I}(q)=\mathcal{I}(r) . \tag{4.4.1}
\end{equation*}
$$

Let $\mathcal{T}$ be any proof tree of $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash q \wedge r F$. By (4.4.1), the last rule applied in $\mathcal{T}$ is of the form

$$
\text { either } \frac{\{s F: q \wedge r \xlongequal{\epsilon} \mid s\}}{q \wedge r F} \text { or } \frac{q \wedge r \stackrel{\alpha}{\longrightarrow} s^{\prime},\{s F: q \wedge r \stackrel{\alpha}{\longrightarrow} s\}}{q \wedge r F} \text {. }
$$

Since both $q$ and $r$ are stable, so is $q \wedge r$. Then, for the first alternative, the label of the node directly above the root of $\mathcal{T}$ is $q \wedge r F$ itself, as desired.

Next we consider the second alternative. In this case, $\tau \neq \alpha \in \mathcal{I}(q \wedge r)$ and

$$
\begin{equation*}
\forall s(q \wedge r \xrightarrow{\alpha} s \text { implies } s \in F) . \tag{4.4,2}
\end{equation*}
$$

Hence $\alpha \in \mathcal{I}(q) \cap \mathcal{I}(r)$. Then $\alpha \in \mathcal{I}(p)$ due to 4.41). Further, since $p \notin F$, by Theorem 4.2 we get

$$
\begin{equation*}
p \xrightarrow{\alpha}_{F} p^{\prime} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid p^{\prime \prime} \text { for some } p^{\prime} \text { and } p^{\prime \prime} . \tag{4.4.3}
\end{equation*}
$$

Then it immediately follows from $p{\underset{\sim}{\square}}_{R S} q$ and $p{\underset{\sim}{~}}_{R S} r$ that

$$
\begin{gather*}
q \xrightarrow{\alpha}_{F} q^{\prime} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid q^{\prime \prime} \text { and } p^{\prime \prime}{\underset{\sim}{\sim_{R S}}} q^{\prime \prime} \text { for some } q^{\prime}, q^{\prime \prime}, \text { and }  \tag{4.4,4}\\
r \xrightarrow{\alpha}_{F} r^{\prime} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid r^{\prime \prime} \text { and } p^{\prime \prime}{\underset{\sim}{\sim}}_{R S} r^{\prime \prime} \text { for some } r^{\prime}, r^{\prime \prime} . \tag{4.4,5}
\end{gather*}
$$

So, $q \wedge r \xrightarrow{\alpha} q^{\prime} \wedge r^{\prime}$. Then $q^{\prime} \wedge r^{\prime} \in F$ by (4.4,2). Moreover, we obtain $q^{\prime} \equiv q_{0} \xrightarrow{\tau}{ }_{F}$ $, \ldots, \xrightarrow{\tau}_{F} \mid q_{n} \equiv q^{\prime \prime}$ for some $q_{i}(0 \leq i \leq n)$, and $r^{\prime} \equiv r_{0} \xrightarrow{\tau}_{F}, \ldots, \xrightarrow{\tau}{ }_{F} \mid r_{m} \equiv r^{\prime \prime}$ for some $r_{j}(0 \leq j \leq m)$. Then

$$
\begin{equation*}
q^{\prime} \wedge r^{\prime} \equiv q_{0} \wedge r_{0} \xrightarrow{\tau}, . ., \xrightarrow[\longrightarrow]{\tau} q_{n} \wedge r_{0} \xrightarrow{\tau} q_{n} \wedge r_{1}, . ., \xrightarrow{\tau} \mid q_{n} \wedge r_{m} \equiv q^{\prime \prime} \wedge r^{\prime \prime} \tag{4.4,6}
\end{equation*}
$$

By Lemma 4.2, it follows from $q^{\prime} \wedge r^{\prime} \in F$ that

$$
\begin{equation*}
q_{i} \wedge r_{j} \in F \text { for each } q_{i} \wedge r_{j} \text { occurring in (4.4) } 6 \text { ). } \tag{4.4,7}
\end{equation*}
$$

It follows from (4.4.3), 4.4.4) and (4.4.5) that $q_{n} \wedge r_{m} \equiv q^{\prime \prime} \wedge r^{\prime \prime} \in \Omega$. Moreover, since one of nodes directly above the root of $\mathcal{T}$ is labelled with $q^{\prime} \wedge r^{\prime} F$, by (4.46), (4.4.7) and Lemma 4.3, it follows from $q_{i} \notin F(0 \leq i \leq n)$ and $r_{j} \notin F(0 \leq j \leq m)$ that $\mathcal{T}$ has a proper subtree with root $q_{n} \wedge r_{m} F$.
Lemma 4.5. If $p{\underset{\sim}{\sqcap}}_{R S} q$ and $p{\underset{\sim}{\sim}}_{R S} r$ then $p{\underset{\sim}{\sim}}_{R S} q \wedge r$.

Proof. Set

$$
\mathcal{R}=\left\{\left(p_{1}, p_{2} \wedge p_{3}\right): p_{1}{\underset{\sim}{\tau}}^{\Sigma_{S}} p_{2} \text { and } p_{1}{\underset{\sim}{\tau}}_{R S} p_{3}\right\} .
$$

It suffices to show that $\mathcal{R}$ is a stable ready simulation relation, which is almost immediate by using Lemma 4.4 to handle (RS2) and (RS3).

We conclude this section with recalling a result obtained in (Lüttgen and Vogler 2010) and (Zhang et al. 2011) in different style, which reveals that $\sqsubseteq_{R S}$ is precongruent w.r.t the operators $\square, \|_{A}, \vee$ and $\wedge$. Formally,

## Theorem 4.3.

(1) For each $\odot \in\left\{\square, \|_{A}, \wedge\right\}$, if $p{\underset{\sim}{\sim}}_{R S} q$ and $s{\underset{\sim}{\sim}}_{R S} r$ then $p \odot s{\underset{\sim}{~}}_{R S} q \odot r$.
(2) For each $\odot \in\left\{\square, \|_{A}, \vee, \wedge\right\}$, if $p \sqsubseteq_{R S} q$ and $s \sqsubseteq_{R S} r$ then $p \odot s \sqsubseteq_{R S} q \odot r$.

Proof. The item (2) follows from item (1). For item (1), the proofs are not much different from ones given in (Zhang et al. 2011). In particular, Lemma 4.5 is applied in the proof for the case $\odot=\wedge$.

## 5. Basic properties of unfolding, context and transitions

This section will provide a number of useful results that will be used in the following sections. Subsection 5.1 will recall the notion of unfolding and give some elementary properties of it. In subsection 5.2, we will be concerned with capturing one-step transitions in terms of contexts and substitutions. A treatment of a more general case involving multi $\tau$-transitions will be considered in subsection 5.3.

### 5.1. Unfolding

The notion of unfolding plays an important role when dealing with recursive operators. This subsection will give a few results concerning it. We begin with recalling the notion of unfolding.

Definition 5.1. Let $X$ be a free variable in a given term $t$. An occurrence of $X$ in $t$ is unfolded, if this occurrence does not occur in the scope of any recursive operation $\langle Y \mid E\rangle$. Moreover, $X$ is unfolded if all occurrences of $X$ in $t$ are unfolded.

Definition 5.2 (Baeten and Bravetti 2008). A series of binary relations $\Rightarrow_{k}$ over terms with $k<\omega$ is defined inductively as:
$-t \Rightarrow_{0} s$ if $t \equiv s$;

- $t \Rightarrow_{1} s$ if $t$ has a subterm $\langle Y \mid E\rangle$ with $Y=t_{Y} \in E$ which is not in the scope of any recursive operation, and $s$ is obtained from $t$ by replacing this subterm by $\left\langle t_{Y} \mid E\right\rangle$;
$-t \Rightarrow_{k+1} s$ if $t \Rightarrow_{k} t^{\prime}$ and $t^{\prime} \Rightarrow_{1} s$ for some term $t^{\prime}$.
Moreover, $\Rightarrow \triangleq \bigcup_{0 \leq k<\omega} \Rightarrow_{k}$. For any $t$ and $s, s$ is a multi-step unfolding of $t$ if $t \Rightarrow s$.

For instance, consider $t \equiv(\langle X \mid X=a . X \square b .\langle Y \mid Y=c . Y\rangle\rangle \square d .0) \square Z$, we have

$$
t \Rightarrow_{1}((a .\langle X \mid X=a . X \square b .\langle Y \mid Y=c . Y\rangle\rangle \square b .\langle Y \mid Y=c . Y\rangle) \square d .0) \square Z
$$

but it does not hold that $t \Rightarrow_{1}(\langle X \mid X=a . X \square b . c .\langle Y \mid Y=c . Y\rangle\rangle \square d .0) \square Z$ because the subterm $\langle Y \mid Y=c . Y\rangle$ is in the scope of the recursive operation $\langle X| X=a . X \square b .\langle Y| Y=$ $c . Y\rangle\rangle$. The simple result below provides an equivalent formulation of the binary relation $\Rightarrow{ }_{1}$.

Lemma 5.1. For any term $t_{1}$ and $t_{2}, t_{1} \Rightarrow{ }_{1} t_{2}$ iff there exists a term $s$ and variable $X$ such that
$\left(1_{\Rightarrow}^{\Rightarrow}\right) X$ is a unfolded variable in $s$,
$(2 \Rightarrow) X$ occurs in $s$ exactly once, and
$\left(3_{\Rightarrow}^{\Rightarrow}\right) t_{1} \equiv s\{\langle Y \mid E\rangle / X\}$ and $t_{2} \equiv s\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\}$ for some $Y, E$ with $Y=t_{Y} \in E$.

Proof. Immediately follows from Def. 5.2 ,

A few trivial but useful properties concerning $\Rightarrow_{n}$ are listed in the next lemma.

Lemma 5.2. For any term $t, s$ and $X \in F V(t)$, if $t \Rightarrow_{n} s$ then
(1) if $X$ is unfolded in $t$ then so it is in $s$ and the number of occurrences of $X$ in $s$ is equal to that in $t$;
(2) the number of unguarded occurrences of $X$ in $s$ is not more than that in $t$;
(3) if $X$ is (strongly) guarded in $t$ then so it is in $s$;
(4) $F V(s) \subseteq F V(t)$;
(5) if $X$ occurs in the scope of conjunction in $s$ (that is, there exists a subterm $t_{1} \wedge t_{2}$ of $s$ such that $X$ occurs in either $t_{1}$ or $t_{2}$ ) then so does it in $t$.

Proof. By Lemma 5.1 and Convention 3.2 it is straightforward by induction on $n$.

Notice that the clause (2) in the above lemma does not always hold for guarded occurrences. For example, consider $t \equiv\langle X \mid X=a . X \wedge b . Y\rangle$, we have $t \Rightarrow_{1} a \cdot\langle X| X=$ $a . X \wedge b . Y\rangle \wedge b . Y$, and $Y$ guardedly occurs in the latter twice but occurs in $t$ only once. Clearly, the clause (2) strongly depends on Convention 3.2. Moreover, the clause (4) cannot be strengthened to " $F V(s)=F V(t)$ ". Consider $t \equiv\left\langle X_{1} \mid\left\{X_{1}=a .0, X_{2}=b . X_{1} \square Y\right\}\right\rangle$ and $t \Rightarrow_{1} a .0$, then we have $F V(t)=\{Y\}$ and $F V(a .0)=\emptyset$.

Given a variable $X$ and term $t$, the folding number of $X$ in $t$, in symbols $F N(t, X)$, is defined as the sum of depths of nested recursive operations surrounding all unguarded occurrences of $X$ in $t$. Formally:

Definition 5.3 (Folding number). Given a term $t$ and $X \in F V(t)$, the folding number of $X$ in $t$, denoted by $F N(t, X)$, is defined recursively below, where $U F V(t)$ is the set of
all free variables which have unguarded occurrence in $t$.

$$
\begin{aligned}
& -F N(0, X)=F N(\perp, X)=F N(Y, X)=F N\left(t_{1} \vee t_{2}, X\right)=F N(\alpha . t, X) \triangleq 0 ; \\
& -F N\left(t_{1} \odot t_{2}, X\right) \triangleq F N\left(t_{1}, X\right)+F N\left(t_{2}, X\right) \text { with } \odot \in\left\{\square, \|_{A}, \wedge\right\} ; \\
& -F N(\langle Y \mid E\rangle, X) \triangleq\left\{\begin{array}{cc}
1+\sum_{Z=t_{Z} \in E} F N\left(t_{Z}, X\right), & \text { if } X \in U F V(\langle Y \mid E\rangle) \\
0, & \text { otherwise. }
\end{array}\right.
\end{aligned}
$$

For instance, consider $t \equiv\left\langle X \mid X=a . X \vee Y_{1}\right\rangle \square\left\langle Z \mid Z=c . Z \square Y_{2}\right\rangle$, then $F N\left(t, Y_{1}\right)=0$ and $F N\left(t, Y_{2}\right)=1$.

Lemma 5.3. For any term $t$, there exits a term $s$ such that $t \Rightarrow s$ and each unguarded occurrence of any free variable in $s$ is unfolded.

Proof. It proceeds by induction on $n=\sum_{X \in U F V(t)} F N(t, X)$. For the induction base $n=0$, it is easy to see that for each $X \in F V(t)$, any unguarded occurrence of $X$ in $t$ must be unfolded. Thus $t$ itself meets our requirement because of $t \Rightarrow t$. For the inductive step $n=k+1$, due to $n=k+1>0, t$ is of the format either $t_{1} \odot t_{2}$ with $\odot \in\left\{\wedge, \|_{A} \square\right\}$ or $\langle Y \mid E\rangle$. In the following, we shall proceed by induction on the structure of $t$. In case $t \equiv t_{1} \odot t_{2}$ with $\odot \in\left\{\wedge, \|_{A}, \square\right\}$, it is straightforward by applying IH on $t_{1}$ and $t_{2}$. Next we consider the case $t \equiv\langle Y \mid E\rangle$ with $Y=t_{Y} \in E$.

Clearly, $U F V(\langle Y \mid E\rangle) \neq \emptyset$ because of $n>0$. Since $\langle Y \mid E\rangle \Rightarrow_{1}\left\langle t_{Y} \mid E\right\rangle$, by Lemma 5.2(2)(4), we have

$$
U F V\left(\left\langle t_{Y} \mid E\right\rangle\right) \subseteq U F V(\langle Y \mid E\rangle)
$$

Moreover, by Convention 3.2 and the definition of $\left\langle t_{Y} \mid E\right\rangle$, it is not difficult to get $F N(\langle Y \mid E\rangle, X)>F N\left(\left\langle t_{Y} \mid E\right\rangle, X\right)$ for each $X \in U F V\left(\left\langle t_{Y} \mid E\right\rangle\right)$. Hence

$$
\sum_{X \in U F V\left(\left\langle t_{Y} \mid E\right\rangle\right)} F N\left(\left\langle t_{Y} \mid E\right\rangle, X\right)<\sum_{X \in U F V(\langle Y \mid E\rangle)} F N(\langle Y \mid E\rangle, X)
$$

Then, by IH on $n$, there exists $s$ such that $\left\langle t_{Y} \mid E\right\rangle \Rightarrow s$ and each unguarded occurrence of any free variable is unfolded in $s$. Moreover, $\langle Y \mid E\rangle \Rightarrow s$ due to $\langle Y \mid E\rangle \nRightarrow_{1}\left\langle t_{Y} \mid E\right\rangle$.

### 5.2. Contexts and transitions

Due to Rules $R p_{12}, R p_{13}$ and $R p_{15}$, in order to obtain further properties of the inconsistency predicate $F$, we often need to capture the connection between the formats of $p$ and $q$ for a given transition $p \xrightarrow{\alpha} q$. Clearly, if $p$ involves recursive operations, $q$ is not always a subterm of $p$ and its format often depends on some unfolding of $p$. This subsection intends to explore this issue.

Definition 5.4 (Context). A context $C_{\tilde{X}}$ is a term whose free variables are among a $n$-tuple distinct variables $\widetilde{X}=\left(X_{1}, \ldots, X_{n}\right)(n \geq 0)$. Given a $n$-tuple processes $\widetilde{p}=$ $\left(p_{1}, \ldots, p_{n}\right)$, the term $C_{\widetilde{X}}\left\{p_{1} / X_{1}, \ldots, p_{n} / X_{n}\right\}\left(C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}\right.$, for short) is obtained from $C_{\widetilde{X}}$ by replacing $X_{i}$ by $p_{i}$ for each $i<n$ simultaneously. In particular, we use $C_{\tilde{X}}\{p / \tilde{X}\}$ to
denote the result of replacing all variables in $\widetilde{X}$ by $p$. A context $C_{\widetilde{X}}$ is said to be stable if $C_{\widetilde{X}}\{0 / \widetilde{X}\} \stackrel{\tau}{\longrightarrow}$.

In the remainder of this paper, whenever the expression $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$ occurs, we always assume that $|\widetilde{p}|=|\widetilde{X}|$ and $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$ is subject to Convention 3.1 (recursive variables occurring in $\widetilde{p}$ may be renamed if it is necessary), where $|\widetilde{X}|$ is the length of the tuple $\widetilde{X}$.

Definition 5.5 (Active). An occurrence of a free variable $X$ in term $t$ is active if such occurrence is unguarded and unfolded. A free variable $X$ in term $t$ is active if all its occurrences are active. A free variable $X$ in term $t$ is 1-active if $X$ occurs in $t$ exactly once and such occurrence is active.

For example, $X$ is 1-active in $\langle Y \mid Y=a . Y\rangle \square X$. Moreover, it is evident that, for any context $C_{\widetilde{X}}$, if there exists an active occurrence of some variable within $C_{\widetilde{X}}$, then $C_{\widetilde{X}}$ is not of the form $\alpha \cdot B_{\tilde{X}}, B_{\tilde{X}} \vee D_{\tilde{X}}$ and $\langle Y \mid E\rangle$. This fact is used in demonstrating the next two lemmas, which give some properties of 1-active place-holder. Before presenting them, for simplicity of notation, we introduce the notation below.

Notation Given $n$-tuple processes $\widetilde{p}=\left(p_{1}, \ldots, p_{n}\right)$ and $p^{\prime}$, we use $\widetilde{p}\left[p^{\prime} / p_{i}\right]$ to denote $\left(p_{1}, \ldots, p_{i-1}, p^{\prime}, p_{i+1}, \ldots, p_{n}\right)$.

Lemma 5.4. For any $C_{\widetilde{X}}$ with 1-active variable $X_{i_{0}}$ and $\widetilde{p}$ with $p_{i_{0}} \xrightarrow{\tau} p^{\prime}, C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau}$ $C_{\widetilde{X}}\left\{\widetilde{p}\left[p^{\prime} / p_{i_{0}}\right] / \widetilde{X}\right\}$.

Proof. Proceed by induction on the structure of $C_{\widetilde{X}}$.
This result does not always hold for visible transitions. For instance, consider $C_{X} \equiv$ $X \square \tau . r$ and $p \equiv a . q$, although $p \xrightarrow{a} q$ and $X$ is 1-active in $C_{X}$, it is false that $C_{X}\{p / X\} \xrightarrow{a}$.

Lemma 5.5. For any $p$ and $C_{X}$ with 1-active variable $X$, if $p \in F$ then $C_{X}\{p / X\} \in F$.
Proof. By a straightforward induction on $C_{X}$.
In order to prove that $\sqsubseteq_{R S}$ still is precongruent in the presence of recursive operations, it is necessary to formally describe the contribution of $C_{\widetilde{X}}$ and $\widetilde{p}$ for a given transition $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\alpha} r$. In the following, we shall provide a few of results concerning this. We begin with considering $\tau$-labelled transitions. Before giving the next lemma formally, we illustrate the intuition behind it by means of an example. Consider $C_{X} \equiv(a .0 \vee X) \square X$, $B_{X} \equiv\langle Y \mid Y=X \square b . Y\rangle \square X, p \equiv c .0 \vee e .0$ and $q \equiv d .0$, then we have two $\tau$-labelled transitions

$$
C_{X}\{q / X\} \xrightarrow{\tau} a .0 \square d .0
$$

and

$$
B_{X}\{p / X\} \xrightarrow{\tau}(e .0 \square b .\langle Y \mid Y=(c .0 \vee e .0) \square b . Y\rangle) \square(c .0 \vee e .0) .
$$

It is not difficult to see that these two $\tau$-labelled transitions depend on the capability of context $C_{X}$ and substitution $p$ respectively. For the former, no matter what $q$ is, the corresponding $\tau$-transition still exists for $C_{X}\{q / X\}$. Moreover, the target has the same pattern. Set $C_{X}^{\prime} \equiv a .0 \square X$. Clearly, $C_{X}\{q / X\} \xrightarrow{\tau} C_{X}^{\prime}\{q / X\}$ for any $q$. The latter is
much more trick. Intuitively, one instance of $p$ first exposes itself and then performs a $\tau$-transition. Since there are multi instances of $p$ and some of them are nested by recursive operations, we should identify the real performer of the $\tau$-transition and this identification is very helpful when we deal with multi $\tau$-transitions. As long as $p$ can perform $\tau$-transition, so can $B_{X}\{p / X\}$. Similarly, the target also has a pattern. Set $B_{X, Z}^{\prime} \equiv$ $(Z \square b .\langle Y \mid Y=X \square b . Y\rangle) \square X$. It is easy to see that $B_{X}\{p / X\} \xrightarrow{\tau} B_{X, Z}^{\prime}\left\{p / X, p^{\prime} / Z\right\}$ for any $p \xrightarrow{\tau} p^{\prime}$. We summarize this observation formally as follows, where two clauses capture $\tau$-transitions exited by contexts and substitutions respectively; moreover, some simple properties on contexts are also listed in (C- $\tau-3$ ) which will be used in the sequel.

Lemma 5.6. For any $C_{\widetilde{X}}$ and $\widetilde{p}$, if $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} r$ then one of conclusions below holds.
(1) There exists $C_{\tilde{X}}^{\prime}$ such that
(C- $\tau$-1) $r \equiv C_{\widetilde{X}}^{\prime}\{\widetilde{p} / \widetilde{X}\} ;$
(C- $\tau$-2) for any processes $\widetilde{q}, C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{\tau} C_{\widetilde{X}}^{\prime}\{\widetilde{q} / \widetilde{X}\}$;
(C- $\tau-3)$ for each $X \in \widetilde{X}$,
( $\mathbf{C}-\tau \mathbf{- 3 - i}$ ) if $X$ is active in $C_{\widetilde{X}}$ then so it is in $C_{\widetilde{X}}^{\prime}$ and the number of occurrences of $X$ in $C_{\widetilde{X}}^{\prime}$ is equal to that in $C_{\widetilde{X}}$;
( $\mathbf{C}-\tau-\mathbf{-} \mathbf{- i i}$ ) if $X$ is unfolded in $C_{\widetilde{X}}$ then so it is in $C_{\widetilde{X}}^{\prime}$ and the number of occurrences of $X$ in $C_{\widetilde{X}}^{\prime}$ is not more than that in $C_{\widetilde{X}}$;
(C- $\tau$-3-iii) if $X$ is strongly guarded in $C_{\widetilde{X}}$ then so it is in $C_{\widetilde{X}}^{\prime}$;
( $\mathbf{C}-\tau$-3-iv) if $X$ does not occur in the scope of any conjunction in $C_{\tilde{X}}$ then neither does it in $C_{\tilde{X}}^{\prime}$.
(2) There exist $C_{\widetilde{X}}^{\prime}, C_{\widetilde{X}, Z}^{\prime \prime}$ with $Z \notin \widetilde{X}$ and $i \leq|\widetilde{X}|$ such that
$(\mathbf{P}-\tau-\mathbf{1}) C_{\widetilde{X}} \Rightarrow C_{\widetilde{X}}^{\prime}$, in particular, if $X_{i}$ is active in $C_{\widetilde{X}}$ then $C_{\widetilde{X}}^{\prime} \equiv C_{\widetilde{X}}$;
$\mathbf{( P - \tau - 2 )} p_{i} \xrightarrow{\tau} p^{\prime}$ and $r \equiv C_{\tilde{X}, Z}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, p^{\prime} / Z\right\}$ for some $p^{\prime}$;
$(\mathbf{P}-\tau \mathbf{- 3}) C_{\tilde{X}, Z}^{\prime \prime}\left\{X_{i} / Z\right\} \equiv C_{\tilde{X}}^{\prime}$ and $Z$ is 1-active in $C_{\widetilde{\sim}, Z}^{\prime \prime} ;$
$\mathbf{( P - \tau - 4 )}$ for any processes $\widetilde{q}$ with $q_{i} \xrightarrow{\tau} q^{\prime}, C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{\tau} C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, q^{\prime} / Z\right\}$.
Proof. It proceeds by induction on the depth of the inference of $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash$ $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} r$. We distinguish six cases based on the form of $C_{\widetilde{X}}$ as follows.

Case $1 C_{\tilde{X}}$ is closed.
Set $C_{\widetilde{X}}^{\prime} \triangleq r$. Then $(\mathrm{C}-\tau-1,2,3)$ hold trivially.
Case $2 C_{\widetilde{X}} \equiv X$ with $X \in \widetilde{X}$.
Put $C_{\widetilde{X}}^{\prime} \triangleq X$ and $C_{\widetilde{X}, Z}^{\prime \prime} \triangleq Z$ with $Z \notin \widetilde{X}$. Then it is easy to check that (P- $\left.\tau-1\right)-$ (P- $\tau-4$ ) hold.

Case $3 C_{\widetilde{X}} \equiv \alpha . B_{\widetilde{X}}$.
Thus $\alpha=\tau$ and $r \equiv B_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\}$. Then it is not difficult to see that (C- $\tau-1,2,3$ ) hold by taking $C_{\widetilde{X}}^{\prime} \triangleq B_{\tilde{X}}$.

Case $4 C_{\widetilde{X}} \equiv B_{\widetilde{X}} \vee D_{\widetilde{X}}$.
Obviously, $r \equiv B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$ or $r \equiv D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$. W.l.o.g, assume that $r \equiv B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$. We set $C_{\widetilde{X}}^{\prime} \triangleq B_{\tilde{X}}$. Then it is straightforward that (C- $\tau-1,2$ ) and (C- $\tau-3$-ii,iii,iv) hold. Moreover, since $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \vee D_{\tilde{X}}$, for each $X \in \widetilde{X}$, each occurrence of $X$ is weakly guarded. Hence ( $\mathrm{C}-\tau-3-\mathrm{i}$ ) holds trivially.

Case $5 C_{\widetilde{X}} \equiv B_{\widetilde{X}} \odot D_{\widetilde{X}}$ with $\odot \in\left\{\square, \wedge, \|_{A}\right\}$.
We consider the case $\odot=\square$, others may be handled similarly and omitted. W.l.o.g, assume the last rule applied in the inference is

$$
\frac{B_{\widetilde{X}}\{\widetilde{p} / \tilde{X}\} \xrightarrow{\tau} r^{\prime}}{B_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \square D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} r^{\prime} \square D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}}
$$

Then $r \equiv r^{\prime} \square D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$. For the $\tau$-labelled transition $B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} r^{\prime}$, by IH, either the clause (1) or (2) holds.

For the former case, there exists $B_{\widetilde{X}}^{\prime}$ that satisfies (C- $\left.\tau-1,2,3\right)$. Put $C_{\widetilde{X}}^{\prime} \triangleq B_{\tilde{X}}^{\prime} \square D_{\tilde{X}}$. It immediately follows that $C_{\widetilde{X}}^{\prime}$ satisfies (C- $\tau-1,2,3$ ).

Next we consider the latter case. In this situation, there exist $B_{\tilde{X}}^{\prime}, B_{\tilde{X}, Z}^{\prime \prime}$ with $Z \notin \widetilde{X}$ and $i_{0} \leq|\widetilde{X}|$ that satisfy $(\mathrm{P}-\tau-1)-(\mathrm{P}-\tau-4)$. Set

$$
C_{\widetilde{X}}^{\prime} \triangleq B_{\widetilde{X}}^{\prime} \square D_{\widetilde{X}} \text { and } C_{\widetilde{X}, Z}^{\prime \prime} \triangleq B_{\widetilde{X}, Z}^{\prime \prime} \square D_{\tilde{X}}
$$

We shall show that, for the $\tau$-labelled transition $C_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} r, C_{\widetilde{X}}^{\prime}, C_{\widetilde{X}, Z}^{\prime \prime}$ and $i_{0}$ realize $(\mathrm{P}-\tau-1)-(\mathrm{P}-\tau-4)$.
$(\mathbf{P}-\tau-\mathbf{1})$ It follows from $B_{\widetilde{X}} \Rightarrow B_{\widetilde{X}}^{\prime}$ that $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \square D_{\tilde{X}} \Rightarrow B_{\tilde{X}}^{\prime} \square D_{\tilde{X}} \equiv C_{\widetilde{X}}^{\prime}$. If $X_{i_{0}}$ is active in $C_{\tilde{X}}$ then so it is in $B_{\tilde{X}}$, and hence $C_{\tilde{X}}^{\prime} \equiv C_{\widetilde{X}}$ due to $B_{\tilde{X}}^{\prime} \equiv B_{\tilde{X}}$.
(P- $\tau-\mathbf{2}$ ) Since $B_{\widetilde{X}}^{\prime}$ satisfies $(\mathrm{P}-\tau-2), p_{i_{0}} \xrightarrow{\tau} p^{\prime}$ and $r^{\prime} \equiv B_{\tilde{X}, Z}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, p^{\prime} / Z\right\}$ for some $p^{\prime}$. Due to $Z \notin \widetilde{X}$, we have $r \equiv B_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, p^{\prime} / Z\right\} \square D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \equiv C_{\widetilde{\widetilde{x}}, Z}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, p^{\prime} / Z\right\}$.
$\mathbf{( P - \tau - 3 )}$ It follows from $B_{\widetilde{X}, Z}^{\prime \prime}\left\{X_{i_{0}} / Z\right\} \equiv B_{\widetilde{X}}^{\prime}$ and $Z \notin \widetilde{X}$ that $C_{\tilde{X}, Z}^{\prime \prime}\left\{X_{i_{0}} / Z\right\} \equiv$ $B_{\tilde{X}, Z}^{\prime \prime}\left\{X_{i_{0}} / Z\right\} \square D_{\tilde{X}} \equiv C_{\tilde{X}}^{\prime}$. Moreover, since $Z$ is 1-active in $B_{\tilde{X}, Z}^{\prime \prime}$, so it is in $C_{\tilde{X}, Z}^{\prime \prime}$.
$\mathbf{( \mathbf { P } - \tau - 4 )}$ Let $\widetilde{q}$ be any tuple with $|\widetilde{q}|=|\widetilde{p}|$ and $q_{i_{0}} \xrightarrow{\tau} q^{\prime}$. It follows from $B_{\widetilde{X}}\{\widetilde{\sim} / \widetilde{\sim} / \widetilde{X}\} \xrightarrow{\tau}$ $B_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, q^{\prime} / Z\right\}$ and $Z \notin \widetilde{X}$ that $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{\tau} C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, q^{\prime} / Z\right\}$.

Case $6 C_{\widetilde{X}} \equiv\langle Y \mid E\rangle$.
Clearly, the last rule applied in the inference is

$$
\frac{\left\langle t_{Y} \mid E\right\rangle\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} r}{\langle Y \mid E\rangle\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} r} \text { with } Y=t_{Y} \in E \text {. }
$$

For the $\tau$-labelled transition $\left\langle t_{Y} \mid E\right\rangle\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} r$, by IH, either the clause (1) or (2) holds.

For the first alternative, there exists $C_{\widetilde{X}}^{\prime}$ satisfying (C- $\left.\tau-1,2,3\right)$. Then it is not difficult to check that, for the transition $\langle Y \mid E\rangle\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} r, C_{\widetilde{X}}^{\prime}$ also realizes the conditions (C-
$\tau-1,2,3)$. Here $\langle Y \mid E\rangle\{\widetilde{p} / \tilde{X}\} \Rightarrow{ }_{1}\left\langle t_{Y} \mid E\right\rangle\{\widetilde{p} / \tilde{X}\}$ and Lemma 5.2 (3)(5) are used to assert (C- $\tau$ - 3 -iii,iv) to be true.

For the second alternative, there exist $C_{\tilde{X}}^{\prime}, C_{\widetilde{X}, Z}^{\prime \prime}$ with $Z \notin \widetilde{X}$ and $i_{0} \leq|\widetilde{X}|$ that satisfy (P- $\tau-1,2,3,4)$. Clearly, $C_{\tilde{X}}^{\prime}, C_{\tilde{X}, Z}^{\prime \prime}$ and $i_{0}$ also realize (P- $\tau-1,2,3,4$ ) for the transition $\langle Y \mid E\rangle\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} r$. In particular, $\langle Y \mid E\rangle \Rightarrow C_{\widetilde{X}}^{\prime}$ follows from $\langle Y \mid E\rangle \Rightarrow_{1}\left\langle t_{Y} \mid E\right\rangle \Rightarrow C_{\widetilde{X}}^{\prime}$.

As an immediate consequence of Lemma 5.6 we have
Lemma 5.7. For any context $C_{\widetilde{X}}, C_{\widetilde{X}}$ is stable iff $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \not f^{\tau}$ for some $\widetilde{p}$.
Proof. Straightforward by Lemma 5.6 .
In the following, we intend to provide an analogue of Lemma 5.6 for transitions labelled with visible actions. To explain intuition behind the next result clearly, it is best to work with an example. Consider $C_{X_{1}, X_{2}} \equiv\left(\left(X_{1} \wedge\langle Y \mid Y=a . Y\rangle\right) \square a . b .0\right) \|_{\{b\}}\left(X_{1} \wedge X_{2}\right), p_{1} \equiv a .0$ and $p_{2} \equiv a . c .0$, we have three $a$-labelled transitions

$$
\begin{gathered}
C_{X_{1}, X_{2}}\left\{p_{1} / X_{1}, p_{2} / X_{2}\right\} \xrightarrow{a}(0 \wedge\langle Y \mid Y=a . Y\rangle) \|_{\{b\}}(a .0 \wedge a . c .0), \\
C_{X_{1}, X_{2}}\left\{p_{1} / X_{1}, p_{2} / X_{2}\right\} \xrightarrow{a} b .0 \|_{\{b\}}(a .0 \wedge a . c .0),
\end{gathered}
$$

and

$$
C_{X_{1}, X_{2}}\left\{p_{1} / X_{1}, p_{2} / X_{2}\right\} \xrightarrow{a}((a .0 \wedge\langle Y \mid Y=a . Y\rangle) \square a . b .0) \|_{\{b\}}(0 \wedge c .0) .
$$

These visible transitions starting from $C_{X_{1}, X_{2}}\left\{p_{1} / X_{1}, p_{2} / X_{2}\right\}$ are activated by three distinct events. Clearly, both the context $C_{X_{1}, X_{2}}$ and the substitution $p_{1}$ contribute to the first transition, while two latter transitions depend merely on the capability of $C_{X_{1}, X_{2}}$ and $\widetilde{p_{1,2}}$ respectively. These three situations may be described uniformly in the lemma below. Here some additional properties on contexts are also listed in (CP-a-4), which will be useful in the sequel.
Lemma 5.8. For any $a \in A c t, C_{\widetilde{X}}$ and $\widetilde{p}$, if $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r$ then there exist $C_{\widetilde{X}}^{\prime}, C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ and $C_{\tilde{X}, \tilde{Y}}^{\prime \prime}$ with $\widetilde{X} \cap \widetilde{Y}=\emptyset$ satisfying the conditions:
(CP-a-1) $C_{\widetilde{X}} \Rightarrow C_{\widetilde{X}}^{\prime}$;
(CP-a-2) for each $Y \in \widetilde{Y}, Y$ is 1-active in $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ and $C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}$;
(CP-a-3) there exist $i_{Y} \leq|\widetilde{X}|$ for each $Y \in \widetilde{Y}$ such that
(CP-a-3-i) $C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{X_{i_{Y}}} / \tilde{Y}\right\} \equiv C_{\widetilde{X}}^{\prime} ;$
(CP-a-3-ii) there exist $p_{Y}^{\prime}$ such that $p_{i_{Y}} \xrightarrow{a} p_{Y}^{\prime}$ for each $Y \in \widetilde{Y}$ and $r \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\}$;
(CP-a-3-iii) for any $\widetilde{q}$ with $|\widetilde{q}|=|\widetilde{X}|$ and $\widetilde{q^{\prime}}$ such that $\left|\widetilde{q^{\prime}}\right|=|\widetilde{Y}|$ and $q_{i_{Y}} \xrightarrow{a} q_{Y}^{\prime}$ for each $Y \in \widetilde{Y}$, if $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\}$ is stable then $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{a} C_{\widetilde{X}, \tilde{Y}}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\}$;
(CP-a-4) for each $X \in \widetilde{X}$,
(CP-a-4-i) the number of occurrences of $X$ in $C_{\widetilde{X}, \tilde{Y}}^{\prime \prime}$ is not more than that in $C_{\tilde{X}, \tilde{Y}}^{\prime}$;
(CP-a-4-ii) if $X$ is active in $C_{\widetilde{X}, \tilde{Y}}^{\prime}$ then so it is in $C_{\tilde{X}, \tilde{Y}}^{\prime \prime}$;
(CP-a-4-iii) if $X$ does not occur in the scope of any conjunction in $C_{\widetilde{X}}$ then neither does it in $C_{\tilde{X}, \tilde{Y}}^{\prime \prime}$.

Proof. It proceeds by induction on the depth of the inference of $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash$ $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r$. Due to $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \not{ }^{\tau}$, it is impossible that $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \vee D_{\widetilde{X}}$. Thus we can distinguish seven cases depending on the form of $C_{\tilde{X}}$.

Case $1 C_{\widetilde{X}}$ is closed.
Set $C_{\widetilde{X}}^{\prime} \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime} \triangleq C_{\widetilde{X}}$ and $C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime} \triangleq r$ with $\widetilde{Y}=\emptyset$. Clearly, these contexts realize conditions (CP-a-i) $(1 \leq i \leq 4)$ trivially.

Case $2 C_{\widetilde{X}} \equiv X_{i_{0}}$ with $i_{0} \leq|\widetilde{X}|$.
Put $C_{\widetilde{X}}^{\prime} \triangleq X_{i_{0}}$ and $C_{\widetilde{X}, \widetilde{Y}}^{\prime} \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime} \triangleq Y$ with $Y \notin \widetilde{X}$. Then $(\mathrm{CP}-a-i)(1 \leq i \leq 4)$ follow immediately, in particular, for (CP-a-3), we take $i_{Y} \triangleq i_{0}$.

Case $3 C_{\widetilde{X}} \equiv \alpha . B_{\widetilde{X}}$.
Then $\alpha=a$ and $r \equiv B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$. Put $C_{\widetilde{X}}^{\prime} \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime} \triangleq \alpha \cdot B_{\widetilde{X}}$ and $C_{\tilde{X}, \widetilde{Y}}^{\prime \prime} \triangleq B_{\widetilde{X}}$ with $\widetilde{Y}=\emptyset$. Obviously, these contexts are what we seek.

Case $4 C_{\widetilde{X}} \equiv B_{\widetilde{X}} \square D_{\tilde{X}}$.
W.l.o.g, suppose that the last rule applied in the inference is

$$
\frac{B_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \stackrel{a}{\longrightarrow} r, D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \not^{\tau}}{B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \square D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r} .
$$

By IH, for the $a$-labelled transition $B_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r$, there exist $B_{\tilde{X}}^{\prime}, B_{\widetilde{X}, \widetilde{Y}}^{\prime}$ and $B_{\widetilde{X}}^{\prime \prime}, \widetilde{Y}$ with $\widetilde{X} \cap \widetilde{Y}=\emptyset$ that satisfy (CP-a-1) - (CP-a-4). Set

$$
C_{\tilde{X}}^{\prime} \triangleq B_{\tilde{X}}^{\prime} \square D_{\widetilde{X}}, C_{\tilde{X}, \widetilde{Y}}^{\prime} \triangleq B_{\tilde{X}, \tilde{Y}}^{\prime} \square D_{\widetilde{X}} \text { and } C_{\tilde{X}, \widetilde{Y}}^{\prime \prime} \triangleq B_{\tilde{X}, \widetilde{Y}}^{\prime \prime}
$$

Then it is not difficult to check that, for the $a$-labelled transition $C_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r$, these contexts above realizes (CP- $a-1$ ) - (CP- $a-4$ ), as desired.

Case $5 C_{\widetilde{X}} \equiv B_{\widetilde{X}} \wedge D_{\tilde{X}}$.
In this situation, the last rule applied in the inference is

$$
\frac{B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r_{1}, D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \stackrel{a}{\longrightarrow} r_{2}}{B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \wedge D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r_{1} \wedge r_{2}}
$$

and $r \equiv r_{1} \wedge r_{2}$. Then by IH , there exist $B_{\tilde{X}}^{\prime}, B_{\tilde{X}, \widetilde{Y}}^{\prime}$ and $B_{\tilde{X}, \widetilde{Y}}^{\prime \prime}$ with $\widetilde{X} \cap \widetilde{Y}=\emptyset$ and, $D_{\widetilde{X}}^{\prime}$, $D_{\widetilde{X}, \widetilde{Z}}^{\prime}$ and $D_{\widetilde{X}, \widetilde{Z}}^{\prime \prime}$ with $\widetilde{X} \cap \widetilde{Z}=\emptyset$ that realize (CP-a-1,2,3,4) for two $a$-labelled transitions involving in premises respectively. W.l.o.g, we may assume $\tilde{Y} \cap \widetilde{Z}=\emptyset$. Then it is straightforward to verify that, for the $a$-labelled transition $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r$, the contexts $C_{\widetilde{X}}^{\prime} \triangleq B_{\widetilde{X}}^{\prime} \wedge D_{\widetilde{X}}^{\prime}, C_{\widetilde{X}, \widetilde{V}}^{\prime} \triangleq B_{\widetilde{X}, \widetilde{Y}}^{\prime} \wedge D_{\widetilde{X}, \widetilde{Z}}^{\prime}$ and $C_{\widetilde{X}, \widetilde{V}}^{\prime \prime} \triangleq B_{\widetilde{X}, \widetilde{Y}}^{\prime \prime} \wedge D_{\widetilde{X}, \widetilde{Z}}^{\prime \prime}$ with $\widetilde{V}=\widetilde{Y} \cup \widetilde{Z}$ realize (CP- $a-1$ ) - (CP- $a-4$ ), as desired.

Case $6 C_{\widetilde{X}} \equiv B_{\tilde{X}} \|_{A} D_{\tilde{X}}$.

Then the last rule applied in the proof tree is one of the following:
(6.1) $\frac{B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r_{1}, D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r_{2}}{B_{\widetilde{X}}\{\widetilde{p} / \tilde{X}\}\left\|_{A} D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r_{1}\right\|_{A} r_{2}}$ with $a \in A$;
(6.2) $\frac{B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r^{\prime}, D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} f^{\top}}{B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}\left\|_{A} D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\rightarrow} r^{\prime}\right\|_{A} D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}}$ with $a \notin A$;
(6.3) $\frac{D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r^{\prime}, B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \mathcal{F}^{\tau}}{B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}\left\|_{A} D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} B_{\widetilde{X}}\{\widetilde{p} / \tilde{X}\}\right\|_{A} r^{\prime}}$ with $a \notin A$.

Among them, the argument for (6.1) is similar to one for Case 5 . We shall consider the case (6.2), and (6.3) may be handled similarly. In this situation, $r \equiv r^{\prime} \|_{A} D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$. Moreover, for the $a$-labelled transition $B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r^{\prime}$, by IH, there exist $B_{\tilde{X}}^{\prime}, B_{\tilde{X}, \widetilde{Y}}^{\prime}$ and $B_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}$ with $\widetilde{X} \cap \widetilde{Y}=\emptyset$ that satisfy (CP-a-1) - (CP-a-4). Put

$$
C_{\widetilde{X}}^{\prime} \triangleq B_{\tilde{X}}^{\prime}\left\|_{A} D_{\tilde{X}}, C_{\tilde{X}, \tilde{Y}}^{\prime} \triangleq B_{\widetilde{X}, \tilde{Y}}^{\prime}\right\|_{A} D_{\widetilde{X}} \text { and } C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime} \triangleq B_{\tilde{X}, \tilde{Y}}^{\prime \prime} \|_{A} D_{\tilde{X}}
$$

Next we want to show that these contexts realize (CP-a-1) - (CP-a-4).
(CP-a-1) It is obvious because of $B_{\tilde{X}} \Rightarrow B_{\tilde{X}}^{\prime}$.
(CP-a-2) For each $Y \in \widetilde{Y}$, since $Y$ is 1-active in $B_{\tilde{X}, \tilde{Y}}^{\prime \prime}$ and $B_{\tilde{X}, \tilde{Y}}^{\prime}$, so it is in $C_{\tilde{X}, \widetilde{Y}}^{\prime \prime}$ and $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ because of $\widetilde{X} \cap \tilde{Y}=\emptyset$.
(CP-a-3) By IH, there exist $i_{Y} \leq|\widetilde{X}|(Y \in \widetilde{Y})$ which realize subclauses (i)(ii)(iii) in (CP-a-3). In the following, we will verify that these $i_{Y}$ also work well for the induction step. Clearly, it follows from $B_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{X_{i_{Y}}} / \widetilde{Y}\right\} \equiv B_{\widetilde{X}}^{\prime}$ and $\widetilde{X} \cap \widetilde{Y}=\emptyset$ that $C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{X_{i_{Y}}} / \widetilde{Y}\right\} \equiv$ $C_{\tilde{X}}^{\prime}$. Hence these $i_{Y}$ satisfy the subclause (CP- $a-3-\mathrm{i}$ ) for the induction step. Moreover, due to $r^{\prime} \equiv B_{\widetilde{X}, \tilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\}$ for some $p_{Y}^{\prime}(Y \in \widetilde{Y})$ with $p_{i_{Y}} \xrightarrow{a} p_{Y}^{\prime}$ for each $Y \in \widetilde{Y}$ and $\widetilde{X} \cap \widetilde{Y}=\emptyset$, we have $r \equiv r^{\prime} \|_{A} D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \equiv C_{\widetilde{X}}^{\prime \prime}, \widetilde{Y}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\}$, that is, they realize (CP-a-3-ii) for the induction step. Finally, to verify that these $i_{Y}$ also meet the challenge of (CP-a-3-iii), we assume that $\widetilde{\sim}$ and $\widetilde{q^{\prime}}$ be any tuple such that $|\widetilde{q}|=|\widetilde{X}|$, $q_{i_{Y}} \xrightarrow{a} q_{Y}^{\prime}$ for each $Y \in \widetilde{Y}$ and $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\}$ is stable. So, $B_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\}$ and $D_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\}$ are stable. Further, since $B_{\tilde{X}, \widetilde{Y}}^{\prime \prime}$ satisfies (CP-a-3-iii) and $a \notin A$, it is easy to obtain that $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{a} C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\}$.
(CP-a-4) All subclauses immediately follow from IH and constructions of $C_{\widetilde{X}}^{\prime}, C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ and $C_{\tilde{X}, \tilde{Y}}^{\prime \prime}$.

Case $7 C_{\widetilde{X}} \equiv\langle Y \mid E\rangle$.
Clearly, the last rule applied in the inference is

$$
\frac{\left\langle t_{Y} \mid E\right\rangle\{\tilde{p} / \tilde{X}\} \xrightarrow{a} r}{\langle Y \mid E\rangle\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r} \text { with } Y=t_{Y} \in E .
$$

For the transition $\left\langle t_{Y} \mid E\right\rangle\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r$, by IH , there exist $C_{\tilde{X}}^{\prime}, C_{\tilde{X}, \tilde{Y}}^{\prime}$ and $C_{\tilde{X}, \tilde{Y}}^{\prime \prime}$ with $\tilde{X} \cap \tilde{Y}=\emptyset$ that satisfy (CP-a-1)-(CP-a-4). It is trivial to check that these contexts are what we need.

Clearly, whenever all free variables occurring in $C_{\widetilde{X}}$ are guarded, any action labelled transition starting from $C_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\}$ must be performed by $C_{\widetilde{X}}$ itself.

Lemma 5.9. Let $C_{\widetilde{X}}$ be a context such that $X$ is guarded for each $X \in \widetilde{X}$. If $C_{\widetilde{X}}\{\widetilde{p} / \tilde{X}\} \xrightarrow{\alpha}$ $r$ then there exists $B_{\widetilde{X}}$ such that $r \equiv B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$ and $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{\alpha} B_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\}$ for any $\widetilde{q}$.

Proof. Firstly, we handle the case $\alpha=\tau$. For the transition $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} r$, either the clause (1) or (2) in Lemma 5.6 holds. It is a simple matter to see that the clause (1) implies what we desire. The task is now to show that the clause (2) does not hold for such transition. On the contrary, assume that the clause (2) holds. Then there exist $C_{\tilde{X}}^{\prime}, C_{\widetilde{X}, Z}^{\prime \prime}$ and $i_{0} \leq|\widetilde{X}|$ satisfying $(\mathrm{P}-\tau-1,2,3,4)$. For each $X \in \widetilde{X}$, since it is guarded in $C_{\tilde{X}}$, by Lemma $5.2(3)$ and $(\mathrm{P}-\tau-1)$, so it is in $C_{\tilde{X}}^{\prime}$. Hence a contradiction arises due to ( $\mathrm{P}-\tau-3$ ), as desired.
Next we treat the other case $\alpha \in$ Act. By Lemma 5.8, for the transition $C_{\widetilde{X}}\{\widetilde{\sim} / \widetilde{X}\} \xrightarrow{\alpha}$ $r$, there exist $C_{\widetilde{X}}^{\prime}, C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ and $C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}$ realizing (CP-a-1)-(CP-a-4). Clearly, if $\widetilde{Y}=\emptyset$ then $C_{\tilde{X}, \tilde{Y}}^{\prime \prime}$ is exactly one that we need. Thus, to complete the proof, it suffices to show that $\widetilde{Y}$ is indeed empty. Since $X$ is guarded in $C_{\widetilde{X}}$ for each $X \in \widetilde{X}$ and $C_{\widetilde{X}} \Rightarrow C_{\widetilde{X}}^{\prime}$ (i.e., (CP-$a-1)$ ), by Lemma [5.2 (3), all occurrences of free variables in $C_{\widetilde{X}}^{\prime}$ are guarded. Moreover, since $C_{\tilde{X}, \tilde{Y}}^{\prime}$ satisfies (CP-a-2) and (CP-a-3-i), we get $\widetilde{Y}=\emptyset$, as desired.

Lemma 5.10. For any $Y, E$ with $Y=t_{Y} \in E$ and context $C_{X}$ with at most one occurrence of the unfolded variable $X$, we have
(1) if $C_{X}\{\langle Y \mid E\rangle / X\} \xrightarrow{\alpha} q$ then there exists $B_{X}$ such that
(1.1) $q \equiv B_{X}\{\langle Y \mid E\rangle / X\}$,
(1.2) $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \xrightarrow{\alpha} B_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\}$, and
(1.3) $X$ occurs in $B_{X}$ at most once; moreover, such occurrence is unfolded;
(2) if $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \xrightarrow{\alpha} q$ then there exists $B_{X}$ such that
(2.1) $q \equiv B_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\}$,
(2.2) $C_{X}\{\langle Y \mid E\rangle / X\} \xrightarrow{\alpha} B_{X}\{\langle Y \mid E\rangle / X\}$, and
(2.3) $X$ occurs in $B_{X}$ at most once; moreover, such occurrence is unfolded.

Proof. We prove only item (1), and the same reasoning applies to item (2). For (1), the argument is splitted into two parts based on $\alpha$.

Case $1 \alpha=\tau$.
Assume $C_{X}\{\langle Y \mid E\rangle / X\} \xrightarrow{\tau} q$. Then, for such transition, by Lemma 5.6, either there exists $C_{X}^{\prime}$ that satisfies (C- $\tau-1,2,3$ ) or there exist $C_{X}^{\prime}, C_{X, Z}^{\prime \prime}$ with $Z \neq X$ that satisfy (P- $\tau-1,2,3,4$ ).

For the first alternative, it follows from $C_{X}^{\prime}$ satisfies (C- $\left.\tau-1,2\right)$ that $q \equiv C_{X}^{\prime}\{\langle Y \mid E\rangle / X\}$ and $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \xrightarrow{\tau} C_{X}^{\prime}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\}$. Moreover, due to (C- $\tau$-3-ii), there is at most one occurrence of the unfolded variable $X$ in $C_{X}^{\prime}$. Consequently, the context $C_{X}^{\prime}$ is exactly one that we seek.

For the second alternative, by (P- $\tau-2$ ), there exists $q^{\prime}$ such that

$$
\langle Y \mid E\rangle \xrightarrow{\tau} q^{\prime} \text { and } q \equiv C_{X, Z}^{\prime \prime}\left\{\langle Y \mid E\rangle / X, q^{\prime} / Z\right\}
$$

Hence $\left\langle t_{Y} \mid E\right\rangle \xrightarrow{\tau} q^{\prime}$. Then it follows from (P- $\tau-4$ ) that

$$
C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \xrightarrow{\tau} C_{X, Z}^{\prime \prime}\left\{\left\langle t_{Y} \mid E\right\rangle / X, q^{\prime} / Z\right\} .
$$

In addition, due to ( $\mathrm{P}-\tau-1$ ), by Lemma 5.2(1), there is at most one occurrence of the unfolded variable $X$ in $C_{X}^{\prime}$. Moreover, since $C_{X}^{\prime}$ and $C_{X, Z}^{\prime \prime}$ satisfy ( $\mathrm{P}-\tau-3$ ), we obtain $X \notin F V\left(C_{X, Z}^{\prime \prime}\right)$. Hence $q \equiv C_{X, Z}^{\prime \prime}\left\{\langle Y \mid E\rangle / X, q^{\prime} / Z\right\} \equiv C_{X, Z}^{\prime \prime}\left\{\left\langle t_{Y} \mid E\right\rangle / X, q^{\prime} / Z\right\}$. Then it is easy to see that $B_{X} \triangleq q$ is what we need.

Case $2 \alpha \in$ Act.
Let $C_{X}\{\langle Y \mid E\rangle / X\} \xrightarrow{\alpha} q$. Then, by Lemma 5.8 there exist $C_{X}^{\prime}, C_{X, \widetilde{Z}}^{\prime}$ and $C_{X, \widetilde{Z}}^{\prime \prime}$ with $X \notin \widetilde{Z}$ that satisfy (CP-a-1) - (CP-a-4). Since $C_{X}\{\langle Y \mid E\rangle / X\}$ is stable, by item (2), so is $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\}$.

If $\widetilde{Z}=\emptyset$, it follows trivially by (CP- $a$-3-iii) that $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \xrightarrow{\alpha} C_{X, \widetilde{Z}}^{\prime \prime}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\}$; moreover, by (CP-a-1), (CP-a-3-i), (CP-a-4-i) and Lemma 5.2 (1), there is at most one occurrence of the unfolded variable $X$ in $C_{X, \widetilde{Z}}^{\prime \prime}$. Therefore, $C_{X, \widetilde{Z}}^{\prime \prime}$ is exactly the context that we need.

We next deal with the other case $\widetilde{Z} \neq \emptyset$. Since $C_{X}^{\prime}$ satisfies (CP- $a-1$ ), by Lemma $5.2(1)$, there is at most one occurrence of the unfolded variable $X$ in $C_{X}^{\prime}$. Then it follows from (CP-a-2), (CP-a-3-i) and (CP-a-4-i) that $|\widetilde{Z}|=1$ and $X \notin F V\left(C_{X, \widetilde{Z}}^{\prime \prime}\right)$. So, due to (CP-$a-3-\mathrm{ii})$, there exists $q^{\prime}$ such that

$$
\langle Y \mid E\rangle \xrightarrow{\alpha} q^{\prime} \text { and } q \equiv C_{X, \widetilde{Z}}^{\prime \prime}\left\{\langle Y \mid E\rangle / X, q^{\prime} / \widetilde{Z}\right\}
$$

Hence $\left\langle t_{Y} \mid E\right\rangle \xrightarrow{\alpha} q^{\prime}$. Then $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \xrightarrow{\alpha} C_{X, \widetilde{Z}}^{\prime \prime}\left\{\left\langle t_{Y} \mid E\right\rangle / X, q^{\prime} / \widetilde{Z}\right\}$ by (CP-a-3-iii) and $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \not{ }^{\tau}$. Thus $q \equiv C_{X, \widetilde{Z}}^{\prime \prime}\left\{\left\langle t_{Y} \mid E\right\rangle / X, q^{\prime} / \widetilde{Z}\right\}$ because of $X \notin F V\left(C_{X, \widetilde{Z}}^{\prime \prime}\right)$. Then it is easy to check that $B_{X} \triangleq q$ is exactly what we seek.

### 5.3. Multi- $\tau$ transitions and more on unfolding

Based on the result obtained in the preceding subsections, we shall give a few further properties of unfolding. We first want to indicate some simple properties.

Lemma 5.11. The relation $\Rightarrow$ satisfies the forward and backward conditions, that is, for any $\alpha \in A c t_{\tau}$ and $p, q$ such that $p \Rightarrow q$, we have
(1) if $p \xrightarrow{\alpha} p^{\prime}$ then $q \xrightarrow{\alpha} q^{\prime}$ and $p^{\prime} \Rightarrow q^{\prime}$ for some $q^{\prime}$;
(2) if $q \xrightarrow{\alpha} q^{\prime}$ then $p \xrightarrow{\alpha} p^{\prime}$ and $p^{\prime} \Rightarrow q^{\prime}$ for some $p^{\prime}$.

Proof. (1) Assume $p \Rightarrow q$ and $p \xrightarrow{\alpha} p^{\prime}$. Clearly, $p \Rightarrow_{n} q$ for some $n$. It proceeds by induction on $n$. For the induction base $n=0$, it holds trivially. For the induction step $n=k+1$, we have $p \Rightarrow_{k} r \Rightarrow_{1} q$ for some $r$. By $\mathrm{IH}, r \xrightarrow{\alpha} r^{\prime}$ and $p^{\prime} \Rightarrow r^{\prime}$ for some $r^{\prime}$. Moreover, for $r \Rightarrow_{1} q$, by Lemma 5.1 5.10(1) and ??(1), there exists $q^{\prime}$ such that $q \xrightarrow{\alpha} q^{\prime}$ and $r^{\prime} \Rightarrow q^{\prime}$. Obviously, we also have $p^{\prime} \Rightarrow q^{\prime}$.
(2) Similar to item (1), but applying Lemmas 5.10 (2) and ??(2) instead of Lemmas 5.10(1) and ??(1).

A similar result also holds w.r.t $\xlongequal{\epsilon} \mid$, that is
Lemma 5.12. For any $p, q$ such that $p \Rightarrow q$, we have
(1) if $p \xlongequal{\epsilon} \mid p^{\prime}$ then $q \xlongequal{\epsilon} \mid q^{\prime}$ and $p^{\prime} \Rightarrow q^{\prime}$ for some $q^{\prime}$;
(2) if $q \xlongequal{\epsilon} \mid q^{\prime}$ then $p \xlongequal{\epsilon} \mid q^{\prime}$ and $p^{\prime} \Rightarrow q^{\prime}$ for some $p^{\prime}$.

Proof. By applying Lemma 5.11 finitely often.
In fact, for any $p, q$ such that $p \Rightarrow q$, it is to be expected that $p=_{R S} q$. To verify it, we need to prove that $p \in F$ if and only if $q \in F$. The next lemma will serve as a stepping stone in proving this.

Convention 5.1. The arguments in the remainder of this paper often proceed by distinguishing some cases based on the last rule applied in an inference. For such argument, since rules about operations $\wedge, \vee, \|_{A}$ and $\square$ are symmetric w.r.t their two operands (for instance, Rules $R p_{11}$ and $R p_{12}, R a_{4}$ and $R a_{5}$, and so on), we shall consider only one of two symmetric rules and omit another one.

Lemma 5.13. For any $Y, E$ with $Y=t_{Y} \in E$ and context $C_{X}$ with at most one occurrence of the unfolded variable $X, C_{X}\{\langle Y \mid E\rangle / X\} \in F$ iff $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \in F$.

Proof. We give proof only for the implication from left to right, the converse implication may be proved similarly and omitted. Assume $C_{X}\{\langle Y \mid E\rangle / X\} \in F$. It proceeds by induction on the depth of the inference $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash C_{X}\{\langle Y \mid E\rangle / X\} F$, which is a routine case analysis on the form of $C_{X}$. We give the proof only for the case $C_{X} \equiv B_{X} \wedge D_{X}$, the other cases are left to the reader. In this situation, the last rule applied in the inference is one of the following.

Case $1 \frac{B_{X}\{\langle Y \mid E\rangle / X\} F}{B_{X}\{\langle Y \mid E\rangle / X\} \wedge D_{X}\{\langle Y \mid E\rangle / X\} F}$.
By IH, we get $B_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \in F$. Hence $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \in F$.
Case $2 \frac{B_{X}\{\langle Y \mid E\rangle / X\} \xrightarrow{a} r, D_{X}\{\langle Y \mid E\rangle / X\} \not{ }^{a}, C_{X}\{\langle Y \mid E\rangle / X\} f^{\top}}{B_{X}\{\langle Y \mid E\rangle / X\} \wedge D_{X}\{\langle Y \mid E\rangle / X\} F}$.
By Lemma 5.10 and ??, we have $B_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \xrightarrow{a}, D_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \not{ }^{a}$ and $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \not{ }^{\top}$. So, $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \in F$.

Case $3 \frac{C_{X}\{\langle Y \mid E\rangle / X\} \xrightarrow{\alpha} s,\left\{r F: C_{X}\{\langle Y \mid E\rangle / X\} \stackrel{ }{\longrightarrow} r\right\}}{C_{X}\{\langle Y \mid E\rangle / X\} F}$.
Then $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \xrightarrow{\alpha}$ by Lemma [5.10(1) and ??(1). Assume $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \xrightarrow{\alpha}$ $q$. Thus it follows from Lemma $5.10(2)$ and ? ? (2) that there exists $C_{X}^{\prime}$ with at most one occurrence of the unfolded variable $X$ such that

$$
C_{X}\{\langle Y \mid E\rangle / X\} \xrightarrow{\alpha} C_{X}^{\prime}\{\langle Y \mid E\rangle / X\} \text { and } q \equiv C_{X}^{\prime}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} .
$$

Then, by IH, $q \equiv C_{X}^{\prime}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \in F$. Hence $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \in F$ by Theorem4.2.

Case $\left.\left.4 \frac{\left\{r F: C_{X}\{\langle Y \mid E\rangle / X\} \xlongequal{\epsilon}\right.}{C_{X}\{\langle Y \mid E\rangle / X\} F} \right\rvert\, r\right\}$.
Assume $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \stackrel{\epsilon}{\Longrightarrow} \mid t$. Repeated application of Lemma 5.10 (2) enables us to get $C_{X}\{\langle Y \mid E\rangle / X\} \stackrel{\epsilon}{\Longrightarrow} \mid r, r \equiv C_{X}^{\prime}\{\langle Y \mid E\rangle / X\}$ and $t \equiv C_{X}^{\prime}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\}$ for some $r$ and context $C_{X}^{\prime}$ with at most one occurrence of the unfolded free variable $X$. Since $r \equiv C_{X}^{\prime}\{\langle Y \mid E\rangle / X\} \in F$, we have $t \in F$ by IH. Then $C_{X}\left\{\left\langle t_{Y} \mid E\right\rangle / X\right\} \in F$ by Theorem4.2,

Next we can show that the relation $\Rightarrow$ preserves and respects the inconsistency.
Lemma 5.14. For any $p, q$, if $p \Rightarrow q$, then $p \in F$ iff $q \in F$.
Proof. Suppose $p \Rightarrow q$. Hence $p \Rightarrow_{n} q$ for some $n$. Then, using Lemma 5.1 and 5.13, the proof is straightforward by induction on $n$.

We now have the assertion below of the equivalence of $p$ and $q$ modulo $={ }_{R S}$ whenever $p \Rightarrow q$.

Lemma 5.15. If $p_{1} \Rightarrow p_{2}$ then $p_{1}={ }_{R S} p_{2}$, in particular, $p_{1} \approx_{R S} p_{2}$ whenever $p_{1} f^{\tau}$.
Proof. We only prove $p_{1} \underset{\sim}{\sqsubset}{ }_{R S} p_{2}$ whenever $p_{1} \stackrel{\tau}{\leftrightarrows}$, other proofs are straightforward and omitted. Set

$$
\mathcal{R}=\left\{(p, q): p \Rightarrow q \text { and } p \not{ }_{\neq}^{\top}\right\} .
$$

It suffices to prove that $\mathcal{R}$ is a stable ready simulation relation. Suppose $(p, q) \in \mathcal{R}$. Then, by Lemma 5.11 and 5.14 it is evident that such pair satisfies (RS1), (RS2) and (RS4). For (RS3), suppose $p{ }^{a}{ }_{F} \mid p^{\prime}$. Then $p \xrightarrow{a}_{F} p^{\prime \prime}{ }^{\epsilon}{ }_{F} \mid p^{\prime}$ for some $p^{\prime \prime}$. By Lemma 5.11 and 5.14, there exists $q^{\prime \prime}$ such that $q \xrightarrow{a}_{F} q^{\prime \prime}$ and $p^{\prime \prime} \Rightarrow q^{\prime \prime}$. Further, by Lemma 4.2 , 5.12 and 5.14 $p^{\prime} \Rightarrow q^{\prime}$ and $q^{\prime \prime}{ }^{\epsilon}{ }_{F} \mid q^{\prime}$ for some $q^{\prime}$. Moreover, $\left(p^{\prime}, q^{\prime}\right) \in \mathcal{R}$, as desired.

In the following, we shall generalize Lemma 5.6 to the situation involving a sequence of $\tau$-labelled transitions. Given a process $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$, by Lemma 5.6, any $\tau$-transition starting from $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$ may be caused by $C_{\widetilde{X}}$ itself or some $p_{i}$. Thus, for a sequence of $\tau$-transitions, these two situations may occur alternately. Based on Lemma 5.6, we can capture this as follows.

Lemma 5.16. For any $C_{\widetilde{X}}$ and $\widetilde{p}$, if $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xlongequal{\epsilon} r$ then there exist $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ and $i_{Y} \leq$ $|\widetilde{X}|, p_{Y}^{\prime}(Y \in \tilde{Y}) \underset{\sim}{\sim}$ such that
(MS- $\tau$-1) $\widetilde{X} \cap \widetilde{Y}=\emptyset$ and $Y$ is 1-active in $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ for each $Y \in \widetilde{Y}$;
(MS- $\tau$-2) $p_{i_{Y}} \xlongequal{\tau} p_{Y}^{\prime}$ for each $Y \in \widetilde{Y}$ and $r \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\}$;
(MS- $\tau-3)$ for any $\widetilde{q}$ and $\widetilde{q_{Y}^{\prime}}$ with $|\widetilde{q}|=|\widetilde{X}|$ and $Y \in \widetilde{Y}$,
(MS- $\tau \mathbf{- 3 - i})$ if $q_{i_{Y}} \xlongequal{\epsilon} q_{Y}^{\prime}$ for each $Y \in \widetilde{Y}$ then $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xlongequal{\epsilon} \Rightarrow C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\}$;
(MS- $\tau$-3-ii) if $q_{i_{Y}} \xlongequal{\tau} q_{Y}^{\prime}$ for each $Y \in \widetilde{Y}$ then $C_{\tilde{X}}\{\widetilde{q} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}, \tilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\}$; (MS- $\tau \mathbf{- 4})$ if $C_{\widetilde{X}}$ is stable then so is $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ and $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \Rightarrow C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{i_{Y}}} / \widetilde{Y}\right\}$ for any $\widetilde{q} ;$
(MS- $\tau$-5) for each $X \in \widetilde{X}$, if $X$ is strongly guarded in $C_{\widetilde{X}}$ then so it is in $C_{\tilde{X}, \widetilde{Y}}^{\prime}$ and $X \not \equiv X_{i_{Y}}$ for each $Y \in \widetilde{Y}$;
(MS- $\tau$ - 6) for each $X \in \widetilde{X}$ (or, $Y \in \widetilde{Y}$ ), if $X\left(X_{i_{Y}}\right.$ respectively) does not occur in the scope of any conjunction in $C_{\widetilde{X}}$ then neither does $X$ ( $Y$ respectively) in $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$;
(MS- $\tau$-7) if $r$ is stable then so are $C_{\widetilde{X}, \tilde{Y}}^{\prime}$ and $p_{Y}^{\prime}$ for each $Y \in \widetilde{Y}$.
Proof. Suppose $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}(\xrightarrow{\tau})^{n} r(n \geq 0)$. We proceed by induction on $n$. For the inductive base $n=0$, the conclusion holds trivially by taking $C_{\widetilde{X}, \widetilde{Y}}^{\prime} \triangleq C_{\widetilde{X}}$ with $\widetilde{Y}=\emptyset$.

For the inductive step, assume $C_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\}(\xrightarrow{\tau})^{k} s \xrightarrow{\tau} r$ for some $s$. For the transition $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}(\xrightarrow{\tau})^{k} s$, by IH, there exist $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ and $i_{Y} \leq|\widetilde{X}|, p_{Y}^{\prime}(Y \in \widetilde{Y})$ that realize (MS-$\tau-l)(1 \leq l \leq 7)$. In particular, we have $s \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\}$ due to (MS- $\left.\tau-2\right)$. Then, for the transition $s \xrightarrow{\tau} r$, either the clause (1) or (2) in Lemma 5.6 holds. The argument splits into two cases.

Case 1 For the transition $s \xrightarrow{\tau} r$, the clause (1) in Lemma 5.6 holds.
That is, for the transition $C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\} \equiv s \xrightarrow{\tau} r$, there exists $C_{\tilde{X}, \widetilde{Y}}^{\prime \prime}$ satisfying (C- $\tau-1,2,3)$ in Lemma 5.6. We shall check that $C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}, \widetilde{i_{Y}}$ and $\widetilde{p_{Y}^{\prime}}$ realize (MS- $\tau-1$ ) - (MS-$\tau$-7) w.r.t $C_{\widetilde{X}}\{\widetilde{p} / \tilde{X}\}(\xrightarrow{\tau})^{k+1} r$.

Since $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ satisfies (MS- $\tau-1,5,6$ ), it follows that $C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}$ and $\widetilde{i_{Y}}$ realize (MS- $\tau-1$ ), (MS-$\tau-5$ ) and (MS- $\tau-6$ ) due to (C- $\tau-3-\mathrm{i})$, (C- $\tau-3$-iii) and (C- $\tau$-3-iv) respectively. Moreover, as $C_{\tilde{X}, \tilde{Y}}^{\prime \prime}$ satisfies (C- $\tau-1$ ) it follows immediately that (MS- $\tau-2$ ) holds. Since $C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}$ satisfies (C- $\tau-2$ ), by Lemma 5.7, $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ is not stable. Then neither is $C_{\widetilde{X}}$ because $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ satisfies (MS- $\tau-4$ ). Thus, $C_{\tilde{X}, \widetilde{Y}}^{\prime \prime}$ satisfies (MS- $\left.\tau-4\right)$ trivially.

Next we verify (MS- $\tau-3$ ). Let $\widetilde{q}$ be any processes with $|\widetilde{q}|=|\widetilde{X}|$ and $q_{i_{Y}} \xlongequal{\epsilon} q_{Y}^{\prime}$ for each $Y \in \widetilde{Y}$.
(MS- $\tau \mathbf{- 3 - i}$ ) Since $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ satisfies (MS- $\tau-3-\mathrm{i}$ ), we have

$$
C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \Rightarrow C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} \text { for some } t .
$$

Moreover, we have $C_{\widetilde{X}, \tilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} \xrightarrow{\tau} C_{\widetilde{X}, \tilde{Y}}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\}$ due to (C- $\left.\tau-2\right)$. Then it follows from Lemma 5.11 that

$$
t \xrightarrow{\tau} t^{\prime} \Rightarrow C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} \text { for some } t^{\prime}
$$

Therefore, $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \xrightarrow{\tau} t^{\prime} \Rightarrow C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\}$, as desired.
(MS- $\tau$-3-ii) It is straightforward as $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ satisfies (MS- $\tau-3$-ii) and $C_{\tilde{X}, \widetilde{Y}}^{\prime \prime}$ satisfies (C-$\tau-2)$.
(MS- $\tau$-7) Suppose $r \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\} \not{ }^{\tau}$. Then, since $C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}$ satisfies (MS- $\tau-1$ ), by Lemmas 5.4 and 5.7 it is easy to see that both $C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}$ and $\widetilde{p_{Y}^{\prime}}$ are stable.

Case 2 For the transition $s \xrightarrow{\tau} r$, the clause (2) in Lemma 5.6 holds.

Then there exist $i_{0} \leq|\widetilde{X}|+|\widetilde{Y}|, C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left(\equiv C_{X_{1}, \ldots, X_{|\widetilde{X}|}, Y_{|\widetilde{X}|+1}, \ldots, Y_{|\widetilde{X}|+|\tilde{Y}|}^{\prime \prime}}^{\prime \prime}\right)$ and $C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}(\equiv$ $C_{X_{1}, \ldots, X_{|\widetilde{X}|}^{\prime \prime}, Y_{|\widetilde{X}|+1}, \ldots, Y_{|\widetilde{X}|+|\tilde{Y}|}, Z}$ ) with $Z \notin \widetilde{X} \cup \tilde{Y}$ satisfying $(\mathrm{P}-\tau-1)$ - ( $\mathrm{P}-\tau-4$ ). In particular, by $(\mathrm{P}-\tau-3)$,

$$
C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime} \equiv \begin{cases}C_{\widetilde{X}, \tilde{Y}, Z}^{\prime \prime \prime}\left\{X_{i_{0}} / Z\right\}, & \text { if } 1 \leq i_{0} \leq|\widetilde{X}| \\ C_{\tilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{Y_{i_{0}} / Z\right\}, & \text { if }|\widetilde{X}|+1 \leq i_{0} \leq|\widetilde{X}|+|\widetilde{Y}|\end{cases}
$$

In case $|\widetilde{X}|+1 \leq i_{0} \leq|\widetilde{X}|+|\widetilde{Y}|$, by (P- $\left.\tau-2\right)$, there exists $p^{\prime}$ such that

$$
p_{Y_{i_{0}}}^{\prime} \xrightarrow{\tau} p^{\prime} \text { and } r \equiv C_{\widetilde{X}, \tilde{Y}, Z}^{\prime \prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}, p^{\prime} / Z\right\}
$$

Moreover, since $Y_{i_{0}}$ is 1-active in $C_{\tilde{X}, \tilde{Y}}^{\prime}$, by (P- $\tau-1$ ), we have $C_{\widetilde{X}, \widetilde{Y}}^{\prime} \equiv C_{\widetilde{X}, \tilde{Y}}^{\prime \prime}$. Further, since $Z$ is 1-active in $C_{\tilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}$ and $C_{\tilde{X}, \widetilde{Y}}^{\prime \prime} \equiv C_{\tilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{Y_{i_{0}} / Z\right\}$, it is easy to see that $Y_{i_{0}}$ does not occur in $C_{\tilde{X}, \tilde{Y}, Z}^{\prime \prime \prime}$. Hence

$$
r \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}}\left[p^{\prime} / p_{Y_{i_{0}}}^{\prime}\right] / \widetilde{Y}\right\} \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}}\left[p^{\prime} / p_{Y_{i_{0}}}^{\prime}\right] / \widetilde{Y}\right\}
$$

Then it is not difficult to check that $C_{\widetilde{X}}^{\prime}, \widetilde{Y}, \widetilde{p_{Y}^{\prime}}\left[p^{\prime} / p_{Y_{i_{0}}}^{\prime}\right]$ and $\widetilde{i_{Y}}$ realize (MS- $\left.\tau-l\right)(1 \leq l \leq 7)$ w.r.t the transition $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}(\xrightarrow{\tau})^{k+1} r$, as desired.

In case $1 \leq i_{0} \leq|\widetilde{X}|$, by (P- $\left.\tau-2\right)$, there exists $p^{\prime \prime}$ such that $p_{i_{0}} \xrightarrow{\tau} p^{\prime \prime}$ and $r \equiv$ $C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}, p^{\prime \prime} / Z\right\}$. Set

$$
i_{Z} \triangleq i_{0} \text { and } p_{Z}^{\prime} \triangleq p^{\prime \prime}
$$

In the following, we intend to verify that $C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}, i_{U}(U \in \widetilde{Y} \cup\{Z\})$ and $|\widetilde{Y}|+$ 1-tuple $\widetilde{p_{U}^{\prime}}$ with $U \in \widetilde{Y} \cup\{Z\}$ realize (MS- $\left.\tau-1\right)$ - (MS- $\tau-7$ ) w.r.t $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}(\xrightarrow{\tau})^{k+1} r$.
(MS- $\tau$-1) By (P- $\tau-1$ ), we have $C_{\widetilde{X}, \widetilde{Y}}^{\prime} \Rightarrow C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}$. Moreover, since $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ satisfy (MS- $\tau-1$ ), by Lemma $5.2(1), Y$ is 1-active in $C_{\tilde{X}, \tilde{Y}}^{\prime \prime}$ for each $Y \in \tilde{Y}$. Further, by (P- $\left.\tau-3\right)$, each $Y \in \widetilde{Y}$ and $Z$ are 1-active in $C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}$.
(MS- $\tau$-2) It is straightforward.
(MS- $\tau \mathbf{- 3})$ Let $\widetilde{q}$ be any processes with $|\widetilde{q}|=|\widetilde{X}|$.
(MS- $\tau$-3-i) Suppose $q_{i} \xlongequal{\epsilon} q_{U}^{\prime}$ for each $U \in \widetilde{Y} \cup\{Z\}$. Since $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ satisfies (MS- $\tau-3-\mathrm{i}$ ), we have

$$
C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \Rightarrow C_{\widetilde{X}, \tilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} \text { for some } t
$$

It follows from $q_{i_{Z}} \xlongequal{\epsilon} q_{Z}^{\prime}$ that $q_{i_{Z}}(\xrightarrow{\tau})^{m} q_{Z}^{\prime}$ for some $m \geq 0$. We shall distinguish two cases based on $m$.

In case $m=0$, we get $q_{i_{Z}} \equiv q_{Z}^{\prime}$. Since $C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}$ satisfies (P- $\tau-1$ ) and (P- $\tau-3$ ), we have

$$
C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} \Rightarrow C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} \equiv C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}, q_{Z}^{\prime} / Z\right\}
$$

Therefore $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \Rightarrow C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} \equiv C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}, q_{Z}^{\prime} / Z\right\}$, as desired.

In case $m>0$, i.e., $q_{i_{Z}} \xrightarrow{\tau} q^{\prime \prime} \xrightarrow{\epsilon} q_{Z}^{\prime}$ for some $q^{\prime \prime}$, by ( $\mathrm{P}-\tau-4$ ), we obtain

$$
C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} \xrightarrow{\tau} C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}, q^{\prime \prime} / Z\right\} .
$$

Moreover, since $Z$ is 1-active, by Lemma 5.4, we get

$$
C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}, q^{\prime \prime} / Z\right\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}, q_{Z}^{\prime} / Z\right\}
$$

Then, by Lemma 5.12, it follows from $t \Rightarrow C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\}$ that there exist $t^{\prime}$ such that

$$
t \xlongequal{\epsilon} t^{\prime} \Rightarrow C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}, q_{Z}^{\prime} / Z\right\}
$$

Consequently, $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \xlongequal{\epsilon} t^{\prime} \Rightarrow C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}, q_{Z}^{\prime} / Z\right\}$.
(MS- $\tau$-3-ii) Suppose $q_{i} \xrightarrow{\tau} q_{U}^{\prime}$ for each $U \in \widetilde{Y} \cup\{Z\}$. Since $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ satisfies (MS- $\tau$-3ii), we have

$$
C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\}
$$

Moreover, $q_{i_{Z}} \xrightarrow{\tau} q^{\prime \prime} \xlongequal{\epsilon} q_{Z}^{\prime}$ for some $q^{\prime \prime}$ because of $q_{i_{Z}} \xrightarrow{\tau} q_{Z}^{\prime}$. Hence by (P- $\tau-4$ )

$$
C_{\tilde{X}, \tilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} \xrightarrow{\tau} C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}, q^{\prime \prime} / Z\right\} .
$$

Further, since $Z$ is 1-active, it follows from Lemma 5.4 that

$$
C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}, q^{\prime \prime} / Z\right\} \xlongequal{\epsilon} C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}, q_{Z}^{\prime} / Z\right\}
$$

Consequently, $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}, q_{Z}^{\prime} / Z\right\}$, as desired.
(MS- $\tau \mathbf{- 4 )}$ Assume $C_{\widetilde{X}}$ is stable. By (MS- $\left.\tau-4\right), C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ is stable and for any $\widetilde{q}, C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \Rightarrow$ $C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{i_{Y}}} / \widetilde{Y}\right\}$. Moreover, by Lemma 5.11] it follows from $C_{\widetilde{X}, \tilde{Y}}^{\prime} \Rightarrow C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}$ (i.e., (P-$\tau-1)$ ) and $C_{\tilde{X}, \widetilde{Y}, Z}^{\prime \prime}\left\{X_{i_{Z}} / Z\right\} \equiv C_{\tilde{X}, \widetilde{Y}}^{\prime \prime}$ (i.e., $\left.(\mathrm{P}-\tau-3)\right)$ that $C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}$ is stable and

$$
C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{i_{Y}}} / \widetilde{Y}\right\} \Rightarrow C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{i_{Y}}} / \widetilde{Y}, q_{i_{Z}} / Z\right\}
$$

(MS- $\tau-\mathbf{5}, 6)$ By Lemma $\left[5.2(3)(5)\right.$, they immediately follow from the fact that $C_{\tilde{X}, \tilde{Y}}^{\prime}$ satisfies (MS- $\tau-5,6)$ and $C_{\widetilde{X}, \widetilde{Y}, Z}^{\prime \prime \prime}$ satisfies (P- $\tau-1,3$ ).
(MS- $\tau-7$ ) Immediately follows from (MS- $\tau-1$ ), (MS- $\tau-2$ ) and Lemmas 5.4 and 5.7.
Lemma 5.17. For any $\widetilde{p}$ and stable context $C_{\tilde{X}}$, if, for each $i \leq|\widetilde{X}|, p_{i} \xlongequal{\epsilon} \mid p_{i}^{\prime}$ then $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} \mid q$ for some $q$.

Proof. By Lemma 5.3 and 5.2(4), $C_{\widetilde{X}} \Rightarrow C_{\widetilde{X}}^{\prime}$ for some $C_{\widetilde{X}}^{\prime}$ such that each unguarded occurrence of any free variable in $C_{\widetilde{X}}^{\prime}$ is unfolded. Moreover, since $C_{\tilde{X}}$ is stable, so is $C_{\widetilde{X}}^{\prime}$ by $C_{\widetilde{X}}\{0 / \widetilde{X}\} \Rightarrow C_{\widetilde{X}}^{\prime}\{0 / \widetilde{X}\}$ and Lemma 5.11.

Let $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ be the context obtained from $C_{\widetilde{X}}^{\prime}$ by replacing simultaneously all unguarded and unfolded occurrences of free variables in $\widetilde{X}$ by distinct and fresh variables $\widetilde{Y}$. Here distinct occurrences are replaced by distinct variables. Clearly, we have
(1) for each $Y \in \widetilde{Y}$, there exists exactly one $i_{Y} \leq|\widetilde{X}|$ such that $C_{\widetilde{X}}^{\prime} \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{X_{i_{Y}}} / \widetilde{Y}\right\}$,
(2) all variables in $\tilde{Y}$ are 1-active in $C_{\tilde{X}, \tilde{Y}}^{\prime}$, and
(3) $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ is stable.

Then $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \Rightarrow C_{\widetilde{X}}^{\prime}\{\widetilde{p} / \widetilde{X}\} \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{i_{Y}}} / \widetilde{Y}\right\}$, and by Lemma 5.4, we obtain $C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{i_{Y}}} / \widetilde{Y}\right\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{i_{Y}}^{\prime}} / \widetilde{Y}\right\}$. Further, since $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ and $\widetilde{p_{i_{Y}}^{\prime}}$ are stable and $\widetilde{Y}$ contains all unguarded occurrences of variables in $C_{\tilde{X}, \tilde{Y}}^{\prime}$, we get $C_{\widetilde{X}, \tilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{i_{Y}^{\prime}}^{\prime}} / \widetilde{Y}\right\} \not{ }^{\top}$ by Lemma 5.6. Hence, by Lemma 5.12, $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xlongequal{\epsilon} \mid q \Rightarrow C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{i_{Y}}^{\prime}} / \widetilde{Y}\right\}$ for some $q$.

Given a process $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$ and its stable $\tau$-descendant $r$ (i.e., $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} \mid r$ ), in general there exist more than one evolution paths from $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$ to $r$. Since each $\tau$ labelled transition in $\mathrm{CLL}_{R}$ activated by a single process, a natural conjecture arises at this point that there exist some "canonical" evolution paths from $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$ to $r$ in which the context $C_{\widetilde{X}}$ first evolves itself into a stable context then $p_{i}$ evolves. A weak version of this conjecture will be verified in Lemma 5.19. To this end, a preliminary result is given:

Lemma 5.18. Let $t_{1}, t_{2}$ be two terms and $\widetilde{X}$ a tuple of variables such that any recursive variable occurring in $t_{i}$ (with $i=1,2$ ) is not in $\widetilde{X}$, and let $\widetilde{a_{X} .0}$ be a tuple of processes with fresh visible action $a_{X}$ for each $X \in \widetilde{X}$. Then
(1) if $t_{1}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv t_{2}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\}$ then $t_{1} \equiv t_{2}$;
(2) if $t_{1}\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}\right\} \Rightarrow t_{2}\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}\right\}$ then $t_{1}\{\widetilde{r} / \widetilde{X}\} \Rightarrow \Rightarrow_{1} t_{2}\{\widetilde{r} / \widetilde{X}\}$ for any $\widetilde{r}$.

Proof. (1) If $F V\left(t_{1}\right) \cap \widetilde{X}=\emptyset$ then $t_{1}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv t_{1} \equiv t_{2}\left\{\widetilde{a_{X} .0} / \tilde{X}\right\}$. Further, since $a_{X}$ is fresh for each $X \in \widetilde{X}$, we have $F \underset{\sim}{F} V\left(t_{2}\right) \cap \widetilde{X}=\emptyset$. Hence $t_{1} \equiv t_{2}$. In the following, we consider the other case $F V\left(t_{1}\right) \cap \widetilde{X} \neq \emptyset$. We proceed by induction on $t_{1}$.

Case $1 t_{1} \equiv X_{i}$.
Then $t_{1}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv a_{X_{i}} .0 \equiv t_{2}\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}\right\}$. Hence $t_{2} \equiv X_{i}$ due to the freshness of $a_{X_{i}}$.

Case $2 t_{1} \equiv \alpha . s$.
So $t_{1}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv \alpha . s\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv t_{2}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\}$. Since $\alpha \neq a_{X}$ for each $X \in \widetilde{X}$, there exists $s^{\prime}$ such that $t_{2} \equiv \alpha . s^{\prime}$ and $s\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}\right\} \equiv s^{\prime}\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}\right\}$. By IH, we have $s \equiv s^{\prime}$. Hence $t_{1} \equiv t_{2}$.

Case $3 t_{1} \equiv s_{1} \odot s_{2}$ with $\odot \in\left\{\vee, \square, \|_{A}, \wedge\right\}$.
Then $t_{1}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv s_{1}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \odot s_{2}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv t_{2}\left\{\widetilde{a_{X} .0} / \tilde{X}\right\}$. Since $\widetilde{a_{X} .0}$ do not contain $\odot$, there exist $s_{1}^{\prime}, s_{2}^{\prime}$ such that $t_{2} \equiv s_{1}^{\prime} \odot s_{2}^{\prime}, s_{1}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv s_{1}^{\prime}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\}$ and $s_{2}\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}\right\} \equiv s_{2}^{\prime}\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}\right\}$. Hence $s_{1} \equiv s_{1}^{\prime}$ and $s_{2} \equiv s_{2}^{\prime}$ by applying IH.

Case $4 t_{1} \equiv\langle Y \mid E\rangle$ for some $E(V)$ with $Y \in V$.
Then $t_{1}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv\left\langle Y \mid E\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\}\right\rangle \equiv t_{2}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\}$. So, $t_{2} \equiv\left\langle Y \mid E^{\prime}\right\rangle$ for some $E^{\prime}(V)$ such that for each $Z \in V, t_{Z}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv t_{Z}^{\prime}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\}$ where $Z=t_{Z} \in E$ and $Z=t_{Z}^{\prime} \in E^{\prime}$. By IH, $t_{Z} \equiv t_{Z}^{\prime}$ for each $Z \in V$. Thus $t_{1} \equiv\langle Y \mid E\rangle \equiv\left\langle Y \mid E^{\prime}\right\rangle \equiv t_{2}$.
(2) In case $F V\left(t_{1}\right) \cap \widetilde{X}=\emptyset$, since one-step unfolding does not introduce fresh actions, we have $F V\left(t_{2}\right) \cap \widetilde{X}=\emptyset$. Thus, $t_{1} \equiv t_{1}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \Rightarrow_{1} t_{2}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv t_{2}$, and hence $t_{1} \equiv$ $t_{1}\{\widetilde{r} / \widetilde{X}\} \Rightarrow_{1} t_{2}\{\widetilde{r} / \widetilde{X}\} \equiv t_{2}$ for any $\widetilde{r}$. Next we consider the other case $F V\left(t_{1}\right) \cap \widetilde{X} \neq \emptyset$. It proceeds by induction on $t_{1}$. This is a routine case analysis on the format of $t_{1}$, we handle only the case $t_{1} \equiv\langle Y \mid E\rangle$.

In this case, $t_{1}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv\langle Y \mid E\rangle\left\{\widetilde{a_{X} .0} / \tilde{X}\right\}$. By Def. 5.2, the unique result of onestep unfolding of $\langle Y \mid E\rangle\left\{\widetilde{a_{X} .0 / \tilde{X}}\right\}$ is $\left\langle t_{Y} \mid E\right\rangle\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\}$ where $Y=t_{Y} \in E$. Thus, we get $\left\langle t_{\tilde{Y}} \mid E\right\rangle\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv t_{2}\left\{\widetilde{a_{X} .0} / \tilde{X}\right\}$. By item (1), we have $\left\langle t_{Y} \mid E\right\rangle \equiv t_{2}$, and hence $t_{1}\{\widetilde{r} / \widetilde{X}\} \Rightarrow_{1}\left\langle t_{Y} \mid E\right\rangle\{\widetilde{r} / \widetilde{X}\} \equiv t_{2}\{\widetilde{r} / \widetilde{X}\}$ for any $\widetilde{r}$.

Having disposed of this preliminary step, we can now verify a weak version of the conjecture mentioned above, which is sufficient for the aim of this paper. At present, we do not know whether this result still holds if the requirement (1) in the next lemma is strengthened to $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\epsilon} D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} \mid r$.

Lemma 5.19. For any $C_{\widetilde{X}}$ and $\widetilde{p}$, if $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xlongequal{\epsilon} \mid r$ then there exists a stable context $D_{\tilde{X}}$ such that
(1) $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xlongequal{\epsilon} D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xlongequal{\epsilon} \mid r^{\prime} \Rightarrow r$ for some $r^{\prime}$, and
(2) $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} D_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\}$ for any $\widetilde{q}$ with $|\widetilde{q}|=|\widetilde{X}|$.

Proof. Suppose $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}(\xrightarrow{\tau})^{n} \mid r$. It proceeds by induction on $n$. For the inductive base $n=0$, it follows from $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \equiv r \stackrel{f^{\tau}}{\longrightarrow}$ that $C_{\widetilde{X}}$ is stable by Lemma 5.7. Then it is straightforward to verify that $C_{\widetilde{X}}$ itself is exactly what we seek. For the inductive step, assume $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} t(\xrightarrow{\tau})^{k} \mid r$ for some $t$. Then, for the $\tau$-labelled transition $C_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} t$, either the clause (1) or (2) in Lemma 5.6 holds. The first alternative is easy to handle and is thus omitted. Next we consider the second alternative.

In this situation, there exist $C_{\tilde{X}}^{\prime}, C_{\tilde{X}, Z}^{\prime \prime}$ with $Z \notin \widetilde{X}$ and $i_{0} \leq|\widetilde{X}|$ that satisfy (P- $\tau-1$ ) $-(\mathrm{P}-\tau-4)$. By (P- $\tau-2)$, we have

$$
t \equiv C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, p^{\prime} / Z\right\} \text { for some } p^{\prime} \text { with } p_{i_{0}} \xrightarrow{\tau} p^{\prime} .
$$

Then, for $C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, p^{\prime} / Z\right\}(\xrightarrow{\tau})^{k} \mid r$, by IH, there exists a stable context $D_{\widetilde{X}, Z}^{\prime}$ such that

$$
\begin{equation*}
C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, p^{\prime} / Z\right\} \stackrel{\epsilon}{\Longrightarrow} D_{\widetilde{X}, Z}^{\prime}\left\{\widetilde{p} / \widetilde{X}, p^{\prime} / Z\right\} \stackrel{\epsilon}{\Longrightarrow} \mid r^{\prime} \Rightarrow r \text { for some } r^{\prime} \tag{5.19,1}
\end{equation*}
$$

and for any $q^{\prime}$ and $\tilde{q}$, we have

$$
\begin{equation*}
C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{q} / \tilde{X}, q^{\prime} / Z\right\} \xlongequal{\epsilon} D_{\widetilde{X}, Z}^{\prime}\left\{\widetilde{q} / \widetilde{X}, q^{\prime} / Z\right\} \tag{5.19}
\end{equation*}
$$

In particular, we have $C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{a_{X} .0} / \widetilde{X}, a_{Z} \cdot 0 / Z\right\} \xlongequal{\epsilon} D_{\widetilde{X}, Z}^{\prime}\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}, a_{Z} \cdot 0 / Z\right\}$ where distinct visible actions $\widetilde{a_{X}}$ and $a_{Z}$ are fresh. For this transition, applying Lemma 5.6 finitely often (notice that, in this procedure, since $\widetilde{a_{X} .0}$ and $a_{Z .} .0$ are stable, the clause (2) in Lemma 5.6 is always false), then by clause (1) in Lemma 5.6, we get the sequence

$$
\begin{aligned}
C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{a_{X} \cdot 0} / \tilde{X}, a_{Z} \cdot 0 / Z\right\} & \equiv C_{\widetilde{X}, Z}^{0}\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}, a_{Z} \cdot 0 / Z\right\} \xrightarrow{\tau} C_{\widetilde{X}, Z}^{1}\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}, a_{Z} \cdot 0 / Z\right\} \stackrel{\tau}{\longrightarrow} \\
& \cdots \xrightarrow{\tau} C_{\widetilde{X}, Z}^{n}\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}, a_{Z} \cdot 0 / Z\right\} \equiv D_{\widetilde{X}, Z}^{\prime}\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}, a_{Z} \cdot 0 / Z\right\} .
\end{aligned}
$$

Here $n \geq 0$ and $C_{\tilde{X}, Z}^{i}$ satisfies $(\mathrm{C}-\tau-1,2,3)$ for each $1 \leq i \leq n$. Moreover, since $Z$ is 1 active in $C_{\widetilde{X}, Z}^{\prime \prime}$, by (C- $\tau-3$-i $)$, so is $Z$ in $C_{\widetilde{X}, Z}^{n}$. We also have $C_{\widetilde{X}, Z}^{n} \equiv D_{\widetilde{X}, Z}^{\prime}$ by Lemma 5.18. Hence we conclude that

$$
\begin{equation*}
Z \text { is 1-active in } D_{\widetilde{X}, Z}^{\prime} \tag{5.19}
\end{equation*}
$$

Since $C_{\widetilde{X}}^{\prime}$ and $C_{\tilde{X}, Z}^{\prime \prime}$ satisfy (P- $\tau-1$ ) and (P- $\left.\tau-3\right)$, for any $\widetilde{s}$, we get

$$
\begin{equation*}
C_{\widetilde{X}}\{\widetilde{s} / \widetilde{X}\} \Rightarrow C_{\widetilde{X}}^{\prime}\{\widetilde{s} / \widetilde{X}\} \equiv C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{s} / \widetilde{X}, s_{i_{0}} / Z\right\} \tag{5.19,4}
\end{equation*}
$$

In order to complete the proof, it suffices to find a stable context $D_{\tilde{X}}$ satisfying conditions (1) and (2). In the following, we shall use $\widetilde{a_{X} .0}$ again to obtain such context.

Since $\widetilde{a_{X} .0}$ and $D_{\widetilde{X}, Z}^{\prime}$ are stable, by ( $5.19,2$ ), we get

$$
C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}, a_{X_{i_{0}}} \cdot 0 / Z\right\} \stackrel{\epsilon}{\Longrightarrow} \mid D_{\widetilde{X}, Z}^{\prime}\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}, a_{X_{i_{0}}} \cdot 0 / Z\right\}
$$

Moreover, by (5.19, 4), we have $C_{\widetilde{X}}^{\prime}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{a_{X} .0} / \widetilde{X}, a_{X_{i_{0}}} \cdot 0 / Z\right\}$. Thus, it follows that

$$
C_{\widetilde{X}}^{\prime}\left\{\widetilde{a_{X} \cdot 0} / \tilde{X}\right\} \stackrel{\epsilon}{\Longrightarrow} \mid D_{\widetilde{X}, Z}^{\prime}\left\{\widetilde{a_{X} \cdot 0} / \tilde{X}, a_{X_{i_{0}}} \cdot 0 / Z\right\}
$$

Then, since $\widetilde{a_{X} .0}$ are stable, by Lemma 5.16, there exists a stable context $B_{\widetilde{X}}$ such that

$$
\begin{equation*}
B_{\tilde{X}}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \equiv D_{\widetilde{X}, Z}^{\prime}\left\{\widetilde{a_{X} .0} / \widetilde{X}, a_{X_{i_{0}}} \cdot 0 / Z\right\} \tag{5.19,5}
\end{equation*}
$$

and

$$
\begin{equation*}
C_{\widetilde{X}}^{\prime}\{\widetilde{s} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} B_{\widetilde{X}}\{\widetilde{s} / \widetilde{X}\} \text { for any } \widetilde{s} \tag{5.19}
\end{equation*}
$$

In addition, by (5.19, 4) and Lemma 5.12, we have $C_{\tilde{X}}\left\{\widetilde{a_{X} .0} / \tilde{X}\right\} \Rightarrow C_{\tilde{X}}^{\prime}\left\{\widetilde{a_{X} .0} / \tilde{X}\right\}$ and $C_{\widetilde{X}}\left\{\widetilde{a_{X} .0} / \widetilde{X}\right\} \xlongequal{\epsilon} \mid t^{\prime} \Rightarrow D_{\widetilde{X}, Z}^{\prime}\left\{\widetilde{a_{X} .0} / \widetilde{X}, a_{X_{i_{0}}} .0 / Z\right\}$ for some $t^{\prime}$. Further, since $\widetilde{a_{X} .0}$ are stable, by Lemma 5.16, there exists a stable context $D_{\tilde{X}}$ such that

$$
\begin{equation*}
t^{\prime} \equiv D_{\widetilde{X}}\left\{\widetilde{a_{X} \cdot 0} / \widetilde{X}\right\} \Rightarrow D_{\widetilde{X}, Z}^{\prime}\left\{\widetilde{a_{X} \cdot 0} / \tilde{X}, a_{X_{i_{0}}} \cdot 0 / Z\right\} \tag{5.19.7}
\end{equation*}
$$

and

$$
\begin{equation*}
C_{\widetilde{X}}\{\widetilde{s} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} D_{\widetilde{X}}\{\widetilde{s} / \widetilde{X}\} \text { for any } \widetilde{s} \tag{5.19}
\end{equation*}
$$

Notice that, 5.19 , follows from (MS- $\tau-3$-ii) with $\widetilde{Y}=\emptyset$. In the following, we intend to prove that $D_{\tilde{X}}$ is what we seek. It immediately follows from (5.198) that $D_{\tilde{X}}$ meets the requirement (2). We are left with the task of verifying that $D_{\tilde{X}}$ satisfies the condition (1). So far, for any $\widetilde{s}$, we have the diagram below, where the first line follows from (5.194),

$$
\begin{array}{cccc}
C_{\widetilde{X}}\{\widetilde{s} / \widetilde{X}\} & \Rightarrow & C_{\widetilde{X}}^{\prime}\{\widetilde{s} / \widetilde{X}\} & \equiv C_{\tilde{X}, Z}^{\prime \prime}\left\{\widetilde{s} / \widetilde{X}, s_{i_{0}} / Z\right\} \\
\Downarrow \epsilon \text { by (5.19) } 8) & \Downarrow \epsilon \text { by (5.19, } 6) & & \Downarrow \epsilon \text { by (5.19) } 2) \\
D_{\widetilde{X}}\{\widetilde{s} / \widetilde{X}\} & \Rightarrow & B_{\widetilde{X}}\{\widetilde{s} / \widetilde{X}\} & \equiv D_{\widetilde{X}, Z}^{\prime}\left\{\widetilde{s} / \widetilde{X}, s_{i_{0}} / Z\right\}
\end{array}
$$

Here the last line in the above follows from (5.19.7) and (5.19.5) using Lemma 5.18.

Further, by Lemma 5.4 and $p_{i_{0}} \xrightarrow{\tau} p^{\prime}$, it follows from (5.191) and (5.19] 3) that

$$
B_{\tilde{X}}\{\widetilde{p} / \tilde{X}\} \equiv D_{\widetilde{X}, Z}^{\prime}\left\{\widetilde{p} / \tilde{X}, p_{i_{0}} / Z\right\} \xrightarrow{\tau} D_{\widetilde{X}, Z}^{\prime}\left\{\widetilde{p} / \tilde{X}, p^{\prime} / Z\right\} \stackrel{\epsilon}{\Longrightarrow} \mid r^{\prime} \Rightarrow r
$$

Finally, since $D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \Rightarrow B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$, by Lemma 5.12 we get $D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xlongequal{\epsilon} \mid r^{\prime \prime} \Rightarrow$ $r^{\prime} \Rightarrow r$ for some $r^{\prime \prime}$, which, together with $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$, implies that the stable context $D_{\tilde{X}}$ also meets the requirement (1), as desired.

The result below asserts that there exist another "canonical" evolution paths from $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$ to a given stable $\tau$-descendant $r$. For these paths, an unstable $p_{i}$ evolves first provided that such $p_{i}$ is located in an active position.

Lemma 5.20. For any $C_{\widetilde{X}}$ and $\widetilde{p}$, if $C_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \xlongequal{\epsilon} \mid q$ and $X_{i}$ is 1-active in $C_{\widetilde{X}}$ for some $i \leq|\widetilde{X}|$, then there exists $p^{\prime}$ such that $p_{i} \xlongequal{\epsilon} \mid p^{\prime}$ and $C_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \xlongequal{\epsilon} C_{\widetilde{X}}\left\{\widetilde{p}\left[p^{\prime} / p_{i}\right] / \widetilde{X}\right\} \xlongequal{\epsilon}$ $\mid q$.

Proof. Suppose $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}(\xrightarrow{\tau})^{n} \mid q$ for some $n \geq 0$. We shall prove it by induction on $n$. For the inductive base $n=0$, we have $p_{i}{ }^{\tau}$ by Lemma 5.4. and hence it holds trivially by taking $p^{\prime} \equiv p_{i}$. For the inductive step $n=k+1$, suppose $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} r(\xrightarrow{\tau})^{k} \mid q$ for some $r$. For the transition $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} r$, either the clause (1) or (2) in Lemma 5.6 holds.

For the first alternative, there exists a context $C_{\widetilde{X}}^{\prime}$ such that
(1.1) $X_{i}$ is 1-active in $C_{\widetilde{X}}^{\prime}$ (by (C- $\left.\left.\tau-3-\mathrm{i}\right)\right)$,
(1.2) $r \equiv C_{\widetilde{X}}^{\prime}\{\widetilde{p} / \widetilde{X}\}$, and
(1.3) $C_{\widetilde{X}}\{\widetilde{s} / \widetilde{X}\} \xrightarrow{\tau} C_{\widetilde{X}}^{\prime}\{\widetilde{s} / \widetilde{X}\}$ for any $\widetilde{s}$.

By (1.1), we can apply IH for the transition $r \equiv C_{\tilde{X}}^{\prime}\{\widetilde{p} / \widetilde{X}\}(\xrightarrow{\tau})^{k} \mid q$, and hence there exists $p^{\prime}$ such that $p_{i} \xlongequal{\epsilon} \mid p^{\prime}$ and $C_{\widetilde{X}}^{\prime}\{\widetilde{p} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}^{\prime}\left\{\widetilde{p}\left[p^{\prime} / p_{i}\right] / \widetilde{X}\right\} \xrightarrow{\epsilon} \mid q$. Moreover, since $X_{i}$ is 1-active in $C_{\widetilde{X}}$ and $p_{i} \xlongequal{\epsilon} \mid p^{\prime}$, we have $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xlongequal{\epsilon} C_{\widetilde{X}}\left\{\widetilde{p}\left[p^{\prime} / p_{i}\right] / \widetilde{X}\right\}$ by Lemma 5.4. We also have $C_{\widetilde{X}}\left\{\widetilde{p}\left[p^{\prime} / p_{i}\right] / \widetilde{X}\right\} \xrightarrow{\tau} C_{\widetilde{X}}^{\prime}\left\{\widetilde{p}\left[p^{\prime} / p_{i}\right] / \widetilde{X}\right\}$ by (1.3). Therefore, $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\left\{\widetilde{p}\left[p^{\prime} / p_{i}\right] / \widetilde{X}\right\} \xrightarrow{\tau} C_{\widetilde{X}}^{\prime}\left\{\widetilde{p}\left[p^{\prime} / p_{i}\right] / \widetilde{X}\right\} \xrightarrow{\epsilon} \mid q$, as desired.

For the second alternative, there exist $C_{\tilde{X}}^{\prime}, C_{\tilde{X}, Z}^{\prime \prime}$ and $i_{0} \leq|\widetilde{X}|$ such that
(2.1) $Z$ is 1-active in $C_{\widetilde{X}, Z}^{\prime \prime}$,
(2.2) $r \equiv C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, p_{i_{0}}^{\prime} / Z\right\}$ for some $p_{i_{0}}^{\prime}$ with $p_{i_{0}} \xrightarrow{\tau} p_{i_{0}}^{\prime}$, and
(2.3) $C_{\widetilde{X}}\{\widetilde{s} / \widetilde{X}\} \xrightarrow{\tau} C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{s} / \widetilde{X}, s^{\prime} / Z\right\}$ for any $\widetilde{s}$ and $s^{\prime}$ with $s_{i_{0}} \xrightarrow{\tau} s^{\prime}$.

In case $i_{0}=i$, we have $C_{\widetilde{X}} \equiv C_{\widetilde{X}}^{\prime}$ by (P- $\tau-1$ ), and hence $r \equiv C_{\widetilde{X}}\left\{\widetilde{p}\left[p_{i_{0}}^{\prime} / p_{i}\right] / \widetilde{X}\right\}$ by (2.2) and (P- $\tau-3$ ). For the transition $r \equiv C_{\widetilde{X}}\left\{\widetilde{p}\left[p_{i_{0}}^{\prime} / p_{i}\right] / \widetilde{X}\right\}(\xrightarrow{\tau})^{k} \mid q$, by IH, there exists $p^{\prime \prime}$ such that $p_{i_{0}}^{\prime} \xlongequal{\epsilon} \mid p^{\prime \prime}$ and $C_{\widetilde{X}}\left\{\widetilde{p}\left[p_{i_{0}}^{\prime} / p_{i}\right] / \widetilde{X}\right\} \xlongequal{\epsilon} C_{\widetilde{X}}\left\{\widetilde{p}\left[p^{\prime \prime} / p_{i}\right] / \widetilde{X}\right\} \xlongequal{\epsilon} \mid q$. Hence $p_{i_{0}} \xrightarrow{\tau} p_{i_{0}}^{\prime} \xrightarrow{\epsilon} \mid p^{\prime \prime}$ and $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau} C_{\widetilde{X}}\left\{\widetilde{p}\left[p_{i_{0}}^{\prime} / p_{i}\right] / \widetilde{X}\right\} \xrightarrow{\epsilon} C_{\widetilde{X}}\left\{\widetilde{p}\left[p^{\prime \prime} / p_{i}\right] / \widetilde{X}\right\} \xrightarrow{\epsilon} \mid q$.

Next we consider the other case $i_{0} \neq i$. Then for the transition $r \equiv C_{\tilde{X}, Z}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, p_{i_{0}}^{\prime} / Z\right\}(\xrightarrow{\tau}$ $)^{k} \mid q$, by IH, there exists $p^{\prime}$ such that $p_{i} \xlongequal{\epsilon} \mid p^{\prime}$ and

$$
C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, p_{i_{0}}^{\prime} / Z\right\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{p}\left[p^{\prime} / p_{i}\right] / \widetilde{X}, p_{i_{0}}^{\prime} / Z\right\} \stackrel{\epsilon}{\Longrightarrow} \mid q .
$$

In addition, since $X_{i}$ is 1-active in $C_{\widetilde{X}}$ and $p_{i} \xlongequal{\epsilon} \mid p^{\prime}$, by Lemma 5.4, we obtain $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\left\{\widetilde{p}\left[p^{\prime} / p_{i}\right] / \widetilde{X}\right\}$. Moreover, $C_{\widetilde{X}}\left\{\widetilde{p}\left[p^{\prime} / p_{i}\right] / \widetilde{X}\right\} \xrightarrow{\tau} C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{p}\left[p^{\prime} / p_{i}\right] / \widetilde{X}, p_{i_{0}}^{\prime} / Z\right\}$ by (2.3). Thus

$$
C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\left\{\widetilde{p}\left[p^{\prime} / p_{i}\right] / \widetilde{X}\right\} \xrightarrow{\tau} C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{p}\left[p^{\prime} / p_{i}\right] / \widetilde{X}, p_{i_{0}}^{\prime} / Z\right\} \stackrel{\epsilon}{\Longrightarrow} \mid q,
$$

as desired.

## 6. Precongruence

This section intends to establish a fundamental property that $\sqsubseteq_{R S}$ is a precongruence, that is, it is substitutive w.r.t all operations in $\mathrm{CLL}_{R}$. This constitutes one of two main results of this paper. Its proof is far from trivial and requires a solid effort. As mentioned in Section 1, a distinguishing feature of LLTS is that it involves consideration of inconsistencies. It is the inconsistency predicate $F$ that make everything become quite troublesome. A crucial part in carrying out the proof is that we need to prove that $C_{X}\{q / X\} \in F$ implies $C_{X}\{p / X\} \in F$ whenever $p \sqsubseteq_{R S} q$. Its argument will be divided into two steps. First, we shall show that, for any stable process $p, C_{X}\{\tau . p / X\} \in F$ iff $C_{X}\{p / X\} \in F$. Second, we intend to prove that $C_{X}\{q / X\} \in F$ implies $C_{X}\{p / X\} \in F$ whenever $p$ and $q$ are uniform w.r.t stability and $p \sqsubseteq_{R S} q$.

Definition 6.1 (Uniform w.r.t stability). Two tuples $\widetilde{p}$ and $\widetilde{q}$ with $|\widetilde{q}|=|\widetilde{p}|$ are uniform w.r.t stability, in symbols $\widetilde{p} \bowtie \widetilde{q}$, if they are component-wise uniform w.r.t stability, that is, $p_{i}$ is stable iff $q_{i}$ is stable for each $i \leq|\widetilde{p}|$.

An elementary property of this notion is given:
Lemma 6.1. The uniformity w.r.t stability are preserved under substitutions. That is, for any $\widetilde{p}, \widetilde{q}$ and $C_{\widetilde{X}}$, if $\widetilde{p} \bowtie \widetilde{q}$ then $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \bowtie C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\}$.

Proof. Immediately follows from Lemma 5.6
Notation For convenience, given tuples $\widetilde{p}$ and $\widetilde{q}$, for $R \in\left\{\sqsubseteq_{R S}, \sqsubset_{\sim}^{\sim}, ~ \xlongequal{\epsilon} \mid, \equiv\right\}$, the notation $\widetilde{p} R \widetilde{q}$ means that $|\widetilde{p}|=|\widetilde{q}|$ and $p_{i} R q_{i}$ for each $i \leq|\widetilde{p}|$.

Lemma 6.2. For any $C_{\widetilde{X}}, \widetilde{p}$ and $\widetilde{q}$ with $\widetilde{p} \sqsubseteq_{R S} \widetilde{q}$, if $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$ and $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\}$ are stable and $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$, then $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a}$ iff $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{a}$ for any $a \in$ Act.

Proof. We give the proof only for the implication from right to left, the same argument applies to the other implication. Assume $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{a} q^{\prime}$. Then there exist $C_{\widetilde{X}}^{\prime}, C_{\widetilde{X}, \tilde{Y}}^{\prime}$ and $C_{\tilde{X}, \tilde{Y}}^{\prime \prime}$ with $\widetilde{X} \cap \widetilde{Y}=\emptyset$ that satisfy (CP-a-1) - (CP-a-4) in Lemma 5.8. Hence, due to (CP-a-1) and (CP-a-3-i), there exist $i_{Y} \leq|\widetilde{X}|(Y \in \widetilde{Y})$ such that for any $\widetilde{r}$ with $|\widetilde{r}|=|\widetilde{X}|$

$$
C_{\widetilde{X}}\{\widetilde{r} / \widetilde{X}\} \Rightarrow C_{\widetilde{X}}^{\prime}\{\widetilde{r} / \widetilde{X}\} \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{r} / \widetilde{X}, \widetilde{r_{i_{Y}}} / \widetilde{Y}\right\}
$$

In particular, by Lemma 5.11, it follows from $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau}$ and $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow[\sim]{\sim}$ that both $C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{i_{Y}}} / \widetilde{Y}\right\}$ and $C_{\widetilde{X}}^{\prime}, \widetilde{Y}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{i_{Y}}} / \widetilde{Y}\right\}$ are stable. Then, for each $Y \in \widetilde{Y}$, both $p_{i_{Y}}$ and $q_{i_{Y}}$ are stable by Lemma 5.4 and (CP-a-2). Moreover, by (6.2,1) with $\widetilde{r} \equiv \widetilde{p}$ and

Lemma 5.14 and 5.5 we have $p_{i_{Y}} \notin F$ for each $Y \in \widetilde{Y}$ due to $C_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$. Therefore, for each $Y \in \widetilde{Y}$, it follows from $\widetilde{p} \sqsubseteq_{R S} \widetilde{q}$ that $p_{i_{Y}} \check{\sim}_{R S} q_{i_{Y}}$, and $\mathcal{I}\left(p_{i_{Y}}\right)=\mathcal{I}\left(q_{i_{Y}}\right)$ because of $p_{i_{Y}} \notin F$. Hence $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a}$ by (CP- $a-3$-iii).

In the following, we intend to show that, for any stable $p, C_{X}\{p / X\}$ and $C_{X}\{\tau . p / X\}$ are undifferentiated w.r.t consistency, which falls naturally into two parts: Lemmas 6.3 and 6.5

Lemma 6.3. For any $C_{X}$ and stable $p, C_{X}\{p / X\} \notin F$ implies $C_{X}\{\tau . p / X\} \notin F$.
Proof. Let $p$ be any stable process. Set

$$
\Omega \triangleq\left\{B_{X}\{\tau . p / X\}: B_{X}\{p / X\} \notin F \text { and } B_{X} \text { is a context }\right\} .
$$

Similar to Lemma 4.4 it suffices to prove that for any $t \in \Omega$, each proof tree of $t F$ has a proper subtree with root $s F$ for some $s \in \Omega$. Suppose that $C_{X}\{\tau . p / X\} \in \Omega$ and $\mathcal{T}$ is any proof tree of $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash C_{X}\{\tau . p / X\} F$. Hence $C_{X}\{p / X\} \notin F$. We distinguish six cases based on the form of $C_{X}$.

Case $1 C_{X}$ is closed or $C_{X} \equiv X$.
In this situation, it is easy to see that $C_{X}\{\tau \cdot p / X\} \notin F$. Hence there is no proof tree of $C_{X}\{\tau . p / X\} F$. Thus the conclusion holds trivially.

Case $2 C_{X} \equiv \alpha . B_{X}$.
Then the last rule applied in $\mathcal{T}$ is $\frac{B_{X}\{\tau, p / X\} F}{\alpha \cdot B_{X}\{\tau, p / X\} F}$. Since $C_{X}\{p / X\} \notin F$, we get $B_{X}\{p / X\} \notin$ $F$. Hence $B_{X}\{\tau . p / X\} \in \Omega$; moreover, the node directly above the root of $\mathcal{T}$ is labelled with $B_{X}\{\tau \cdot p / X\} F$, as desired.

Case $3 C_{X} \equiv B_{X} \vee D_{X}$.
Clearly, the last rule applied in $\mathcal{T}$ is $\frac{B_{X}\{\tau, p / X\} F, D_{X}\{\tau, p / X\} F}{B_{X}\{\tau \cdot p / X\} \vee D_{X}\{\tau, p / X\} F}$. Since $C_{X}\{p / X\} \notin F$, either $B_{X}\{p / X\} \notin F$ or $D_{X}\{p / X\} \notin F$. W.l.o.g, assume $B_{X}\{p / X\} \notin F$. Then $B_{X}\{\tau . p / X\} \in$ $\Omega$. Moreover, it is obvious that $\mathcal{T}$ has a proper subtree with root $B_{X}\{\tau \cdot p / X\} F$.

Case $4 C_{X} \equiv B_{X} \odot D_{X}$ with $\odot \in\left\{\square, \|_{A}\right\}$.
W.l.o.g, assume the last rule applied in $\mathcal{T}$ is $\frac{B_{X}\{\tau, p / X\} F}{B_{X}\{\tau \cdot p / X\} \odot D_{X}\{\tau \cdot p / X\} F}$. It is evident that $B_{X}\{p / X\} \notin F$ due to $C_{X}\{p / X\} \notin F$. Hence $B_{X}\{\tau \cdot p / X\} \in \Omega$, as desired.

Case $5 C_{X} \equiv\langle Y \mid E\rangle$.
Then the last rule applied in $\mathcal{T}$ is

$$
\text { either } \frac{\left\langle t_{Y} \mid E\right\rangle\{\tau . p / X\} F}{\langle Y \mid E\rangle\{\tau \cdot p / X\} F} \text { with } Y=t_{Y} \in E \text { or } \frac{\{r F:\langle Y \mid E\rangle\{\tau . p / X\} \xlongequal{\epsilon} \mid r\}}{\langle Y \mid E\rangle\{\tau \cdot p / X\} F} \text {. }
$$

For the first alternative, since $C_{X}\{p / X\} \equiv\langle Y \mid E\rangle\{p / X\} \notin F$, by Lemma 4.1 (8), we get $\left\langle t_{Y} \mid E\right\rangle\{p / X\} \notin F$. Hence $\left\langle t_{Y} \mid E\right\rangle\{\tau . p / X\} \in \Omega$.

For the second alternative, since $C_{X}\{p / X\} \notin F$, we get $C_{X}\{p / X\} \xlongequal{\epsilon}{ }_{F} \mid q$ for some $q$. Moreover, by Lemma 5.16] it follows from $p \not{ }_{\nmid}^{\top}$ that there exists a stable context $C_{X}^{\prime}$
such that

$$
\begin{equation*}
q \equiv C_{X}^{\prime}\{p / X\} \text { and } C_{X}\{\tau \cdot p / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X}^{\prime}\{\tau \cdot p / X\} . \tag{6.3,1}
\end{equation*}
$$

Further, by Lemma 5.17 and $\tau . p \xrightarrow{\tau} \mid p$, we get

$$
\begin{equation*}
C_{X}^{\prime}\{\tau \cdot p / X\} \stackrel{\epsilon}{\Longrightarrow} \mid s \text { for some } s \tag{6.3,2}
\end{equation*}
$$

For the above transition, by Lemma 5.16 again, there exists $C_{X, \widetilde{Z}}^{\prime \prime}$ with $X \notin \widetilde{Z}$ such that

$$
s \equiv C_{X, \widetilde{Z}}^{\prime \prime}\{\tau \cdot p / X, p / \widetilde{Z}\} \text { and } C_{X}^{\prime}\{p / X\} \Rightarrow C_{X, \widetilde{Z}}^{\prime \prime}\{p / X, p / \widetilde{Z}\}
$$

Thus, by Lemma 5.14, we have $C_{X, \widetilde{Z}}^{\prime \prime}\{p / X, p / \widetilde{Z}\} \notin F$ because of $q \equiv C_{X}^{\prime}\{p / X\} \notin F$. Set

$$
C_{X}^{\prime \prime \prime} \triangleq C_{X, \widetilde{Z}}^{\prime \prime}\{p / \widetilde{Z}\}
$$

Then it follows from $C_{X}^{\prime \prime \prime}\{p / X\} \equiv C_{X, \widetilde{Z}}^{\prime \prime}\{p / X, p / \widetilde{Z}\} \notin F$ that $s \equiv C_{X}^{\prime \prime \prime}\{\tau . p / X\} \in \Omega$. Moreover, $\mathcal{T}$ contains a proper subtree with root $s F$ due to (6.3,1) and (6.3,2).

Case $6 C_{X} \equiv B_{X} \wedge D_{X}$.
Clearly, the last rule applied in $\mathcal{T}$ has one of the following formats.
Case $6.1 \frac{B_{X}\{\tau \cdot p / X\} F}{B_{X}\{\tau \cdot p / X\} \wedge D_{X}\{\tau \cdot p / X\} F}$.
Similar to Case 4, omitted.
Case $6.2 \xrightarrow{B_{X}\{\tau \cdot p / X\} \xrightarrow{a} r, D_{X}\{\tau . p / X\} \not{ }^{a} \rightarrow, B_{X}\{\tau \cdot p / X\} \wedge D_{X}\{\tau \cdot p / X\} \not{ }^{\tau}} B_{X}\{\tau \cdot p / X\} \wedge D_{X}\{\tau \cdot p / X\} F$.
In this situation, $B_{X}\{\tau \cdot p / X\}, C_{X}$ and $B_{X}$ are stable. Moreover, since $p$ is stable, so is $B_{X}\{p / X\}$. Due to $C_{X}\{p / X\} \notin F$, we obtain $B_{X}\{p / X\} \notin F$. Then, by Lemma 6.2, it follows from $p=_{R S} \tau . p$ and $B_{X}\{\tau . p / X\} \xrightarrow{a}$ that

$$
\begin{equation*}
B_{X}\{p / X\} \xrightarrow{a} . \tag{6.3,3}
\end{equation*}
$$

Similarly, it follows from $D_{X}\{\tau \cdot p / X\} \xrightarrow{a}$ that

$$
\begin{equation*}
D_{X}\{p / X\} \xrightarrow{a} . \tag{6.3,4}
\end{equation*}
$$

In addition, since $B_{X} \wedge D_{X}$ and $p$ are stable, so is $B_{X}\{p / X\} \wedge D_{X}\{p / X\}$. So, by (6.3,3) and (6.3, 4), we get $C_{X}\{p / X\} \equiv B_{X}\{p / X\} \wedge D_{X}\{p / X\} \in F$ by Rule $R p_{10}$, which contradicts that $C_{X}\{\tau . p / X\} \in \Omega$. Hence this case is impossible.

Case $6.3 \frac{C_{X}\{\tau \cdot p / X\} \xrightarrow{\alpha} s,\left\{r F: C_{X}\{\tau \cdot p / X\} \xrightarrow{\alpha} r\right\}}{C_{X}\{\tau \cdot p / X\} F}$.
The argument splits into two cases based on $\alpha$.

Case 6.3.1 $\alpha=\tau$.
We distinguish two cases depending on whether $C_{X}$ is stable.
Case 6.3.1.1 $C_{X}$ is not stable.
Since $C_{X}\{p / X\} \notin F$, we have $C_{X}\{p / X\} \stackrel{\epsilon}{\Longrightarrow} F_{F} \mid p^{\prime}$ for some $p^{\prime}$. Moreover, by Lemma 5.19 ,
there exist $p^{\prime \prime}$ and stable $C_{X}^{*}$ such that

$$
C_{X}\{p / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X}^{*}\{p / X\} \stackrel{\epsilon}{\Longrightarrow} \mid p^{\prime \prime} \Rightarrow p^{\prime}
$$

and

$$
C_{X}\{t / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X}^{*}\{t / X\} \text { for any } t
$$

Further, since $C_{X}$ is not stable and $p \overbrace{}^{\tau}$, by Lemma 5.6, there exists $C_{X}^{\prime}$ such that

$$
C_{X}\{p / X\} \xrightarrow{\tau} C_{X}^{\prime}\{p / X\} \xrightarrow{\epsilon} C_{X}^{*}\{p / X\} \text { and } C_{X}\{\tau \cdot p / X\} \xrightarrow{\tau} C_{X}^{\prime}\{\tau \cdot p / X\} .
$$

Since $p^{\prime} \notin F$ and $p^{\prime \prime} \Rightarrow p^{\prime}$, by Lemma 5.14, we get $p^{\prime \prime} \notin F$. Together with the transitions $C_{X}^{\prime}\{p / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X}^{*}\{p / X\} \xlongequal{\epsilon} \mid p^{\prime \prime}$, by Lemma 4.2, this implies $C_{X}^{\prime}\{p / X\} \notin F$. Hence $C_{X}^{\prime}\{\tau . p / X\} \in \Omega$, and $\mathcal{T}$ has a proper subtree with root $C_{X}^{\prime}\{\tau . p / X\} F$.

Case 6.3.1.2 $C_{X}$ is stable.
Due to $C_{X}\{\tau \cdot p / X\} \xrightarrow{\tau} s$, either the clause (1) or (2) in Lemma 5.6 holds. Since $C_{X}$ is stable, by (C- $\tau-2$ ) in Lemma [5.6] it is easy to see that the clause (1) does not hold, and hence the clause (2) holds, that is, there exists $C_{X, Z}^{\prime}$ with $X \neq Z$ such that

$$
C_{X}\{\tau . p / X\} \xrightarrow{\tau} C_{X, Z}^{\prime}\{\tau . p / X, p / Z\} \text { and } C_{X}\{p / X\} \Rightarrow C_{X, Z}^{\prime}\{p / X, p / Z\}
$$

Set

$$
C_{X}^{\prime \prime} \triangleq C_{X, Z}^{\prime}\{p / Z\}
$$

Hence $\mathcal{T}$ has a proper subtree with root $C_{X}^{\prime \prime}\{\tau \cdot p / X\} F$. Moreover, by Lemma 5.14 it follows from $C_{X}\{p / X\} \notin F$ that $C_{X, Z}^{\prime}\{p / X, p / Z\} \notin F$. Thus $C_{X}^{\prime \prime}\{\tau . p / X\} \equiv C_{X, Z}^{\prime}\{\tau . p / X, p / Z\} \in$ $\Omega$, as desired.

Case 6.3.2 $\alpha \in$ Act.
Then it is not difficult to know that both $C_{X}$ and $C_{X}\{p / X\}$ are stable. Moreover, since $C_{X}\{\tau . p / X\} \xrightarrow{\alpha}, \tau . p={ }_{R S} p$ and $C_{X}\{p / X\} \notin F$, by Lemma 6.2, we get $C_{X}\{p / X\} \xrightarrow{\alpha}$. Further, by Theorem4.2, it follows from $C_{X}\{p / X\} \notin F$ that $C_{X}\{p / X\} \xrightarrow{\alpha}{ }_{F} q$ for some $q$. For such $\alpha$-labelled transition, by Lemma [5.8 there exist $C_{X}^{\prime}, C_{X, \widetilde{Z}}^{\prime}$ and $C_{X, \widetilde{Z}}^{\prime \prime}$ with $X \notin \widetilde{Z}$ that realize (CP-a-1) - (CP-a-4).

In order to complete the proof, we intend to prove that $\widetilde{Z}=\emptyset$. On the contrary, suppose $\widetilde{Z} \neq \emptyset$. Then, by (CP-a-2) and (CP-a-3-i), there exists an active occurrence of the variable $X$ in $C_{X}^{\prime}$. So, by Lemma 5.4, $C_{X}^{\prime}\{\tau . p / X\} \xrightarrow{\tau}$. Then, by Lemma 5.11, it follows from $C_{X}\{\tau . p / X\} \Rightarrow C_{X}^{\prime}\{\tau . p / X\}$ (i.e., $(\mathrm{CP}-a-1)$ ) that $C_{X}\{\tau . p / X\} \xrightarrow{\tau}$, which contradicts $C_{X}\{\tau . p / X\} \xrightarrow{\alpha}$.

Thus $\widetilde{Z}=\emptyset$, and hence $q \equiv C_{X, \widetilde{Z}}^{\prime \prime}\{p / X\}$ by (CP- $\left.a-3-\mathrm{ii}\right)$. Since $C_{X}\{\tau \cdot p / X\}$ is stable, by (CP-a-3-iii), we get $C_{X}\{\tau \cdot p / X\} \xrightarrow{\alpha} C_{X, Z}^{\prime \prime}\{\tau \cdot p / X\}$. Thus, $\mathcal{T}$ contains a proper subtree with root $C_{X, \widetilde{Z}}^{\prime \prime}\{\tau \cdot p / X\} F ;$ moreover, $C_{X, \widetilde{Z}}^{\prime \prime}\{\tau \cdot p / X\} \in \Omega$ due to $C_{X, \widetilde{Z}}^{\prime \prime}\{p / X\} \equiv q \notin F$.
Case $6.4 \frac{\left\{r F: B_{X}\{\tau \cdot p / X\} \wedge D_{X}\{\tau \cdot p / X\} \xlongequal{\epsilon} \mid r\right\}}{B_{X}\{\tau \cdot p / X\} \wedge D_{X}\{\tau \cdot p / X\} F}$.
Analogous to the second alternative in Case 5, omitted.

In order to show the converse of the above result, the preliminary result below is given. Here, for any finite set $S$ of processes, by virtue of the commutative and associative laws of external choice (Zhang et al. 2011), we may introduce the notation of a generalized external choice (denoted by $\square_{p \in S} p$ ) by the standard method.

Lemma 6.4. Let $t_{1}, t_{2}$ be two terms and $\{X\} \cup \widetilde{Z}$ a tuple of variables such that none of recursive variable occurring in $t_{i}($ with $i=1,2)$ is in $\{X\} \cup \widetilde{Z}$. Suppose that $Z$ is active in $t_{1}, t_{2}$ for each $Z \in \widetilde{Z}$ and

$$
T \triangleq \begin{cases}\square_{Z \in \widetilde{Z}} \alpha \cdot a_{Z} \cdot 0 & \text { if } \widetilde{Z} \neq \emptyset \\ a_{X} \cdot 0 & \text { otherwise }\end{cases}
$$

where $a_{X}$ and $\widetilde{a_{Z}}$ are distinct fresh visible actions and $\alpha \in A c t$. Then
(1) if $t_{1}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \equiv t_{2}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}$ then $t_{1}\{p / X, \widetilde{q} / \widetilde{Z}\} \equiv t_{2}\{p / X, \widetilde{q} / \widetilde{Z}\}$ for any $p$ and $\widetilde{q}$;
(2) if $t_{1}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \Rightarrow \Rightarrow_{1} t_{2}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}$ then $t_{1}\{p / X, \widetilde{q} / \widetilde{Z}\} \Rightarrow_{1} t_{2}\{p / X, \widetilde{q} / \widetilde{Z}\}$ for any $p$ and $\widetilde{q}$.

Proof. (1) It proceeds by induction on $t_{1}$. We distinguish three cases as follows.

Case $1 t_{1}$ is closed or $t_{1}$ is of the format $X$ or $\beta . s$ or $s_{1} \vee s_{2}$ or $\langle Y \mid E\rangle$.
Since $Z$ is active in $t_{1}$ for each $Z \in \widetilde{Z}$, we get $\widetilde{Z}=\emptyset$. Then it follows by Lemma 5.18.

Case $2 t_{1} \equiv s_{1} \odot s_{2}$ with $\odot \in\left\{\|_{A}, \wedge\right\}$.
Then $t_{1}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \equiv s_{1}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \odot s_{2}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \equiv t_{2}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}$. Since neither $\widetilde{a_{Z} \cdot 0}$ nor $T$ contain $\odot$, there exist $s_{1}^{\prime}, s_{2}^{\prime}$ such that $s_{1}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \equiv$ $s_{1}^{\prime}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}, s_{2}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \equiv s_{2}^{\prime}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}$ and $t_{2} \equiv s_{1}^{\prime} \odot s_{2}^{\prime}$. Hence it immediately follows that $t_{1}\{p / X, \widetilde{q} / \widetilde{Z}\} \equiv t_{2}\{p / X, \widetilde{q} / \widetilde{Z}\}$ for any $p$ and $\widetilde{q}$ by IH.

Case $3 t_{1} \equiv s_{1} \square s_{2}$.
Then $t_{1}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \equiv s_{1}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \square s_{2}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \equiv t_{2}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}$. Hence the topmost operator of $t_{2}\left\{T / X, \widetilde{\left.a_{Z} \cdot 0 / \widetilde{Z}\right\}}\right.$ is an external choice $\square$. Clearly, such operator comes from either $T$ or $t_{2}$. For the former, we get $t_{2} \equiv X$. If $|\widetilde{Z}| \leq 1$ then $t_{2}\left\{T / X, \widetilde{a_{Z} .0} / \widetilde{Z}\right\}(\equiv T)$ does not contain the operator $\square$ at all, a contradiction. Next we treat the other case $|\widetilde{Z}|>1$. Clearly, $a_{Z} .0$ is guarded in $T$ for each $Z \in \widetilde{Z}$. So $F V\left(t_{1}\right) \cap \widetilde{Z}=$ $\emptyset$ and the numbers of visible actions of $s_{1}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}$ and $s_{2}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}$ are different from ones of two operands of the topmost external choice of $T$. Hence this case is impossible and $t_{2} \equiv s_{1}^{\prime} \square s_{2}^{\prime}$ for some $s_{1}^{\prime}$ and $s_{2}^{\prime}$. The rest of the proof is as in Case 2.
(2) If $F V\left(t_{1}\right) \cap \widetilde{Z}=\emptyset$, it follows by Lemma 5.18. Next we consider the other case $F V\left(t_{1}\right) \cap \widetilde{Z} \neq \emptyset$. It proceeds by induction on $t_{1}$. Since $Z$ is active in $t_{1}$ for each $Z \in \widetilde{Z}$, we get either $t_{1} \equiv Z$ or $t_{1} \equiv s_{1} \odot s_{2}$ for some $s_{1}$ and $s_{2}$, where $Z \in \widetilde{Z}$ and $\odot \in\left\{\wedge, \|_{A}, \square\right\}$. We give the proof only for the case $t_{1} \equiv s_{1} \square s_{2}$, the proofs for the remaining cases are straightforward and omitted.

It follows from $t_{1} \equiv s_{1} \square s_{2}$ that

$$
t_{1}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \equiv s_{1}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \square s_{2}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \Rightarrow \Rightarrow_{1} t_{2}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}
$$

So the topmost operator of $t_{2}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}$ is an external choice $\square$ which comes from either $T$ or $t_{2}$ ．Similar to Case 3 in the proof for item（1），we can make the conclu－ sion that there exist $s_{1}^{\prime}, s_{2}^{\prime}$ such that $t_{2} \equiv s_{1}^{\prime} \square s_{2}^{\prime}$ ．Moreover，it is easily seen that either $s_{1}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}$ or $s_{2}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}$ triggers the unfolding from $t_{1}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}$ to $t_{2}\left\{T / X, \widetilde{a_{Z} .0} / \widetilde{Z}\right\}$ ．W．l．o．g，we consider the first alternative．Then $s_{1}\left\{T / X, \widetilde{a_{Z} .0} / \widetilde{Z}\right\} \Rightarrow_{1}$ $s_{1}^{\prime}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}$ and $s_{2}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \equiv s_{2}^{\prime}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}$ ．Hence，by IH and item （1），for any $p$ and $\widetilde{q}$ ，we have $s_{1}\{p / X, \widetilde{q} / \widetilde{Z}\} \Rightarrow \Rightarrow_{1} s_{1}^{\prime}\{p / X, \widetilde{q} / \widetilde{Z}\}$ and $s_{2}\{p / X, \widetilde{q} / \widetilde{Z}\} \equiv$ $s_{2}^{\prime}\{p / X, \widetilde{q} / \widetilde{Z}\}$ ．Therefore，$t_{1}\{p / X, \widetilde{q} / \widetilde{Z}\} \equiv s_{1}\{p / X, \widetilde{q} / \widetilde{Z}\} \square s_{2}\{p / X, \widetilde{q} / \widetilde{Z}\} \Rightarrow_{1} t_{2}\{p / X, \widetilde{q} / \widetilde{Z}\}$ ．

The next lemma establishes the converse of Lemma 6．3．
Lemma 6．5．For any $C_{X}$ and stable process $p, C_{X}\{\tau . p / X\} \notin F$ implies $C_{X}\{p / X\} \notin F$ ．
Proof．Let $p$ be any stable process．Set

$$
\Omega \triangleq\left\{B_{X}\{p / X\}: B_{X}\{\tau \cdot p / X\} \notin F \text { and } B_{X} \text { is a context }\right\} .
$$

Assume $t \in \Omega$ ．Then $t \equiv C_{X}\{p / X\}$ for some $C_{X}$ such that $C_{X}\{\tau . p / X\} \notin F$ ．Let $\mathcal{T}$ be any proof tree of $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash C_{X}\{p / X\} F$ ．Similar to Lemma 6．3 it is sufficient to prove that $\mathcal{T}$ has a proper subtree with root $s F$ for some $s \in \Omega$ ，which is a routine case analysis based on the last rule applied in $\mathcal{T}$ ．Here we treat only three primary cases．

Case $\left.\left.1 \frac{\left\{r F: C_{X}\{p / X\} ⿳ ⺈ ⿴ 囗 十 一\right.}{C_{X}\{p / X\} F} \right\rvert\, r\right\}$ with $C_{X} \equiv\langle Y \mid E\rangle$ ．
Since $C_{X}\{\tau . p / X\} \notin F$ ，we get $C_{X}\{\tau . p / X\} \stackrel{\epsilon}{\Longrightarrow} \mid q$ for some $q$ ．By Lemma 5．16，for this transition，there exists a stable context $C_{X, \widetilde{Z}}^{\prime}$ satisfying（MS－$\left.\tau-1\right)-(\mathrm{MS}-\tau-7)$ ．In particular，since $p$ and $q$ are stable，by（MS－$\tau-2,7$ ），we have

$$
q \equiv C_{X, \widetilde{Z}}^{\prime}\{\tau \cdot p / X, p / \widetilde{Z}\} \notin F
$$

Moreover，since each $Z(\in \widetilde{Z})$ is 1－active in $C_{X, \tilde{Z}}^{\prime}$（i．e．，（MS－$\left.\left.\tau-1\right)\right)$ and $\tau . p \xrightarrow{\tau} p$ ，by Lemma 5．4，we get $C_{X, \widetilde{Z}}^{\prime}\{\tau \cdot p / X, \tau \cdot p / \widetilde{Z}\} \stackrel{\epsilon}{\Longrightarrow} C_{X, \widetilde{Z}}^{\prime}\{\tau \cdot p / X, p / \widetilde{Z}\} \equiv q \notin F$ ，which，by Lemma 4．2，implies

$$
\begin{equation*}
C_{X, \widetilde{Z}}^{\prime}\{\tau \cdot p / X, \tau \cdot p / \widetilde{Z}\} \notin F \tag{6.5,1}
\end{equation*}
$$

Let $a_{X}$ be any fresh visible action．By（MS－$\tau-3-\mathrm{i}$ ），it follows from $a_{X} .0 \xlongequal{\epsilon} \mid a_{X} .0$ that there exists $s$ such that

$$
C_{X}\left\{a_{X} .0 / X\right\} \stackrel{\epsilon}{\Longrightarrow} s \Rightarrow C_{X, \widetilde{Z}}^{\prime}\left\{a_{X} .0 / X, a_{X} .0 / \widetilde{Z}\right\}
$$

Since $a_{X} .0$ and $C_{X, \widetilde{Z}}^{\prime}$ are stable，so is $C_{X, \widetilde{Z}}^{\prime}\left\{a_{X} .0 / X, a_{X} .0 / \widetilde{Z}\right\}$ by Lemma 5．6．Then，by Lemma 5．11，$s$ is stable．Thus，for the transition in（6．5．2），by Lemma 5．16 there exists
a stable context $C_{X}^{*}$ such that

$$
\begin{equation*}
s \equiv C_{X}^{*}\left\{a_{X} .0 / X\right\} \text { and } C_{X}\{r / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X}^{*}\{r / X\} \text { for any } r . \tag{6.5,3}
\end{equation*}
$$

Then, by Lemma 5.18, it follows from $s \equiv C_{X}^{*}\left\{a_{X} .0 / X\right\} \Rightarrow C_{X, \widetilde{Z}}^{\prime}\left\{a_{X} .0 / X, a_{X} .0 / \widetilde{Z}\right\}$ that

$$
C_{X}^{*}\{\tau \cdot p / X\} \Rightarrow C_{X, \widetilde{Z}}^{\prime}\{\tau \cdot p / X, \tau \cdot p / \widetilde{Z}\}
$$

Hence $C_{X}^{*}\{\tau . p / X\} \notin F$ by (6.51) and Lemma 5.14 which implies $C_{X}^{*}\{p / X\} \in \Omega$. Moreover, since $C_{X}^{*}$ and $p$ are stable, so is $C_{X}^{*}\{p / X\}$ by Lemma 5.6. which implies $C_{X}\{p / X\} \stackrel{\epsilon}{\Longrightarrow} \mid C_{X}^{*}\{p / X\}$ by (6.5).3). Therefore, $\mathcal{T}$ has a proper subtree with root $C_{X}^{*}\{p / X\} F$.

Case $2 \frac{B_{X}\{p / X\} \xrightarrow{a} r, D_{X}\{p / X\} \stackrel{A}{a}^{a}, C_{X}\{p / X\} f^{\tau}}{C_{X}\{p / X\} F}$ with $C_{X} \equiv B_{X} \wedge D_{X}$.
Clearly, in this situation, both $B_{X}$ and $D_{X}$ are stable. Since $C_{X}\{\tau . p / X\} \notin F$, we have $C_{X}\{\tau . p / X\} \stackrel{\epsilon}{\Longrightarrow}{ }_{F} \mid q$ for some $q$. So, there exist $s$ and $t$ such that $q \equiv s \wedge t$ and

$$
B_{X}\{\tau \cdot p / X\} \stackrel{\epsilon}{\epsilon}_{F} \mid s \text { and } D_{X}\{\tau \cdot p / X\} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid t .
$$

Then, for these two transitions, by Lemma 5.16, there exist $B_{X, \widetilde{Y}}^{\prime}$ and $D_{X, \widetilde{Z}}^{\prime}$ satisfying (MS- $\tau-1)-(\mathrm{MS}-\tau-7)$ respectively. In particular, since $p, B_{X}$ and $D_{X}$ are stable, by (MS-$\tau-2,4,7)$, we have
(1) $s \equiv B_{X, \widetilde{Y}}^{\prime}\{\tau \cdot p / X, p / \widetilde{Y}\}$ and $B_{X}\{p / X\} \Rightarrow B_{X, \tilde{Y}}^{\prime}\{p / X, p / \widetilde{Y}\}$;
(2) $t \equiv D_{X, \widetilde{Z}}^{\prime}\{\tau \cdot p / X, p / \widetilde{Z}\}$ and $D_{X}\{p / X\} \Rightarrow D_{X, \widetilde{Z}}^{\prime}\{p / X, p / \widetilde{Z}\}$.

Hence, by Lemma 5.11 it follows from $B_{X}\{p / X\} \xrightarrow{a}$ and $D_{X}\{p / X\} \stackrel{a}{\longrightarrow}$ that

$$
B_{X, \tilde{Y}}^{\prime}\{p / X, p / \widetilde{Y}\} \xrightarrow{a} \text { and } D_{X, \widetilde{Z}}^{\prime}\{p / X, p / \widetilde{Z}\} \not{ }^{a} .
$$

Further, since $B_{X, \tilde{Y}}^{\prime}\{p / X, p / \widetilde{Y}\}$ and $B_{X, \tilde{Y}}^{\prime}\{\tau \cdot p / X, p / \widetilde{Y}\}$ are stable, by Lemma 6.2 it follows from $\tau . p={ }_{R S} p$ and $s \equiv B_{X, \widetilde{Y}}^{\prime}\{\tau \cdot p / X, p / \widetilde{Y}\} \notin F$ that $B_{X, \widetilde{Y}}^{\prime}\{\tau \cdot p / X, p / \widetilde{Y}\} \xrightarrow{a}$. Similarly, we also have $D_{X, \widetilde{Z}}^{\prime}\{\tau \cdot p / X, p / \widetilde{Z}\} \xrightarrow{a}$. Hence $q \equiv s \wedge t \in F$ by Rule $R p_{10}$, a contradiction. Thus this case is impossible.

Case $3 \frac{C_{X}\{p / X\} \stackrel{\alpha}{\rightarrow} r^{\prime},\left\{r F: C_{X}\{p / X\} \stackrel{\alpha}{\longrightarrow} r\right\}}{C_{X}\{p / X\} F}$ with $C_{X} \equiv B_{X} \wedge D_{X}$.
Since $C_{X}\{\tau . p / X\} \notin F$, we have

$$
C_{X}\{\tau \cdot p / X\} \xlongequal{\epsilon}_{F} \mid q \text { for some } q
$$

Next we distinguish two cases based on $\alpha$.

Case $3.1 \alpha=\tau$.
By (6.5.4) and Lemma 5.19, there exist $t$ and stable context $C_{X}^{*}$ such that

$$
C_{X}\{\tau . p / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X}^{*}\{\tau . p / X\} \stackrel{\epsilon}{\Longrightarrow} \mid t \Rightarrow q \notin F
$$

and

$$
C_{X}\{p / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X}^{*}\{p / X\} .
$$

 such that

$$
C_{X}\{p / X\} \xrightarrow{\tau} C_{X}^{\prime}\{p / X\} \stackrel{\epsilon}{\longrightarrow} C_{X}^{*}\{p / X\}
$$

and

$$
C_{X}\{\tau \cdot p / X\} \xrightarrow{\tau} C_{X}^{\prime}\{\tau \cdot p / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X}^{*}\{\tau \cdot p / X\} \stackrel{\epsilon}{\Longrightarrow} \mid t .
$$

Further, by Lemma 5.14, it follows from $q \notin F$ and $t \Rightarrow q$ that $t \notin F$. Then, by Lemma4.2 and the transition above, we have $C_{X}^{\prime}\{\tau \cdot p / X\} \notin F$. Hence $C_{X}^{\prime}\{p / X\} \in \Omega$ and one of nodes directly above the root of $\mathcal{T}$ is labelled with $C_{X}^{\prime}\{p / X\} F$, as desired.

Case $3.2 \alpha \in$ Act.
In this case, $C_{X}$ is stable by Lemma 5.7. By (6.5,4) and Lemma 5.16 there exists a stable context $C_{X, \tilde{Y}}^{\prime}$ with $X \notin \widetilde{Y}$ that satisfies (MS- $\left.\tau-1\right)-($ MS- $\tau-7)$. Then $q \equiv$ $C_{X, \tilde{Y}}^{\prime}\{\tau \cdot p / X, p / \tilde{Y}\}$ due to $p \nrightarrow^{\tau}$ and (MS- $\left.\tau-2\right)$. Moreover, since $C_{X}$ is stable, by (MS- $\left.\tau-4\right)$, we have

$$
C_{X}\{r / X\} \Rightarrow C_{X, \tilde{Y}}^{\prime}\{r / X, r / \tilde{Y}\} \text { for any } r
$$

Then, by $C_{X}\{p / X\} \xrightarrow{\alpha}$ and Lemma 5.11, we get

$$
C_{X, \tilde{Y}}^{\prime}\{p / X, p / \tilde{Y}\} \xrightarrow{\alpha} .
$$

Further, by Lemma 6.2, we also have $C_{X, \widetilde{Y}}^{\prime}\{\tau . p / X, p / \widetilde{Y}\} \xrightarrow{\alpha}$ because of $\tau . p={ }_{R S} p$ and $q \equiv C_{X, \widetilde{Y}}^{\prime}\{\tau . p / X, p / \widetilde{Y}\} \notin F$. Thus, by Theorem 4.2, we obtain

$$
C_{X, \tilde{Y}}^{\prime}\{\tau . p / X, p / \widetilde{Y}\} \xrightarrow{\alpha}_{F} t \text { for some } t
$$

For such $\alpha$-labelled transition, by Lemma 5.8, there exist $C_{X, \widetilde{Y}}^{\prime \prime}, C_{X, \widetilde{Y}, \widetilde{Z}}^{\prime \prime}$ and $C_{X, \widetilde{Y}, \widetilde{Z}}^{\prime \prime \prime}$ with $(\{X\} \cup \tilde{Y}) \cap \widetilde{Z}=\emptyset$ that realize $(\mathrm{CP}-a-1)-(\mathrm{CP}-a-4)$. In particular, due to $\tau . p \xrightarrow{\mu}$ and (CP-a-3-ii), there exist $p_{Z}^{\prime}(Z \in \widetilde{Z})$ such that

$$
\begin{equation*}
p \xrightarrow{\alpha} p_{Z}^{\prime} \text { for each } Z \in \widetilde{Z} \text { and } t \equiv C_{X, \widetilde{Y}, \widetilde{Z}}^{\prime \prime \prime}\left\{\tau \cdot p / X, p / \widetilde{Y}, \widetilde{p_{Z}^{\prime}} / \widetilde{Z}\right\} \notin F \tag{6.5.7}
\end{equation*}
$$

Moreover, by (CP-a-3-iii), for any $r, s$ and $s_{Z}^{\prime}(Z \in \widetilde{Z})$ such that $s \xrightarrow{\alpha} s_{Z}^{\prime}$ for each $Z \in \widetilde{Z}$, we have

$$
\begin{equation*}
C_{X, \widetilde{Y}}^{\prime}\{r / X, s / \widetilde{Y}\} \xrightarrow{\alpha} C_{X, \widetilde{Y}, \widetilde{Z}}^{\prime \prime \prime}\left\{r / X, s / \widetilde{Y}, \widetilde{s_{Z}^{\prime}} / \widetilde{Z}\right\} \text { whenever } C_{X, \widetilde{Y}}^{\prime}\{r / X, s / \widetilde{Y}\} \text { is stable. } \tag{6.5.8}
\end{equation*}
$$

For each $Z \in \widetilde{Z} \cup\{X\}$, we fix a fresh and distinct visible action $a_{Z}$ and set

$$
T \triangleq \begin{cases}\square \square_{Z \in \widetilde{Z}} \alpha \cdot a_{Z} \cdot 0, & \text { if } \widetilde{Z} \neq \emptyset \\ a_{X} \cdot 0, & \text { otherwise }\end{cases}
$$

Since $T$ and $C_{X, \widetilde{Y}}^{\prime}$ are stable, so is $C_{X, \widetilde{Y}}^{\prime}\{T / X, T / \widetilde{Y}\}$ by Lemma 5.6. Then, by (6.5 8), we have

$$
C_{X, \widetilde{Y}}^{\prime}\{T / X, T / \widetilde{Y}\} \xrightarrow{\alpha} C_{X, \widetilde{Y}, \widetilde{Z}}^{\prime \prime \prime}\left\{T / X, T / \widetilde{Y}, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}
$$

So, by Lemma 5.11, it follows from (6.55) that there exists $t^{\prime}$ such that

$$
\begin{equation*}
C_{X}\{T / X\} \xrightarrow{\alpha} t^{\prime} \text { and } t^{\prime} \Rightarrow C_{X, \widetilde{Y}, \widetilde{Z}}^{\prime \prime \prime}\left\{T / X, T / \widetilde{Y}, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\} \tag{6.5,9}
\end{equation*}
$$

Then, by Lemma [5.8, it is not difficult to see that there exists a context $B_{X, \widetilde{Z}}$ that satisfies the conditions:
(a) $t^{\prime} \equiv B_{X, \widetilde{Z}}\left\{T / X, \widetilde{a_{Z} \cdot 0} / \widetilde{Z}\right\}$;
(b) none of $a_{Z}$ with $Z \in \widetilde{Z}$ occurs in $B_{X, \widetilde{Z}}$;
(c) for any $s$ and $s_{Z}^{\prime}(Z \in \widetilde{Z})$ such that $s \xrightarrow{\alpha} s_{Z}^{\prime}$ for each $Z \in \widetilde{Z}$,

$$
C_{X}\{s / X\} \xrightarrow{\alpha} B_{X, \widetilde{Z}}\left\{s / X, \widetilde{s_{Z}^{\prime}} / \widetilde{Z}\right\} \text { whenever } C_{X}\{s / X\} \text { is stable. }
$$

Now we obtain the diagram
by (6.55)

$$
C_{X}\{p / X\}
$$

$\downarrow \alpha$ by (c)
$\Rightarrow$
$C_{X, \widetilde{Y}}^{\prime}\{p / X, p / \tilde{Y}\}$
$\downarrow \alpha$ by (6.5) 6) and (6.5 8)
by (6.5.9), (a) and Lemma 6.4
$B_{X, \widetilde{Z}}\left\{p / X, \widetilde{p_{Z}^{\prime}} / \widetilde{Z}\right\}$ $\Rightarrow$

$$
C_{X, \widetilde{Y}, \widetilde{Z}}^{\prime \prime \prime}\left\{p / X, p / \widetilde{Y}, \widetilde{p_{Z}^{\prime}} / \widetilde{Z}\right\}
$$

By Lemma 6.4 we also have

$$
\begin{equation*}
B_{X, \widetilde{Z}}\left\{\tau \cdot p / X, \widetilde{p_{Z}^{\prime}} / \widetilde{Z}\right\} \Rightarrow C_{X, \widetilde{Y}, \widetilde{Z}}^{\prime \prime \prime}\left\{\tau \cdot p / X, \tau \cdot p / \widetilde{Y}, \widetilde{p_{Z}^{\prime}} / \widetilde{Z}\right\} \tag{6.5}
\end{equation*}
$$

For each $Y \in \tilde{Y}$, since $Y$ is 1-active in $C_{X, \widetilde{Y}}^{\prime}$, by Lemma $5.2(1)(2)$ and $C_{X, \widetilde{Y}}^{\prime} \Rightarrow C_{X, \widetilde{Y}}^{\prime \prime}$ (i.e., (CP-a-1)), so it is in $C_{X, \widetilde{Y}}^{\prime \prime}$. Moreover, by (CP-a-4-i,ii), for each $Y \in \widetilde{Y} \cap F V\left(C_{X, \widetilde{Y}, \tilde{Z}}^{\prime \prime \prime}\right), Y$ is 1-active in $C_{X, \widetilde{Y}, \widetilde{Z}}^{\prime \prime \prime}$. Then, by Lemma 5.4] we have

$$
C_{X, \widetilde{Y}, \widetilde{Z}}^{\prime \prime \prime}\left\{\tau \cdot p / X, \tau \cdot p / \widetilde{Y}, \widetilde{p_{Z}^{\prime}} / \widetilde{Z}\right\} \stackrel{\epsilon}{\Longrightarrow} C_{X, \widetilde{Y}, \widetilde{Z}}^{\prime \prime \prime}\left\{\tau \cdot p / X, p / \widetilde{Y}, \widetilde{p_{Z}^{\prime}} / \widetilde{Z}\right\}
$$

which, together with (6.5.7), implies $C_{X, \widetilde{Y}, \widetilde{Z}}^{\prime \prime \prime}\left\{\tau \cdot p / X, \tau \cdot p / \widetilde{Y}, \widetilde{p_{Z}^{\prime}} / \widetilde{Z}\right\} \notin F$ by Lemma 4.2 , Hence, by Lemma 5.14, it follows from (6.510) that $B_{X, \widetilde{Z}}\left\{\tau \cdot p / X, \widetilde{p_{Z}^{\prime}} / \widetilde{Z}\right\} \notin F$. Thus, $B_{X, \widetilde{Z}}\left\{p / X, \widetilde{p_{Z}^{\prime}} / \widetilde{Z}\right\} \in \Omega ;$ moreover, $\mathcal{T}$ has a proper subtree with root $B_{X, \widetilde{Z}}\left\{p / X, \widetilde{p_{Z}^{\prime}} / \widetilde{Z}\right\} F$ due to (c) and (6.57).

Hitherto we have completed the first step mentioned at the beginning of this section. Now we return to carry out the second step. Before proving Lemma 6.7 a result concerning proof tree is given first.

Lemma 6.6. Let $C_{\tilde{X}, \widetilde{Z}}$ be any context such that for each $Z \in \widetilde{Z}, Z$ is active and occurs at most once. If $\widetilde{p}, \widetilde{q}, \widetilde{t}, \widetilde{s}$ and $\widetilde{r}$ are any processes such that
(a) $\tilde{p} \sqsubseteq_{R S} \widetilde{q}$,
(b) $\widetilde{p} \bowtie \widetilde{q}$,
(c) $\widetilde{r} \xlongequal{\epsilon} \mid \widetilde{t}$,
(d) $\widetilde{s}{\underset{\sim}{\sim}}^{\sqsubset_{S}} \tilde{t}$, and
(e) $C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{p} / \widetilde{X}, \widetilde{s} / \widetilde{Z}\} \notin F$,
then, for any proof tree $\mathcal{T}$ for $\operatorname{Strip}\left(\mathcal{P}_{\operatorname{CLL}_{R}}, M_{\operatorname{CLL}_{R}}\right) \vdash C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} F$, there exist $C_{\widetilde{X}, \widetilde{Z}, \widetilde{Y}}^{*}$ and $p_{Y}^{\prime \prime}, q_{Y}^{\prime \prime}(Y \in \widetilde{Y})$ such that
(1) $\mathcal{T}$ has a subtree with $\operatorname{root} C_{\widetilde{X}, \widetilde{Z}, \widetilde{Y}}^{*}\left\{\widetilde{q} / \widetilde{X}, \widetilde{t} / \widetilde{Z}, \widetilde{q_{Y}^{\prime \prime}} / \widetilde{Y}\right\} F$,
(2) $C_{\widetilde{X}, \widetilde{Z}, \widetilde{Y}}^{*}\left\{\widetilde{p} / \widetilde{X}, \widetilde{s} / \widetilde{Z}, \widetilde{p_{Y}^{\prime \prime}} / \widetilde{Y}\right\} \notin F$, and
(3) $\widetilde{p_{Y}^{\prime \prime}}{\underset{\sim}{Z S}}^{\square_{Y}} \widetilde{q_{Y}^{\prime \prime}}$.

Proof. It proceeds by induction on the depth of $\mathcal{T}$. We distinguish different cases depending on the form of $C_{\widetilde{X}, \widetilde{Z}}$.

Case $1 C_{\widetilde{X}, \widetilde{Z}}$ is closed or $C_{\widetilde{X}, \widetilde{Z}} \equiv X_{i}$ or $C_{\widetilde{X}, \widetilde{Z}} \equiv Z_{j}$ for some $i \leq|\widetilde{X}|$ and $j \leq|\widetilde{Z}|$.
It is straightforward to show that this lemma holds trivially for such case. As a example, we consider the case $C_{\widetilde{X}, \widetilde{Z}} \equiv Z_{j}$. Since $C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{p} / \widetilde{X}, \widetilde{s} / \widetilde{Z}\} \equiv s_{j} \notin F$ and $\widetilde{s}{\underset{\sim}{r}}_{R S} \widetilde{t}$, we have $t_{j} \notin F$. Hence $r_{j}{ }^{\epsilon}{ }_{F} \mid t_{j}$ by Lemma 4.2. So $C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} \equiv r_{j} \notin F$. That is, there is no proof tree of $\operatorname{Strip}\left(\mathcal{P}_{\operatorname{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} F$. Thus the conclusion holds trivially.

Case $2 C_{\tilde{X}, \tilde{Z}}$ is of the format $\alpha \cdot B_{\tilde{X}, \tilde{Z}}$ or $B_{\tilde{X}, \widetilde{Z}} \vee D_{\tilde{X}, \tilde{Z}}$ or $\langle Y \mid E\rangle$.
For these three formats, since each $Z(\in \widetilde{Z})$ is active in $C_{\widetilde{X}}, \widetilde{Z}$, it is obvious that $\widetilde{Z}=\emptyset$. Thus $C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} \equiv C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{t} / \widetilde{Z}\}$. So, $\mathcal{T}$ has the root labelled with $C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{t} / \widetilde{Z}\} F$. Therefore, the conclusion holds by setting $C_{\widetilde{X}, \widetilde{Z}, \widetilde{Y}}^{*} \triangleq C_{\widetilde{X}, \widetilde{Z}}$ with $\widetilde{Y}=\emptyset$.

Case $3 C_{\widetilde{X}, \widetilde{Z}} \equiv B_{\widetilde{X}, \widetilde{Z}} \odot D_{\widetilde{X}, \widetilde{Z}}$ with $\odot \in\left\{\square, \|_{A}\right\}$.
W.l.o.g, assume the last rule applied in $\mathcal{T}$ is

$$
\frac{B_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} F}{B_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} \odot D_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} F}
$$

Then $\mathcal{T}$ has a proper subtree $\mathcal{T}^{\prime}$ with root $B_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} F$. Since $C_{\widetilde{X}}, \widetilde{Z}\{\widetilde{p} / \widetilde{X}, \widetilde{s} / \widetilde{Z}\} \notin$ $F$, we get $B_{\widetilde{X}, \widetilde{Z}}\{\widetilde{p} / \widetilde{X}, \widetilde{s} / \widetilde{Z}\} \notin F$. Then the conclusion immediately follows by applying IH on $\mathcal{T}^{\prime}$.

Case $4 C_{\widetilde{X}, \widetilde{Z}} \equiv B_{\widetilde{X}, \widetilde{Z}} \wedge D_{\widetilde{X}, \widetilde{Z}}$.
The argument splits into four cases based on the last rule applied in $\mathcal{T}$.
Case 4.1 $\frac{B_{\widetilde{X}}, \tilde{Z}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} F}{B_{\widetilde{X}, \tilde{Z}}\{\widetilde{q} / \tilde{X}, \widetilde{r} / \widetilde{Z}\} \wedge D_{\widetilde{X}, \tilde{Z}}\{\widetilde{q} / \tilde{X}, \widetilde{r} / \widetilde{Z}\} F}$.
Similar to Case 3, omitted.
Case $4.2 \frac{B_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} \xrightarrow{a} r^{\prime}, D_{\widetilde{X}}\left\{\widetilde{Z}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\}{ }^{a} \rightarrow C_{\widetilde{X}, \tilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} \digamma^{\tau}\right.}{B_{\widetilde{X}, \tilde{Z}}\{\widetilde{q} / \tilde{X}, \widetilde{r} / \widetilde{Z}\} \wedge D_{\widetilde{X}, \tilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} F}$.

For any $Z(\in \widetilde{Z})$ occurring in $C_{\widetilde{X}, \widetilde{Z}}$, since $Z$ is active and $C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} \stackrel{f^{\tau}}{ }$, by Lemma 5.4. we have $r_{Z} \xrightarrow{\tau}$, and hence $r_{Z} \equiv t_{Z}$ because of (c). So, $C_{\widetilde{X}}, \widetilde{Z}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} \equiv$ $C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{t} / \widetilde{Z}\}$. Hence $\mathcal{T}$ has the root labelled with $C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{t} / \widetilde{Z}\} F$. Clearly, the conclusion holds by setting $C_{\widetilde{X}, \widetilde{Z}, \widetilde{Y}}^{*} \triangleq C_{\widetilde{X}, \widetilde{Z}}$ with $\widetilde{Y}=\emptyset$.

Case $4.3 \frac{C_{\widetilde{X}, \tilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} \stackrel{\alpha}{\longrightarrow} s^{\prime},\left\{r F: C_{\widetilde{X}, \tilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} \xrightarrow{\alpha} r\right\}}{C_{\widetilde{X}, \tilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} F}$.
If $\alpha \in A c t$, the argument is similar to one of Case 4.2 and omitted. In the following, we handle the case $\alpha=\tau$. If $r_{Z}{ }^{\tau}$ for any $Z(\in \widetilde{Z})$ occurring in $C_{\widetilde{X}, \widetilde{Z}}$, then the conclusion holds trivially by putting $C_{\widetilde{X}}^{*}, \widetilde{Z}, \widetilde{Y}, ~ \triangleq C_{\widetilde{X}, \widetilde{Z}}$ with $\widetilde{Y}=\emptyset$. Next we consider the other case where $r_{Z_{0}} \xrightarrow{\tau}$ for some $Z_{0}(\in \widetilde{Z})$ occurring in $C_{\tilde{X}, \widetilde{Z}}$. Then $r_{Z_{0}} \xrightarrow{\tau} r^{\prime} \xlongequal{\epsilon} \mid t_{Z_{0}}$ for some $r^{\prime}$ by (c); moreover, $Z_{0}$ is 1-active in $C_{\tilde{X}, \widetilde{Z}}$. Thus, by Lemma 5.4] we get

$$
C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} \xrightarrow{\tau} C_{\widetilde{X}, \widetilde{Z}}\left\{\widetilde{q} / \widetilde{X}, \widetilde{r}\left[r^{\prime} / r_{Z_{0}}\right] / \widetilde{Z}\right\} .
$$

So, $\mathcal{T}$ has a proper subtree $\mathcal{T}^{\prime}$ with $\operatorname{root} C_{\widetilde{X}, \widetilde{Z}}\left\{\widetilde{q} / \widetilde{X}, \widetilde{r}\left[r^{\prime} / r_{Z_{0}}\right] / \widetilde{Z}\right\} F$. Since $\widetilde{r}\left[r^{\prime} / r_{Z_{0}}\right] \xlongequal{\epsilon} \mid \widetilde{t}$ and $C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{p} / \widetilde{X}, \widetilde{s} / \widetilde{Z}\} \notin F$, by $\mathrm{IH}, \mathcal{T}^{\prime}$ has a subtree with root $C_{\widetilde{X}}^{*}, \widetilde{Z}, \widetilde{Y}\left\{\widetilde{q} / \widetilde{X}, \widetilde{t} / \widetilde{Z}, \widetilde{q_{Y}^{\prime \prime}} / \widetilde{Y}\right\} F$ for some $C_{\widetilde{X}}^{*}, \widetilde{Z}, \widetilde{Y}, \widetilde{p_{Y}^{\prime \prime}}$ and $\widetilde{q_{Y}^{\prime \prime}}$ such that $\left.C_{\widetilde{X}}^{*}, \widetilde{Z}, \widetilde{Y}, \widetilde{p} / \widetilde{X}, \widetilde{s} / \widetilde{Z}, \widetilde{p_{Y}^{\prime \prime}} / \widetilde{Y}\right\} \notin F$ and $\widetilde{p_{Y}^{\prime \prime}} \sqsubset_{R S} \widetilde{q_{Y}^{\prime \prime}}$.
Case $4.4 \frac{\left\{r F: C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} \neq \mid r\right\}}{C_{\widetilde{X}, \tilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} F}$.
It follows from $C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{p} / \widetilde{X}, \widetilde{s} / \widetilde{Z}\} \notin F$ that

$$
C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{p} / \widetilde{X}, \widetilde{s} / \widetilde{Z}\} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid p^{\prime} \text { for some } p^{\prime}
$$

Then, by Lemma 5.16 for such transition, there exist a stable context $C_{\widetilde{X}, \widetilde{Z}, \widetilde{Y}}^{\prime}$ and $i_{Y}, p_{Y}^{\prime \prime \prime}(Y \in \widetilde{Y})$ that realize (MS- $\left.\tau-1\right)-(\mathrm{MS}-\tau-7)$. In particular, since each $s(\in \widetilde{s})$ is stable, by (MS- $\tau-2,7$ ), we have $i_{Y} \leq|\widetilde{X}|$ for each $Y \in \widetilde{Y}$ and

$$
p_{i_{Y}} \xlongequal{\tau} \mid p_{Y}^{\prime \prime \prime} \text { for each } Y \in \widetilde{Y} \text { and } p^{\prime} \equiv C_{\widetilde{X}, \widetilde{Z}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{s} / \widetilde{Z}, \widetilde{p_{Y}^{\prime \prime \prime}} / \widetilde{Y}\right\} \notin F
$$

Then, by Lemma 5.5, it follows from (MS- $\tau-1$ ) that, for each $Y \in \widetilde{Y}, p_{Y}^{\prime \prime \prime} \notin F$ and hence $p_{i_{Y}}{ }^{\tau}{ }_{F} \mid p_{Y}^{\prime \prime \prime}$ by Lemma 4.2. Further, since $\widetilde{p} \bowtie \widetilde{q}$ and $\widetilde{p} \sqsubseteq_{R S} \widetilde{q}$, there exist $q_{Y}^{\prime \prime \prime}(Y \in \widetilde{Y})$ such that

$$
q_{i_{Y}}{ }^{\tau}{\underset{F}{F}} \mid q_{Y}^{\prime \prime \prime} \text { and } p_{Y}^{\prime \prime \prime} \sqsubset_{R S} q_{Y}^{\prime \prime \prime} \text { for each } Y \in \widetilde{Y} \text {. }
$$

Then it follows from (MS- $\tau$-3-ii) that

$$
\begin{equation*}
C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{t} / \widetilde{Z}\} \xlongequal{\epsilon} C_{\widetilde{X}, \widetilde{Z}, \widetilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{t} / \widetilde{Z}, \widetilde{q_{Y}^{\prime \prime \prime}} / \widetilde{Y}\right\} \tag{6.6,1}
\end{equation*}
$$

Moreover, since $Z$ is active and occurs at most once in $C_{\widetilde{X}, \widetilde{Z}}$ for each $Z \in \widetilde{Z}$, by Lemma 5.4, it follows from $\widetilde{r} \xlongequal{\epsilon} \widetilde{t}$ that

$$
\begin{equation*}
C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{r} / \widetilde{Z}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}, \widetilde{Z}}\{\widetilde{q} / \widetilde{X}, \widetilde{t} / \widetilde{Z}\} \tag{6.6,2}
\end{equation*}
$$

Since $\widetilde{p} \bowtie \widetilde{q}, \widetilde{s}{\underset{\sim}{r}} \widetilde{t}$ and $\widetilde{p_{Y}^{\prime \prime \prime}}{\underset{\sim}{r}}_{R S} \widetilde{q_{Y}^{\prime \prime \prime}}$, by $p^{\prime} \equiv C_{\widetilde{X}, \widetilde{Z}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{s} / \widetilde{Z}, \widetilde{\left.p_{Y}^{\prime \prime \prime} / \widetilde{Y}\right\}} f^{\tau}\right.$ and

Lemma 5.6. we can conclude that $C_{\widetilde{X}, \widetilde{Z}, \widetilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{t} / \widetilde{Z}, \widetilde{q_{Y}^{\prime \prime \prime}} / \widetilde{Y}\right\}$ is stable. Hence $\mathcal{T}$ has a proper subtree with root $C_{\underset{\widetilde{X}}{\prime}, \widetilde{Z}, \widetilde{Y}}\left\{\widetilde{q} / \widetilde{X}, \widetilde{t} / \widetilde{Z}, \widetilde{q_{Y}^{\prime \prime \prime}} / \widetilde{Y}\right\} F$ by (6.6 1) and (6.6.2); moreover, we also have $p^{\prime} \equiv C_{\widetilde{X}, \widetilde{Z}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{s} / \widetilde{Z}, \widetilde{p_{Y}^{\prime \prime \prime}} / \widetilde{Y}\right\} \notin F$ and $\widetilde{p_{Y}^{\prime \prime \prime}} \sqsubset_{R S} \widetilde{q_{Y}^{\prime \prime \prime}}$. Consequently, $C_{\widetilde{X}, \widetilde{Z}, \widetilde{Y}}^{\prime}$, $\widetilde{p_{Y}^{\prime \prime \prime}}$ and $\widetilde{q_{Y}^{\prime \prime \prime}}$ are what we seek.

Lemma 6.7. For any $C_{\widetilde{X}}$ and processes $\widetilde{r}$ and $\widetilde{s}$, if $\widetilde{r} \bowtie \widetilde{s}$ and $\widetilde{r} \sqsubseteq_{R S} \widetilde{s}$, then $C_{\widetilde{X}}\{\widetilde{r} / \widetilde{X}\} \notin$ $F$ implies $C_{\widetilde{X}}\{\widetilde{s} / \widetilde{X}\} \notin F$.

Proof. Set

$$
\Omega=\left\{B_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\}: \widetilde{p} \bowtie \widetilde{q}, \widetilde{p} \sqsubseteq_{R S} \widetilde{q}, B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F \text { and } B_{X} \text { is a context }\right\}
$$

Let $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \in \Omega$ and $\mathcal{T}$ be any proof tree of $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash C_{\widetilde{X}}\{\widetilde{q} / \tilde{X}\} F$. Similar to Lemma 6.3, it suffices to show that $\mathcal{T}$ has a proper subtree with root $s F$ for some $s \in \Omega$. We distinguish six cases based on the form of $C_{\widetilde{X}}$.

Case $1 C_{\widetilde{X}}$ is closed or $C_{\widetilde{X}} \equiv X_{i}$.
In this situation, $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{\sim}\} \notin F$ because of $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$ and $\widetilde{p} \sqsubseteq_{R S} \widetilde{q}$. Thus there is no proof tree of $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} F$. Hence the conclusion holds trivially.

Case $2 C_{\widetilde{X}} \equiv \alpha . B_{\widetilde{X}}$.
Then the last rule applied in $\mathcal{T}$ is $\frac{B_{\widetilde{X}}\{\widetilde{q} / \tilde{X}\} F}{\alpha \cdot B_{\widetilde{X}}\{\widetilde{q} / \tilde{X}\} F}$. Moreover $B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$ due to $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$. Hence $B_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \in \Omega$, as desired.

Case $3 C_{\widetilde{X}} \equiv B_{\tilde{X}} \vee D_{\tilde{X}}$.
Obviously, the last rule applied in $\mathcal{T}$ is $\frac{B_{\widetilde{X}}\{\widetilde{q} / \tilde{X}\} F, D_{\widetilde{\widetilde{X}}}\{\widetilde{q} / \tilde{X}\} F}{B_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \vee D_{\widetilde{X}}\{\tilde{q} / \widetilde{X}\} F}$. Due to $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$, we have either $B_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$ or $D_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$, which implies $B_{\tilde{X}}\{\widetilde{q} / \widetilde{X}\} \in \Omega$ or $D_{\tilde{X}}\{\widetilde{q} / \widetilde{X}\} \in \Omega$. Thus $\mathcal{T}$ contains a proper subtree with root $s F$ for some $s \in \Omega$.

Case $4 C_{\tilde{X}} \equiv B_{\tilde{X}} \odot D_{\tilde{X}}$ with $\odot \in\left\{\square, \|_{A}\right\}$.
W.l.o.g, assume the last rule applied in $\mathcal{T}$ is $\frac{B_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} F}{B_{\widetilde{X}}\{\widetilde{q} / \tilde{X}\} \odot D_{\widetilde{X}}\{\widetilde{q} / \tilde{X}\} F}$. Since $C_{\widetilde{X}}\{\widetilde{p} / \tilde{X}\} \notin F$, we get $B_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$, which implies $B_{\tilde{X}}\{\widetilde{q} / \widetilde{X}\} \in \Omega$, as desired.

Case $5 C_{\widetilde{X}} \equiv\langle Y \mid E\rangle$.
Clearly, the last rule applied in $\mathcal{T}$ is

$$
\text { either } \frac{\left\langle t_{Y} \mid E\right\rangle\{\widetilde{q} / \widetilde{X}\} F}{\langle Y \mid E\rangle\{\widetilde{q} / \widetilde{X}\} F} \text { with } Y=t_{Y} \in E \text { or } \frac{\{r F:\langle Y \mid E\rangle\{\widetilde{q} / \widetilde{X}\} \xlongequal{\epsilon} \Rightarrow \mid r\}}{\langle Y \mid E\rangle\{\widetilde{q} / \widetilde{X}\} F}
$$

For the first alternative, we have $\left\langle t_{Y} \mid E\right\rangle\{\widetilde{p} / \widetilde{X}\} \notin F$ because of $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$, and hence $\left\langle t_{Y} \mid E\right\rangle\{\widetilde{q} / \widetilde{X}\} \in \Omega$.

For the second alternative, due to $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$, we get

$$
C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid s \text { for some } s
$$

For such transition, by Lemma 5.16, there exist $C_{\widetilde{X}, \widetilde{Z}}^{\prime}$ and $i_{Z} \leq|\widetilde{X}|, p_{Z}^{\prime}(Z \in \widetilde{Z})$ that realize (MS- $\tau-1$ ) - (MS- $\tau-7$ ). Amongst them, by (MS- $\tau-2,7$ ), we have

$$
\begin{equation*}
p_{i_{Z}} \xlongequal{\tau} \mid p_{Z}^{\prime} \text { for each } Z \in \widetilde{Z} \text { and } s \equiv C_{\widetilde{X}, \widetilde{Z}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Z}^{\prime}} / \widetilde{Z}\right\} \notin F \text {. } \tag{6.7.1}
\end{equation*}
$$

Thus, for each $Z \in \widetilde{Z}$, by (MS- $\tau-1$ ) and Lemma 5.5, it follows that $p_{Z}^{\prime} \notin F$, and hence $p_{i_{Z}} \stackrel{\tau}{\Longrightarrow}_{F} \mid p_{Z}^{\prime}$ by Lemma 4.2. Further, since $\widetilde{p} \bowtie \widetilde{q}$, it follows from $\widetilde{p} \sqsubseteq_{R S} \widetilde{q}$ that for each $Z \in \widetilde{Z}$,

$$
\begin{equation*}
q_{i_{Z}} \stackrel{\tau}{\Longrightarrow}_{F} \mid q_{Z}^{\prime} \text { and } p_{Z}^{\prime}{\underset{\sim}{\sim}}_{R S} q_{Z}^{\prime} \text { for some } q_{Z}^{\prime} \tag{6.7,2}
\end{equation*}
$$

Then $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}, \widetilde{Z}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Z}^{\prime}} / \widetilde{Z}\right\}$ by (MS- $\tau$-3-ii). In addition, since $\widetilde{p} \bowtie \widetilde{q}$ and $\widetilde{p_{Z}^{\prime}}{\underset{\sim}{R S}} \widetilde{q_{Z}^{\prime}}$, by Lemma 5.6, it follows from $s \equiv C_{\widetilde{X}, \widetilde{Z}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Z}^{\prime}} / \widetilde{Z}\right\} \not{ }^{\tau}$ that $C_{\widetilde{X}}^{\prime}, \widetilde{Z}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Z}^{\prime}} / \widetilde{Z}\right\}$ is stable. Therefore

$$
C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} \mid C_{\widetilde{X}, \widetilde{Z}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Z}^{\prime}} / \widetilde{Z}\right\}
$$

Hence $\mathcal{T}$ has a proper subtree with root $C_{\widetilde{X}, \widetilde{Z}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Z}^{\prime}} / \widetilde{Z}\right\} F ;$ moreover $C_{\widetilde{X}, \widetilde{Z}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Z}^{\prime}} / \widetilde{Z}\right\} \in$ $\Omega$ due to 6.7.1) and 6.7.2).

Case $6 C_{\widetilde{X}} \equiv B_{\widetilde{X}} \wedge D_{\widetilde{X}}$.
The argument splits into five subcases depending on the last rule applied in $\mathcal{T}$.
Case 6.1 $\frac{B_{\widetilde{\widetilde{X}}}\{\widetilde{q} / \widetilde{X}\} F}{\left.B_{\widetilde{X}}\{\widetilde{q} / X)\right\} D_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} F}$.
Similar to Case 4, omitted.
Case $6.2 \frac{B_{\widetilde{X}}\{\widetilde{q} / \tilde{X}\} \xrightarrow{a} r, D_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} f^{a}, C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} f^{\top}}{B_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \wedge D_{\widetilde{X}}\{\widetilde{q} / \tilde{X}\} F}$.
Clearly, $B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$ and $D_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$ due to $C_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$. Moreover, by $\widetilde{p} \bowtie \widetilde{q}$ and Lemma 5.6, we have $C_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \stackrel{\digamma^{\top}}{\longrightarrow}$ because of $C_{\tilde{X}}\{\widetilde{q} / \widetilde{X}\} \quad{ }^{\top} \rightarrow$. Further, by Lemma 6.2, we also have $B_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a}$ and $D_{\tilde{X}}\{\widetilde{p} / \tilde{X}\} \xrightarrow{a}$. So, $C_{\tilde{X}}\{\widetilde{p} / \tilde{X}\} \equiv$ $B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \wedge D_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \in F$ by Rule $R p_{10}$, which contradicts $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \in \Omega$. Hence such case is impossible.

Case $6.3 \frac{C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{\tau} r^{\prime},\left\{r F: C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{\tau} r\right\}}{C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} F}$.
It follows from $C_{\widetilde{X}}\{\widetilde{p} / \tilde{X}\} \notin F$ that

$$
\begin{equation*}
C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid s \text { for some } s \tag{6.7.3}
\end{equation*}
$$

Since $\widetilde{p} \bowtie \widetilde{q}$ and $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{\tau}$, by Lemma 5.6, we get $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau}$. Then, by (6.7,3), we have

$$
C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{\tau}_{F} t \stackrel{\epsilon}{\Longrightarrow}_{F} \mid s \text { for some } t
$$

For the $\tau$-labelled transition leading to $t$, either the clause (1) or (2) in Lemma 5.6 holds.

For the former, there exists $C_{\widetilde{X}}^{\prime}$ such that $t \equiv C_{\widetilde{X}}^{\prime}\{\widetilde{p} / \widetilde{X}\}$ and $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{\tau} C_{\widetilde{X}}^{\prime}\{\widetilde{q} / \widetilde{X}\}$. Hence $C_{\widetilde{X}}^{\prime}\{\widetilde{q} / \widetilde{X}\} F$ is one of premises of the last inferring step in $\mathcal{T}$. Moreover, it is evident that $C_{\widetilde{X}}^{\prime}\{\widetilde{q} / \widetilde{X}\} \in \Omega$.

For the latter, there exist $C_{\widetilde{X}}^{\prime}, C_{\widetilde{X}, Z}^{\prime \prime}$ with $Z \notin \widetilde{X}$ and $i_{0} \leq|\widetilde{X}|$ that realize (P- $\left.\tau-1\right)-$ ( $\mathrm{P}-\tau-4$ ). In particular, by ( $\mathrm{P}-\tau-2$ ), we have

$$
t \equiv C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, p^{\prime} / Z\right\} \text { for some } p^{\prime} \text { with } p_{i_{0}} \xrightarrow{\tau} p^{\prime} .
$$

Further, since $t \stackrel{\epsilon}{\Longrightarrow} F \mid s$ and $Z$ is 1-active in $C_{\widetilde{X}, Z}^{\prime \prime}$, by Lemma 5.20 and 4.2 there exists $p^{\prime \prime}$ such that $p^{\prime} \xlongequal{\epsilon} \mid p^{\prime \prime}$ and

$$
t \equiv C_{\tilde{X}, Z}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, p^{\prime} / Z\right\} \stackrel{\epsilon}{\Longrightarrow}_{F} C_{\tilde{X}, Z}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, p^{\prime \prime} / Z\right\} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid s .
$$

Moreover, $p^{\prime \prime} \notin F$ by Lemma [5.5. Hence $p_{i_{0}} \xrightarrow{\tau_{F}} p^{\prime} \xrightarrow{\epsilon_{F}} \mid p^{\prime \prime}$ by Lemma 4.2, Since $\widetilde{p} \bowtie \widetilde{q}$, it follows from $\widetilde{p} \sqsubseteq_{R S} \widetilde{q}$ that

$$
\begin{equation*}
q_{i_{0}} \xrightarrow{\tau}_{F} q^{\prime} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid q^{\prime \prime} \text { and } p^{\prime \prime} \sqsubset_{R S} q^{\prime \prime} \text { for some } q^{\prime} \text { and } q^{\prime \prime} \tag{6.7,4}
\end{equation*}
$$

Then $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{\tau} C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, q^{\prime} / Z\right\}$ by $(\mathrm{P}-\tau-4)$. Therefore, $\mathcal{T}$ contains a proper subtree $\mathcal{T}^{\prime}$ with root $C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, q^{\prime} / Z\right\} F$. In order to complete the proof, it is sufficient to show that $\mathcal{T}^{\prime}$ contains a node labelled with $s^{\prime} F$ for some $s^{\prime} \in \Omega$. Since $Z$ is 1-active, $\widetilde{p} \sqsubseteq_{R S} \widetilde{q}$, $\widetilde{p} \bowtie \widetilde{q}, q^{\prime} \xlongequal{\epsilon} \mid q^{\prime \prime}, p^{\prime \prime} \sqsubset_{R S} q^{\prime \prime}$ and $C_{\widetilde{X}, Z}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, p^{\prime \prime} / Z\right\} \notin F$, by Lemma 6.6] there exist $C_{\widetilde{X}, Z, \widetilde{Y}}^{*}$ and $q_{Y}^{\prime \prime \prime}, p_{Y}^{\prime \prime \prime}(Y \in Y)$ such that
(a.1) $\mathcal{T}^{\prime}$ has a subtree with root $C_{\widetilde{X}, Z, \widetilde{Y}}^{*}\left\{\widetilde{q} / \widetilde{X}, q^{\prime \prime} / Z, \widetilde{q_{Y}^{\prime \prime \prime}} / \widetilde{Y}\right\} F$,
(a.2) $C_{\tilde{X}, Z, \tilde{Y}}^{*}\left\{\widetilde{p} / \widetilde{X}, p^{\prime \prime} / Z, \widetilde{p_{Y}^{\prime \prime \prime}} / \widetilde{Y}\right\} \notin F$, and
(a.3) $\widetilde{p_{Y}^{\prime \prime \prime}} \sqsubset_{R S} \widetilde{q_{Y}^{\prime \prime \prime}}$.

Clearly, $C_{\widetilde{X}, Z, \widetilde{Y}}^{*}\left\{\widetilde{q} / \widetilde{X}, q^{\prime \prime} / Z, \widetilde{q_{Y}^{\prime \prime \prime}} / \widetilde{Y}\right\} \in \Omega$ due to (a.2), (a.3) and (6.7,4), as desired.
Case 6.4 $\frac{C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{a} r^{\prime},\left\{r F: C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{a} r\right\}}{C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} F}(a \in A c t)$.
Since $\widetilde{p} \bowtie \widetilde{q}$, by Lemma 5.6, it follows from $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{a}$ that $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$ is stable. Further, since $\widetilde{p} \sqsubseteq_{R S} \widetilde{q}$ and $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$, we get $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a}$ by Lemma 6.2, So, by Theorem 4.2 and $C_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$, we have

$$
\begin{equation*}
C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a}_{F} t{\underset{ }{\epsilon}}_{F} \mid s \text { for some } t \text { and } s . \tag{6.7.5}
\end{equation*}
$$

On the one hand, for the $a$-labelled transition in (6.75), by Lemma 5.8, there exist $C_{\tilde{X}}^{\prime}, C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ and $C_{\tilde{X}, \widetilde{Y}}^{\prime \prime}$ that satisfy (CP-a-1) - (CP-a-4). In particular, by (CP-a-3-ii), there exist $i_{Y} \leq|\tilde{X}|, p_{Y}^{\prime}(Y \in \tilde{Y})$ such that

$$
p_{i_{Y}} \xrightarrow{a} p_{Y}^{\prime} \text { for each } Y \in \widetilde{Y} \text { and } t \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\} .
$$

Moreover, by (CP-a-1) and (CP-a-3-i), we have

$$
C_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \Rightarrow C_{\widetilde{X}}^{\prime}\{\widetilde{p} / \widetilde{X}\} \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\tilde{p} / \widetilde{X}, \widetilde{p_{i_{Y}}} / \tilde{Y}\right\}
$$

Hence $C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{i_{Y}}} / \widetilde{Y}\right\} \notin F$ by $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$ and Lemma 5.14 Further, for each $Y \in \tilde{Y}$, since $Y$ is 1-active in $C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ (i.e., (CP-a-2)), by Lemma 5.5, $p_{i_{Y}} \notin F$.

On the other hand, for the transition $t \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid s$ in (6.75), by Lemma 5.20, it follows from each $Y(\in \tilde{Y})$ that is 1-active in $C_{\tilde{X}, \tilde{Y}}^{\prime \prime}$ (i.e., (CP-a-2)) that there exist $p_{Y}^{\prime \prime}(Y \in \widetilde{Y})$ such that $p_{Y}^{\prime} \xlongequal{\epsilon} \mid p_{Y}^{\prime \prime}$ for each $Y \in \widetilde{Y}$ and

$$
t \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}^{\prime \prime}, \widetilde{Y}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime \prime}} / \widetilde{Y}\right\} \stackrel{\epsilon}{\Longrightarrow} \mid s
$$

Since $s \notin F$, we obtain $C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime \prime}} / \widetilde{Y}\right\} \notin F$ by Lemma4.2, which implies that $p_{Y}^{\prime \prime} \notin F$ for each $Y \in \widetilde{Y}$ due to Lemma 5.5 Thus

$$
p_{i_{Y}} \xrightarrow{a}_{F} p_{Y}^{\prime} \stackrel{\epsilon}{\longrightarrow}_{F} \mid p_{Y}^{\prime \prime} \text { for each } Y \in \widetilde{Y}
$$

Since $\widetilde{p} \bowtie \widetilde{q}$, it follows from $\widetilde{p} \sqsubseteq_{R S} \widetilde{q}$ that, for each $Y \in \widetilde{Y}$, there exist $q_{Y}^{\prime}$ and $q_{Y}^{\prime \prime}$ such that

$$
q_{i_{Y}} \stackrel{a}{\longrightarrow}_{F} q_{Y}^{\prime} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid q_{Y}^{\prime \prime} \text { and } p_{Y}^{\prime \prime}{\underset{\sim}{r}}_{R S} q_{Y}^{\prime \prime}
$$

Then $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{a} C_{\widetilde{X}}^{\prime \prime}, \widetilde{Y}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\}$ by (CP-a-3-iii). Hence $\mathcal{T}$ has a proper subtree $\mathcal{T}^{\prime}$ with root $C_{\widetilde{X}, \tilde{Y}}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} F$. In order to complete the proof, it suffices to show that $\mathcal{T}^{\prime}$ contains a node labelled with $s^{\prime} F$ for some $s^{\prime} \in \Omega$. Since each $Y(\in \widetilde{Y})$ is 1-active in $C_{\tilde{X}, \widetilde{Y}}^{\prime \prime}, \widetilde{p} \sqsubseteq_{R S} \widetilde{q}, \widetilde{p} \bowtie \widetilde{q}, \widetilde{q_{Y}^{\prime}} \stackrel{\epsilon}{\Longrightarrow} \mid \widetilde{q_{Y}^{\prime \prime}}, \widetilde{p_{Y}^{\prime \prime}} \sqsubset_{R S} \widetilde{q_{Y}^{\prime \prime}}$ and $C_{\tilde{X}, \tilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime \prime}} / \widetilde{Y}\right\} \notin F$, by Lemma 6.6, there exist $C_{\widetilde{X}, \widetilde{Y}, \widetilde{Z}}^{*}$ and $q_{Z}^{\prime \prime \prime}, p_{Z}^{\prime \prime \prime}(Z \in \widetilde{Z})$ such that
(b.1) $\mathcal{T}^{\prime}$ has a subtree with $\operatorname{root} C_{\widetilde{X}}^{*}, \widetilde{Y}, \tilde{Z}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime \prime}} / \widetilde{Y}, \widetilde{q_{Z}^{\prime \prime \prime}} / \widetilde{Z}\right\} F$,
(b.2) $C_{\widetilde{X}, \widetilde{Y}, \widetilde{Z}}^{*}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime \prime}} / \widetilde{Y}, \widetilde{p_{Z}^{\prime \prime \prime}} / \widetilde{Z}\right\} \notin F$, and
(b.3) $\widetilde{p_{Z}^{\prime \prime \prime}} \check{\sim}_{R S} \widetilde{q_{Z}^{\prime \prime \prime}}$.

Obviously, $C_{\widetilde{X}}^{*}, \widetilde{Y}, \widetilde{Z}$, $\left.\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime \prime}} / \widetilde{Y}, \widetilde{q_{Z}^{\prime \prime \prime}} / \widetilde{Z}\right\} \in \Omega$ due to (b.2), (b.3) and (6.76), as desired.
Case $6.5 \frac{\left\{r F: B_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \wedge D_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xlongequal{\epsilon} \mid r\right\}}{B_{\widetilde{X}}\{\widetilde{q} / \tilde{X}\} \wedge D_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} F}$.
Similar to the second alternative in Case 5, omitted.
In the remainder of this section, we shall prove that $\sqsubseteq_{R S}$ is indeed precongruent. Let us first recall a distinct but equivalent formulation of $\sqsubseteq_{R S}$ due to Van Glabbeek (Lüttgen and Vogler 2010).

Definition 6.2. A relation $\mathcal{R} \subseteq T\left(\Sigma_{\mathrm{CLL}_{R}}\right) \times T\left(\Sigma_{\mathrm{CLL}_{R}}\right)$ is an alternative ready simulation relation, if for any $(p, q) \in \mathcal{R}$ and $a \in A c t$
$\mathbf{( R S i )} p{\underset{ }{\epsilon}}_{F} \mid p^{\prime}$ implies $\exists q^{\prime} \cdot q \stackrel{\epsilon}{\epsilon}_{F} \mid q^{\prime}$ and $\left(p^{\prime}, q^{\prime}\right) \in \mathcal{R}$;
(RSiii) $p{ }^{a}{ }_{F} \mid p^{\prime}$ and $p, q$ stable implies $\exists q^{\prime} \cdot q{ }^{a}{ }_{F} \mid q^{\prime}$ and $\left(p^{\prime}, q^{\prime}\right) \in \mathcal{R}$;
(RSiv) $p \notin F$ and $p, q$ stable implies $\mathcal{I}(p)=\mathcal{I}(q)$.
We write $p \sqsubseteq_{A L T} q$ if there exists an alternative ready simulation relation $\mathcal{R}$ with $(p, q) \in \mathcal{R}$.

The next proposition reveals that this definition agrees with the one given in Def. 2.3 ,
Proposition 6.1. $\sqsubseteq_{R S}=\sqsubseteq_{A L T}$.
Proof. See Prop. 13 in (Lüttgen and Vogler 2010).
One advantage of Def. 6.2 is that, given $p$ and $q$, we can prove $p \sqsubseteq_{R S} q$ by means of giving an alternative ready simulation relation relating them. It is well known that up-to technique is a tractable way for such coinduction proof. Here we introduce the notion of an alternative ready relation up to $\sqsubset_{\sim}^{■_{R S}}$ as follows.
\left. Definition 6.3 (ALT up to ${\underset{\sim}{\sim}}_{R S}\right)$. A relation $\mathcal{R} \subseteq T\left(\Sigma_{\mathrm{CLL}_{R}}\right) \times T\left(\Sigma_{\mathrm{CLL}_{R}}\right)$ is an alternative ready simulation relation up to ${\underset{\sim}{\sim}}_{R S}$, if for any $(p, q) \in \mathcal{R}$ and $a \in$ Act

(ALT-upto-2) $p{ }^{a}{ }_{F} \mid p^{\prime}$ and $p, q$ stable implies $\exists q^{\prime} \cdot q{\underset{F}{a}}_{{ }_{F}} \mid q^{\prime}$ and $p^{\prime}{\underset{\sim}{\sim}}_{R S} \mathcal{R}{\underset{\sim}{\sim}}_{R S} q^{\prime} ;$ (ALT-upto-3) $p \notin F$ and $p, q$ stable implies $\mathcal{I}(p)=\mathcal{I}(q)$.

As usual, given a relation $\mathcal{R}$ satisfying the conditions (ALT-upto-1,2,3), in general, $\mathcal{R}$ in itself is not an alternative ready simulation relation. But simple result below ensures that up-to technique based on the above notion is sound.

Lemma 6.8. If a relation $\mathcal{R}$ is an alternative ready simulation relation up to ${\underset{\sim}{\sim}}_{R S}$ then $\mathcal{R} \subseteq \sqsubseteq_{R S}$.

Proof. By Prop. 6.1, it is sufficient to prove that the relation $\sqsubseteq_{R S} \circ \mathcal{R} \circ \sqsubseteq_{R S}$ is an alternative ready simulation. We leave it to the reader.

Now we are ready to prove the main result of this section: $\sqsubseteq_{R S}$ is precongruent w.r.t all operations in $\mathrm{CLL}_{R}$. We shall divide the proof into the next two lemmas.

Lemma 6.9. $C_{X}\{p / X\}={ }_{R S} C_{X}\{\tau \cdot p / X\}$ for any context $C_{X}$ and stable process $p$.
Proof. Let $p$ be any stable process. First, we shall show that $C_{X}\{p / X\} \sqsubseteq_{R S} C_{X}\{\tau . p / X\}$. Set

$$
\mathcal{R} \triangleq\left\{\left(B_{X}\{p / X\}, B_{X}\{\tau \cdot p / X\}\right): B_{X} \text { is a context }\right\}
$$

By Prop. 6.1 and Lemma 6.8, it is sufficient to prove that $\mathcal{R}$ is an alternative ready simulation relation up to $\underset{\sim}{\sqsubset_{R S}}$. Let $\left(C_{X}\{p / X\}, C_{X}\{\tau . p / X\}\right) \in \mathcal{R}$.
(ALT-upto-1) Assume that $C_{X}\{p / X\} \stackrel{\epsilon}{\Longrightarrow}{ }_{F} \mid p^{\prime}$. For this transition, since $p$ is stable, by Lemma 5.16, there exists a stable context $C_{X}^{\prime}$ such that

$$
\begin{equation*}
p^{\prime} \equiv C_{X}^{\prime}\{p / X\} \text { and } C_{X}\{\tau \cdot p / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X}^{\prime}\{\tau . p / X\} \tag{6.9}
\end{equation*}
$$

Moreover, by Lemma 5.17 it follows from $\tau . p \xrightarrow{\tau} \mid p$ that

$$
\begin{equation*}
C_{X}^{\prime}\{\tau \cdot p / X\} \stackrel{\epsilon}{\Longrightarrow} \mid r \text { for some } r \tag{6.9}
\end{equation*}
$$

For this transition, by Lemma 5.16, there exists a context $C_{X, \widetilde{Y}}^{\prime \prime}$ with $X \notin \widetilde{Y}$ such that
$r \equiv C_{X, \widetilde{Y}}^{\prime \prime}\{\tau . p / X, p / \widetilde{Y}\}$ and

$$
\begin{equation*}
p^{\prime} \equiv C_{X}^{\prime}\{p / X\} \Rightarrow C_{X, \widetilde{Y}}^{\prime \prime}\{p / X, p / \widetilde{Y}\} \tag{6.9,3}
\end{equation*}
$$

Since $p^{\prime} \notin F$, by Lemma 5.14, we get $C_{X, \tilde{Y}}^{\prime \prime}\{p / X, p / \tilde{Y}\} \notin F$. Further, by Lemma 6.3, $r \equiv C_{X, \widetilde{Y}}^{\prime \prime}\{\tau . p / X, p / \widetilde{Y}\} \notin F$. So, by (6.9, 1), (6.9,2) and Lemma 4.2, we obtain

$$
C_{X}\{\tau \cdot p / X\} \stackrel{\epsilon}{\epsilon}_{F} \mid C_{X, \tilde{Y}}^{\prime \prime}\{\tau \cdot p / X, p / \tilde{Y}\} .
$$

Moreover, by Lemma 5.15, it follows from (6.9,3) that

$$
p^{\prime}{\underset{\sim}{\square}}^{\overbrace{S}} C_{X, \widetilde{Y}}^{\prime \prime}\{p / X, p / \widetilde{Y}\} \mathcal{R} C_{X, \widetilde{Y}}^{\prime \prime}\{\tau \cdot p / X, p / \widetilde{Y}\}
$$

(ALT-upto-2) Assume that $C_{X}\{p / X\}$ and $C_{X}\{\tau . p / X\}$ are stable and $C_{X}\{p / X\} \stackrel{a}{\Longrightarrow}{ }_{F}$ $\mid p^{\prime}$. Hence $C_{X}\{p / X\} \xrightarrow{a}_{F} r \stackrel{\epsilon}{\Longrightarrow}_{F} \mid p^{\prime}$ for some $r$. Moreover, by Lemma6.3 and $C_{X}\{p / X\} \notin$ $F$, we have

$$
\begin{equation*}
C_{X}\{\tau \cdot p / X\} \notin F . \tag{6.9,4}
\end{equation*}
$$

Next we intend to prove that $p$ does not involve in the transition $C_{X}\{p / X\}{ }^{a}{ }_{F} r$. For this transition, by Lemma [5.8, there exist $C_{X}^{\prime}, C_{X, \widetilde{Y}}^{\prime}$ and $C_{X, \widetilde{Y}}^{\prime \prime}$ that realize (CP-a-1)-(CP-a-4). By (CP-a-1) and (CP-a-3-i), we have

$$
C_{X}\{\tau \cdot p / X\} \Rightarrow C_{X}^{\prime}\{\tau \cdot p / X\} \equiv C_{X, \widetilde{Y}}^{\prime}\{\tau \cdot p / X, \tau \cdot p / \widetilde{Y}\}
$$

If $\tilde{Y} \neq \emptyset$ then, by (CP-a-2) and Lemma 5.4 we have $C_{X, \tilde{Y}}^{\prime}\{\tau \cdot p / X, \tau \cdot p / \widetilde{Y}\} \xrightarrow{\tau}$, and hence $C_{X}\{\tau . p / X\} \xrightarrow{\tau}$ by Lemma 5.11, which contradicts that $C_{X}\{\tau . p / X\}$ is stable. Thus $\widetilde{Y}=\emptyset$, as desired. So, $r \equiv C_{X, \widetilde{Y}}^{\prime \prime}\{p / X\}$ by (CP- $a-3$-ii) and

$$
\begin{equation*}
C_{X}\{\tau \cdot p / X\} \xrightarrow{a} C_{X, \tilde{Y}}^{\prime \prime}\{\tau \cdot p / X\} \text { by }\left(\text { CP-a-3-iii) and } C_{X}\{\tau \cdot p / X\} \not f^{\tau}\right. \tag{6.9,5}
\end{equation*}
$$

Moreover, by (ALT-upto-1), it follows from $\left(C_{X, \widetilde{Y}}^{\prime \prime}\{p / X\}, C_{X, \widetilde{Y}}^{\prime \prime}\{\tau \cdot p / X\}\right) \in \mathcal{R}$ and $r \equiv$ $C_{X, \tilde{Y}}^{\prime \prime}\{p / X\} \stackrel{\epsilon}{\Longrightarrow} F \mid p^{\prime}$ that $C_{X, \tilde{Y}}^{\prime \prime}\{\tau . p / X\}{\underset{\sim}{\epsilon}}_{F} \mid q^{\prime}$ and $p^{\prime}{\underset{\sim}{\square}}_{R S} \mathcal{R}{\underset{\sim}{\sim}}_{R S} q^{\prime}$ for some $q^{\prime}$. Moreover, we also have $C_{X}\{\tau \cdot p / X\} \stackrel{a}{\Longrightarrow}{ }_{F} \mid q^{\prime}$ due to (6.9.4) and (6.9,5), as desired.
(ALT-upto-3) Immediately follows from Lemma 6.2
Next we intend to prove $C_{X}\{\tau \cdot p / X\} \sqsubseteq_{R S} C_{X}\{p / X\}$. Set

$$
\mathcal{R} \triangleq\left\{\left(B_{X}\{\tau \cdot p / X\}, B_{X}\{p / X\}\right): B_{X} \text { is a context }\right\}
$$

Similarly, it is sufficient to prove that $\mathcal{R}$ is an alternative ready simulation relation up to ${\underset{\sim}{R S}}^{\sqsubset}$. Let $\left(C_{X}\{\tau \cdot p / X\}, C_{X}\{p / X\}\right) \in \mathcal{R}$. (ALT-upto-3) immediately follows from Lemma 6.2, In the following, we prove the other two conditions.
(ALT-upto-1) Assume that $C_{X}\{\tau \cdot p / X\} \stackrel{\epsilon}{\Longrightarrow}{ }_{F} \mid p^{\prime}$. For this transition, by Lemma 5.19, there exist $r$ and stable context $C_{X}^{*}$ such that $C_{X}\{p / X\} \xlongequal{\epsilon} C_{X}^{*}\{p / X\}$ and

$$
C_{X}\{\tau . p / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X}^{*}\{\tau . p / X\} \stackrel{\epsilon}{\Longrightarrow} \mid r \Rightarrow p^{\prime}
$$

Moreover, since $p$ is stable, so is $C_{X}^{*}\{p / X\}$ by Lemma 5.6. Due to $r \Rightarrow p^{\prime}$ and $p^{\prime} \notin F$, by Lemma 5.14, we get $r \notin F$. Hence $C_{X}^{*}\{\tau . p / X\} \notin F$ by (6.9, 6) and Lemma 4.2, Then $C_{X}^{*}\{p / X\} \notin F$ by Lemma 6.5. Thus

$$
C_{X}\{p / X\}{\underset{ }{\epsilon}}_{F} \mid C_{X}^{*}\{p / X\} .
$$

To complete the proof, it remains to prove that $p^{\prime}{\underset{\sim}{\sim}}_{R S} \mathcal{R} \underset{\sim_{R S}}{\sqsubset_{X}} C_{X}^{*}\{p / X\}$. For the transition $C_{X}^{*}\{\tau \cdot p / X\} \stackrel{\epsilon}{\Longrightarrow} \mid r$ in (6.9] 6 ), by Lemma 5.16 , there exists a stable context $C_{X, \widetilde{Y}}^{\prime *}$ such that $r \equiv C_{X, \widetilde{Y}}^{* *}\{\tau \cdot p / X, p / \widetilde{Y}\} \Rightarrow p^{\prime}$ and $C_{X}^{*}\{p / X\} \Rightarrow C_{X, \widetilde{Y}}^{\prime *}\{p / X, p / \widetilde{Y}\}$, which, by Lemma 5.15, implies

$$
p^{\prime} \sqsubset_{\sim}^{\sqsubset_{R S}} C_{X, \tilde{Y}}^{\prime *}\{\tau \cdot p / X, p / \widetilde{Y}\} \mathcal{R} C_{X, \tilde{Y}}^{\prime *}\{p / X, p / \widetilde{Y}\} \underset{\sim}{\sqsubset_{R S}} C_{X}^{*}\{p / X\}
$$

(ALT-upto-2) Assume that $C_{X}\{\tau \cdot p / X\}$ and $C_{X}\{p / X\}$ are stable and $C_{X}\{\tau . p / X\} \xlongequal{a}{ }_{F}$ $\mid p^{\prime}$. Hence $C_{X}\{\tau \cdot p / X\} \xrightarrow{a}{ }_{F} r \stackrel{\epsilon}{\Longrightarrow} \mid p^{\prime}$ for some $r$. Moreover, by Lemma 6.5 and $C_{X}\{\tau . p / X\} \notin F$, we have

$$
C_{X}\{p / X\} \notin F .
$$

For the $a$-labelled transition $C_{X}\{\tau \cdot p / X\} \xrightarrow{a} F r$, by Lemma 5.8, it is not difficult to see that there exists $C_{X}^{\prime}$ such that

$$
C_{X}\{\tau \cdot p / X\} \xrightarrow{a} C_{X}^{\prime}\{\tau \cdot p / X\} \equiv r \text { and } C_{X}\{p / X\} \xrightarrow{a} C_{X}^{\prime}\{p / X\} .
$$

Moreover, by (ALT-upto-1), it follows from $\left(C_{X}^{\prime}\{\tau \cdot p / X\}, C_{X}^{\prime}\{p / X\}\right) \in \mathcal{R}$ and $r \equiv$ $C_{X}^{\prime}\{\tau . p / X\} \stackrel{\epsilon}{\Longrightarrow} F \mid p^{\prime}$ that $C_{X}^{\prime}\{p / X\} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid q^{\prime}$ and $p^{\prime}{\underset{\sim}{\sim}}_{R S} \mathcal{R}_{\sim_{R S}}^{\sqsubset_{R}} q^{\prime}$ for some $q^{\prime}$. Clearly, we have $C_{X}\{p / X\}{ }^{a}{ }_{F} \mid q^{\prime}$, as desired.

Lemma 6.10. If $\widetilde{p} \bowtie \widetilde{q}$ and $\widetilde{p} \sqsubseteq_{R S} \widetilde{q}$ then $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \sqsubseteq_{R S} C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\}$ for any $C_{\widetilde{X}}$.
Proof. Set

$$
\mathcal{R} \triangleq\left\{\left(B_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}, B_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\}\right): \widetilde{p} \bowtie \widetilde{q}, \widetilde{p} \sqsubseteq_{R S} \widetilde{q} \text { and } B_{\widetilde{X}} \text { is a context }\right\} .
$$

Similarly, it suffices to prove that $\mathcal{R}$ is an alternative ready simulation relation up to ${\underset{\sim}{\sim}}_{R S}$. Suppose $\left(C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}, C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\}\right) \in \mathcal{R}$. Then, by Lemma 6.2 it is obvious that such pair satisfies the condition (ALT-upto-3). In the following, we consider two remaining conditions in turn.
(ALT-upto-1) Assume that $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow}{ }_{F} \mid s$. For this transition, by Lemma 5.16, there exist $C_{\tilde{X}, \widetilde{Y}}^{\prime}$ and $i_{Y} \leq|\widetilde{X}|, p_{Y}^{\prime}(Y \in \widetilde{Y})$ that satisfy (MS- $\left.\tau-1\right)$ - (MS- $\tau-7$ ). In particular, by (MS- $\tau-2,7$ ), we have

$$
p_{i_{Y}} \xlongequal{\tau} \mid p_{Y}^{\prime} \text { for each } Y \in \widetilde{Y} \text { and } s \equiv C_{\tilde{X}, \tilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\} \notin F
$$

Then, by (MS- $\tau-1$ ) and Lemma 5.5 $p_{Y}^{\prime} \notin F$ and hence $p_{i_{Y}}{ }^{\tau}{ }_{F} \mid p_{Y}^{\prime}$ by Lemma 4.2 for each $Y \in \widetilde{Y}$. Since $\widetilde{p} \bowtie \widetilde{q}$, it follows from $\widetilde{p} \sqsubseteq_{R S} \widetilde{q}$ that there exist $q_{Y}^{\prime}(Y \in \widetilde{Y})$ such that

$$
\begin{equation*}
q_{i_{Y}} \stackrel{\tau}{\Longrightarrow}_{F} \mid q_{Y}^{\prime} \text { and } p_{Y}^{\prime}{\underset{\sim}{\sqcap}}_{R S} q_{Y}^{\prime} \text { for each } Y \in \widetilde{Y} . \tag{6.10.1}
\end{equation*}
$$

So, by (MS- $\tau$-3-ii), we get

$$
C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}, \tilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} .
$$

Moreover, by Lemma55.6, it follows from $s \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\} \not{ }^{\tau} \xrightarrow{\longrightarrow}, \widetilde{p} \bowtie \widetilde{q}$ and $\widetilde{p_{Y}^{\prime}}{\underset{\sim}{\sim}}_{R S}$ $\widetilde{q_{Y}^{\prime}}$ that

$$
C_{\widetilde{X}, \tilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} \not \not^{\tau}
$$

In addition, by Lemma 6.7 and $C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\} \notin F$, we get $C_{\widetilde{X}, \tilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} \notin F$. Hence, by Lemma 4.2, we obtain

$$
C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} .
$$

Clearly, $\left(C_{\widetilde{X}, \tilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \tilde{Y}\right\}, C_{\widetilde{X}, \tilde{Y}}^{\prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\}\right) \in{\underset{\sim}{R S}} \mathcal{R}{\underset{\sim}{r}}_{R S}$ due to (6.10, 1) and the reflexivity of ${\underset{\sim}{\sim}}_{R S}$.
(ALT-upto-2) Let $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\}$ and $C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\}$ be stable and $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \stackrel{a}{\Longrightarrow}{ }_{F} \mid s$. Then

$$
\begin{equation*}
C_{\tilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a}_{F} r \xrightarrow{\epsilon}_{F} \mid s \text { for some } r . \tag{6.10,2}
\end{equation*}
$$

Moreover, by Lemma 6.7 it follows from $\widetilde{p} \bowtie \widetilde{q}, \widetilde{p} \sqsubseteq_{R S} \widetilde{q}$ and $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$ that

$$
C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \notin F
$$

(6.10 3)

For the transition $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \xrightarrow{a} r$, by Lemma 5.8, there exist $C_{\widetilde{X}}^{\prime}, C_{\widetilde{X}, \widetilde{Y}}^{\prime}$ and $C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}$ that satisfy (CP-a-1) - (CP-a-4). In particular, by (CP-a-3-ii), there exist $i_{Y} \leq|\widetilde{X}|, p_{Y}^{\prime}(Y \in$ $\widetilde{Y})$ such that $p_{i_{Y}} \xrightarrow{a} p_{Y}^{\prime}$ for each $Y \in \widetilde{Y}$ and $r \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\}$. Moreover, by (CP-a-1) and (CP-a-3-i), we have

$$
C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \Rightarrow C_{\widetilde{X}}^{\prime}\{\widetilde{p} / \widetilde{X}\} \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{i_{Y}}} / \tilde{Y}\right\}
$$

Hence $C_{\widetilde{X}, \widetilde{Y}}^{\prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{i_{Y}}} / \widetilde{Y}\right\} \notin F$ by $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \notin F$ and Lemma 5.14, Further, since each $Y(\in \widetilde{Y})$ is 1-active in $C_{\tilde{X}, \tilde{Y}}^{\prime}$, by Lemma 5.5, we get

$$
\begin{equation*}
p_{i_{Y}} \notin F \text { for each } Y \in \widetilde{Y} \tag{6.10,4}
\end{equation*}
$$

For the transition $r \equiv C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\} \stackrel{\epsilon}{\Longrightarrow} \mid s$ in (6.10, 2), by Lemma 5.20, it follows that for each $Y \in \widetilde{Y}$, there exists $p_{Y}^{\prime \prime}$ such that $p_{Y}^{\prime} \xlongequal{\epsilon} \mid p_{Y}^{\prime \prime}$ and

$$
C_{\widetilde{X}, \tilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}, \tilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime \prime}} / \widetilde{Y}\right\} \stackrel{\epsilon}{\Longrightarrow} \mid s
$$

Then $C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime \prime}} / \widetilde{Y}\right\} \notin F$ due to $s \notin F$ and Lemma4.2, and hence $p_{Y}^{\prime \prime} \notin F$ for each $Y \in \tilde{Y}$ by Lemma 5.5. Therefore, by (6.10, 4) and Lemma 4.2 we have

$$
p_{i_{Y}} \xrightarrow{a}_{F} p_{Y}^{\prime} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid p_{Y}^{\prime \prime} \text { for each } Y \in \widetilde{Y} .
$$

So it follows from $\widetilde{p} \bowtie \widetilde{q}$ and $\widetilde{p} \sqsubseteq_{R S} \widetilde{q}$ that for each $Y \in \widetilde{Y}$, there exist $q_{Y}^{\prime}$ and $q_{Y}^{\prime \prime}$ such
that $q_{i_{Y}} \xrightarrow{a}_{F} q_{Y}^{\prime} \stackrel{\epsilon}{\epsilon}_{F} \mid q_{Y}^{\prime \prime}$ and $p_{Y}^{\prime \prime}{\underset{\sim}{\sim}}_{R S} q_{Y}^{\prime \prime}$. By (CP-a-3-iii), we get

$$
\begin{equation*}
C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\} \xrightarrow{a} C_{\widetilde{X}, \tilde{Y}}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} . \tag{6.105}
\end{equation*}
$$

Further, by Lemma 5.4 and (CP-a-2), we obtain

$$
\begin{equation*}
C_{\widetilde{X}, \tilde{Y}}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime}} / \widetilde{Y}\right\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}, \tilde{Y}}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime \prime}} / \widetilde{Y}\right\} \tag{6.106}
\end{equation*}
$$

Clearly, $\left(C_{\tilde{X}, \tilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime \prime}} / \widetilde{Y}\right\}, C_{\tilde{X}, \tilde{Y}}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime \prime}} / \widetilde{Y}\right\}\right) \in \mathcal{R}$. So, by $C_{\widetilde{X}, \tilde{Y}}^{\prime \prime}\left\{\widetilde{p} / \widetilde{X}, \widetilde{p_{Y}^{\prime \prime}} / \widetilde{Y}\right\} \xlongequal{\epsilon}{ }_{F}$ $\mid s$ and (ALT-upto-1), there exists $t$ such that $C_{\widetilde{X}, \widetilde{Y}}^{\prime \prime}\left\{\widetilde{q} / \widetilde{X}, \widetilde{q_{Y}^{\prime \prime}} / \widetilde{Y}\right\}{\underset{F}{\epsilon}}_{F} \mid t$ and $s{\underset{\sim}{~}}_{R S}$ $\mathcal{R}{\underset{\sim}{\sim}}_{R S} t$; moreover, we also have $C_{\widetilde{X}}\{\widetilde{q} / \tilde{X}\} \stackrel{a}{\Longrightarrow}_{F} \mid t$ due to (6.10 3), (6.10, 5), (6.10, 6) and Lemma 4.2, as desired.

We are now in a position to state the main result of this section.
Theorem 6.1 (Precongruence). If $p \sqsubseteq_{R S} q$ then $C_{X}\{p / X\} \sqsubseteq_{R S} C_{X}\{q / X\}$ for any context $C_{X}$.

Proof. By Lemma 6.9 and 6.10 it immediately follows from $\tau . p={ }_{R S} p \sqsubseteq_{R S} q={ }_{R S} \tau . q$.

As an immediate consequence of this theorem, we also have
Corollary 6.1. If $\widetilde{p} \sqsubseteq_{R S} \widetilde{q}$ then $C_{\widetilde{X}}\{\widetilde{p} / \widetilde{X}\} \sqsubseteq_{R S} C_{\widetilde{X}}\{\widetilde{q} / \widetilde{X}\}$ for any context $C_{\widetilde{X}}$.
Proof. Applying Theorem 6.1 finitely many times.

## 7. Unique solution of equations

This section focuses on the solutions of equations. Especially, we shall prove that the equation $X={ }_{R S} t_{X}$ has at most one consistent solution modulo $=_{R S}$ provided that $X$ is strongly guarded and does not occur in the scope of any conjunction in $t_{X}$; moreover, the process $\left\langle X \mid X=t_{X}\right\rangle$ is indeed the unique consistent solution whenever such equation has a consistent solution. We begin with giving two results on the inconsistency predicate $F$.

Lemma 7.1. For any stable processes $p, q \notin F$ and context $C_{X}$ such that $X$ does not occur in the scope of any conjunction, if $C_{X}\{p / X\} \in F$ then $C_{X}\{q / X\} \in F$.

Proof. Assume that $C_{X}\{p / X\} \in F$ and $\mathcal{T}$ is any proof tree of $\operatorname{Strip}\left(\mathcal{P}_{\operatorname{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash$ $C_{X}\{p / X\} F$. We proceed by induction on the depth of $\mathcal{T}$. The argument is a routine case analysis on $C_{X}$. Moreover, since $X$ does not occur in the scope of any conjunction, in addition to that $C_{X}$ is closed, the form of $C_{X}$ is one of the following: $X, \alpha . B_{X}, B_{X} \odot D_{X}$ with $\odot \in\left\{\vee, \square, \|_{A}\right\}$ and $\langle Y \mid E\rangle$. Here, we give the proof only for the case $C_{X} \equiv\langle Y \mid E\rangle$, the other cases are straightforward and omitted.

In case $C_{X} \equiv\langle Y \mid E\rangle$, the last rule applied in $\mathcal{T}$ is

$$
\text { either } \frac{\left\langle t_{Y} \mid E\right\rangle\{p / X\} F}{\langle Y \mid E\rangle\{p / X\} F} \text { with } Y=t_{Y} \in E \text { or } \frac{\{r F:\langle Y \mid E\rangle\{p / X\} \xlongequal{\epsilon} \mid r\}}{\langle Y \mid E\rangle\{p / X\} F} \text {. }
$$

For the first alternative, we have $\left\langle t_{Y} \mid E\right\rangle\{q / X\} \in F$ by IH, and hence $C_{X}\{q / X\} \equiv$ $\langle Y \mid E\rangle\{q / X\} \in F$.

For the second alternative, assume $\langle Y \mid E\rangle\{q / X\} \stackrel{\epsilon}{\Longrightarrow} \mid s$. Since $q$ is stable, by Lemma 5.16, $s \equiv C_{X}^{\prime}\{q / X\}$ for some stable $C_{X}^{\prime}$ such that $X$ does not occur in the scope of any conjunction in $C_{X}^{\prime}$ and $\langle Y \mid E\rangle\{p / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X}^{\prime}\{p / X\}$. Moreover, since $p$ is stable, so is $C_{X}^{\prime}\{p / X\}$. Thus there exists a proper subtree of $\mathcal{T}$ with root $C_{X}^{\prime}\{p / X\} F$. So, by IH, $s \equiv C_{X}^{\prime}\{q / X\} \in F$. Hence $C_{X}\{q / X\} \in F$ by Theorem4.2, as desired.

This result is of independent interest, but its principal use is that it will serve as an important step in demonstrating the next lemma, which reveals that the above result still holds if it is deleted from the hypotheses that $q$ and $p$ are stable.

Lemma 7.2. For any processes $p, q \notin F$ and context $C_{X}$ such that $X$ does not occur in the scope of any conjunction, if $C_{X}\{p / X\} \in F$ then $C_{X}\{q / X\} \in F$.

Proof. Suppose that $C_{X}\{p / X\} \in F$. We proceed by induction on the depth of the proof tree $\mathcal{T}$ of $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash C_{X}\{p / X\} F$. Similar to the preceding lemma, we handle only the case $C_{X} \equiv\langle Y \mid E\rangle$. In this situation, the last rule applied in $\mathcal{T}$ is

$$
\text { either } \frac{\left\langle t_{Y} \mid E\right\rangle\{p / X\} F}{\langle Y \mid E\rangle\{p / X\} F} \text { with } Y=t_{Y} \in E \text { or } \frac{\{r F:\langle Y \mid E\rangle\{p / X\} \xlongequal{\epsilon} \mid r\}}{\langle Y \mid E\rangle\{p / X\} F} \text {. }
$$

The argument for the former is the same as the one in Lemma 7.1 and omitted. In the following, we consider the latter and suppose $\langle Y \mid E\rangle\{q / X\} \xlongequal{\epsilon} \mid s$. By Theorem 4.2, it is not difficult to see that, to complete the proof, it suffices to prove that $s \in F$. By Lemma 5.19, there exist $t$ and stable context $C_{X}^{*}$ such that

$$
\langle Y \mid E\rangle\{q / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X}^{*}\{q / X\} \stackrel{\epsilon}{\Longrightarrow} \mid t \Rightarrow s
$$

and

$$
\begin{equation*}
\langle Y \mid E\rangle\{r / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X}^{*}\{r / X\} \text { for any } r . \tag{7.2,1}
\end{equation*}
$$

In particular, we have $\langle Y \mid E\rangle\left\{a_{X} .0 / X\right\} \xlongequal{\epsilon} C_{X}^{*}\left\{a_{X} .0 / X\right\}$ where $a_{X}$ is a fresh visible action. For this transition, applying Lemma 5.6 finitely many times (notice that, in this procedure, since $a_{X} .0$ is stable, the clause (2) in Lemma 5.6 is always false), and by the clause (1) in Lemma 5.6, we get the sequence

$$
\begin{aligned}
\langle Y \mid E\rangle\left\{a_{X} .0 / X\right\} \equiv C_{X}^{0}\left\{a_{X} .0 / X\right\} & \xrightarrow{\tau} C_{X}^{1}\left\{a_{X} .0 / X\right\} \\
& \xrightarrow{\tau} \\
& \ldots \xrightarrow{\tau} C_{X}^{n}\left\{a_{X} .0 / X\right\} \equiv C_{X}^{*}\left\{a_{X} .0 / X\right\} .
\end{aligned}
$$

Here $n \geq 0$ and for each $1 \leq i \leq n, C_{X}^{i}$ satisfies (C- $\tau-1,2,3$ ) in Lemma 5.6. Since $X$ does not occur in the scope of any conjunction in $\langle Y \mid E\rangle$, by (C- $\tau$-3-iv), neither does $X$ in $C_{X}^{n}$. In addition, by Lemma 5.18, we have $C_{X}^{n} \equiv C_{X}^{*}$. Hence $X$ does not occur in the scope of any conjunction in $C_{X}^{*}$.

If $p$ is stable then so is $C_{X}^{*}\{p / X\}$ by Lemma 5.6. Thus, by (7.2 1 ), $C_{X}^{*}\{p / X\} F$ is one of premises in the last inferring step in $\mathcal{T}$. Hence $C_{X}^{*}\{q / X\} \in F$ by applying IH. Then $t \in F$ by Lemma 4.2. Further, by Lemma 5.14 it follows from $t \Rightarrow s$ that $s \in F$, as desired.

Next we consider the other case where $p$ is not stable. In this situation, due to $p \notin F$, we have

$$
\begin{equation*}
p \xlongequal{\tau}_{F} \mid p^{*} \text { for some } p^{*} . \tag{7.2,2}
\end{equation*}
$$

In the following, we distinguish two cases based on whether $q$ is stable.

Case $1 q$ is stable.
Then, for the transition $\langle Y \mid E\rangle\{q / X\} \stackrel{\epsilon}{\Longrightarrow} \mid s$, by Lemma 5.16, we have $s \equiv C_{X}^{\prime}\{q / X\}$ for some stable $C_{X}^{\prime}$ such that $X$ does not occur in the scope of any conjunction and $C_{X}\{p / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X}^{\prime}\{p / X\}$. Moreover, by Lemma 5.17, it follows from (7.2, 2) that

$$
C_{X}^{\prime}\{p / X\} \xlongequal{\epsilon} \mid p^{\prime} \text { for some } p^{\prime}
$$

For this transition, by Lemma5.16, there exist a stable context $C_{X, \widetilde{Y}}^{* *}$ and stable processes $p_{Y}^{\prime}(Y \in \widetilde{Y})$ that realize (MS- $\left.\tau-1\right)-(\mathrm{MS}-\tau-7)$. In particular, by (MS- $\tau$ - 3 -ii), it follows from (7.2,2) that

$$
C_{X}^{\prime}\{p / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X, \tilde{Y}}^{* *}\left\{p / X, p^{*} / \tilde{Y}\right\} .
$$

Then, since $C_{X, \tilde{Y}}^{* *}, p$ and $p^{*}$ are stable, by Lemma 5.6, so is $C_{X, \tilde{Y}}^{* *}\left\{p / X, p^{*} / \widetilde{Y}\right\}$. Thus, $C_{X, \widetilde{Y}}^{* *}\left\{p / X, p^{*} / \widetilde{Y}\right\} F$ is one of premises of the last inferring step in $\mathcal{T}$. Moreover, by (MS-$\tau-2), p^{\prime} \equiv C_{X, \widetilde{Y}}^{*}\left\{p / X, \widetilde{p_{Y}^{\prime}} / \widetilde{Y}\right\}$. Then, by (MS- $\left.\tau-6\right)$ and IH, we obtain

$$
C_{X, \tilde{Y}}^{\prime *}\left\{q / X, p^{*} / \widetilde{Y}\right\} \in F
$$

Further, by (MS- $\tau-6$ ) and Lemma 7.1, we get

$$
C_{X, \tilde{Y}}^{\prime *}\{q / X, q / \widetilde{Y}\} \in F
$$

In addition, due to the stableness of $C_{X}^{\prime}$, by (MS- $\left.\tau-4\right)$, we have

$$
C_{X}^{\prime}\{q / X\} \Rightarrow C_{X, \widetilde{Y}}^{\prime *}\{q / X, q / \widetilde{Y}\}
$$

Hence $s \equiv C_{X}^{\prime}\{q / X\} \in F$ by Lemma 5.14, as desired.
Case $2 q$ is not stable.
By Lemma 5.16, for the transition $\langle Y \mid E\rangle\{q / X\} \xlongequal{\epsilon} \mid s$, there exist a stable context $C_{X, \widetilde{Z}}^{\prime}$ and $q_{Z}^{\prime}(Z \in \widetilde{Z})$ that satisfy (MS- $\left.\tau-1\right)-(\mathrm{MS}-\tau-7)$. Amongst them, by (MS- $\left.\tau-2,7\right)$,

$$
q \xlongequal{\tau} \mid q_{Z}^{\prime} \text { for each } Z \in \widetilde{Z} \text { and } s \equiv C_{X, \widetilde{Z}}^{\prime}\left\{q / X, \widetilde{q_{Z}^{\prime}} / \widetilde{Z}\right\}
$$

If $q_{Z}^{\prime} \in F$ for some $Z \in \widetilde{Z}$ then by Lemma 5.5, we get $s \in F$ (notice that each $Z$ in $\widetilde{Z}$ is 1-active), as desired. In the following, we handle the other case where

$$
\begin{equation*}
q_{Z}^{\prime} \notin F \text { for each } Z \in \widetilde{Z} \tag{7.2,4}
\end{equation*}
$$

By (MS- $\tau$-3-ii), it follows from (7.2,2) that

$$
C_{X}\{p / X\} \stackrel{\epsilon}{\Longrightarrow} C_{X, \widetilde{Z}}^{\prime}\left\{p / X, p^{*} / \widetilde{Z}\right\}
$$

Since $p \xrightarrow{\tau}, q \xrightarrow{\tau}, p^{*} f^{\tau}, q_{Z}^{\prime} f^{\tau}$ for each $Z \in \widetilde{Z}$ and $s \equiv C_{X, \widetilde{Z}}^{\prime}\left\{q / X, \widetilde{q_{Z}^{\prime}} / \widetilde{Z}\right\}{ }^{\tau}$,
by Lemma 5.6, $C_{X, \widetilde{Z}}^{\prime}\left\{p / X, p^{*} / \widetilde{Z}\right\}$ is stable. Hence $\mathcal{T}$ has a proper subtree with root $C_{X, \widetilde{Z}}^{\prime}\left\{p / X, p^{*} / \widetilde{Z}\right\} F$. Then $C_{X, \widetilde{Z}}^{\prime}\left\{q / X, p^{*} / \widetilde{Z}\right\} \in F$ by (MS- $\tau-6$ ) and IH. Further, by Lemma 7.1, it follows from (7.2, 3) and (7.2, 4) that $s \equiv C_{X, \widetilde{Z}}^{\prime}\left\{q / X, \widetilde{q_{Z}^{\prime}} / \widetilde{Z}\right\} \in F$, as desired.

We shall use the notation $\operatorname{Dep}(\mathcal{T})$ to denote the depth of a given proof tree $\mathcal{T}$. Given $p, q$ and $\alpha \in A c t_{\tau}$, for any proof tree $\mathcal{T}$ of $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash p \xrightarrow{\alpha} q$, it is evident that $\mathcal{T}$ involves only rules in Table 11 Moreover, since each rule in Table 1 has only finitely many premises, it is not difficult to show that $\operatorname{Dep}(\mathcal{T})<\omega$ by induction on the depth of $\mathcal{T}$. This makes it legitimate to use arithmetical expressions with the form like $\sum_{\mathcal{T} \in \Omega} \operatorname{Dep}(\mathcal{T})$ where $\Omega$ is a finite set and each $\mathcal{T} \in \Omega$ is a proof tree for some labelled transition $p \xrightarrow{\alpha} r$.

Definition 7.1. Given $p{ }_{F}^{\epsilon} q$ and a finite set $\Omega$ of proof trees, we say that $\Omega$ is a proof forest for $p{ }^{\epsilon}{ }_{F} q$ if there exist $p_{i}(0 \leq i \leq n)$ such that
(1) $p \equiv p_{0} \xrightarrow{\tau}_{F} p_{1} \xrightarrow{\tau}_{F} \cdots \xrightarrow{\tau}{ }_{F} p_{n} \equiv q$,
(2) for each $i<n, \Omega$ contains exactly one proof tree for $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash p_{i} \xrightarrow{\tau}$ $p_{i+1}$, and
(3) for each $\mathcal{T} \in \Omega, \mathcal{T}$ is a proof tree for $\operatorname{Strip}\left(\mathcal{P}_{\operatorname{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash p_{i} \xrightarrow{\tau} p_{i+1}$ for some $i<n$.

The depth of $\Omega$ is defined as $\operatorname{Dep}(\Omega) \triangleq \sum_{\mathcal{T} \in \Omega} \operatorname{Dep}(\mathcal{T})$. Similarly, we may define the notion of a proof forest for $p{ }^{a}{ }_{F} q$.

It is obvious that $p{ }^{\epsilon}{ }_{F} q$ (or, $p{ }^{a}{ }_{F} q$ ) holds if and only if there exists a proof forest for it. The following lemma will prove extremely useful in establishing the main result in this section and its proof involves induction on the depths of proof forests.

Lemma 7.3. Let $C_{X}$ be any context where $X$ is strongly guarded and does not occur in the scope of any conjunction. For any processes $p, q \notin F$ with $p \bowtie q$, if $p={ }_{R S} C_{X}\{p / X\}$ and $q={ }_{R S} C_{X}\{q / X\}$ then $p={ }_{R S} q$.

Proof. Suppose $p, q \notin F$ with $p \bowtie q, p={ }_{R S} C_{X}\{p / X\}$ and $q={ }_{R S} C_{X}\{q / X\}$. It is sufficient to prove that $p \sqsubseteq_{R S} q$. Put

$$
\mathcal{R} \triangleq\left\{\left(B_{X}\{p / X\}, B_{X}\{q / X\}\right): X \text { does not occur in the scope of any conjunction in } B_{X}\right\} .
$$

By Prop. 6.1 and Lemma 6.8, it suffices to prove that $\mathcal{R}$ is an alternative ready simulation relation up to ${\underset{\sim}{\sim}}_{R S}$. Let $\left(B_{X}\{p / X\}, B_{X}\{q / X\}\right) \in \mathcal{R}$.
(ALT-upto-1) Assume that $B_{X}\{p / X\} \xrightarrow{\epsilon}{ }_{F} \mid p^{\prime}$ and $\Omega$ is any proof forest for it. Hence $B_{X}\{p / X\} \equiv p_{0} \xrightarrow{\tau}{ }_{F} p_{1} \xrightarrow{\tau} \ldots p_{n-1} \xrightarrow{\tau}_{F} \mid p_{n} \equiv p^{\prime}$ for some $p_{i}(0 \leq i \leq n)$, and $\Omega$ exactly consists of proof trees $\mathcal{T}_{i}(0 \leq i<n)$ for $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash p_{i} \xrightarrow{\tau} p_{i+1}$. We intend to prove that there exists $q^{\prime}$ such that $B_{X}\{q / X\}{\underset{\sim}{\epsilon}}_{F} \mid q^{\prime}$ and $p^{\prime}{\underset{\sim}{\sim}}_{R S} \mathcal{R}{\underset{\sim}{\sim}}_{R S} q^{\prime}$ by induction on $\operatorname{Dep}(\Omega)$. It is a routine case analysis on $B_{X}$. We treat only three cases

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as examples.
Case $1 B_{X} \equiv X$.
Then $B_{X}\{p / X\} \equiv p \xrightarrow{\epsilon}{ }_{F} \mid p^{\prime}$. Thus it follows from $p={ }_{R S} C_{X}\{p / X\}$ that

$$
C_{X}\{p / X\} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid s \text { and } p^{\prime}{\underset{\sim}{\square}}^{\sqsubset_{R}} \text { s for some } s .
$$

Since $X$ is strongly guarded and does not occur in the scope of any conjunction in $C_{X}$, by Lemma 5.16. there exists a stable context $C_{X}^{\prime}$ such that
(a.1) $s \equiv C_{X}^{\prime}\{p / X\}$,
(a.2) $X$ is strongly guarded and does not occur in the scope of any conjunction in $C_{X}^{\prime}$, and
(a.3) $C_{X}\{q / X\} \xlongequal{\epsilon} C_{X}^{\prime}\{q / X\}$.

Since $s \equiv C_{X}^{\prime}\{p / X\} \stackrel{\tau}{\not}$, by (a.2) and Lemma 5.9, we have $C_{X}^{\prime}\{q / X\} \stackrel{\tau}{\nrightarrow}$. Moreover, by Lemma 7.2, $C_{X}^{\prime}\{q / X\} \notin F$ follows from $C_{X}^{\prime}\{p / X\} \notin F$ and $p, q \notin F$. Hence $C_{X}\{q / X\}{ }^{\epsilon}{ }_{F} \mid C_{X}^{\prime}\{q / X\}$ by (a.3) and Lemma 4.2. Further, it follows from $q={ }_{R S}$ $C_{X}\{q / X\}$ that

$$
q \stackrel{\epsilon}{\Longrightarrow}_{F} \mid q^{\prime} \text { and } C_{X}^{\prime}\{q / X\} \underset{\sim}{\sqsubset_{R S}} q^{\prime} \text { for some } q^{\prime} .
$$

Therefore, $B_{X}\{q / X\} \equiv q{\underset{\sim}{\epsilon}}_{F} \mid q^{\prime}$ and $p^{\prime}{\underset{\sim}{\square}}_{R S} s \equiv C_{X}^{\prime}\{p / X\} \mathcal{R} C_{X}^{\prime}\{q / X\} \underset{\sim}{\sqsubset_{R S}} q^{\prime}$.
Case $2 B_{X} \equiv\langle Y \mid E\rangle$.
If $\langle Y \mid E\rangle\{p / X\}$ is stable then so is $\langle Y \mid E\rangle\{q / X\}$ by $p \bowtie q$ and Lemma 5.6. Moreover, by Lemma 7.2, we have $\langle Y \mid E\rangle\{q / X\} \notin F$ because of $\langle Y \mid E\rangle\{p / X\} \notin F$. Hence $\langle Y \mid E\rangle\{q / X\} \xlongequal{\epsilon}_{F} \mid\langle Y \mid E\rangle\{q / X\}$ and $(\langle Y \mid E\rangle\{p / X\},\langle Y \mid E\rangle\{q / X\}) \in_{\sim_{R S}} \mathcal{R}{\underset{\sim}{\sim}}_{R S}$ due to the reflexivity of ${\underset{\sim}{\sim}}_{R S}$.

Next we handle the other case where $\langle Y \mid E\rangle\{p / X\}$ is not stable. Clearly, the last rule applied in $\mathcal{T}_{0}$ is

$$
\frac{\left\langle t_{Y} \mid E\right\rangle\{p / X\} \xrightarrow{\tau} p_{1}}{\langle Y \mid E\rangle\{p / X\} \xrightarrow{\tau} p_{1}} \text { with } Y=t_{Y} \in E .
$$

Thus, $\mathcal{T}_{0}$ contains a proper subtree, say $\mathcal{T}_{0}^{\prime}$, which is a proof tree of $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash$ $\left\langle t_{Y} \mid E\right\rangle\{p / X\} \xrightarrow{\tau} p_{1}$ and $\operatorname{Dep}\left(\mathcal{T}_{0}^{\prime}\right)<\operatorname{Dep}\left(\mathcal{T}_{0}\right)$. Thus $\Omega^{\prime} \triangleq\left\{\mathcal{T}_{0}^{\prime}, \mathcal{T}_{i}: 1 \leq i \leq n-1\right\}$ is a proof forest for $\left\langle t_{Y} \mid E\right\rangle\{p / X\} \stackrel{\epsilon}{\Longrightarrow}{ }_{F} \mid p^{\prime}$; moreover

$$
\operatorname{Dep}\left(\Omega^{\prime}\right)<\operatorname{Dep}(\Omega)
$$

Then, by Lemma $\boxed{5.2}(5)$ and IH, we have $\left\langle t_{Y} \mid E\right\rangle\{q / X\} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid q^{\prime}$ and $p^{\prime}{\underset{\sim}{\sim}}_{R S} \mathcal{R}{\underset{\sim}{\sim}}_{R S} q^{\prime}$ for some $q^{\prime}$. Moreover, we also have $B_{X}\{q / X\} \equiv\langle Y \mid E\rangle\{q / X\}{ }^{\epsilon}{ }_{F} \mid q^{\prime}$, as desired.

Case $3 B_{X} \equiv D_{X} \square D_{X}^{\prime}$.
If $B_{X}\{p / X\}$ is stable then we can proceed analogously to Case 2 with $\langle Y \mid E\rangle\{p / X\} \not{ }^{\tau} \rightarrow$. In the following, we consider the case $B_{X}\{p / X\} \xrightarrow{\tau}$.

For the transitions $D_{X}\{p / X\} \square D_{X}^{\prime}\{p / X\} \equiv p_{0} \xrightarrow{\tau}_{F} \cdots \xrightarrow{\tau}_{F} \mid p_{n} \equiv p^{\prime}(n \geq 1)$, there exist two sequences of processes $t_{0}\left(\equiv D_{X}\{p / X\}\right), \ldots, t_{n}$ and $s_{0}\left(\equiv D_{X}^{\prime}\{p / X\}\right), \ldots, s_{n}$
such that $t_{n}, s_{n}$ are consistent and stable, $p_{n} \equiv t_{n} \square s_{n}$, and for each $0 \leq i<n, p_{i} \equiv t_{i} \square s_{i}$ and the last rule applied in $\mathcal{T}_{i}$ is

$$
\text { either } \frac{t_{i} \xrightarrow{\tau} t_{i+1}}{t_{i} \square s_{i} \xrightarrow{\tau} t_{i+1} \square s_{i+1}} \text { or } \frac{s_{i} \xrightarrow{\tau} s_{i+1}}{t_{i} \square s_{i} \xrightarrow{\tau} t_{i+1} \square s_{i+1}} \text {. }
$$

For the former, $s_{i+1} \equiv s_{i}$ and $\mathcal{T}_{i}$ contains a proper subtree $\mathcal{T}_{i}^{\prime}$ which is a proof tree for $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash t_{i} \xrightarrow{\tau} t_{i+1}$. We use $\Omega_{1}$ to denote the (finite) set of all these proof trees $\mathcal{T}_{i}^{\prime}$. Similarly, for the latter, $t_{i+1} \equiv t_{i}$ and $\mathcal{T}_{i}$ contains a proper subtree $\mathcal{T}_{i}^{\prime \prime}$ which is a proof tree for $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash s_{i} \xrightarrow{\tau} s_{i+1}$. We use $\Omega_{2}$ to denote the (finite) set of all these proof trees $\mathcal{T}_{i}^{\prime \prime}$. It is obvious that $\Omega_{1}$ is a proof forest for $D_{X}\{p / X\} \xlongequal{\epsilon}{ }_{F} \mid t_{n}$; moreover,

$$
\operatorname{Dep}\left(\Omega_{1}\right)<\operatorname{Dep}(\Omega)
$$

Thus, by IH, we have $D_{X}\{q / X\} \stackrel{\epsilon}{\Longrightarrow} F \mid q_{1}^{\prime}$ and $t_{n}{\underset{\sim}{\sim}}_{\sqsubset_{R S}} \mathcal{R}{\underset{\sim}{\square}}_{R S} q_{1}^{\prime}$ for some $q_{1}^{\prime}$. Similarly, for the transition $D_{X}^{\prime}\{p / X\} \stackrel{\epsilon}{\Longrightarrow} F \mid s_{n}$, we also have $D_{X}^{\prime}\{q / X\} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid q_{2}^{\prime}$ and $s_{n}{\underset{\sim}{\sim}}_{R S}$ $\mathcal{R} \sqsubset_{\sim}^{\sim_{R S}} q_{2}^{\prime}$ for some $q_{2}^{\prime}$. Then, by Theorem4.3. it is easy to check that $p^{\prime} \equiv t_{n} \square s_{n}{\underset{\sim}{\sim}}_{\sim}^{\sim}{ }_{R S}$ $\mathcal{R} \sqsubset_{R S} q_{1}^{\prime} \square q_{2}^{\prime}$. Moreover, we also have $B_{X}\{q / X\} \equiv D_{X}\{q / X\} \square D_{X}^{\prime}\{q / X\} \stackrel{\epsilon}{\Longrightarrow}{ }_{F} \mid q_{1}^{\prime} \square q_{2}^{\prime}$.
(ALT-upto-2) Suppose that $B_{X}\{p / X\}$ and $B_{X}\{q / X\}$ are stable. Let $B_{X}\{p / X\}{ }^{a}{ }_{F}$ $\mid p^{\prime}$ and $\Omega$ be its proof forest. So, there exist $p_{0}, \ldots, p_{n}(n \geq 1)$ such that

$$
\begin{equation*}
B_{X}\{p / X\} \equiv p_{0} \xrightarrow{a}_{F} p_{1} \xrightarrow{\tau}_{F} \cdots \xrightarrow{\tau}_{F} \mid p_{n} \equiv p^{\prime}, \tag{7.3,1}
\end{equation*}
$$

and $\Omega$ exactly consists of proof trees $\mathcal{T}_{i}$ for $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash p_{i} \xrightarrow{\alpha_{i}} p_{i+1}$ for $i<n$, where $\alpha_{0}=a$ and $\alpha_{j}=\tau(1 \leq j<n)$. We want to prove that there exists $q^{\prime}$ such that $B_{X}\{q / X\} \stackrel{a}{\Longrightarrow}{ }_{F} \mid q^{\prime}$ and $p^{\prime}{\underset{\sim}{\sim}}_{R S} \mathcal{R}{\underset{\sim}{\sim}}_{R S} q^{\prime}$ by induction on $\operatorname{Dep}(\Omega)$. Since $B_{X}\{p / X\}$ is stable and $X$ does not occur in the scope of any conjunction in $B_{X}$, the topmost operator of $B_{X}$ is neither disjunction nor conjunction. Thus, we distinguish five cases based on the form of $B_{X}$.

Case $1 B_{X} \equiv X$.
Due to $B_{X}\{p / X\} \equiv p \stackrel{a}{\Longrightarrow}{ }_{F} \mid p^{\prime}$, we have $p \notin F$. Moreover, since $p\left(\equiv B_{X}\{p / X\}\right)$ is stable, we get $p{ }^{\epsilon}{ }_{F} \mid p$. Hence it follows from $p={ }_{R S} C_{X}\{p / X\}$ that

$$
C_{X}\{p / X\} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid s \text { and } p{\underset{\sim}{\square}}_{R S} s \text { for some } s .
$$

Further, since $X$ is strongly guarded and does not occur in the scope of any conjunction in $C_{X}$, by Lemma 5.16. there exists a stable context $C_{X}^{\prime}$ such that
(b.1) $X$ is strongly guarded and does not occur in the scope of any conjunction in $C_{X}^{\prime}$,
(b.2) $s \equiv C_{X}^{\prime}\{p / X\}$, and
(b.3) $C_{X}\{q / X\} \xlongequal{\epsilon} C_{X}^{\prime}\{q / X\}$.

Then it follows from $p{\underset{\sim}{\sqsubset}}_{R S} s \equiv C_{X}^{\prime}\{p / X\}$ and $p \Longrightarrow_{F}^{a} \mid p^{\prime}$ that

$$
C_{X}^{\prime}\{p / X\} \stackrel{a}{\Longrightarrow}_{F} \mid s^{\prime} \text { and } p^{\prime}{\underset{\sim}{r}}_{R S} s^{\prime} \text { for some } s^{\prime} .
$$

Since $p{ }^{\tau} \xrightarrow{\tau}$, by (b.1), Lemma 5.9 and 5.16, there exists a stable context $C_{X}^{\prime \prime}$ such that (c.1) $s^{\prime} \equiv C_{X}^{\prime \prime}\{p / X\}$,
(c.2) $X$ does not occur in the scope of any conjunction in $C_{X}^{\prime \prime}$, and
(c.3) $C_{X}^{\prime}\{q / X\} \xrightarrow{a} \xlongequal{\epsilon} C_{X}^{\prime \prime}\{q / X\}$.

Moreover, since $q\left(\equiv B_{X}\{q / X\}\right)$ is stable, so is $C_{X}^{\prime \prime}\{q / X\}$. Then, by (b.3) and (c.3), we have

$$
C_{X}\{q / X\} \stackrel{\epsilon}{\Longrightarrow}\left|C_{X}^{\prime}\{q / X\} \stackrel{a}{\Longrightarrow}\right| C_{X}^{\prime \prime}\{q / X\}
$$

Further, by Lemma 7.2 and 4.2 it follows from $p, q, C_{X}\{p / X\}, C_{X}^{\prime}\{p / X\}, C_{X}^{\prime \prime}\{p / X\} \notin F$ that

$$
\begin{equation*}
C_{X}\{q / X\} \stackrel{\epsilon}{\Longrightarrow}_{F}\left|C_{X}^{\prime}\{q / X\} \stackrel{a}{\Longrightarrow}_{F}\right| C_{X}^{\prime \prime}\{q / X\} \tag{7.3}
\end{equation*}
$$

Then, since $C_{X}\{q / X\}={ }_{R S} q$ and $q \stackrel{\tau}{\longrightarrow}$, we get

$$
C_{X}^{\prime}\{q / X\}{\underset{\sim}{R S}}^{\sqsubset_{R}}
$$

Further, due to (7.32), it follows that

$$
B_{X}\{q / X\}(\equiv q) \stackrel{a}{\Longrightarrow}_{F} \mid q^{\prime} \text { and } C_{X}^{\prime \prime}\{q / X\}{\underset{\sim}{R S}}^{q^{\prime}} \text { for some } q^{\prime} .
$$

Moreover, $p^{\prime}{\underset{\sim}{\sim}}_{R S} s^{\prime} \equiv C_{X}^{\prime \prime}\{p / X\} \mathcal{R} C_{X}^{\prime \prime}\{q / X\} \underset{\sim}{{\underset{\sim}{R S}}}{ }^{\prime} q^{\prime}$, as desired.
Case $2 B_{X} \equiv \alpha . D_{X}$.
So $\alpha=a$ and $D_{X}\{p / X\} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid p^{\prime}$. Clearly, $\left(D_{X}\{p / X\}, D_{X}\{q / X\}\right) \in R$. By (ALT-upto-1), there exists $q^{\prime}$ such that $D_{X}\{q / X\} \stackrel{\epsilon}{\Longrightarrow}{ }_{F} \mid q^{\prime}$ and $p^{\prime}{\underset{\sim}{~}}_{R S} \mathcal{R}{\underset{\sim}{R S}}^{\sqsubset_{R S}} q^{\prime}$. Moreover, it is evident that $\alpha . D_{X}\{q / X\} \stackrel{a}{\Longrightarrow} F \mid q^{\prime}$.

Case $3 B_{X} \equiv D_{X} \square D_{X}^{\prime}$.
W.l.o.g, assume that the last rule applied in $\mathcal{T}_{0}$ is $\frac{D_{X}\{p / X\} \xrightarrow{a} p_{1}, D_{X}^{\prime}\{p / X\} \not^{\tau}}{D_{X}\{p / X\} \square D_{X}^{\prime}\{p / X\} \xrightarrow{a} p_{1}}$. Then $\mathcal{T}_{0}$ has a proper subtree, say $\mathcal{T}_{0}^{\prime}$, which is a proof tree for $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash$ $D_{X}\{p / X\} \xrightarrow{a} p_{1}$. Clearly, $\Omega^{\prime} \triangleq\left\{\mathcal{T}_{0}^{\prime}, \mathcal{T}_{i}: 1 \leq i \leq n-1\right\}$ is a proof forest for $D_{X}\{p / X\} \xrightarrow{a}{ }_{F}$ $\mid p^{\prime}$ and $\operatorname{Dep}\left(\Omega^{\prime}\right)<\operatorname{Dep}(\Omega)$. Moreover, since $B_{X}\{q / X\}$ is stable, so are $D_{X}\{q / X\}$ and $D_{X}^{\prime}\{q / X\}$. Then, by IH, we have $D_{X}\{q / X\} \Longrightarrow_{F}^{a} \mid q^{\prime}$ and $p^{\prime}{\underset{\sim}{\sim}}_{R S} \mathcal{R}{\underset{\sim}{\sim}}_{R S} q^{\prime}$ for some $q^{\prime}$. Moreover, $D_{X}^{\prime}\{p / X\} \notin F$ because of $B_{X}\{p / X\} \notin F$, which, by Lemma 7.2, implies $D_{X}^{\prime}\{q / X\} \notin F$. Hence $B_{X}\{q / X\} \equiv D_{X}\{q / X\} \square D_{X}^{\prime}\{q / X\} \notin F$, and $B_{X}\{q / X\} \equiv$ $D_{X}\{q / X\} \square D_{X}^{\prime}\{q / X\} \stackrel{a}{\Longrightarrow}_{F} \mid q^{\prime}$, as desired.

Case $4 B_{X} \equiv D_{X} \|_{A} D_{X}^{\prime}$.
Then the last rule applied in $\mathcal{T}_{0}$ is one of the following three formats:
(1) $\frac{D_{X}\{p / X\} \xrightarrow{a} t_{1}, D_{X}^{\prime}\{p / X\} \xrightarrow{a} s_{1}}{B_{X}\{p / X\}\left\|_{A} D_{X}\{p / X\} \xrightarrow{a} t_{1}\right\|_{A} s_{1}}$ with $a \in A$ and $p_{1} \equiv t_{1} \|_{A} s_{1}$;
(2) $\frac{D_{X}\{p / X\} \xrightarrow{a} t_{1}, D_{X}^{\prime}\{p / X\} \AA^{\top}}{D_{X}\{p / X\}\left\|_{A} D_{X}^{\prime}\{p / X\} \xrightarrow{a} t_{1}\right\|_{A} D_{X}^{\prime}\{p / X\}}$ with $a \notin A$ and $p_{1} \equiv t_{1} \|_{A} D_{X}^{\prime}\{p / X\}$;
(3) $\frac{D_{X}^{\prime}\{p / X\} \xrightarrow{a} s_{1}, D_{X}\{p / X\} \AA^{\top}}{D_{X}\{p / X\}\left\|_{A} D_{X}^{\prime}\{p / X\} \xrightarrow{a} D_{X}\{p / X\}\right\|_{A} s_{1}}$ with $a \notin A$ and $p_{1} \equiv D_{X}\{p / X\} \|_{A} s_{1}$.

We treat the first one, and the proof of the later two runs, as in Case 3. Clearly, $\mathcal{T}_{0}$
has two proper subtrees $\mathcal{T}_{0}^{\prime}$ and $\mathcal{T}_{0}^{\prime \prime}$, which are proof trees for $D_{X}\{p / X\} \xrightarrow{a} t_{1}$ and $D_{X}^{\prime}\{p / X\} \xrightarrow{a} s_{1}$ respectively. Moreover, for the transitions $p_{1} \xrightarrow{\tau} \cdots \xrightarrow{\tau} \mid p_{n}$, there exist two processes sequences $t_{1}, \ldots, t_{n}$ and $s_{1}, \ldots, s_{n}$ such that $t_{n}, s_{n}$ are stable, $p_{n} \equiv$ $t_{n} \|_{A} s_{n}$, and for each $1 \leq i<n, p_{i} \equiv t_{i} \|_{A} s_{i}$ and the last rule applied in $\mathcal{T}_{i}$ is

$$
\text { either } \frac{t_{i} \xrightarrow{\tau} t_{i+1}}{t_{i}\left\|_{A} s_{i} \xrightarrow{\tau} t_{i+1}\right\|_{A} s_{i+1}} \text { or } \frac{s_{i} \xrightarrow{\tau} s_{i+1}}{t_{i}\left\|_{A} s_{i} \xrightarrow{\tau} t_{i+1}\right\|_{A} s_{i+1}} \text {. }
$$

For the former, $s_{i+1} \equiv s_{i}$ and $\mathcal{T}_{i}$ contains a proper subtree $\mathcal{T}_{i}^{\prime}$ which is a proof tree for $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash t_{i} \xrightarrow{\tau} t_{i+1}$. We use $\Omega_{1}$ to denote the (finite) set of all these proof tree $\mathcal{T}_{i}^{\prime}$. Similarly, for the latter, $t_{i+1} \equiv t_{i}$ and $\mathcal{T}_{i}$ contains a proper subtree $\mathcal{T}_{i}^{\prime \prime}$ which is a proof tree for $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash s_{i} \xrightarrow{\tau} s_{i+1}$. We use $\Omega_{2}$ to denote the (finite) set of all these proof tree $\mathcal{T}_{i}^{\prime \prime}$. Clearly, $\Omega^{\prime} \triangleq\left\{\mathcal{T}_{0}^{\prime}\right\} \cup \Omega_{1}$ is a proof forest for $D_{X}\{p / X\}{ }^{a}{ }_{F} \mid t_{n}$ and $\operatorname{Dep}\left(\Omega^{\prime}\right)<\operatorname{Dep}(\Omega)$. Thus, by IH, we have $D_{X}\{q / X\}{ }_{F}^{a} \mid q_{1}^{\prime}$ and $t_{n}{\underset{\sim}{\sim}}_{R S} \mathcal{R}{\underset{\sim}{\sim}}_{R S} q_{1}^{\prime}$ for some $q_{1}^{\prime}$. Similarly, for the transition $D_{X}^{\prime}\{p / X\} \xrightarrow{a}{ }_{F} \mid s_{n}$, we also have $D_{X}^{\prime}\{q / X\} \stackrel{a}{\Longrightarrow} F \mid q_{2}^{\prime}$ and $s_{n} \check{\sim}_{R S} \mathcal{R} \sqsubset_{\sim_{R S}} q_{2}^{\prime}$ for some $q_{2}^{\prime}$. Therefore, by Theorem 4.3, we obtain $p^{\prime} \equiv t_{n}\left\|_{A} s_{n}{\underset{\sim}{\sim}}_{R S} \mathcal{R}{\underset{\sim}{\sim}}_{R S} q_{1}^{\prime}\right\|_{A} q_{2}^{\prime}$. Moreover, it is not difficult to see that $B_{X}\{q / X\} \equiv D_{X}\{q / X\}\left\|_{A} D_{X}^{\prime}\{q / X\} \stackrel{a}{\Longrightarrow}{ }_{F} \mid q_{1}^{\prime}\right\|_{A} q_{2}^{\prime}$ because of $B_{X}\{q / X\} \not{ }^{\top}$, $D_{X}\{q / X\} \stackrel{a}{\Longrightarrow}_{F} \mid q_{1}^{\prime}$ and $D_{X}^{\prime}\{q / X\}{ }^{a}{ }_{F} \mid q_{2}^{\prime}$.

Case $5 B_{X} \equiv\langle Y \mid E\rangle$.
Clearly, the last rule applied in $\mathcal{T}_{0}$ is $\frac{\left\langle t_{Y} \mid E\right\rangle\{p / X\} \xrightarrow{a} p_{1}}{\langle Y \mid E\rangle\{p / X\} \xrightarrow{a} p_{1}}$. Hence $\mathcal{T}_{0}$ contains a proper subtree, say $\mathcal{T}_{0}^{\prime}$, which is a proof tree for $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash\left\langle t_{Y} \mid E\right\rangle\{p / X\} \xrightarrow{a} p_{1}$, and $\operatorname{Dep}\left(\mathcal{T}_{0}^{\prime}\right)<\operatorname{Dep}\left(\mathcal{T}_{0}\right)$. So, $\Omega^{\prime} \triangleq\left\{\mathcal{T}_{0}^{\prime}, \mathcal{T}_{i}: 1 \leq i<n\right\}$ is a proof forest for $\left\langle t_{Y} \mid E\right\rangle\{p / X\} \xlongequal{a}{ }_{F}$ $\mid p^{\prime}$ and $\operatorname{Dep}\left(\Omega^{\prime}\right)<\operatorname{Dep}(\Omega)$. Then, by IH, we have $\left\langle t_{Y} \mid E\right\rangle\{q / X\} \stackrel{a}{\Longrightarrow}_{F} \mid q^{\prime}$ and $p^{\prime} \sqsubset_{R S}$ $\mathcal{R} \sqsubset_{\sim} q_{R} q^{\prime}$ for some $q^{\prime} ;$ moreover, $B_{X}\{q / X\} \equiv\langle Y \mid E\rangle\{q / X\}{ }_{\Longrightarrow}^{a}{ }_{F} \mid q^{\prime}$, as desired.
(ALT-upto-3) Let $B_{X}\{p / X\}$ and $B_{X}\{q / X\}$ be stable and $B_{X}\{p / X\} \notin F$. We shall prove $\mathcal{I}\left(B_{X}\{p / X\}\right) \supseteq \mathcal{I}\left(B_{X}\{q / X\}\right)$, the converse inclusion may be proved in a similar manner and is omitted. Assume that $B_{X}\{q / X\} \xrightarrow{a} q^{\prime}$. Then, for such $a$-labelled transition, by Lemma [5.8, there exist $B_{X}^{\prime}, B_{X, \widetilde{Y}}^{\prime}$ and $B_{X, \widetilde{Y}}^{\prime \prime}$ with $X \notin \widetilde{Y}$ that satisfy (CP-a-1) - (CP-a-4). In case $\tilde{Y}=\emptyset$, it immediately follows from (CP-a-3-iii) that $B_{X}\{p / X\} \xrightarrow{a} B_{X, \widetilde{Y}}^{\prime \prime}\{p / X\}$.

Next we handle the case $\widetilde{Y} \neq \emptyset$. In this situation, by (CP-a-3-iii), to complete the proof, it suffices to prove that $\mathcal{I}(p)=\mathcal{I}(q)$. By (CP-a-1) and (CP-a-3-i), we have

$$
B_{X}\{r / X\} \Rightarrow B_{X, \widetilde{Y}}^{\prime}\{r / X, r / \widetilde{Y}\} \text { for any } r
$$

Then, since $B_{X}\{p / X\}$ and $B_{X}\{q / X\}$ are stable, by $\tilde{Y} \neq \emptyset,($ CP- $a-2)$ and Lemmas 5.11 and 5.4 it follows that both $p$ and $q$ are stable. Hence $p \stackrel{\epsilon}{\Longrightarrow}{ }_{F} \mid p$ by $p \notin F$. Then, due to $p={ }_{R S} C_{X}\{p / X\}$, we have

$$
C_{X}\{p / X\} \stackrel{\epsilon}{\Longrightarrow}_{F} \mid s \text { and } p{\underset{\sim}{R S}} s \text { for some } s .
$$

For the transition above, since $X$ is strongly guarded in $C_{X}$, by Lemma 5.16, there exists a stable context $D_{X}$ such that
(d.1) $s \equiv D_{X}\{p / X\} \stackrel{\tau}{\longrightarrow}$,
(d.2) $X$ is strongly guarded and does not occur in the scope of any conjunction in $D_{X}$, and
(d.3) $C_{X}\{q / X\} \xlongequal{\epsilon} D_{X}\{q / X\}$.

Hence $\mathcal{I}(p)=\mathcal{I}\left(D_{X}\{p / X\}\right)$ by (d.1), $p{\underset{\sim}{\sim}}_{R S} s$ and $p \notin F$. Moreover, by (d.1), (d.2) and
Lemma [5.9, we have $D_{X}\{q / X\} \stackrel{\tau}{\nrightarrow}$ and

$$
\mathcal{I}(p)=\mathcal{I}\left(D_{X}\{p / X\}\right)=\mathcal{I}\left(D_{X}\{q / X\}\right)
$$

We also obtain $D_{X}\{q / X\} \notin F$ by $p \notin F, q \notin F, s \equiv D_{X}\{p / X\} \notin F$ and Lemma 7.2 , So, $C_{X}\{q / X\}{ }^{\epsilon}{ }_{F} \mid D_{X}\{q / X\}$ by Lemma4.2. Further, it follows from $q={ }_{R S} C_{X}\{q / X\}$
 Therefore, $\mathcal{I}(p)=\mathcal{I}\left(D_{X}\{p / X\}\right)=\mathcal{I}\left(D_{X}\{q / X\}\right)=\mathcal{I}(q)$, as desired.

The next lemma is the crucial step in the demonstrating the assertion that $\left\langle X \mid X=t_{X}\right\rangle$ is a consistent solution of a given equation $X={ }_{R S} t_{X}$ whenever consistent solutions exist.

Lemma 7.4. For any term $t_{X}$ where $X$ is strongly guarded and does not occur in the scope of any conjunction, if $q={ }_{R S} t_{X}\{q / X\}$ for some $q \notin F$ then $\left\langle X \mid X=t_{X}\right\rangle \notin F$.

Proof. Assume $p={ }_{R S} t_{X}\{p / X\}$ for some $p \notin F$. Then $t_{X}\{p / X\} \notin F$. Set
$\Omega=\left\{B_{Y}\left\{\left\langle X \mid X=t_{X}\right\rangle / Y\right\}: \begin{array}{l}B_{Y}\{p / Y\} \notin F \text { and } Y \text { does not occur in the scope of } \\ \text { any conjunction in } B_{Y}\end{array}\right\}$.
It is obvious that $\left\langle X \mid X=t_{X}\right\rangle \in \Omega$ by taking $B_{Y} \triangleq Y$. Thus we intend to show that $\Omega \cap F=\emptyset$. Assume $C_{Y}\left\{\left\langle X \mid X=t_{X}\right\rangle / Y\right\} \in \Omega$. Let $\mathcal{T}$ be any proof tree for $\operatorname{Strip}\left(\mathcal{P}_{\mathrm{CLL}_{R}}, M_{\mathrm{CLL}_{R}}\right) \vdash C_{Y}\left\{\left\langle X \mid X=t_{X}\right\rangle / Y\right\} F$. Similar to Lemma 6.3 it is sufficient to prove that $\mathcal{T}$ has a proper subtree with root $s F$ for some $s \in \Omega$, which is a routine case analysis based on the last rule applied in $\mathcal{T}$. Here we treat only two cases as examples.

Case $1 C_{Y} \equiv Y$.
Then $C_{Y}\left\{\left\langle X \mid X=t_{X}\right\rangle / Y\right\} \equiv\left\langle X \mid X=t_{X}\right\rangle$. Clearly, the last rule applied in $\mathcal{T}$ is

$$
\text { either } \frac{\left\langle t_{X} \mid X=t_{X}\right\rangle F}{\left\langle X \mid X=t_{X}\right\rangle F} \text { or } \frac{\left\{r F:\left\langle X \mid X=t_{X}\right\rangle \stackrel{\epsilon}{\Longrightarrow} \mid r\right\}}{\left\langle X \mid X=t_{X}\right\rangle F}
$$

For the former, $\mathcal{T}$ has a proper subtree with root $\left\langle t_{X} \mid X=t_{X}\right\rangle F$; moreover, $\left\langle t_{X}\right| X=$ $\left.t_{X}\right\rangle \equiv t_{X}\left\{\left\langle X \mid X=t_{X}\right\rangle / X\right\} \in \Omega$ due to $t_{X}\{p / X\} \notin F$, as desired.

For the latter, if $\left\langle X \mid X=t_{X}\right\rangle \stackrel{\tau}{\longrightarrow}$, then, in $\mathcal{T}$, the unique node directly above the root is labelled with $\left\langle X \mid X=t_{X}\right\rangle F$; moreover $\left\langle X \mid X=t_{X}\right\rangle \in \Omega$, as desired. In the following, we consider the nontrivial case $\left\langle X \mid X=t_{X}\right\rangle \xrightarrow{\tau}$. Since $t_{X}\{p / X\} \notin F$, by Theorem4.2, we get $t_{X}\{p / X\} \stackrel{\epsilon}{\Longrightarrow} \mid p^{\prime}$ for some $p^{\prime}$. For this transition, since $X$ is strongly guarded and does not occur in the scope of any conjunction, by Lemma 5.16. there exists a stable context $B_{X}$ such that
(a.1) $X$ is strongly guarded and does not occur in the scope of any conjunction, (a.2) $p^{\prime} \equiv B_{X}\{p / X\}$, and
(a.3) $t_{X}\left\{\left\langle X \mid X=t_{X}\right\rangle / X\right\} \xlongequal{\epsilon} B_{X}\left\{\left\langle X \mid X=t_{X}\right\rangle / X\right\}$.

Since $p^{\prime} \equiv B_{X}\{p / X\} \stackrel{\tau}{\not}$, by (a.1) and Lemma 5.9, $B_{X}\left\{\left\langle X \mid X=t_{X}\right\rangle / X\right\} \stackrel{{ }^{\tau}}{\longrightarrow}$. Then it follows from (a.3) and $\left\langle X \mid X=t_{X}\right\rangle \xrightarrow{\tau}$ that $\left\langle X \mid X=t_{X}\right\rangle \xrightarrow{\epsilon} \mid B_{X}\left\{\left\langle X \mid X=t_{X}\right\rangle / X\right\}$. Hence $\mathcal{T}$ has a proper subtree with root $B_{X}\left\{\left\langle X \mid X=t_{X}\right\rangle / X\right\} F$; moreover, $B_{X}\{\langle X| X=$ $\left.\left.t_{X}\right\rangle / X\right\} \in \Omega$ because of $p^{\prime} \notin F$, (a.1) and (a.2).

Case $2 C_{Y} \equiv\langle Z \mid E\rangle$.
Here, $C_{Y}\left\{\left\langle X \mid X=t_{X}\right\rangle / Y\right\} \equiv\left\langle Z \mid E\left\{\left\langle X \mid X=t_{X}\right\rangle / Y\right\}\right\rangle$. Then the last rule applied in $\mathcal{T}$ is
either $\frac{\left\langle t_{Z} \mid E\right\rangle\left\{\left\langle X \mid X=t_{X}\right\rangle / Y\right\} F}{\langle Z \mid E\rangle\left\{\left\langle X \mid X=t_{X}\right\rangle / Y\right\} F}\left(Z=t_{Z} \in E\right)$ or $\frac{\left\{r F:\langle Z \mid E\rangle\left\{\left\langle X \mid X=t_{X}\right\rangle / Y\right\} \xlongequal{\epsilon} \mid r\right\}}{\langle Z \mid E\rangle\left\{\left\langle X \mid X=t_{X}\right\rangle / Y\right\} F}$.
For the first alternative, by Lemma 4.1(8), it follows from $\langle Z \mid E\rangle\{p / Y\} \notin F$ that $\left\langle t_{Z} \mid E\right\rangle\{p / Y\} \notin F$. Since $Y$ does not occur in the scope of any conjunction in $\langle Z \mid E\rangle$, by Lemma 5.2(5), neither does it in $\left\langle t_{Z} \mid E\right\rangle$. Therefore $\left\langle t_{Z} \mid E\right\rangle\left\{\left\langle X \mid X=t_{X}\right\rangle / Y\right\} \in \Omega$, as desired.

For the second alternative, since $\langle Z \mid E\rangle\{p / Y\} \notin F$ and $p={ }_{R S} t_{X}\{p / X\}$, we get $\langle Z \mid E\rangle\left\{t_{X}\{p / X\} / Y\right\} \notin F$ by Theorem 6.1] So $\langle Z \mid E\rangle\left\{t_{X}\{p / X\} / Y\right\} \xrightarrow{\epsilon}{ }_{F} \mid p^{\prime}$ for some $p^{\prime}$. Then, for this transition, by Lemma 5.16, there exist processes $q_{W}(W \in \widetilde{W})$ and a context $D_{Y, \widetilde{W}}$ with $Y \notin \widetilde{W}$ such that
(b.1) $t_{X}\{p / X\} \xlongequal{\tau} \mid q_{W}$ for each $W \in \widetilde{W}$ and $p^{\prime} \equiv D_{Y, \widetilde{W}}\left\{t_{X}\{p / X\} / Y, \widetilde{q_{W}} / \widetilde{W}\right\}$,
(b.2) $Y$ and each $W(\in \widetilde{W})$ are strongly guarded and do not occur in the scope of any conjunction in $D_{Y, \widetilde{W}}$, and
(b.3) $\langle Z \mid E\rangle\{r / Y\} \xlongequal{\epsilon} D_{Y, \widetilde{W}}\left\{r / Y, \widetilde{r_{W}} / \widetilde{W}\right\}$ for any $r$ and $r_{W}(W \in \widetilde{W})$ such that $r \xlongequal{\tau}$ $r_{W}$ for each $W \in \widetilde{W}$.
Then, since $X$ is strongly guarded and does not occur in the scope of any conjunction in $t_{X}$, by Lemma 5.16 and 5.9, for each transition $t_{X}\{p / X\} \stackrel{\tau}{\Longrightarrow} \mid q_{W}$, there exists a stable context $t_{X}^{W}$ such that
(c.1) $X$ is strongly guarded and does not occur in the scope of any conjunction in $t_{X}^{W}$,
(c.2) $q_{W} \equiv t_{X}^{W}\{p / X\}$, and
(c.3) $t_{X}\left\{\left\langle X \mid X=t_{X}\right\rangle / X\right\} \xlongequal{\tau} \mid t_{X}^{W}\left\{\left\langle X \mid X=t_{X}\right\rangle / X\right\}$.

For the simplicity of notation, we let $Q_{W}$ stand for $t_{X}^{W}\left\{\left\langle X \mid X=t_{X}\right\rangle / X\right\}$ for each $W \in \widetilde{W}$. So, by (c.3), $\left\langle X \mid X=t_{X}\right\rangle \stackrel{\tau}{\Longrightarrow} \mid Q_{W}$ for each $W \in \widetilde{W}$. Hence it follows from (b.3) that

$$
\begin{equation*}
\langle Z \mid E\rangle\left\{\left\langle X \mid X=t_{X}\right\rangle / Y\right\} \stackrel{\epsilon}{\Longrightarrow} D_{Y, \widetilde{W}}\left\{\left\langle X \mid X=t_{X}\right\rangle / Y, \widetilde{Q_{W}} / \widetilde{W}\right\} \tag{7.4,1}
\end{equation*}
$$

By (b.2) and (c.1), it is not difficult to see that $X$ is strongly guarded and does not occur in the scope of any conjunction in $D_{Y, \widetilde{W}}\left\{t_{X} / Y, \widetilde{t_{X}^{W}} / \widetilde{W}\right\}$. So, by Lemma 5.9 and $p^{\prime} \equiv D_{Y, \widetilde{W}}\left\{t_{X} / Y, \widetilde{t_{X}^{W}} / \widetilde{W}\right\}\{p / X\} \not{ }^{\tau}$, we get

$$
D_{Y, \widetilde{W}}\left\{t_{X} / Y, \widetilde{t_{X}^{W}} / \widetilde{W}\right\}\left\{\left\langle X \mid X=t_{X}\right\rangle / X\right\} \not{ }^{\tau}
$$

Hence $D_{Y, \widetilde{W}}\left\{\left\langle X \mid X=t_{X}\right\rangle / Y, \widetilde{Q_{W}} / \widetilde{W}\right\} \xrightarrow{\tau}$ by Lemma 5.6 and $\mathcal{I}\left(\left\langle X \mid X=t_{X}\right\rangle\right)=$ $\mathcal{I}\left(t_{X}\left\{\left\langle X \mid X=t_{X}\right\rangle / X\right\}\right)$. Then $\mathcal{T}$ has a proper subtree with root $D_{Y, \widetilde{W}}\left\{\left\langle X \mid X=t_{X}\right\rangle / Y, \widetilde{Q_{W}} / \widetilde{W}\right\} F$ due to (7.4,1). Moreover, by Theorem 6.1 and $p={ }_{R S} t_{X}\{p / X\}$, it follows from $p^{\prime} \equiv$ $\left.D_{Y, \widetilde{W}}\left\{t_{X}\{p / X\} / Y, t_{X}^{\widehat{W}\{p / X}\right\} / \widetilde{W}\right\} \notin F$ that $D_{Y, \widetilde{W}}\left\{p / Y, t_{X}^{\widehat{W}\{p / X\}} / \widetilde{W}\right\} \notin F$. Set

$$
D_{Y}^{\prime} \triangleq D_{Y, \widetilde{W}}\left\{t_{X}^{W} \widetilde{\left\{_{Y / X}\right\}} / \widetilde{W}\right\}
$$

Therefore, $D_{Y, \widetilde{W}}\left\{\left\langle X \mid X=t_{X}\right\rangle / Y, \widetilde{Q_{W}} / \widetilde{W}\right\} \equiv D_{Y}^{\prime}\left\{\left\langle X \mid X=t_{X}\right\rangle / Y\right\} \in \Omega$, as desired.
We now have the assertion below which states that given an equation $X={ }_{R S} t_{X}$ satisfying some conditions, $\left\langle X \mid X=t_{X}\right\rangle$ is the unique consistent solution whenever consistent solutions exist.

Theorem 7.1 (Unique solution). For any $p, q \notin F$ and $t_{X}$ where $X$ is strongly guarded and does not occur in the scope of any conjunction, if $p={ }_{R S} t_{X}\{p / X\}$ and $q={ }_{R S} t_{X}\{q / X\}$ then $p={ }_{R S} q$. Moreover, $\left\langle X \mid X=t_{X}\right\rangle$ is the unique consistent solution modulo $={ }_{R S}$ for the equation $X={ }_{R S} t_{X}$ whenever consistent solutions exist.

Proof. If $p \bowtie q$ then $p=R S q$ follows from Lemma 7.3, otherwise, w.l.o.g, we assume that $p$ is stable and $q$ is not. By Theorem6.1 $\tau \cdot p={ }_{R S} p={ }_{R S} t_{X}\{p / X\}={ }_{R S} t_{X}\{\tau \cdot p / X\}$. Then, by Lemma 7.3, it follows from $\tau . p, q \notin F, \tau . p \bowtie q, \tau . p={ }_{R S} t_{X}\{\tau . p / X\}$ and $q={ }_{R S} t_{X}\{q / X\}$ that $\tau \cdot p={ }_{R S} q$. Hence $p={ }_{R S} q$.

Suppose that $X={ }_{R S} t_{X}$ has consistent solutions. It is obvious that $\left\langle X \mid X=t_{X}\right\rangle={ }_{R S}$ $t_{X}\left\{\left\langle X \mid X=t_{X}\right\rangle / X\right\}$ due to $\left\langle X \mid X=t_{X}\right\rangle \Rightarrow_{1}\left\langle t_{X} \mid X=t_{X}\right\rangle \equiv t_{X}\left\{\left\langle X \mid X=t_{X}\right\rangle / X\right\}$ and Lemma 5.15, Further, by Lemma [7.4, $\left\langle X \mid X=t_{X}\right\rangle$ is the unique consistent solution of the equation $X={ }_{R S} t_{X}$.

As an immediate consequence, we have
Corollary 7.1. For any term $t_{X}$ where $X$ is strongly guarded and does not occur in the scope of any conjunction, then the equation $X={ }_{R S} t_{X}$ has consistent solutions iff $\left\langle X \mid X=t_{X}\right\rangle \notin F$.

Proof. Immediately by Theorem 7.1.
We conclude this section with providing a brief discussion. For Theorem 7.1 the condition that $X$ is strongly guarded can not be relaxed to that $X$ is weakly guarded. For instance, consider the equation $X={ }_{R S} \tau . X$, it has infinitely many consistent solutions. In fact, for any $p$, it always holds that $p={ }_{R S} \tau . p$. Moreover, the condition that $p, q \notin F$ is also necessary. For example, both $\langle X \mid X=a . X\rangle$ and $\perp$ are solutions of the equation $X={ }_{R S} a \cdot X$, but they are not equivalent modulo $=_{R S}$.

## 8. Conclusions and future work

This paper considers recursive operations over LLTSs in pure process-algebraic style and a process calculus $\mathrm{CLL}_{R}$, which is obtained from CLL by adding recursive operations, is proposed. We show that the behavioral relation $\sqsubseteq_{R S}$ is precongruent w.r.t all operations
in $\mathrm{CLL}_{R}$, which reveals that this calculus supports compositional reasoning. Moreover, we also provide a theorem on the uniqueness of consistent solution of a given equation $X={ }_{R S} t_{X}$ where $X$ is required to be strongly guarded and does not occur in the scope of any conjunction in $t_{X}$.

Although CLL contains logic operators $\wedge$ and $\vee$ over processes, due to lack of modal operators, it does not afford describing abstract properties of concurrent systems. As we know, some modal operators could be characterized by equations and fixpoints (Bradfield and Stirling 2001). Fortunately, under the mild condition that the set of actions Act is finite, we can integrate standard temporal operators always and unless into $\mathrm{CLL}_{R}$ (Zhu et al. 2013) but this requires us to strengthen Theorem 7.1 by removing the restriction that "recursive variables do not occur in the scope of any conjunction in recursive specifications". We leave the strengthened Theorem 7.1] as a open problem.

In this paper, we adopt a proof method well-ordered proof tree contradiction to obtain properties of $F$-predicate. It reflects a kind of principle negation as failure and it is different from proof method witnesses adopted by Lüttgen and Vogler. Their method requires one to find proofs (i.e., witness set) to illustrate the existence of properties. However, the way of constructing witnesses is similar to our method.

Future work could proceed along two directions. Firstly, we will add hiding operator to $\mathrm{CLL}_{R}$. As an important feature of LLTS (Lüttgen and Vogler 2010), hiding in the presence of recursion may lead to divergence and introduce inconsistency by (LTS2). For example, $\langle X \mid X=a . X\rangle \notin F$ but $\langle X \mid X=a . X\rangle \backslash a \in F$. The other direction of future work is to find a (ground) complete proof system for regular processes in CLL $_{R}$ along lines adopted in (Milner 1989; Baeten and Bravetti 2008). Here a process is regular if its LTS has only finitely many states and transitions.

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