Under consideration for publication in Math. Struct. in Comp. Science

The Infinitary Lambda Calculus of the Infinite Eta Böhm Trees

Paula Severi and Fer-Jan de Vries

Department of Computer Science, University of Leicester

Received 6 May 2013

In this paper we introduce a strong form of eta reduction called *etabang* that we use to construct a confluent and normalising infinitary lambda calculus, of which the normal forms correspond to Barendregt's infinite eta Böhm trees. This new infinitary perspective on the set of infinite eta Böhm trees allows us to prove that the set of infinite eta Böhm trees is a model of the lambda calculus. The model is of interest because it has the same local structure as Scott's D_∞-models, i.e. two finite lambda terms are equal in the infinite eta Böhm model if and only if they have the same interpretation in Scott's D_∞-models.

Keywords: Infinitary Lambda Calculus, Böhm trees, Models of Lambda Calculus, Etabang Reduction, Infinite Eta Böhm trees.

1. Introduction

In the classical finitary lambda calculus (Barendregt, 1984), one can express that the fixpoint combinator Y $(= \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))$ can reduce to terms of the form $\lambda f.f^n((\lambda x.f(xx))(\lambda x.f(xx)))$, for any $n > 0$, but not that Y has an infinite reduction to the to the infinite term $\lambda f.f^\omega$, where f^ω is convenient shorthand for the infinite term $f(f(f(\ldots)))$. In the infinitary lambda calculus, the set Λ of finite λ-terms is extended to explicitly include infinite terms such as $\lambda f.f^\omega$ and the notation allows for finite and infinite reductions. This makes it possible to define the concept of Böhm tree directly in the notational framework of the infinitary lambda calculus, in contrast to (Barendregt, 1984) where Böhm trees are defined with their own notational machinery.

Infinitary lambda calculus allows an alternative definition of the notion of tree as normal form. Figure 1 summarises the correspondences between the infinitary lambda calculi and the trees which have been studied so far. All these calculi include a notion of $\perp$-reduction and they are all proved to be confluent and normalising before except for the one on the last row (Berarducci, 1996; Kennaway et al., 1995a; Kennaway et al., 1997; Kennaway and de Vries, 2003; Severi and de Vries, 2002; Severi and de Vries, 2011). From any infinitary lambda calculus which is confluent and normalising, we can construct a model of the finite lambda calculus by defining the interpretation of a term to be exactly the (infinite) normal form of that term (or equivalently the tree of that term).

The infinitary lambda calculi sketched in the first four rows of Figure 1 are variations of $\lambda^\infty_{\beta\perp} = (\Lambda^\infty_\perp, \longrightarrow_{\beta\perp})$. By changing the $\perp$-rule, we obtain different notions of trees. If we take the terms without head normal form as meaningless terms, then we obtain an infinitary lambda calculus which is confluent and normalising. The normal form of a term in this calculus

Reduction Rules	**Normal forms**
β-rule $\perp$-rule for terms without head normal form	Böhm trees
β-rule $\perp$-rule for terms without weak head normal form	Lévy-Longo trees
β-rule $\perp$-rule for terms without top normal form	Berarducci trees
β-rule $\perp$-rule parametrised by a set of weakly meaningless terms	Parametric trees
β-rule η-rule $\perp$-rule for terms without head normal form	η-Böhm trees
β-rule $\eta!$-rule $\perp$-rule for terms without head normal form	$\infty\eta$-Böhm trees

Fig. 1. **Trees as infinite normal forms**

correspond to the Böhm tree of this term. The collection of normal forms of this calculus forms a model of the lambda beta calculus, better known as Barendregt's Böhm model (Barendregt, 1984). Similarly, by reducing terms without weak head normal form to $\perp$, we capture the notion of Lévy-Longo tree (Kennaway et al., 1995a; Kennaway et al., 1997; Kennaway and de Vries, 2003) and this gives rise to the model of Lévy-Longo trees. Also by reducing terms without top head normal form to $\perp$, we capture the notion of Berarducci tree (Berarducci, 1996; Kennaway et al., 1995a; Kennaway et al., 1997; Kennaway and de Vries, 2003) which gives rise to the model of Berarducci trees.

The infinitary lambda calculi $\lambda^{\infty}_{\beta\perp}$ with a $\perp$-rule parametric on a set of (weakly) meaningless terms encompasses the previous three cases (Kennaway and de Vries, 2003; Severi and de Vries, 2011). This method to construct models of the lambda beta calculus is quite flexible as there is ample choice for the set of meaningless terms (Severi and de Vries, 2005a; Severi and de Vries, 2005b; Severi and de Vries, 2011). Because the collection of sets of weakly meaningless terms is uncountable, we get an uncountable class of models which are not continuous (Severi and de Vries, 2005a).

The infinitary lambda calculus $\lambda^{\infty}_{\beta\perp\eta} = (\Lambda^{\infty}_{\perp}, \longrightarrow_{\beta\perp\eta})$ sketched in the last but one row incorporates the η-rule (Severi and de Vries, 2002). This calculus captures the notion of η-Böhm tree, which can be described as the eta-normal form of a Böhm tree, and gives rise to an extensional model of the lambda calculus that has the same local structure as Coppo, Dezani and Zacchi's filter model D^{*}_{∞} (Coppo et al., 1987).

The last row in Figure 1 represents the contribution of this paper. The infinitary lambda calculus $\lambda^{\infty}_{\beta\perp\eta!} = (\Lambda^{\infty}_{\perp}, \longrightarrow_{\beta\perp\eta!})$ is constructed with the $\eta!$-rule, a strengthening of the η-rule. The notion of $\eta!$-reduction is based on the observation that the explicit syntactic characterisation of infinite eta expansions in the definition of infinite eta Böhm trees in (Barendregt, 1984) can be succinctly redefined as *strongly converging* eta-expansions in the terminology of infinitary rewriting. The power of $\eta!$-reduction is such that it reduces the Böhm tree of J to I, see Figure 2. The main complication of this paper will be to prove that $\lambda^{\infty}_{\beta\perp\eta!}$ is confluent and normalising. As direct consequences of this result, we will first obtain an alternative def-

Finite λ-term	Böhm tree	η-Böhm tree	$\infty\eta$-Böhm tree
$\mathsf{I} = \lambda x.x$	$\mathsf{BT}(\mathsf{I}) = \mathsf{I}$	$\eta\mathsf{BT}(\mathsf{I}) = \mathsf{I}$	$\infty\eta\mathsf{BT}(\mathsf{I}) = \mathsf{I}$
$\mathsf{1} = \lambda xy.xy$	$\mathsf{BT}(\mathsf{1}) = \mathsf{1}$	$\eta\mathsf{BT}(\mathsf{1}) = \mathsf{I}$	$\infty\eta\mathsf{BT}(\mathsf{1}) = \mathsf{I}$
$\mathsf{J} = \mathsf{Y}(\lambda fxy.x(fy))$	$\mathsf{BT}(\mathsf{J}) = \begin{array}{c}\lambda xy_0.x \\ \mid \\ \lambda y_1.y_0 \\ \mid \\ \lambda y_2.y_1 \\ \vdots \end{array}$	$\eta\mathsf{BT}(\mathsf{J}) = \begin{array}{c}\lambda xy_0.x \\ \mid \\ \lambda y_1.y_0 \\ \mid \\ \lambda y_2.y_1 \\ \vdots \end{array}$	$\infty\eta\mathsf{BT}(\mathsf{J}) = \mathsf{I}$

Fig. 2. Difference between the notions of Böhm, η-Böhm and $\infty\eta$-Böhm trees

inition of the notion of $\infty\eta$-Böhm tree of a lambda term as its normal form in $\lambda^{\infty}_{\beta\perp\eta!}$ which is more compact than the one in (Barendregt, 1984; Barbanera et al., 1998; Bakel et al., 2002). Second, we can show that the set of $\infty\eta$-Böhm trees is an extensional model of the finite lambda calculus. The model of $\infty\eta$-Böhm trees is of interest because it has the same local structure as Scott's D_∞-models, i.e. two finite lambda terms have the same normal form in $\lambda^{\infty}_{\beta\perp\eta!}$ if and only if they are equal in D_∞ (Hyland, 1975; Wadsworth, 1976).
It may appear at first sight that extending an infinitary lambda calculus with η or $\eta!$ should not be complicated. However, the two lambda calculi of Lévy-Longo and Berarducci trees do not seem to accept any variations on the $\perp$-rule without losing confluence. There is a critical pair between the η-rule ($\eta!$-rule) and the $\perp$-rule for terms without weak head normal form:

$$\begin{tikzcd} & \lambda x.\Omega x \arrow[dl, twoheadrightarrow, "\perp"'] \arrow[dr, twoheadrightarrow, "\eta"] & \\ \lambda x.\perp & & \perp \end{tikzcd}$$

The $\perp$-step follows from the fact that the term Ωx has no weak head normal form. This pair can be completed only if $\lambda x.\perp \longrightarrow_\perp \perp$ which is true only for the $\perp$-rule that equates terms without head normal form. For a counterexample of confluence for $\beta\perp\eta$ and $\beta\perp\eta!$ where the $\perp$-rule equates terms without top normal form, we use the term $\Omega_\eta = \lambda x_0.(\lambda x_1.(\ldots)x_1)x_0$. Similar to Ω which β-reduces to itself in only one step, this term η-reduces to itself in only one step. The term Ω_η can be obtained as the fixed point of $\mathsf{1} = \lambda xy.xy$. The body of the outermost abstraction in Ω_η is rootactive (it always reduces to a β-redex) and hence $\Omega_\eta \longrightarrow_\perp \lambda x.\perp$. The span

$$\begin{tikzcd} & & \mathsf{Y1} \arrow[dl, twoheadrightarrow, "\beta\eta"'] \arrow[dr, twoheadrightarrow, "\eta"] & & \\ & \Omega \arrow[dl, twoheadrightarrow, "\perp"'] & & \Omega_\eta \arrow[dr, twoheadrightarrow, "\perp"] & \\ \perp & & & & \lambda x.\perp \end{tikzcd}$$

can only be joined if $\lambda x.\perp \longrightarrow_\perp \perp$.
Section 2 recalls some notions of infinitary lambda calculus and introduces the definition of $\eta!$-reduction. Section 3 studies properties of mainly $\longrightarrow_{\eta!}$ and $\longrightarrow_{\eta^{-1}}$ on their own. Section 4

proves two strip lemmas for $\eta!$ and β. Section 5 proves that outermost $\perp$-reduction commutes with $\eta!$. Section 6 proves confluence and normalisation of the infinitary lambda calculus $\lambda^{\infty}_{\beta\perp\eta!}$. Section 7 explains in detail the connection between the infinite eta Böhm trees and the normal forms in $\lambda^{\infty}_{\beta\perp\eta!}$. Section 8 shows that the set of the normal forms of $\lambda^{\infty}_{\beta\perp\eta!}$ is an extensional model of the finite lambda calculus.

2. Infinitary Lambda Calculus

2.1. *The set $\Lambda^{\infty}_{\perp}$ of finite and infinite terms*

Infinitary lambda calculus provides a single framework for finite lambda terms and infinite terms. Infinite extensions of finite lambda calculus were introduced around 1994 following similar developments in first order term rewriting initiated by Dershowitz and Kaplan (Berarducci, 1996; Kennaway et al., 1997). As starting point for this paper we are interested in one particular extension $\lambda^{\infty}_{\beta\perp}$ of the finite lambda calculus defined in (Kennaway et al., 1997), namely the extension in which the normal forms correspond to the Böhm trees of (Barendregt, 1984). The set $\Lambda^{\infty}_{\perp}$ of finite and infinite terms of $\lambda^{\infty}_{\beta\perp}$ can conveniently be defined as metric completion of the finite terms for a suitable chosen metric. In spirit, this construction goes back at least to Arnold and Nivat (Arnold and Nivat, 1980). The metric context will also be used to define transfinite converging reductions.
We will now briefly recall this construction from (Kennaway et al., 1997). Throughout we assume familiarity with basic notions and notations from (Barendregt, 1984).

2.1 Definition [Set $\Lambda_{\perp}$ of finite lambda terms with $\perp$]: Let $\Lambda_{\perp}$ be the set of finite λ-terms given by the inductive grammar:

$$M ::= \perp \mid x \mid (\lambda x M) \mid (MM)$$

where x is a variable from some fixed countable set of variables $\mathcal{V}$.

We follow the usual conventions on syntax. Terms and variables will respectively be written with (super- and subscripted) letters M, N and x, y, z. Terms of the form $(M_1 M_2)$ and $(\lambda x M)$ will respectively be called applications and abstractions. A context $C[\ ]$ is a term with a hole in it, and $C[M]$ denotes the result of filling the hole by the term M, possibly capturing some free variables of M.

2.2 Notation: We will also use the following abbreviations for terms in $\Lambda_{\perp}$:

$$\begin{array}{ll} \mathsf{I} = \lambda x.x & \Omega = (\lambda x.xx)\lambda x.xx \\ \mathsf{K} = \lambda xy.x & \mathsf{Y} = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)) \\ \mathsf{1} = \lambda xy.xy & \mathsf{J} = \mathsf{Y}(\lambda fxy.x(fy)) \end{array}$$

2.3 Definition [Subterm at a certain position]: Let $M \in \Lambda_{\perp}$ and p be any finite sequence of 0, 1 and 2's. We will use $\langle\rangle$ for the empty sequence. The subterm $M|_p$ of a term $M \in \Lambda_{\perp}$ at position p (if there is one) is defined by induction as usual:

$$M|_{\langle\rangle} = M \quad (\lambda x M)|_{0p} = M|_p \quad (MN)|_{1p} = M|_p \quad (MN)|_{2p} = N|_p$$

2.4 Definition [Depth of a subterm at a certain position]: Let $M \in \Lambda_{\perp}$. The length $L(p)$ of a position p is the number of 2's in p. The depth at which a subterm N occurs in M is the length of the position p of N in M.

$$\begin{array}{cccccl} (\lambda xy.x & \bullet &) & \bullet & & \text{depth } 0 \\ & | & & | & & \\ & y & & z & & \text{depth } 1 \end{array}$$

Fig. 3. Graph representation of $(\lambda x.xy)z$ that respects our notion of depth

Figure 3 shows a graph representation of $(\lambda xy.xy)z$ that respects our notion of depth.
We define now the truncation of a term M at depth n as the term obtained by replacing all subterms at depth n by $\perp$.

2.5 Definition [Truncation]: Let $M \in \Lambda_\perp$. The truncation of M at depth n is defined by induction on M as follows.

$$\begin{array}{lcllcl} M^0 & = & \perp & (\lambda x.M)^{n+1} & = & \lambda x.M^{n+1} \\ \perp^{n+1} & = & \perp & (MN)^{n+1} & = & M^{n+1}N^n \\ x^{n+1} & = & x & & & \end{array}$$

Note that for truncating an abstraction $\lambda x.M$ at depth $n+1$, we truncate the body M at the same depth $n+1$. For the application MN, we truncate the argument N at depth n but the operator M is truncated at depth $n+1$. For example, $(\lambda x.xy)^1 = \lambda x.x\perp$.

2.6 Definition [Metric]: Let $M, N \in \Lambda_\perp$. We define a metric on $\Lambda_\perp$ as follows: $d(M,N) = 0$, if $M = N$ and $d(M,N) = 2^{-m}$, where $m = max\{n \in \mathbb{N} \mid M^n = N^n\}$.

For example, if $M = (x(y(zu))$ and $N = (x(y(zv))$ then $d(M,N) = 2^{-3}$.

2.7 Definition [Set $\Lambda_\perp^\infty$ of finite and infinite terms]: The set $\Lambda_\perp^\infty$ is defined as the *metric completion* of the set of finite lambda terms $\Lambda_\perp$ with respect to the metric d.

From now on, $M, N, \ldots$ will be assumed to belong to $\Lambda_\perp^\infty$ unless we state otherwise.
The set $\Lambda_\perp^\infty$ is constructed to contain all Böhm trees. For example, the term $x(x(x\ldots))$ belongs to $\Lambda_\perp^\infty$. It does not contain the terms $\lambda x.\lambda x.\ldots$ or $((\ldots)x)x$ which are typical Lévy-Longo or Berarducci trees (Longo, 1983; Lévy, 1976; Abramsky and Ong, 1993; Berarducci, 1996; Kennaway et al., 1997).

2.8 Notation: We will also use the following abbreviations for terms in $\Lambda_\perp^\infty$:

$$\mathsf{J}_\infty = \lambda xy_0.x(\lambda y_1.y_0(\lambda y_2.y_1(\ldots))) \quad \mathsf{E}_y = \lambda y_1.y(\lambda y_2.y_1(\ldots)) \text{ for any } y \in \mathcal{V}$$

Note that $\mathsf{J}_\infty = \lambda xy_0.x\mathsf{E}_{y_0}$ and $\mathsf{E}_{y_0} = \lambda y_1.y_0\mathsf{E}_{y_1}$.
Definitions 2.3, 2.4 and 2.5 can all be extended to infinite terms in $\Lambda_\perp^\infty$ in the obvious way. The notion of depth of the hole in $C[\ \ ]$ can be defined in the same way as the depth of a subterm at a certain position (see Definition 2.4).

2.9 Definition [Prefix]: Let $M, N \in \Lambda_\perp^\infty$. We say that M is a prefix of N (we write $M \preceq N$) if M is obtained from N by replacing some subterms of N by $\perp$

2.2. *Converging Reductions*

In this section we define the notion of strongly converging reduction.

2.10 Definition [Reduction]: We call a binary relation $\longrightarrow_\rho$ on $\Lambda_\perp^\infty$ a reduction relation, if $\longrightarrow_\rho$ is closed under contexts, that is, if $M \longrightarrow_\rho N$ implies $C[M] \longrightarrow_\rho C[N]$.

2.11 Definition [Infinitary Lambda Calculus]: Let $\longrightarrow_\rho$ be a reduction relation on $\Lambda^\infty_\perp$. We call the pair $(\Lambda^\infty_\perp, \longrightarrow_\rho)$ an infinitary lambda calculus. Instead of $(\Lambda^\infty_\perp, \longrightarrow_\rho)$, we simply write λ^∞_ρ.

For an infinitary lambda calculus λ^∞_ρ and ordinals α we define reduction sequences of any transfinite ordinal length α.

2.12 Definition [Strongly Converging Reductions (Kennaway and de Vries, 2003)]: A strongly convergent ρ-reduction sequence of length α (an ordinal) is a sequence $\{M_\beta \mid \beta \leq \alpha\}$ of terms in $\Lambda^\infty_\perp$ such that $M_\beta \longrightarrow_\rho M_{\beta+1}$ for all $\beta < \alpha$, besides $M_\lambda = \lim_{\beta<\lambda} M_\beta$ for every limit ordinal $\lambda \leq \alpha$ and $\lim_{i\to\lambda} d_i = \infty$ where d_i is the depth of the redex contracted at $M_i \longrightarrow_\rho M_{i+1}$ for every limit ordinal $\lambda \leq \alpha$.

Strongly converging reduction is a key concept of infinite rewriting (Kennaway et al., 1995b; Kennaway et al., 1997) that generalises and includes finite reduction. Intuitively, an infinite reduction is strongly converging when the depth of the position of the application of the reduction rules goes to infinity along the reduction sequence. Cauchy converging reduction sequence do not behave so nicely as strongly converging reductions (Kennaway et al., 1997; Simonsen, 2004). Hence strongly converging reduction is the natural notion of reduction to study. This preference is reflected in the next notation.

2.13 Notation:
$M \longrightarrow_\rho N$ denotes a one step reduction from M to N;
$M \twoheadrightarrow_\rho N$ denotes a finite reduction from M to N;
$M \twoheadrightarrow\!\!\!\!\rightarrow_\rho N$ denotes a strongly converging reduction from M to N.
$M \longrightarrow_{=,\rho} N$ denotes equality or one step reduction from M to N.

We will sometimes write the depth of the contracted redex on top of the arrows. For example, $M \xrightarrow{m}_\rho N$ denotes a reduction step where the contracted redex is at depth m.
Many notions of finite lambda calculus apply and/or extend now more or less straightforwardly to an infinitary lambda calculus λ^∞_ρ. A term M in λ^∞_ρ is in *ρ-normal form* if there is no N in λ^∞_ρ such that $M \longrightarrow_\rho N$.

2.14 Definition: Let $\lambda^\infty_\rho = (\Lambda^\infty_\perp, \longrightarrow_\rho)$.
λ^∞_ρ is *confluent* if ${}_\rho\!\twoheadleftarrow\!\!\!\!\leftarrow \circ \twoheadrightarrow\!\!\!\!\rightarrow_\rho \subseteq \twoheadrightarrow\!\!\!\!\rightarrow_\rho \circ\, {}_\rho\!\twoheadleftarrow\!\!\!\!\leftarrow$.
λ^∞_ρ is *normalising* if for all $M \in \Lambda^\infty_\perp$ there exists an N in ρ-normal form such that $M \twoheadrightarrow\!\!\!\!\rightarrow_\rho N$.
Let α be an ordinal. λ^∞_ρ is *α-compressible* if for all M, N such that $M \twoheadrightarrow\!\!\!\!\rightarrow_\rho N$ there exists a reduction from M to N of length at most α.

The *ρ-normal form* of M is a term N in ρ-normal form such that $M \twoheadrightarrow\!\!\!\!\rightarrow_\rho N$. If λ^∞_ρ is confluent and normalising, it induces a total function, denoted by nf_ρ, from $\Lambda^\infty_\perp$ to $\Lambda^\infty_\perp$ such that $\mathsf{nf}_\rho(M)$ gives *the ρ-normal form* of M. The set of ρ-normal forms over $\Lambda^\infty_\perp$ is denoted by $\mathsf{nf}_\rho(\Lambda^\infty_\perp)$ and the set of ρ-normal forms over Λ is denoted by $\mathsf{nf}_\rho(\Lambda)$.

2.3. *The Basic Reductions: β, η, η^{-1} and $\perp$-reductions*

In this section we extend several notions of reductions on finite lambda calculus to infinite terms. The β-reduction, denoted by $\longrightarrow_\beta$, is the smallest reduction on $\Lambda^\infty_\perp$ closed under the β-rule:

$$(\lambda x.M)N \longrightarrow M[x := N] \quad (\beta)$$

The β_h-reduction, denoted by $\longrightarrow_{\beta_h}$, is the β-reduction restricted to head redexes, i.e. $\lambda x_1 \dots x_n.(\lambda x.P)QN_1 \dots N_k \longrightarrow_{\beta_h} \lambda x_1 \dots x_n.P[x := Q]N_1 \dots N_k$.
The η-reduction, denoted by $\longrightarrow_\eta$, is the smallest reduction on $\Lambda^\infty_\perp$ closed under the η-rule:

$$\frac{x \notin FV(M)}{\lambda x.Mx \to M}\ (\eta)$$

The η^{-1}-reduction (or the η-expansion), denoted by $\longrightarrow_{\eta^{-1}}$, is the smallest reduction on $\Lambda^\infty_\perp$ closed under the η^{-1}-rule:

$$\frac{x \notin FV(M)}{M \to \lambda x.Mx}\ (\eta^{-1})$$

We now define the $\perp$-rule. The variant that we will use in this paper is the one that equates terms that have no head normal form. The $\perp$-rule is necessary because the infinitary lambda calculus with only β-reduction is not confluent. For example (Berarducci, 1996),

$$\begin{array}{ccccccc}
\Omega_\mathsf{I} & \xrightarrow[\beta]{} \!\!\!\!\! \twoheadrightarrow & \mathsf{I}(\Omega_\mathsf{I}) & \twoheadrightarrow_\beta & \mathsf{I}(\mathsf{I}(\Omega_\mathsf{I})) & \twoheadrightarrow_\beta \quad \dots & \mathsf{I}^\omega \\
\downarrow\!\!\downarrow_\beta & & & & & & \\
\Omega & & & & & &
\end{array}$$

where $\Omega = (\lambda x.xx)\lambda x.xx$, $\mathsf{I} = \lambda x.x$ and $\Omega_\mathsf{I} = (\lambda x.\mathsf{I}(xx))(\lambda x.\mathsf{I}(xx))$.
Let $M \in \Lambda^\infty_\perp$. We say that M is in *head normal form* (*hnf*) if M is of the form $\lambda x_1 \dots x_n.yN_1 \dots N_k$. We say that M *has a head normal form* (or is *head normalising*) if there exists N in head normal form such that $M \twoheadrightarrow_\beta N$. The terms Ω and $\lambda x.\perp x$ are examples of terms without head normal form.
The $\perp$-reduction, denoted by $\longrightarrow_\perp$, is the smallest reduction on $\Lambda^\infty_\perp$ closed under the $\perp$-rule:

$$\frac{M \text{ has no head normal form}}{M \longrightarrow \perp}\ (\perp)$$

Next we define the notion of outermost $\perp$-redex as a maximal subterm without head normal form. For example, the term $M = x((\lambda y.\Omega y)z)$ has four $\perp$-redexes, i.e. Ω, Ωy, $\lambda y.\Omega y$ and $(\lambda y.\Omega y)z$ but only the latter is an outermost $\perp$-redex.
We will also need a variation of the $\perp$-reduction, called $\perp_{out}$-reduction, that contracts only outermost $\perp$-redexes and which is not closed under contexts. The $\perp_{out}$-reduction, denoted by $\longrightarrow_{\perp_{out}}$, is defined as the smallest binary relation on $\Lambda^\infty_\perp$ such that $C[M] \longrightarrow_{\perp_{out}} C[\perp]$ whenever M is an outermost $\perp$-redex of $C[M]$.

2.4. *The New Reduction: $\eta!$-reduction*

We will now introduce the notion of $\eta!$-rule. It is inspired by Barendregt's $\infty\eta$ construction on Böhm trees (Barendregt, 1984). With the current knowledge of infinite rewriting we see that this relation $\leq_\eta$ on Böhm trees is nothing else but an alternative definition for strongly converging η^{-1}-reduction. For η-expansions strong convergence ensures that the expanded terms remain within $\Lambda^\infty_\perp$ and are finitely branching.
We define the $\eta!$-rule on $\Lambda^\infty_\perp$ as follows:

$$\frac{x \twoheadrightarrow\!\!\!\!\rightarrow_{\eta^{-1}} N \qquad x \notin FV(M)}{\lambda x.MN \to M}\ (\eta!)$$

where $\twoheadrightarrow_{\eta^{-1}}$ denotes *strongly converging* η-expansion. The $\eta!$-reduction, denoted by $\longrightarrow_{\eta!}$, is the smallest reduction on $\Lambda_{\perp}^{\infty}$ closed under the $\eta!$-rule.
The $\eta!$-rule does not appear in the finite lambda calculus. Note that the original notion $\leq_\eta$ in (Barendregt, 1984) is defined on $\beta\perp$-normal forms (Böhm trees) only, while η-expansion $\twoheadrightarrow_{\eta^{-1}}$ applies to any term in $\Lambda_{\perp}^{\infty}$. It is easy to see that $\leq_\eta$ and $\twoheadrightarrow_{\eta^{-1}}$ coincide on the set of $\beta\perp$-normal forms. Hence $\twoheadrightarrow_{\eta^{-1}}$ is an extension of $\leq_\eta$ to the set of (possible infinite) terms containing redexes.
The strength of the new $\eta!$-reduction can be demonstrated on the Böhm tree of Wadsworth's term J mentioned above. The Böhm tree of J is represented by the term $\mathsf{J}_\infty = \lambda x y_0.x(\lambda y_1.y_0(\lambda y_2.y_1(\ldots)))$. We see that J_∞ is of the form $\lambda x y_0.x\mathsf{E}_{y_0}$ where $\mathsf{E}_{y_0} = \lambda y_1.y_0(\lambda y_2.y_1(\ldots))$. The term E_{y_0} is the limit of a strongly converging η-expansion of the variable y_0:

$$y_0 \longrightarrow_{\eta^{-1}} (\lambda y_1.y_0y_1) \longrightarrow_{\eta^{-1}} (\lambda y_1.y_0(\lambda y_2.y_1y_2)) \longrightarrow_{\eta^{-1}} \ldots \mathsf{E}_{y_0}$$

Therefore J_∞ reduces to I in a single $\eta!$-step, while J_∞ is not even a η-redex.

2.5. *The Infinitary Calculus $\lambda_{\perp}^{\infty}$*

The infinitary calculus $\lambda_{\perp}^{\infty}$ has some straightforward properties worthwhile to state on their own which have not been stated explicitly before.

Theorem 2.1 (Confluence, normalisation and compression of $\perp$). The infinitary lambda calculus $\lambda_{\perp}^{\infty}$ is confluent, normalising and ω-compressible. Moreover, $M \twoheadrightarrow_{\perp_{out}} \mathsf{nf}_{\perp}(M)$ for all $M \in \Lambda_{\perp}^{\infty}$.

Proof: Confluence follows from Lemma 26 in (Kennaway et al., 1999). Depth-first left-most $\perp$-reduction is clearly a normalising strategy. Since the depth-first left-most strategy contracts only outermost redexes, we have that $M \twoheadrightarrow_{\perp_{out}} \mathsf{nf}_{\perp}(M)$. It is not difficult to show ω-compression by adapting the proof of the compression lemma for λ_{β}^{∞} in (Kennaway et al., 1997) (quite similar to our later proof of Lemma 3.4). □

2.6. *The Infinitary Lambda Calculus $\lambda_{\beta\perp}^{\infty}$*

In this section we recall some properties of the infinitary lambda calculus $\lambda_{\beta\perp}^{\infty}$ that will be used later.

Theorem 2.2 (Confluence, normalisation and compression of $\beta\perp$ (Kennaway et al., 1997; Kennaway and de Vries, 2003)). The infinitary lambda calculus $\lambda_{\beta\perp}^{\infty}$ is confluent, normalising, ω-compressible and satisfies $\beta\perp$-postponement: if $M \twoheadrightarrow_{\beta\perp} N$, then $M \twoheadrightarrow_{\beta} Q \twoheadrightarrow_{\perp} N$ for some $Q \in \Lambda_{\perp}^{\infty}$.

Corollary 2.3 (Existence of $\perp_{out}$-reduction). For all $M \in \Lambda_{\perp}^{\infty}$, we have that $M \twoheadrightarrow_{\beta} N \twoheadrightarrow_{\perp_{out}} \mathsf{nf}_{\beta\perp}(M)$.

Proof: The previous theorem implies that for any $M \in \Lambda_{\perp}^{\infty}$ there has a reduction to normal form $\mathsf{nf}_{\beta\perp}(M)$ in $\lambda_{\beta\perp}^{\infty}$. By postponement this reduction factors into $M \twoheadrightarrow_{\beta} N \twoheadrightarrow_{\perp} \mathsf{nf}_{\beta\perp}(M)$. This implies that $\mathsf{nf}_{\beta\perp}(M)$ is a normal form of N in $\lambda_{\perp}^{\infty}$. Hence, $N \twoheadrightarrow_{\perp_{out}} \mathsf{nf}_{\perp}(N)$ and $\mathsf{nf}_{\perp}(N) = \mathsf{nf}_{\beta\perp}(M)$ by Theorem 2.1. □

Lemma 2.4 (β-reducing a prefix). Let $M, N \in \Lambda^{\infty}_{\perp}$. If $M \preceq N$ and $M \twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\beta} M'$ then there exists N' such that $N \twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\beta} N'$ and $M' \preceq N'$.

Proof: By induction on the length of the reduction sequence. □

Theorem 2.5 (Monotonicity). Let $M, N \in \Lambda^{\infty}_{\perp}$. If $M \preceq N$ then $\mathsf{nf}_{\beta\perp}(M) \preceq \mathsf{nf}_{\beta\perp}(N)$.

Proof: Let $M, N \in \Lambda^{\infty}_{\perp}$ such that $M \preceq N$. We prove that $\mathsf{nf}_{\beta\perp}(M) \preceq \mathsf{nf}_{\beta\perp}(N)$. By normalisation of $\beta\perp$ and postponement of $\perp$ over β (Theorem 2.2), we have that there exists M' such that $M \twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\beta} M' \twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\perp} \mathsf{nf}_{\beta\perp}(M)$. By Lemma 2.4 we have that $N \twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\beta} N'$ and $M' \preceq N'$ for some N'. Next we prove that for all n, $(\mathsf{nf}_{\beta\perp}(M'))^n \preceq (\mathsf{nf}_{\beta\perp}(N'))^n$ by induction on n. The base case $n = 0$ is trivial. Suppose $n = h + 1$. We have three cases:
Case $M' = \perp$. Then $(\mathsf{nf}_{\beta\perp}(M'))^n = \perp \preceq (\mathsf{nf}_{\beta\perp}(N'))^n$.
Case $M' = \lambda x_1 \ldots x_n.y\ P_1 \ldots P_k$. Then $N' = \lambda x_1 \ldots x_n.y\ Q_1 \ldots Q_k$.

$$\begin{aligned}(\mathsf{nf}_{\beta\perp}(M'))^n &= \lambda x_1 \ldots x_n.y\ (\mathsf{nf}_{\beta\perp}(P_1))^{h-k} \ldots (\mathsf{nf}_{\beta\perp}(P_k))^h \\ &\preceq \lambda x_1 \ldots x_n.y\ (\mathsf{nf}_{\beta\perp}(Q_1))^{h-k} \ldots (\mathsf{nf}_{\beta\perp}(Q_k))^h \quad \text{by induction hypothesis} \\ &= (\mathsf{nf}_{\beta\perp}(M'))^n\end{aligned}$$

Case $M' = \lambda x_1 \ldots x_n.(\lambda y.R)SQ_1 \ldots Q_k$. Since $M' \twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\perp} \mathsf{nf}_{\beta\perp}(M)$, M' cannot have head normal form. Hence $(\mathsf{nf}_{\beta\perp}(M'))^n = \perp \preceq (\mathsf{nf}_{\beta\perp}(N'))^n$.

□

Lemma 2.6 (Increasing Truncations). Let $M, N \in \Lambda^{\infty}_{\perp}$. If $M \longrightarrow_{\beta\perp} N$ then for all n there exists $m \geq n$ such that $\mathsf{nf}_{\beta\perp}(M^m) \succeq \mathsf{nf}_{\beta\perp}(N^n)$.

Proof in the appendix.

Theorem 2.7 (Approximation). Let $M \in \Lambda^{\infty}_{\perp}$. For all n, there exists $m \geq n$ such that $\mathsf{nf}_{\beta\perp}(M^m) \succeq (\mathsf{nf}_{\beta\perp}(M))^n$.

Proof: By Theorem 2.2, there exists a strongly convergent reduction sequence of length ω from M to $\mathsf{nf}_{\beta\perp}(M)$:

$$M = M_0 \longrightarrow_{\beta\perp} M_1 \longrightarrow_{\beta\perp} M_2 \ldots \mathsf{nf}_{\beta\perp}(M)$$

Since this reduction sequence is strongly convergent, for all n there exists M_i such that $(\mathsf{nf}_{\beta\perp}(M))^n = (M_i)^n$. By Lemma 2.6, there exists $m = m_i \geq m_{i-1} \geq \ldots m_0 = n$ such that:

$$\begin{aligned}\mathsf{nf}_{\beta\perp}(M^m) &= \mathsf{nf}_{\beta\perp}(M_0^{m_i}) \\ &\succeq \mathsf{nf}_{\beta\perp}(M_1^{m_{i-1}}) \\ &\ldots \\ &\succeq \mathsf{nf}_{\beta\perp}(M_i^{m_0}) = \mathsf{nf}_{\beta\perp}((\mathsf{nf}_{\beta\perp}(M))^n) = \mathsf{nf}_{\beta\perp}(M^n)\end{aligned}$$

□

Theorem 2.8. Let $M \in \Lambda^{\infty}_{\perp}$. The following statements are equivalent:

1 There exists a head normal form N such that $M \twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\beta} N$.
2 There exists a head normal form N' such that $M \twoheadrightarrow_{\beta} N'$.
3 There exists a head normal form N'' such that $M \twoheadrightarrow_{\beta_h} N''$.
4 There exists a head normal form N''' such that $M \xrightarrow{0}\!\!\!\!\twoheadrightarrow_{\beta} N'''$.

Proof in the appendix.

3. Properties of $\eta!$-reduction

Before we will deal with the interaction of $\eta!$-reduction with β- and $\bot$-reduction in the further sections, we will study a number of useful properties of $\eta!$-reduction and η-expansion. First we show that any $\eta!$-reduction is strongly converging. Next we will demonstrate that $\lambda^{\infty}_{\eta!}$ and $\lambda^{\infty}_{\eta^{-1}}$ are dual calculi in the sense that strongly converging η-expansion and strongly converging $\eta!$-reduction are each others inverse (cf. Lemma 3.2). This allows us to prove that strongly converging $\eta!$-reductions and strongly converging η^{-1}-reductions can be compressed to reductions of length at most ω. It also permits us to prove that the steps of a strongly converging η^{-1}-reduction can be ordered according to their depth. Finally we will show in this section that $\lambda^{\infty}_{\eta!}$ is confluent and normalising.

3.1. *Strong Convergence of $\eta!$*

We will prove that any $\eta!$-reduction (and hence η-reduction) starting from a term in $\Lambda^{\infty}_{\bot}$ is strongly converging. This is a direct result of our choice of depth used in the metric completion $\Lambda^{\infty}_{\bot}$ of $\Lambda_{\bot}$. The infinite term Ω_η is an example of a term that is not in $\Lambda^{\infty}_{\bot}$. Clearly Ω_η η-reduces to itself by contraction of the η-redex at its root. Therefore Ω_η can perform infinitely many η-reductions at depth zero, and hence, it is not strongly converging.

Lemma 3.1 (Strong convergence of $\eta!$). Any $\eta!$-reduction sequence in $\Lambda^{\infty}_{\bot}$ is strongly convergent.

Proof: Strong convergence of $\eta!$ reduction follows by a counting argument. For $M \in \Lambda^{\infty}_{\bot}$, let $|M^n|$ denote the number of abstractions in M^n. The number $|M^n|$ decreases by one if we contract an $\eta!$-redex in M at depth n and it remains equal if we contract an $\eta!$-redex at depth $m > n$. Suppose by contradiction that we have a transfinite $\eta!$-reduction sequence that is not strongly convergent, that is, suppose we have a reduction $M_0 \longrightarrow_{\eta!} M_1 \longrightarrow_{\eta!} \dots$ in which infinitely many reductions occur at depth n. Then infinitely many inequalities in the sequence

$$|M^n_0| \geq |M^n_1| \geq |M^n_2| \geq \dots$$

are strict, which is impossible. Hence the limit of the depth of the contracted redexes in any sequence $M_0 \longrightarrow_{\eta!} M_1 \longrightarrow_{\eta!} \dots$ goes to infinity at each limit ordinal $\leq \alpha$. This implies that all $\eta!$-reduction sequences are strongly converging. $\square$

In contrast to $\eta!$-reduction, η-expansion need not be strongly converging. For instance the following infinite sequence of η-expansions is not Cauchy, as the distance between any two terms in this sequence in this sequence is always 1.

$$M \longrightarrow_{\eta^{-1}} \lambda y_0.My_0 \longrightarrow_{\eta^{-1}} \lambda y_0y_1.My_0y_1 \longrightarrow_{\eta^{-1}} \dots$$

3.2. *Relation between η^{-1} and $\eta!$*

Next, we will show that strongly converging η-expansion is the inverse of strongly converging $\eta!$-reduction: $(\twoheadrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}})^{-1} = \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta!}$. In general $\eta!$-reduction may need less steps than its inverse. For example, while an infinite number of eta expansions is necessary to reach E_x

starting from x, the reverse $\eta!$-reduction can be done in only one step.

$$\begin{tikzcd} x \arrow[r, "\eta^{-1}"'] & (\lambda y_1.xy_1) \arrow[r, "\eta^{-1}"'] & (\lambda y_1.x\lambda y_2.y_1y_2) \arrow[r, twoheadrightarrow, "\eta^{-1}"'] & \mathsf{E}_x \arrow[lll, bend right, "\eta!"'] \end{tikzcd}$$

We will make frequent use of this inverse relationship. The proof of the inverse relationship (Theorem 3.5) will follow from some smaller results and ω-compression lemmas for $\eta!$ and η^{-1}. These compression lemmas will simplify many later proofs.

Lemma 3.2 (Inverse of one step reduction). Let M, N in $\Lambda_\perp^\infty$.

1 If $M \xrightarrow{n}_{\eta^{-1}} N$, then $N \xrightarrow{n}_{\eta!} M$.

2 If $M \xrightarrow{n}_{\eta!} N$, then $N \xrightarrow{\geq n}\!\!\!\twoheadrightarrow_{\eta^{-1}} M$.

Proof: The first statement is trivial. The second statement follows directly from the definitions of $\eta!$ and η^{-1} as illustrated in the next diagram. If the depth of the $\eta!$-redex in $M \longrightarrow_{\eta!} N$ is n then the η^{-1}-redexes in $N \twoheadrightarrow_{\eta^{-1}} M$ occur at least at depth n.

$$\begin{tikzcd} C[\lambda x.PQ] \arrow[r, "n", "\eta!"'] & C[P] \arrow[d, dotted, "\eta^{-1}"', "n"] \\ & C[\lambda x.Px] \arrow[ul, dotted, twoheadrightarrow, ">n"', "\eta^{-1}"] \end{tikzcd}$$

□

Lemma 3.3 (Inverse reductions restricted to ω-length). Let M, N in $\Lambda_\perp^\infty$.

1 If $M \twoheadrightarrow_{\eta!} N$ is of length at most ω then $N \twoheadrightarrow_{\eta^{-1}} M$.

2 If $M \twoheadrightarrow_{\eta^{-1}} N$ is of length at most ω then $N \twoheadrightarrow_{\eta!} M$.

Proof: We only prove the first item using induction on the length α of the reduction sequence from M to N. The proof of the second item is similar.
The base case $\alpha = 0$ is trivial. The successor case $\alpha = n + 1$ follows easily from Lemma 3.2 and the induction hypothesis, as shown in the next diagram:

$$\begin{tikzcd} M_0 \arrow[rr, bend left, twoheadrightarrow, "n \text{ steps}", "\eta!"'] \arrow[rr, phantom, "\text{IH}"] & & M_1 \arrow[ll, bend left, dotted, twoheadrightarrow, "\eta^{-1}"] \arrow[rr, bend left, "1 \text{ step}", "\eta!"'] \arrow[rr, phantom, "\text{Lem 3.2}"] & & M_2 \arrow[ll, bend left, dotted, twoheadrightarrow, "\eta^{-1}"] \end{tikzcd}$$

Limit case $\alpha = \omega$. By strong convergence, the number of steps at certain depth n is finite. We can, then, always split the sequence by depth as follows.

$$M = M_0 \xrightarrow[\eta!]{\geq 0}\!\!\!\twoheadrightarrow M_1 \xrightarrow[\eta!]{\geq 1}\!\!\!\twoheadrightarrow M_2 \twoheadrightarrow M_\omega = N$$

Now consider the last step occurring at depth 0 in this sequence. The position of its redex is still present in all terms that follow M_1, including M_ω. By reversing this last $\eta!$-step at depth

0 in the limit M_ω we construct the following diagram:

$$\begin{tikzcd}
M = M_0 \arrow[r, two heads, "\geq 0", "\eta!"'] & C[\lambda x.PQ] \arrow[r, "0", "\eta!"'] \arrow[dr, equal] & C[P] = M_1 \arrow[r, two heads, "> 0", "\eta!"'] \arrow[d, dotted, two heads, "\geq 0"', "\eta^{-1}"] & M_\omega = C'[P'] \arrow[d, dotted, two heads, "\geq 0"', "\eta^{-1}"] \\
& & C[\lambda x.PQ] \arrow[r, two heads, "> 0", "\eta!"'] & C'[\lambda x.P'Q]
\end{tikzcd}$$

We repeat this process for each step at depth 0 and obtain a term N_1 such that:

$$\begin{tikzcd}
M \arrow[r, two heads, "\geq 0", "\eta!"'] \arrow[drr, dotted, two heads, "> 0", "\eta!"'] & M_1 \arrow[r, two heads, "> 0", "\eta!"'] & N \arrow[d, dotted, two heads, "\geq 0"', "\eta^{-1}"] \\
& & N_1
\end{tikzcd}$$

Since all steps in the $\eta!$-reductions sequence from M to N_1 occur at depth greater than 0, the terms M and N_1 coincide at depth 0.
Repeating the above argument on the reduction sequence $M \xrightarrow{\geq 1}\!\!\twoheadrightarrow_{\eta!} N_1$, we find a term N_2 such that $N_1 \xrightarrow{\geq 1}\!\!\twoheadrightarrow_{\eta^{-1}} N_2$ and $M \xrightarrow{> 1}\!\!\twoheadrightarrow_{\eta!} N_2$. Moreover, M and N_2 coincide up to depth 1. In this way we can construct an η^{-1}-reduction sequence starting from N as indicated in the next diagram:

$$\begin{tikzcd}
M \arrow[d, two heads, "\eta!"'] \arrow[dr, dotted, two heads, "> 0", "\eta!"'] \arrow[drr, dotted, two heads, "> 1", "\eta!"'] \arrow[drrr, dotted, two heads, "> 2", "\eta!"'] \arrow[drrrr, dotted, dash, "="] & & & & \\
N \arrow[r, dotted, two heads, "\geq 0", "\eta^{-1}"'] & N_1 \arrow[r, dotted, two heads, "\geq 1", "\eta^{-1}"'] & N_2 \arrow[r, dotted, two heads, "\geq 2", "\eta^{-1}"'] & N_3 \arrow[r, dotted, dash] & N_\omega
\end{tikzcd}$$

Because the reduction sequence that start from N is strongly converging, it has a limit, say N_ω. Since each term N_i coincides with M up to depth i, the limit N_ω of this sequence is exactly M. □

Lemma 3.4 (Compression for η^{-1} and $\eta!$). Strongly converging reduction is ω-compressible in $\lambda^\infty_{\eta^{-1}}$ and $\lambda^\infty_{\eta!}$.

Proof: First we consider $\lambda^\infty_{\eta!}$. The proof proceeds by transfinite induction on the length of the reduction sequence. By a general argument (Kennaway et al., 1995b; Kennaway and de Vries, 2003) it is sufficient to prove that a sequence of length $\omega + 1$ can be compressed into one of length ω. Without loss of generality, we may suppose that we have a strongly convergent $\eta!$-reduction sequence of length $\omega + 1$ as follows:

$$\begin{tikzcd}
\lambda x.M_0N_0 \arrow[r, "\eta!"'] \arrow[d, dotted, "\eta!"'] & \lambda x.M_1N_1 \arrow[r, "\eta!"'] \arrow[d, dotted, "\eta!"'] & \lambda x.M_2N_2 \arrow[r, two heads, "\eta!"'] \arrow[d, dotted, two heads, "\eta!"'] & \lambda x.M_\omega N_\omega \arrow[d, "\eta!"'] \\
M_0 \arrow[r, dotted, "{=,\eta!}"'] & M_1 \arrow[r, dotted, "{=,\eta!}"'] & M_2 \arrow[r, dotted, two heads, "{=,\eta!}"'] & M_\omega
\end{tikzcd}$$

Note that $M_i \twoheadrightarrow_{\eta!} M_\omega$ and $N_i \twoheadrightarrow_{\eta!} N_\omega$ for all i. By Lemma 3.3, $N_\omega \twoheadrightarrow_{\eta^{-1}} N_i$. Since $\lambda x.M_\omega N_\omega$ is an $\eta!$-redex, we have that $x \twoheadrightarrow_{\eta^{-1}} N_\omega$. Hence $x \twoheadrightarrow_{\eta^{-1}} N_\omega \twoheadrightarrow_{\eta^{-1}} N_i$ and all terms $\lambda x.M_iN_i$ in the top row are $\eta!$-redexes. Contracting them, we obtain the terms of

the bottom row. The reduction in the bottom row is the projection of the reduction in the top row. This way we obtain a sequence of length ω from $\lambda x.M_0 N_0$ to M_ω.
The proof of compression for $\lambda^{\infty}_{\eta^{-1}}$ follows a similar pattern, but without the appeal to Lemma 3.3. $\square$

Lemma 3.4 allows us to remove the conditions on length in Lemma 3.3.

Theorem 3.5 (Inverse reductions). $M \twoheadrightarrow_{\eta!} N$ if and only if $N \twoheadrightarrow_{\eta^{-1}} M$ for any M, N in $\Lambda^{\infty}_{\perp}$.

Thus we have shown at the main result of this section that strongly converging $\eta!$-reduction is the inverse of strongly converging η^{-1}-reduction.

3.3. *Confluence of* $\eta!$

In this section we will show that $\eta!$-reduction is confluent. The main ingredients of the proof are Local Confluence and the Strip Lemma for $\eta!$.

Lemma 3.6 (Preservation of $\eta!$-redexes by $\eta!$). If $\lambda x.MN$ is an $\eta!$-redex and $N \twoheadrightarrow_{\eta!} N'$, then $\lambda x.MN'$ is also an $\eta!$-redex.

In order to prove the previous lemma, one proves that:

Lemma 3.7 (Preservation of η-expansions of x after $\eta!$). If $x \twoheadrightarrow_{\eta^{-1}} M$ and $M \twoheadrightarrow_{\eta!} M'$, then $x \twoheadrightarrow_{\eta^{-1}} M'$.

Proof in the appendix.

Lemma 3.8 (Local $\eta!$-Confluence). Given $M \longrightarrow_{\eta!} M_1$ and $M \longrightarrow_{\eta!} M_2$, there exists M_3 such that the following diagram holds:

$$\begin{tikzcd} M_0 \arrow[r, "m", "\eta!"'] \arrow[d, "n"', "\eta!"] & M_1 \arrow[d, dotted, "n"', "{=,\eta!}"] \\ M_2 \arrow[r, dotted, "m", "{=,\eta!}"'] & M_3 \end{tikzcd}$$

Proof: A case analysis on the relative positioning of the $\eta!$-redexes. We prove the case when the $\eta!$-redex $\lambda y.PQ$ is inside the argument N of the other $\eta!$-redex $\lambda x.MN$, i.e. $M_0 = C_1[\lambda x.MC_2[\lambda y.PQ]]$. By Lemma 3.1, $C_2[P]$ is an η-expansion of x and we can construct the following diagram:

$$\begin{tikzcd} C_1[\lambda x.M[C_2[\lambda y.PQ]] \arrow[r, "m", "\eta!"'] \arrow[d, "n"', "\eta!"] & C_1[\lambda x.M[C_2[P]] \arrow[d, dotted, "n"', "\eta!"] \\ C_1[M] \arrow[r, dotted, equal] & C_1[M] \end{tikzcd}$$

$\square$

Note that the annotation of the reduction depths in the above local confluence diagram implies that the depth of an $\eta!$-redex in a term does not change when we contract an $\eta!$-redex elsewhere in the term.

Lemma 3.9 (Strip Lemma for $\eta!$). Given a strongly converging reduction $M \twoheadrightarrow_{\eta!} P$ and a one step reduction $M \longrightarrow_{\eta!} N$, then we can construct the following diagram with elementary local confluence tiles:

$$\begin{tikzcd} M \arrow[rr, twoheadrightarrow, "\eta!"'] \arrow[d, "m"', "\eta!"] & & P \arrow[d, dotted, "m"', "{=,\eta!}"] \\ N \arrow[rr, dotted, twoheadrightarrow, "{=,\eta!}"'] & & Q \end{tikzcd}$$

Proof: By Lemma 3.4 (Compression Lemma) we can assume that the sequence has length ω.

$$\begin{tikzcd} M_0 \arrow[r, "n_0", "\eta!"'] \arrow[d, "m"', "\eta!"] & M_1 \arrow[r, "n_1", "\eta!"'] \arrow[d, dotted, "m"', "{=,\eta!}"] & M_2 \arrow[r, "n_2", "\eta!"'] \arrow[d, dotted, "m"', "{=,\eta!}"] & M_3 \arrow[d, dotted, "m"', "{=,\eta!}"] & \dots & M_\omega \arrow[d, dotted, "m"', "{=,\eta!}"] \\ N_0 \arrow[r, dotted, "n_0", "\eta!"'] & N_1 \arrow[r, dotted, "n_1", "\eta!"'] & N_2 \arrow[r, dotted, "n_2", "\eta!"'] & N_3 & \dots & ? \end{tikzcd}$$

Using Lemma 3.6, we can complete all the subdiagrams except for the limit case. The constructed reduction $N_0 \longrightarrow_{\eta!} N_1 \longrightarrow_{\eta!} N_2 \longrightarrow_{\eta!} \dots$ is strongly converging, say with limit N_ω. Either the vertical $\eta!$-reduction $M_0 \longrightarrow_{\eta!} N_0$ got cancelled out in one of the applications of Local Confluence or not. If it gets cancelled out, then, from that moment on, all vertical reductions are reductions of length 0, implying that M_ω is equal to the limit N_ω. Or the vertical $\eta!$-reduction $M_0 \longrightarrow_{\eta!} N_0$ did not get cancelled out, implying that its residual is present in M_k, for all $k \geq 0$. That is, all M_k with $k \geq 0$ are of the form $C_k[\lambda x.S_k T_k]$, where all the $C_k[\ \ ]$ have the hole at the same position at depth m, and all N_k with $k \geq 0$ are of the form $C_k[S_k]$. The limit term M_ω is of the form $C_\omega[\lambda x.S_\omega T_\omega]$ and the hole of C_ω is also at depth m. By Lemma 3.1, $\lambda x.S_\omega T_\omega$ is an $\eta!$-redex. Contracting this redex in the limit M_ω we obtain $C_\omega[S_\omega]$ which is equal to the limit N_ω of the bottom sequence. □

Theorem 3.10 ($\eta!$-Confluence). The infinitary calculus $\lambda^\infty_{\eta!}$ is confluent.

Proof: Confluence of $\lambda^\infty_{\eta!}$ can be shown by a simultaneous induction on the length of the two given coinitial $\eta!$-reductions. By compression (Lemma 3.4) we may assume that these reductions are at most of length ω, so here we don't need transfinite induction.

The induction proof makes use of so called tiling diagrams (Kennaway and de Vries, 2003), which can be constructed using the induction hypothesis, Lemma 3.6 (Local Confluence) and Lemma 3.7 (Strip Lemma). The important thing to note is that the depth of an $\eta!$-redex in a term does not change when we contract an $\eta!$-redex elsewhere in the term.

The double limit case is more involved. In that case we can construct the tiling diagram shown below. The induction hypothesis allows us to construct all proper subtiling diagrams. It remains to show that the bottom row reduction and the right-most column reduction strongly converge to the same limit.

Clearly, by the fact that all subtiles are depth preserving, both the bottom row reduction and the right-most column reduction inherit the strong convergence property from respectively the top row and the left-most column reductions. Using strong convergence we can show that for any k there exists k_1, k_2 such that for all $i \geq k_1$ and $j \geq k_2$ the terms $M_{i,j}$ have the same prefix up to depth k. Hence the limits of the bottom row reduction and the right-most column reduction are the same.

Alternatively, one can check that the conditions of the general tiling diagram theorem in (Kennaway and de Vries, 2003) are satisfied to conclude that both limits are the same.

$$
\begin{tikzcd}
M_{0,0} \arrow[r, "m_0", "\eta!"'] \arrow[d, "n_0"', "\eta!"] & M_{0,1} \arrow[r, "m_1", "\eta!"'] \arrow[d, dotted, "n_0"', "\eta!"] & M_{0,2} \arrow[r, dotted, dash] \arrow[d, dotted, "n_0"', "\eta!"] & M_{0,\omega} \arrow[d, dotted, "n_0"', "\eta!"] \\
M_{1,0} \arrow[r, dotted, "m_0", "\eta!"'] \arrow[d, "n_1"', "\eta!"] & M_{1,1} \arrow[r, dotted, "m_1", "\eta!"'] \arrow[d, dotted, "n_1"', "\eta!"] & M_{1,2} \arrow[r, dotted, dash] \arrow[d, dotted, "n_1"', "\eta!"] & M_{1,\omega} \arrow[d, dotted, "n_1"', "\eta!"] \\
M_{2,0} \arrow[r, dotted, "m_0", "\eta!"'] \arrow[d, dotted, dash] & M_{2,1} \arrow[r, dotted, "m_1", "\eta!"'] \arrow[d, dotted, dash] & M_{2,2} \arrow[r, dotted, dash] \arrow[d, dotted, dash] & M_{2,\omega} \arrow[d, dotted, dash] \\
M_{\omega,0} \arrow[r, dotted, "m_0", "\eta!"'] & M_{\omega,1} \arrow[r, dotted, "m_1", "\eta!"'] & M_{\omega,2} \arrow[r, dotted, dash] & ?
\end{tikzcd}
$$

□

In a similar way one can prove that strongly converging η-expansion is confluent. We skip the proof, as we don't need this result in this paper.

3.4. *Normalisation of $\eta!$-reduction*

We finish this section showing that $\lambda^{\infty}_{\eta!}$-calculus is normalising in contrast to $\lambda^{\infty}_{\eta^{-1}}$-calculus which is not normalising.

Theorem 3.11 (Normalisation of $\lambda^{\infty}_{\eta!}$). The infinitary lambda calculus $\lambda^{\infty}_{\eta!}$ is depth-first left-most normalising.

Proof: Let M_0 be some lambda term in $\lambda^{\infty}_{\eta!}$. Consider the reduction $M_0 \longrightarrow_{\eta!} M_1 \longrightarrow_{\eta!} M_2 \ldots$ in which each M_{i+1} is obtained from its predecessor M_i by contracting the depth-first left-most $\eta!$-redex in M_i. By Lemma 3.1 this reduction is strongly converging. If it is finite, then the last term is an $\eta!$-normal form. If it is infinite, then by strong convergence it has a limit M_ω. By a reductio ad absurdum M_ω must be an $\eta!$-normal form as well: For suppose M_ω contains an $\eta!$-redex $\lambda x.PX$ at some position p. Then, by strong convergence, there is an M_n in the reduction that contains a subterm of the form$\lambda x.P'X'$ at position p, while all reduction steps after M_n take place at depth greater than the depth of $\lambda x.P'X'$. Hence $X' \twoheadrightarrow_{\eta!} X$, and so $X \twoheadrightarrow_{\eta^{-1}} X'$ by Lemma 3.5. We also have that $x \twoheadrightarrow_{\eta^{-1}} X$, because $\lambda x.PX$ is an $\eta!$-redex. Therefore $x \twoheadrightarrow_{\eta^{-1}} X'$. Thus $\lambda x.P'X'$ must also be an $\eta!$-redex in M_n. Since the later reductions steps in $M_n \twoheadrightarrow_{\eta!} M_\omega$ take place at greater depth than $\lambda x.P'X'$. This contradicts the fact that the reduction $M_0 \twoheadrightarrow_{\eta!} M_\omega$ is depth-first left-most. □

The combination of the previous result with the confluence of $\eta!$-reduction give us uniqueness of normal forms as corollary:

Corollary 3.12 (Uniqueness of $\eta!$-normal forms). Each lambda term in $\lambda^{\infty}_{\eta!}$ has a unique $\eta!$-normal form.

4. Commutation Properties for β and $\eta!$

In this section we will prove various instances of commutation of β and $\eta!$ to be used in the proof of confluence of $\lambda^{\infty}_{\beta\bot\eta!}$.

4.1. *Local Commutation for One Step β and One Step $\eta!$*

The first commutation property that we consider is local commutation for one step β and one step $\eta!$.

Lemma 4.1 (Preservation of $\eta!$-redexes by β). If $\lambda x.MN$ is an $\eta!$-redex and $N \twoheadrightarrow_\beta N'$, then $\lambda x.MN'$ is also an $\eta!$-redex.

In order to prove the previous lemma, one proves that:

4.1 Lemma: if $x \twoheadrightarrow_{\eta^{-1}} N$ and $N \twoheadrightarrow_\beta N'$, then $x \twoheadrightarrow_{\eta^{-1}} N'$.

Proof in the appendix.

Lemma 4.2 (Local Commutation of β and $\eta!$). If $M_0 \longrightarrow_{\eta!} M_1$ and $M_0 \longrightarrow_\beta M_2$, then there exists an M_3 such that the following diagram holds:

$$\begin{tikzcd} M_0 \arrow[r, "m", "\eta!"'] \arrow[d, "n"', "\beta"] & M_1 \arrow[d, dotted, "n"', "{=,\beta}"] \\ M_2 \arrow[r, dotted, two heads, "\geq m-1", "\eta!"'] & M_3 \end{tikzcd}$$

Proof: Suppose M_0 can do both a β-reduction and an $\eta!$-reduction at respectively depths n and m. We prove only one case. The β-redex is inside the expanded variable term of the $\eta!$-redex, that is M_0 is of the form $C_1[\lambda x.MN]$ and $N = C_2[(\lambda y.P)Q]$. By Lemma 4.1, we have that if $N \longrightarrow_\beta N'$ then $\lambda x.MN'$ is also an $\eta!$-redex.

$$\begin{tikzcd} C_1[\lambda x.MN] \arrow[r, "m", "\eta!"'] \arrow[d, "n"', "\beta"] & C_1[M] \arrow[d, dash, dotted] \\ C_1[\lambda x.MN'] \arrow[r, dotted, "m", "\eta!"'] & C_1[M] \end{tikzcd}$$

$\square$

4.2. *Strip Lemma for One step β over $\eta!$*

Next we prove the strip lemma for one step β over $\eta!$.

Lemma 4.3 (Strip Lemma for $\longrightarrow_\beta$ over $\twoheadrightarrow_{\eta!}$). Given $M \longrightarrow_\beta P$ and $M \twoheadrightarrow_{\eta!} N$, then there exists Q such that:

$$\begin{tikzcd} M \arrow[rr, two heads, "\eta!"] \arrow[d, "m"', "\beta"] & & N \arrow[d, dotted, "m"', "{=,\beta}"] \\ P \arrow[rr, dotted, two heads, "\eta!"'] & & Q \end{tikzcd}$$

Proof: The proof is similar to the proof of the strip lemma for $\eta!$. By Lemma 3.4 (Compression Lemma) we can assume that the sequence has length at most ω.

$$\begin{tikzcd}
M = M_0 \arrow[r, "n_0", "\eta!"'] \arrow[d, "m"', "\beta"] & M_1 \arrow[r, "n_1", "\eta!"'] \arrow[d, dotted, "m"', "{=,\beta}"] & M_2 \arrow[r, "n_2", "\eta!"'] \arrow[d, dotted, "m"', "{=,\beta}"] & M_3 \arrow[d, dotted, "m"', "{=,\beta}"] & \dots & M_\omega = N \arrow[d, dotted, "m"', "{=,\beta}"] \\
N_0 \arrow[r, dotted, twoheadrightarrow, "\geq n_0 - 1", "\eta!"'] & N_1 \arrow[r, dotted, twoheadrightarrow, "\geq n_1 - 1", "\eta!"'] & N_2 \arrow[r, dotted, twoheadrightarrow, "\geq n_2 - 1", "\eta!"'] & N_3 & \dots & ?
\end{tikzcd}$$

Using Lemma 4.1, we can complete all the subdiagrams except for the limit case. Either the vertical β-reduction got cancelled out in one of the applications of Local Confluence or not. If it gets cancelled out, then from that moment on all vertical reductions are reductions of length 0, implying that M_ω is equal to the limit N_ω. Or, if the vertical β-reduction did not get cancelled out, then a residual of the β-redex in M is present in all terms M_k. Hence $M_k = C_k[(\lambda x.P_k)Q_k]$ for all $k \geq 0$, and in all $C_k[\;]$ the hole occurs at the same position at depth m, so that all N_k are of the form $C_k[P_k]$. This holds for the limit terms as well. Contracting this residual in the limit M_ω gives us the limit N_ω. $\square$

Lemma 4.4 (Strip Lemma for $\longrightarrow_{\eta!}$ over $\twoheadrightarrow_\beta$). Let X be in $\beta\bot$-normal form. If $M = C[\lambda x.M_0 X] \longrightarrow_{\eta!} C[M_0] = P$ and $M \twoheadrightarrow_\beta N$, then there exists Q such that:

$$\begin{tikzcd}
M \arrow[r, twoheadrightarrow, "\beta"] \arrow[d, "\eta!"] & N \arrow[d, dotted, twoheadrightarrow, "\eta!"] \\
P \arrow[r, dotted, twoheadrightarrow, "\beta"'] & Q
\end{tikzcd}$$

Proof in the appendix.

5. Commutation of $\eta!$ and $\bot_{out}$

Full commutation of $\eta!$ and $\bot$ does not hold. Already local commutation of η and $\bot$ goes wrong (cf. (Severi and de Vries, 2002)) when the contracted $\bot$-redex is not outermost. Take for instance $\Omega \;{}_{\eta!}\!\!\longleftarrow \lambda x.\Omega x \to_\bot \lambda x.\bot$. However, for proving confluence of $\lambda^\infty_{\beta\bot\eta!}$ it is sufficient that $\eta!$-reduction commutes with $\bot_{out}$-reduction. Recall that $\longrightarrow_{\bot_{out}}$ is the reduction that replaces an outermost subterm without head normal form by $\bot$. The proof of this commutation property then follows the familiar pattern.

Lemma 5.1 (Local $\eta!\bot_{out}$-Commutation). If $M_0 \longrightarrow_{\eta!} M_1$ and $M_0 \longrightarrow_{\bot_{out}} M_2$, there exists M_3 such that

$$\begin{tikzcd}
M_0 \arrow[r, "m", "\eta!"'] \arrow[d, "\bot_{out}"', "n"] & M_1 \arrow[d, dotted, "\bot_{out}"', "n"] \\
M_2 \arrow[r, dotted, "m", "{=,\eta!}"'] & M_3
\end{tikzcd}$$

Proof in the appendix.

Lemma 5.2 (Strip Lemma for $\longrightarrow_{\bot_{out}}$ over $\twoheadrightarrow_{\eta!}$). Given a one step reduction $M \longrightarrow_{\bot_{out}} N$ and a strongly converging reduction $M \twoheadrightarrow_{\eta!} P$, then there exists Q such that:

$$\begin{tikzcd} M \arrow[r, two heads, "\eta!"'] \arrow[d, "\bot_{out}"', "n"] & P \arrow[d, dotted, "\bot_{out}"', "n"] \\ N \arrow[r, dotted, two heads, "\eta!"'] & Q \end{tikzcd}$$

Proof in the appendix.

Lemma 5.3 (Strip Lemma for $\longrightarrow_{\eta!}$ over $\twoheadrightarrow_{\bot_{out}}$). Given a one step reduction $M \longrightarrow_{\eta!} N$ and a strongly converging reduction $M \twoheadrightarrow_{\bot_{out}} P$, then there exists Q such that:

$$\begin{tikzcd} M \arrow[r, two heads, "\bot_{out}"'] \arrow[d, "\eta!"', "n"] & P \arrow[d, dotted, "{=,\eta!}"', "n"] \\ N \arrow[r, dotted, two heads, "\bot_{out}"'] & Q \end{tikzcd}$$

Proof in the appendix.

Theorem 5.4 ($\eta!\bot_{out}$-Commutation). Strongly converging $\eta!$-reduction commutes with strongly converging $\bot_{out}$-reduction: $\twoheadleftarrow_{\eta!} \circ \twoheadrightarrow_{\bot_{out}} \subseteq \twoheadrightarrow_{\bot_{out}} \circ \twoheadleftarrow_{\eta!}$

Proof: Similar to the confluence proof of $\eta!$ (Theorem 3.8), but using Lemmas 5.1, 5.2 and 5.3 instead. □

6. Confluence and Normalisation of $\beta\bot\eta!$

We are now ready to prove the main results of this paper concerning confluence and normalisation of the infinite extensional lambda calculus $\lambda^{\infty}_{\beta\bot\eta!}$.

Theorem 6.1 (Preservation of $\beta\bot$-normal forms by $\eta!$). If $M \twoheadrightarrow_{\eta!} N$ and M is a $\beta\bot$-normal form, then N is a $\beta\bot$-normal form.

Proof in the appendix.

Theorem 6.2 (β-normalization of an η-expansion of x). If $x \twoheadrightarrow_{\eta^{-1}} X$, then $x \twoheadrightarrow_{\eta^{-1}} \mathsf{nf}_{\beta}(X)$.

Proof in the appendix.

Theorem 6.3 (Projecting $\beta\bot\eta!$-reductions onto $\eta!$-reductions via $\mathsf{nf}_{\beta\bot}$). If $M \twoheadrightarrow_{\beta\bot\eta!} N$ then $\mathsf{nf}_{\beta\bot}(M) \twoheadrightarrow_{\eta!} \mathsf{nf}_{\beta\bot}(N)$ and the following diagram commutes:

$$\begin{tikzcd} M_0 \arrow[r, two heads, "\beta\bot\eta!"] \arrow[d, two heads, "\beta\bot"'] & M_\alpha \arrow[d, two heads, "\beta\bot"'] \\ \mathsf{nf}_{\beta\bot}(M_0) \arrow[r, two heads, "\eta!"] & \mathsf{nf}_{\beta\bot}(M_\alpha) \end{tikzcd}$$

Proof: We prove for all α, that if $M_0 \twoheadrightarrow_{\beta\bot\eta!} M_\alpha$ then $\mathsf{nf}_{\beta\bot}(M_0) \twoheadrightarrow_{\eta!} \mathsf{nf}_{\beta\bot}(M_\alpha)$, by induction on α.
Case $\alpha = 0$. This is trivial.

Case $\alpha = \gamma + 1$. By the induction hypothesis, we have that $\mathsf{nf}_{\beta\bot}(M_0) \twoheadrightarrow_{\eta!} \mathsf{nf}_{\beta\bot}(M_\gamma)$ We distinguish two cases depending on the last step:

1 If the last step is a $\beta\bot$-reduction step, we have that $\mathsf{nf}_{\beta\bot}(M_\gamma) = \mathsf{nf}_{\beta\bot}(M_{\gamma+1})$ by Theorem 2.2.

$$\begin{tikzcd}
M_0 \arrow[rr, two heads, "\beta\bot\eta!"] \arrow[d, two heads, "\beta\bot"'] & & M_\gamma \arrow[r, "\beta\bot"] \arrow[d, two heads, "\beta\bot"'] & M_{\gamma+1} \arrow[d, "\beta\bot"'] \\
\mathsf{nf}_{\beta\bot}(M_0) \arrow[rr, two heads, "\eta!"] & (\text{IH}) & \mathsf{nf}_{\beta\bot}(M_\delta) \arrow[r, equal, "(\text{Thm 2.2})"] & \mathsf{nf}_{\beta\bot}(M_{\delta+1})
\end{tikzcd}$$

2 If the last step is an $\eta!$-reduction step, then we first normalise the term X of the $\eta!$-redex in $M_\gamma = C[\lambda x.PX]$. By Theorem 6.1, $x \twoheadrightarrow_{\eta^{-1}} \mathsf{nf}_\beta(X)$. Hence $C[\lambda x.P\ \mathsf{nf}_\beta(X)]$ $\eta!$-reduces to $C[P]$. Then we split the $\beta\bot$-reduction sequence from $C[\lambda x.P\mathsf{nf}_\beta(X)]$ to the normal form $\mathsf{nf}_{\beta\bot}(M_\gamma)$ into a β-reduction sequence followed by a $\bot_{out}$-reduction sequence using Theorems 2.2 and 2.3. This is depicted in the following diagram:

$$\begin{tikzcd}
M_0 \arrow[rr, two heads, "\beta\bot\eta!"] \arrow[ddd, two heads, "\beta\bot"'] & & M_\gamma = C[\lambda x.PX] \arrow[r, "\eta!"] \arrow[d, two heads, "\beta"'] \arrow[dr, phantom, "(\text{Thm 6.1})"] & C[P] = M_{\gamma+1} \arrow[d, dotted, equal] \\
& & C[\lambda x.P\ \mathsf{nf}_\beta(X)] \arrow[r, "\eta!"] \arrow[d, two heads, "\beta"'] \arrow[dr, phantom, "(\text{Lem 4.3})"] & C[P] = M_{\gamma+1} \arrow[d, dotted, two heads, "\beta"] \\
& (\text{IH}) & N_1 \arrow[r, dotted, two heads, "\eta!"] \arrow[d, two heads, "\bot_{out}"'] \arrow[dr, phantom, "(\text{Thm 5.4})"] & N_2 \arrow[d, dotted, two heads, "\bot_{out}"] \\
\mathsf{nf}_{\beta\bot}(M_0) \arrow[rr, dotted, two heads, "\eta!"'] & & \mathsf{nf}_{\beta\bot}(M_\gamma) \arrow[r, dotted, two heads, "\eta!"'] & \mathsf{nf}_{\beta\bot}(M_{\gamma+1})
\end{tikzcd}$$

Next, applying first the Strip Lemma 4.3 for $\longrightarrow_{\eta!}$ with $\twoheadrightarrow_\beta$ and secondly the full $\eta!\bot_{out}$-Commutation Theorem 5.4, we find a term Q such that $\mathsf{nf}_{\beta\bot}(M_\gamma) \twoheadrightarrow_{\eta!} Q$ and $C[P] \twoheadrightarrow_{\eta!} \circ \twoheadrightarrow_{\bot_{out}} Q$. Since $\mathsf{nf}_{\beta\bot}(M_\gamma)$ is a $\beta\bot$-normal form, so is Q by Theorem E.3. Hence by the unicity of $\beta\bot$-normal forms in $\lambda^\infty_{\beta\bot}$ implied by Theorem 2.2 we find that $Q = \mathsf{nf}_{\beta\bot}(M_{\gamma+1})$.

Case $\alpha = \lambda$. By the induction hypothesis, we have that $\mathsf{nf}_{\beta\bot}(M_0) \twoheadrightarrow_{\eta!} \mathsf{nf}_{\beta\bot}(M_\gamma)$ for all $\gamma < \lambda$.

$$\begin{tikzcd}
M_0 \arrow[r, "\beta\bot\eta!"'] \arrow[d, two heads, "\beta\bot"] & M_1 \arrow[r, "\beta\bot\eta!"'] \arrow[d, two heads, "\beta\bot"] & M_2 \arrow[r, two heads] \arrow[d, two heads, "\beta\bot"] & M_\lambda \arrow[d, two heads, "\beta\bot"] \\
\mathsf{nf}_{\beta\bot}(M_0) \arrow[r, "\eta!"'] & \mathsf{nf}_{\beta\bot}(M_1) \arrow[r, "\eta!"'] & \mathsf{nf}_{\beta\bot}(M_2) \arrow[r, two heads] & N \stackrel{?}{=} \mathsf{nf}_{\beta\bot}(M_\lambda)
\end{tikzcd}$$

Since all $\eta!$-reduction sequences are strongly convergent (Theorem 3.1), the bottom reduction sequence is strongly convergent, and hence has a limit, say N. To conclude that N is in fact $\mathsf{nf}_{\beta\bot}(M_\lambda)$, it suffices to prove that for all n, $N^n = (\mathsf{nf}_{\beta\bot}(M_\lambda))^n$.
By the Approximation Theorem 2.7, we have that there exists $m \geq n$ such that $(M_\lambda)^m \twoheadrightarrow_{\beta\bot} \mathsf{nf}_{\beta\bot}((M_\lambda)^m) = P \succeq (\mathsf{nf}_{\beta\bot}(M_\lambda))^n$. Since $M_0 \twoheadrightarrow_{\beta\bot\eta!} M_\lambda$ is strongly convergent, we have

for some large enough γ_0 that for all $\gamma \geq \gamma_0$, $(M_\gamma)^m = (M_\lambda)^m$. Hence, for all $\gamma \geq \gamma_0$, we have that $P = \mathsf{nf}_{\beta\bot}((M_\lambda)^m) = \mathsf{nf}_{\beta\bot}((M_\gamma)^m) \preceq \mathsf{nf}_{\beta\bot}(M_\gamma)$ by Monotonicity of $\mathsf{nf}_{\beta\bot}$ (Theorem 2.5). Hence the terms of the bottom reduction sequence have all the same prefix P from $\mathsf{nf}_{\beta\bot}(M_{\gamma_0})$ onwards. Hence P is a prefix of their limit N. Therefore $N^n = P^n = (\mathsf{nf}_{\beta\bot}(M_\lambda))^n$.

□

Theorem 6.4 (Confluence and normalization of $\beta\bot\eta!$). The infinite extensional lambda calculus $\lambda^\infty_{\beta\bot\eta!}$ is confluent and normalising.

Proof: We first prove confluence. Suppose $M_0 \twoheadrightarrow_{\beta\bot\eta!} M_1$ and $M_0 \twoheadrightarrow_{\beta\bot\eta!} M_2$. Then by Theorem 6.2 we project these these $\beta\bot\eta!$-reductions onto $\mathsf{nf}_{\beta\bot}(M_0) \twoheadrightarrow_\eta \mathsf{nf}_{\beta\bot}(M_1)$ and $\mathsf{nf}_{\beta\bot}(M_0) \twoheadrightarrow_\eta \mathsf{nf}_{\beta\bot}(M_2)$. The η!-Confluence Theorem 3.8 then gives us the term M_3 such that $\mathsf{nf}_{\beta\bot}(M_1) \twoheadrightarrow_\eta M_3$ and $\mathsf{nf}_{\beta\bot}(M_1) \twoheadrightarrow_\eta M_3$. The following diagram illustrates this proof.

$$\begin{tikzcd}
 & & M_0 \arrow[dll, ->>, "\beta\bot\eta!"'] \arrow[d, ->>, dotted, "\beta\bot"] \arrow[drr, ->>, "\beta\bot\eta!"] & & \\
M_1 \arrow[d, ->>, dotted, "\beta\bot"'] \arrow[rr, phantom, "\text{(Thm 6.2)}"] & & \mathsf{nf}_{\beta\bot}(M_0) \arrow[dll, ->>, dotted, "\eta!"] \arrow[drr, ->>, dotted, "\eta!"'] \arrow[rr, phantom, "\text{(Thm 6.2)}"] & & M_2 \arrow[d, ->>, dotted, "\beta\bot"] \\
\mathsf{nf}_{\beta\bot}(M_1) \arrow[drr, ->>, dotted, "\eta!"'] \arrow[rrrr, phantom, "\text{(Thm 3.8)}"] & & & & \mathsf{nf}_{\beta\bot}(M_2) \arrow[dll, ->>, dotted, "\eta!"] \\
 & & M_3 & &
\end{tikzcd}$$

Second, normalisation of $\lambda^\infty_{\beta\bot\eta!}$ follows from normalisation of $\beta\bot$-reduction (Theorem 2.2) and normalisation of η!-reduction (Theorem 3.9): given a term M we $\beta\bot$-reduce first to $\mathsf{nf}_{\beta\bot}(M)$ and then we η!-reduce further to $\mathsf{nf}_{\eta!}(\mathsf{nf}_{\beta\bot}(M))$. □

As a consequence of the previous theorem, we have that $\beta\bot\eta!$- reduction is $\omega{+}\omega$-compressible. We also have that:

Corollary 6.5 (Uniqueness of $\beta\bot\eta!$-normal forms). The extensional infinite lambda calculus $\lambda^\infty_{\beta\bot\eta!}$ has unique normal forms.

7. Infinite eta Böhm trees as Normal Forms

In this section we will see that the infinite eta Böhm tree of a lambda term M denoted by $\infty\eta\mathsf{BT}(M)$ is nothing else than the η!-normal form of $\mathsf{BT}(M)$, the Böhm tree of M, which in turn is nothing else than $\mathsf{nf}_{\beta\bot}(M)$.

We begin with the definition of Böhm tree formulated as a term in $\Lambda^\infty_\bot$. The original notion of Böhm tree defined in (Barendregt, 1984) for finite terms applies to infinite terms as well.

7.1 Definition [Böhm trees]: Let $M \in \Lambda^\infty_\bot$. The Böhm tree of a term M (denoted by $\mathsf{BT}(M)$) is defined by co-recursion as follows.

$$\begin{array}{lll}
\mathsf{BT}(M) & = \bot & \text{if } M \text{ has no head normal form} \\
\mathsf{BT}(M) & = \lambda x_1 \ldots \lambda x_n.y\ \mathsf{BT}(M_1) \ldots \mathsf{BT}(M_m) & \text{if } M \twoheadrightarrow_\beta \lambda x_1 \ldots \lambda x_n.y M_1 \ldots M_m
\end{array}$$

We can this read from this definition an algorithm that starting from the root of a term M calculates the Böhm tree of M layer by layer. Since the subterms of the Böhm tree are either a head normal form or $\bot$, it is clear that possibly infinite output $\mathsf{BT}(M)$ of the algorithm is the (unique) $\beta\bot$- normal form of M.

Remark 7.1. We define Böhm trees as terms in the infinitary lambda calculus and this definition is given co-recursively. In (Barendregt, 1984) Definition 10.1.4, a Böhm tree is defined as a function from a set of sequences or positions to a set Σ of labels. Up to a change of representation, Definition 7.1 is very similar to the informal definition of Böhm trees given in Definition 10.1.3 in (Barendregt, 1984).

As stated before, the infinitary lambda calculus $\lambda^\infty_{\beta\bot} = (\Lambda^\infty_\bot, \longrightarrow_{\beta\bot})$ captures the notion of Böhm trees as normal forms (see Figure 1). We recall the proof of this fact in the following theorem.

Theorem 7.2 (Böhm trees as $\beta\bot$-normal forms). Suppose $M \in \Lambda^\infty_\bot$. Then, $\mathsf{BT}(M) = \mathsf{nf}_{\beta\bot}(M)$.

Proof: It is easy to see that $M \twoheadrightarrow_{\beta\bot} \mathsf{BT}(M)$ and that $\mathsf{BT}(M)$ is in $\beta\bot$-normal form. It follows from Theorem 2.2 that $\mathsf{BT}(M) = \mathsf{nf}_{\beta\bot}(M)$. □

Next we redefine the $\infty\eta$-construction of (Barendregt, 1984) using the notation of strongly converging reduction.

7.2 Definition [Infinite eta Böhm trees]: We define $\infty\eta$ on Böhm trees in $\mathsf{BT}(\Lambda^\infty_\bot)$ co-inductively as follows:

$$\begin{aligned}
\infty\eta(\bot) &= \ \bot \\
\infty\eta(\lambda x_1 \ldots \lambda x_n.y\ M_1 \ldots M_m) &= \ \infty\eta(\lambda x_1 \ldots \lambda x_{n-1}.y\ M_1 \ldots M_{m-1}) \\
&\quad\ \text{if } x_n \twoheadrightarrow_{\eta^{-1}} M_m \text{ and } \quad x_n \notin \mathrm{FV}(y\ M_1 \ldots M_{m-1}) \\
\infty\eta(\lambda x_1 \ldots \lambda x_{n-1}.y\ M_1 \ldots M_{m-1}) &= \ \lambda x_1 \ldots \lambda x_n.y\ \infty\eta(M_1) \ldots \infty\eta(M_m) \quad \text{otherwise}
\end{aligned}$$

This $\infty\eta$-construction contracts layer by layer all the $\eta!$-redexes in the Böhm tree $\mathsf{BT}(M)$ of a term M, so that the result is its $\beta\bot\eta$-normal form. We may write $\infty\eta(\mathsf{BT}(M))$ for $\infty\eta(BTM)$.
Barendregt's original definition in (Barendregt, 1984) Proposition 10.2.15 of the infinite eta expansion differs slightly from the above definition.
It uses the order $\leq_\eta$ on Böhm trees instead of $\twoheadrightarrow_{\eta^{-1}}$. To prove that our definition of infinite eta Böhm trees coincides with the definition in (Barendregt, 1984), it suffices to prove that the relations $\leq_\eta$ and $\twoheadrightarrow_{\eta^{-1}}$ coincide on $\beta\bot$-normal forms.

Lemma 7.3. Let M, N be in $\beta\bot$-normal form. Then, $M \leq_\eta N$ if and only if $M \twoheadrightarrow_{\eta^{-1}} N$.

We leave the proof of the above lemma to the reader.

Theorem 7.4 (Infinite eta Böhm trees are $\beta\bot\eta!$-normal forms). Let $M \in \Lambda^\infty_\bot$.

1 If M is in $\beta\bot$-normal form then $\mathsf{nf}_{\eta!}(M) = \infty\eta(M)$.
2 $\mathsf{nf}_{\beta\bot\eta!}(M) = \infty\eta(\mathsf{BT}(M))$.

Proof: For the first part, it is not difficult to prove that M $\eta!$-reduces to $\infty\eta(M)$ and that $\infty\eta(M)$ is in $\eta!$-normal form. By Corollary 3.10, the $\eta!$-normal form is unique, hence

$\infty\eta(M) = \mathsf{nf}_{\eta!}(M)$. The second part follows from Theorem 6.3, Theorem 7.2 and the previous part. $\square$

8. The $\infty\eta$-Böhm trees as a Model of the Lambda Calculus

In (Barendregt, 1984) the Böhm trees got a role on their own, when with help of the continuity theorem it was shown that the set of Böhm trees can be enriched to become a model of the finite lambda calculus λ_β. The original definition of Böhm tree used the language of labelled trees. In this approach, lambda terms and Böhm trees live in different worlds, because lambda terms got defined with the usual syntax definition. Using infinitary lambda calculus (Kennaway et al., 1997) this separation is not only no longer necessary but the fact that the Böhm trees are a model of the finite lambda calculus λ_β is shown directly from the confluence and normalisation properties of $\lambda^\infty_{\beta\perp}$. Having proved that $\lambda^\infty_{\beta\perp\eta!}$ is confluent and normalising, we will show that the infinite eta Böhm trees can be made into a model of $\lambda^\infty_{\beta\eta}$. This is a new result that could not have been proved with a variation of continuity argument used in (Barendregt, 1984) to prove that the Böhm trees for a model of the finite lambda calculus. In (Barendregt, 1984) the proof that the set $\mathfrak{B} = \mathsf{BT}(\Lambda)$ of Böhm-like trees is a λ-model uses continuity of the context operator. However, this appeal to continuity is not possible for $\infty\eta$-Böhm trees, because neither the abstraction nor the application are continuous (Severi and de Vries, 2005a). For instance, take $\lambda x.y\perp$ and $\lambda x.yx$. Then $\lambda x.y\perp \preceq \lambda x.yx$, but $\infty\eta\mathsf{BT}(\lambda x.y\perp) = \lambda x.y\perp \not\preceq y = \infty\eta\mathsf{BT}(\lambda x.yx)$.
From the $\infty\eta$-Böhm trees of the finite lambda terms in Λ we will now construct an extensional model of the finitary lambda calculus, using the properties of $\lambda^\infty_{\beta\perp\eta!}$.

8.1 Definition: We denote the set $\infty\eta\mathsf{BT}(\Lambda)$ of $\infty\eta$-Böhm trees over Λ by $\mathfrak{B}_{\infty\eta}$.

We define the model $\mathfrak{B}_{\infty\eta}$ as follows:

8.2 Definition: The $\infty\eta$-Böhm model is a tuple $(\mathfrak{B}_{\infty\eta}, \bullet, [\![\ ]\!])$ where $\mathfrak{B}_{\infty\eta}$ denotes the set $\infty\eta\mathsf{BT}(\Lambda)$ of $\infty\eta$-Böhm trees over Λ or equivalently the set $\mathsf{nf}_{\beta\perp\eta!}(\Lambda)$ of $\beta\perp\eta!$-normal forms of finite lambda terms, the application $\bullet : \mathfrak{B}_{\infty\eta} \times \mathfrak{B}_{\infty\eta} \to \mathfrak{B}_{\infty\eta}$ is defined by $M\bullet N = \infty\eta\mathsf{BT}(MN)$ for all M, N in $\mathfrak{B}_{\infty\eta}$ and for each map ρ from variables to $\mathfrak{B}_{\infty\eta}$, the interpretation $[\![M]\!]_\rho : \mathfrak{B}_{\infty\eta} \to \mathfrak{B}_{\infty\eta}$ is defined by $[\![M]\!]_\rho = \infty\eta\mathsf{BT}(M^\rho)$, where M^ρ is the result of simultaneously replacing each free variable x in M by $\rho(x)$.

8.3 Definition: Let ρ be a map from variables to $\mathfrak{B}_{\infty\eta}$. We define $\rho(x := N)$ as a map from variables to $\mathfrak{B}_{\infty\eta}$ such that $\rho(x := N)(x) = N$ and $\rho(x := N)(y) = \rho(y)$ for every $y \neq x$.

We will now show that $\mathfrak{B}_{\infty\eta}$ is a syntactic λ-model in the sense of (Barendregt, 1984), Definition 5.3.1. and 5.3.2.

Theorem 8.1. $(\mathfrak{B}_{\infty\eta}, \bullet, [\![\ ]\!])$ is a syntactical model of the lambda calculus, that is, it satisfies for all ρ:

1 $[\![x]\!]_\rho = \rho(x)$
2 $[\![MN]\!]_\rho = [\![M]\!]_\rho \bullet [\![N]\!]_\rho$
3 $[\![\lambda x.M]\!]_\rho \bullet P = [\![M]\!]_{\rho(x:=P)}$
4 $\rho \mid FV(M) = \rho' \mid FV(M)$ implies $[\![M]\!]_\rho = [\![M]\!]_{\rho'}$
5 if $[\![M]\!]_{\rho(x:=P)} = [\![N]\!]_{\rho(x:=P)}$ for all $P \in \mathfrak{B}_{\infty\eta}$, then $[\![\lambda x.M]\!]_\rho = [\![\lambda x.N]\!]_\rho$
6 $[\![\lambda xy.xy]\!]_\rho = [\![\lambda x.x]\!]_\rho$

Proof in the appendix.

9. Remarks and Future Research

Next, let us briefly compare our proofs of confluence of $\beta\bot\eta$(Severi and de Vries, 2002) and $\beta\bot\eta!$ with the confluence proof of the finite $\beta\eta\bot$ lambda calculus (Theorem 15.2.15(ii) in (Barendregt, 1984): note the different notation: the symbol Ω is used in (Barendregt, 1984) for $\bot$). The diagram of the latter proof is:

$$\begin{tikzcd}
 & & & M_0 \arrow[dlll, two heads, "\beta\bot\eta!"'] \arrow[dl, two heads, dotted, "\beta\bot"] \arrow[dr, two heads, dotted, "\beta\bot"'] \arrow[drrr, two heads, "\beta\bot\eta!"] & & & \\
M_1 \arrow[d, two heads, dotted, "\eta"'] & & L_1 \arrow[ll, two heads, dotted, "\eta"] \arrow[dr, two heads, dotted, "\beta\bot"'] & & L_2 \arrow[rr, two heads, dotted, "\eta"'] \arrow[dl, two heads, dotted, "\beta\bot"] & & M_2 \arrow[d, two heads, dotted, "\eta"] \\
\mathsf{nf}_\eta(M_1) \arrow[drrr, two heads, dotted, "\beta\bot\eta"'] & & & L \arrow[d, two heads, dotted, "\eta"] & & & \mathsf{nf}_\eta(M_2) \arrow[dlll, two heads, dotted, "\beta\bot\eta"] \\
 & & & \mathsf{nf}_\eta(L) & & &
\end{tikzcd}$$

At the heart of both proofs is confluence of $\beta\bot$-reduction. But what makes these proofs different is that the proof in (Barendregt, 1984) uses η-normal forms, whereas our proof uses $\beta\bot$-normal forms. Our proof sidesteps the use of postponement of $\eta!$ over $\beta\bot$.

We leave it as an interesting future task to see whether proofs of confluence for finite $\lambda_{\beta\bot\eta}$ that use complete developments can be generalised to the infinitary setting, as for example, the older proof of Barendregt, Bergstra, Klop and Volken (Barendregt et al., 1976) and the proof by van Oostrom (van Oostrom, 1997).

It may be an interesting challenge to see to what extent it is possible to make confluent and normalising $\bot$-extensions of weakly orthogonal systems, be it first or higher order.

We have now seen that the equality relations induced by D_∞, D^*_∞ and $\mathcal{P}\omega$ can be characterised with help of respectively the $\infty\eta$Böhm trees, the ηBöhm trees and the Böhm trees, that is, with help of the normal forms of respectively the confluent and normalising calculi $\lambda^\infty_{\beta\bot\eta!}$, $\lambda^\infty_{\beta\bot\eta}$ and $\lambda^\infty_{\beta\bot}$. Which other models of the lambda calculus allow such characterisations?

References

Abramsky, S. and Ong, C.-H. L. (1993). Full abstraction in the lazy lambda calculus. *Inform. and Comput.*, 105(2):159–267.

Arnold, A. and Nivat, M. (1980). The metric space of infinite trees. Algebraic and topological properties. *Fundamenta Informaticae*, 4:445–476.

Bakel, S. v., Barbanera, F., Dezani-Ciancaglini, M., and de Vries, F. J. (2002). Intersection types for λ-trees. *Theoretical Computer Science*, 272(1-2):3–40.

Barbanera, F., Dezani-Ciancaglini, M., and de Vries, F. J. (1998). Types for trees. In *PROCOMET'98 (Shelter Island, 1998)*, pages 6–29. Chapman & Hall, London.

Barendregt, H. P. (1984). *The Lambda Calculus: Its Syntax and Semantics.* North-Holland, Amsterdam, Revised edition.

Barendregt, H. P. (1992). Lambda calculi with types. In Abramsky, S., Gabbay, D. M., and Maibaum, T. S. E., editors, *Handbook of Logic in Computer Science*, volume 2, pages 117–309. Oxford University Press, Oxford.

Barendregt, H. P., Bergstra, J., Klop, J. W., and Volken., H. (1976). Degrees, reductions and representability in the lambda calculus. Technical report, Department of Mathematics, Utrecht University.

Berarducci, A. (1996). Infinite λ-calculus and non-sensible models. In *Logic and algebra (Pontignano, 1994)*, pages 339–377. Dekker, New York.

Coppo, M., Dezani-Ciancaglini, M., and Zacchi, M. (1987). Type theories, normal forms, and D_∞-lambda-models. *Information and Computation*, 72(2):85–116.

Hyland, J. M. E. (1975). A survey of some useful partial order relations on terms of the lambda calculus. In Böhm, C., editor, *Lambda Calculus and Computer Science Theory*, volume 37 of *LNCS*, pages 83–93. Springer-Verlag.

Kennaway, J. R. and de Vries, F. J. (2003). Infinitary rewriting. In Terese, editor, *Term Rewriting Systems*, volume 55 of *Cambridge Tracts in Theoretical Computer Science*, pages 668–711. Cambridge University Press.

Kennaway, J. R., Klop, J. W., Sleep, M. R., and de Vries, F. J. (1995a). Infinite lambda calculus and Böhm models. In *Rewriting Techniques and Applications*, volume 914 of *LNCS*, pages 257–270. Springer-Verlag.

Kennaway, J. R., Klop, J. W., Sleep, M. R., and de Vries, F. J. (1995b). Transfinite reductions in orthogonal term rewriting systems. *Information and Computation*, 119(1):18–38.

Kennaway, J. R., Klop, J. W., Sleep, M. R., and de Vries, F. J. (1997). Infinitary lambda calculus. *Theoretical Computer Science*, 175(1):93–125.

Kennaway, J. R., van Oostrom, V., and de Vries, F. J. (1999). Meaningless terms in rewriting. *J. Funct. Logic Programming*, Article 1:35 pp.

Lévy, J.-J. (1976). An algebraic interpretation of the $\lambda\beta K$-calculus, and an application of a labelled λ-calculus. *Theoretical Computer Science*, 2(1):97–114.

Longo, G. (1983). Set-theoretical models of λ-calculus: theories, expansions, isomorphisms. *Ann. Pure Appl. Logic*, 24(2):153–188.

Severi, P. G. and de Vries, F. J. (2002). An extensional Böhm model. In *Rewriting Techniques and Applications*, volume 2378 of *LNCS*, pages 159–173. Springer-Verlag.

Severi, P. G. and de Vries, F. J. (2005a). Continuity and Discontinuity in Lambda Calculus. In *Typed Lambda Calculus and Applications*, volume 3461 of *LNCS*, pages 103–116. Springer-Verlag.

Severi, P. G. and de Vries, F. J. (2005b). Order Structures for Böhm-like models. In *Computer Science Logic*, volume 3634 of *LNCS*, pages 103–116. Springer-Verlag.

Severi, P. G. and de Vries, F. J. (2011). Weakening the Axiom of Overlap in the Infinitary Lambda Calculus. In *Rewriting Techniques and Applications*, volume 10 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 313–328.

Simonsen, J. G. (2004). On confluence and residuals in cauchy convergent transfinite rewriting. *Inf. Process. Lett.*, 91(3):141–146.

van Oostrom, V. (1997). Developing developments. *Theoretical Computer Science*, 175(1):159–181.

Wadsworth, C. P. (1976). The relation between computational and denotational properties for Scott's D_∞-models of the lambda-calculus. *SIAM Journal on Computing*, 5(3):488–521.

Appendix A. Omitted Proofs for the Section on Infinitary Lambda Calculus

These sections in the appendix contain all omitted proofs.

The following lemma is proved by induction on the depth of the hole in the context.

Lemma A.1. Let $C[M] \in \Lambda^\infty_\perp$ and d the depth of the hole in C. If $n > d$ then $(C[M])^n = C^n[M^{n-d}]$. Otherwise $(C[M])^n = C^n$ is a term without a hole.

Lemma A.2. Let $P, Q \in \Lambda^\infty_\perp$. Then, $P^n[x := Q^n] \succeq (P[x := Q])^n$.

Proof: This is proved by induction on the lexicographically ordered pair $(n, \|P^n\|)$ where

$\|P^n\|$ is the number of symbols of P^n. Suppose $n > 0$ and $P = P_1P_2$. Then

$$\begin{array}{lll} P^n[x := Q^n] & = P_1^n[x := Q^n]P_2^{n-1}[x := Q^n] & \text{by Definition 2.5} \\ & \succeq P_1^n[x := Q^n]P_2^{n-1}[x := Q^{n-1}] & \\ & \succeq (P_1[x := Q])^n(P_2[x := Q])^{n-1} & \text{by induction hypothesis} \\ & = (P[x := Q])^n & \text{by Definition 2.5} \end{array}$$

□

Back to Lemma 2.6.

Proof: Suppose $M = C[P] \to_\perp C[\perp] = N$. Let d be the position of the hole in $C[[\ ]]$. By Lemma A.1, if $n \leq d$ then $M^n = C^n = N^n$. Otherwise, $M^n = C^n[P^{n-d}] \succeq C^n[\perp] = N^n$. By Monotonicity (Theorem 2.5), we have that $\mathsf{nf}_{\beta\perp}(M^n) \succeq \mathsf{nf}_{\beta\perp}(\mathsf{nf}_{\beta\perp}(N^n))$.
Suppose $M = C[(\lambda x.P)Q] \longrightarrow_\beta C[P[x := Q]] = N$. Let d be the position of the hole in $C[[\ ]]$ and assume $k = n - d > 0$.

$$\begin{array}{llll} (C[(\lambda x.P)Q])^{n+2} & = & C^{n+2}[(\lambda x.P^k)Q^{k+1}] & \text{by Lemma A.1} \\ & \longrightarrow_\beta & C^{n+2}[P^k[x := Q^{k+1}] & \\ & \succeq & C^n[P^k[x := Q^k]] & \\ & \succeq & C^n[(P[x := Q])^k] & \text{by Lemma A.2} \\ & = & (C[P[x := Q]])^n & \text{by Lemma A.1} \end{array}$$

By Confluence (Theorem 2.2) and Monotonicity (Theorem 2.5), we have that $\mathsf{nf}_{\beta\perp}(M^{n+2}) \succeq \mathsf{nf}_{\beta\perp}(\mathsf{nf}_{\beta\perp}(N^n))$. □

We also need a variation of Lemma 2.4 with β_h-reduction instead of $\perp_{out}$-reduction.

Lemma A.3 (β_h-reducing a prefix). Let $M, N \in \Lambda_\perp^\infty$. If $M \preceq N$ and $M \twoheadrightarrow_{\beta_h} M'$ then there exists N' such that $N \twoheadrightarrow_{\beta_h} N'$ and $M' \preceq N'$.

Proof: By induction on $M \twoheadrightarrow_{\beta_h} M'$. □

Back to Lemma 2.8.

Proof: $(i \Rightarrow ii)$. (Kennaway et al., 1997) Suppose there exists N in head normal form such that $M \twoheadrightarrow\!\!\!\!\rightarrow_\beta N$. We can assume that the length of this reduction is ω by Theorem 2.2. Since $\twoheadrightarrow\!\!\!\!\rightarrow_\beta$ is strongly convergent, we have that there exists N'' such that:

$$M \xrightarrow{\geq 0}\!\!\!\!\rightarrow_\beta N'' \xrightarrow{>0}\!\!\!\!\twoheadrightarrow_\beta N$$

It is easy to show that N'' is in head normal form.
$(ii \Rightarrow iii)$. By Theorem 2.2, $M \twoheadrightarrow_\beta N'' \twoheadrightarrow\!\!\!\!\rightarrow_{\beta\perp} \mathsf{nf}_{\beta\perp}(M)$. Since N'' is in head normal form, so is $\mathsf{nf}_{\beta\perp}(M)$. We truncate the normal form of M at depth 1 and apply the well-known results on head normalisation in finite lambda calculus. By Theorem 2.7, there exists $m > 1$ such that $\mathsf{nf}_{\beta\perp}(M^m) \succeq (\mathsf{nf}_{\beta\perp}(M))^1$. Hence, $\mathsf{nf}_{\beta\perp}(M^m)$ is in head normal form because $(\mathsf{nf}_{\beta\perp}(M))^1$ is in head normal form. By Theorem 2.2 and the fact that $\twoheadrightarrow\!\!\!\!\rightarrow_{\beta\perp}$ is strongly convergent, we have that

$$M^m \xrightarrow{\geq 0}\!\!\!\!\rightarrow_\beta P \xrightarrow{>0}\!\!\!\!\twoheadrightarrow_\beta Q \twoheadrightarrow\!\!\!\!\rightarrow_\perp \mathsf{nf}_{\beta\perp}(M^m) \succeq (\mathsf{nf}_{\beta\perp}(M))^1$$

Since the term $M^m \in \Lambda_\perp$ is a finite λ-term and the reduction $M^m \xrightarrow{\geq 0}\!\!\!\!\rightarrow_\beta P$ is finite, we can now apply Theorem 8.3.11 of (Barendregt, 1984). We have that $M^m \twoheadrightarrow_{\beta_h} N$ for some N in head normal form. By Lemma A.3, there exists N'' such that $M \twoheadrightarrow_{\beta_h} N''$ and $N \preceq N''$. Since N is in head normal form, so is N''.

$(iii \Rightarrow iv)$ and $(iv \Rightarrow i)$ are trivial. □

Appendix B. Preservation of η-expansions of x after $\eta!$

By postponing the η^{-1} (or the $\eta!$) steps at greater depth, we can re-order the steps in an η^{-1} (or an $\eta!$) reduction sequence by increasing order of depth.

Lemma B.1 (Postponing η^{-1}-steps at greater depth). Let $j < i$. Then, an η^{-1}-reduction step at depth i can be postponed over an η^{-1}-reduction step at depth j, that is:

$$\begin{tikzcd}
M_1 \arrow[r, "i", "\eta^{-1}"'] \arrow[d, dotted, "\eta^{-1}"', "j"] & M_2 \arrow[d, "\eta^{-1}"', "j"] \\
M_3 \arrow[r, dotted, "i", "\eta^{-1}"'] & M_4
\end{tikzcd}$$

Proof: Suppose $M = C[P] \xrightarrow{i}_{\eta^{-1}} C[\lambda x.Px]$. Since $j < i$, the η^{-1}-redex at depth j occurs either in C or in P but it cannot occur in x. We have that

$$\begin{tikzcd}
C[P] \arrow[r, "i", "\eta^{-1}"'] \arrow[d, dotted, "\eta^{-1}"', "j"] & C[\lambda x.Px] \arrow[d, "\eta^{-1}"', "j"] \\
C'[P'] \arrow[r, dotted, "i", "\eta^{-1}"'] & C'[\lambda x.P'x]
\end{tikzcd}$$

□

As a consequence of the previous lemma, we obtain:

Lemma B.2 (Sorting η^{-1}-reduction sequences by order of depth). If $M \twoheadrightarrow_{\eta^{-1}} N$, then there is either a finite reduction

$$M = M_0 \stackrel{0}{\twoheadrightarrow}_{\eta^{-1}} M_1 \stackrel{1}{\twoheadrightarrow}_{\eta^{-1}} M_2 \stackrel{2}{\twoheadrightarrow}_{\eta^{-1}} \ldots M_{n-1} \stackrel{n-1}{\twoheadrightarrow}_{\eta^{-1}} M_n = N$$

or an infinite reduction

$$M = M_0 \stackrel{0}{\twoheadrightarrow}_{\eta^{-1}} M_1 \stackrel{1}{\twoheadrightarrow}_{\eta^{-1}} M_2 \stackrel{2}{\twoheadrightarrow}_{\eta^{-1}} \ldots M_\omega = N$$

Lemma B.3 (Postponing $\eta!$-steps at greater depth). Let $j < i$. There are only two possible situations that can occur when we postpone an $\eta!$-reduction step at depth i over an $\eta!$-reduction step at depth j:

$$\begin{tikzcd}
M_1 \arrow[r, "i", "\eta!"'] \arrow[d, dotted, "\eta!"', "j"] & M_2 \arrow[d, "\eta!"', "j"] \\
M_3 \arrow[r, dotted, "i", "\eta!"'] & M_4
\end{tikzcd}
\qquad
\begin{tikzcd}
M_1 \arrow[r, "i", "\eta!"'] \arrow[dr, dotted, "j", "\eta!"'] & M_2 \arrow[d, "\eta!"', "j"] \\
 & M_4
\end{tikzcd}$$

Proof: The situation of the second tile occurs when the term M_1 contains a term $\lambda x.PQ$ at depth j and the $\eta!$-redex at depth i is inside Q:

$$\begin{tikzcd}
C[\lambda x.PQ] \arrow[r, "i", "\eta!"'] \arrow[dr, dotted, "j", "\eta!"'] & C[\lambda x.PQ'] \arrow[d, "\eta!"', "j"] \\
 & C[P]
\end{tikzcd}$$

We know that Q' is an infinite η-expansion of x and we have to show that so is Q. Since Q' is obtained from Q by applying only one step of $\eta!$-reduction, by Lemma 3.2, we can reverse the reduction from Q to Q'. Hence $x \twoheadrightarrow_{\eta^{-1}} Q' \twoheadrightarrow_{\eta^{-1}} Q$. □

As a consequence of the previous lemma, we obtain:

Lemma B.4 (Sorting $\eta!$-reduction sequences by order of depth). If $M \twoheadrightarrow_{\eta!} N$, then there is either a finite reduction

$$M = M_0 \xrightarrow{0}\!\!\!\!\twoheadrightarrow_{\eta!} M_1 \xrightarrow{1}\!\!\!\!\twoheadrightarrow_{\eta!} M_2 \xrightarrow{2}\!\!\!\!\twoheadrightarrow_{\eta!} \dots M_{n-1} \xrightarrow{n-1}\!\!\!\!\twoheadrightarrow_{\eta!} M_n = N$$

or an infinite reduction

$$M = M_0 \xrightarrow{0}\!\!\!\!\twoheadrightarrow_{\eta!} M_1 \xrightarrow{1}\!\!\!\!\twoheadrightarrow_{\eta!} M_2 \xrightarrow{2}\!\!\!\!\twoheadrightarrow_{\eta!} \dots M_\omega = N$$

If $x \twoheadrightarrow_{\eta^{-1}} M$ then not all abstractions in M have to be $\eta!$-redexes. For example,

$$x \twoheadrightarrow_{\eta^{-1}} \lambda yz.(\lambda u.x\mathsf{E}_y\mathsf{E}_u)\mathsf{E}_z = M$$

The first abstraction λy is not an $\eta!$-redex. In spite of this, it is possible to undo all the η^{-1}-steps by doing only a finite number of steps of $\eta!$ at depth 0. This is proved in the following lemma which will be used to prove Commutation of β and $\eta!$.

Lemma B.5 (Inverting the expansion of a variable). If $x \twoheadrightarrow_{\eta^{-1}} M$, then $M \xrightarrow{0}\!\!\!\!\twoheadrightarrow_{\eta!} x$ and there is a single free occurrence of x in M at depth 0.

Proof: By Theorem 3.5, $x \twoheadrightarrow_{\eta^{-1}} M$ implies that $M \twoheadrightarrow_{\eta!} x$. By Lemma 3.4 (Compression Lemma) and Lemma B.4 (Sorting by order of depth), we have that there is either a finite reduction

$$M = M_0 \xrightarrow{0}\!\!\!\!\twoheadrightarrow_{\eta!} M_1 \xrightarrow{1}\!\!\!\!\twoheadrightarrow_{\eta!} M_2 \xrightarrow{2}\!\!\!\!\twoheadrightarrow_{\eta!} \dots M_{n-1} \xrightarrow{n-1}\!\!\!\!\twoheadrightarrow_{\eta!} M_n = N = x$$

or an infinite reduction

$$M_0 \xrightarrow{0}\!\!\!\!\twoheadrightarrow_{\eta!} M_1 \xrightarrow{1}\!\!\!\!\twoheadrightarrow_{\eta!} M_2 \xrightarrow{2}\!\!\!\!\twoheadrightarrow_{\eta!} \dots M_\omega = N = x$$

In both cases, the following case analysis allows us to conclude that $M_1 = x$.

1 If M_1 is a variable, then the rest of the reduction sequence from M_1 onwards is empty and $M_1 = x$.
2 Suppose $M_1 = (PQ)$. Then all $\eta!$-reducts from N_1 are applications (including N). This contradicts the fact that N is a variable.
3 Suppose $M_1 = \lambda x.P$. Then either all reducts from N_1 are abstractions (including N) or this abstraction disappears because it is contracted by an $\eta!$-redex. Neither case is possible. The first option contradicts the fact that N is a variable. The second option contradicts the fact that in the reduction from M_1 to N we contract only redexes at depth strictly greater than 0.

□

Local Confluence and the Strip Lemma for $\eta!$ depend on the following lemma that says that $\eta!$-redexes are preserved by certain $\eta!$-reduction sequences.

Lemma B.6. If $x \xrightarrow{<n}\!\!\!\!\twoheadrightarrow_{\eta^{-1}} N \xrightarrow{\geq n}\!\!\!\!\twoheadrightarrow_{\eta^{-1}} M$ and $M \xrightarrow{n}\!\!\!\!\twoheadrightarrow_{\eta!} M'$, then $N \xrightarrow{\geq n}\!\!\!\!\twoheadrightarrow_{\eta^{-1}} M'$.

Proof: Assume $x \xrightarrow{<n}\!\!\twoheadrightarrow_{\eta^{-1}} N \xrightarrow{\geq n}\!\!\twoheadrightarrow_{\eta^{-1}} M$ and $M \xrightarrow{n}_{\eta!} M'$. We will now show that $N \xrightarrow{\geq n}\!\!\twoheadrightarrow_{\eta^{-1}} M'$.

From depth considerations it follows that the abstraction of the $\eta!$-redex contracted in $M \longrightarrow_{\eta!} M'$ got created after N in the reduction $x \xrightarrow{<n}\!\!\twoheadrightarrow_{\eta^{-1}} N \xrightarrow{\geq n}\!\!\twoheadrightarrow_{\eta^{-1}} M \xrightarrow{n}_{\eta!} M'$. Since η^{-1} does not change the depth of any subterm, once the $\eta!$-redex is created, its depth remains fixed. By omitting the η-expansion step that created the abstraction of the $\eta!$-redex plus all the subsequent η-expansions from y to Q, we construct the reduction sequence at the bottom:

$$\begin{tikzcd}
x \arrow[r, two heads, "<n", "\eta^{-1}"'] & N \arrow[r, two heads, "n", "\eta^{-1}"'] & C'[P'] \arrow[r, "n", "\eta^{-1}"'] \arrow[drr, dotted, two heads, "\geq n", "\eta^{-1}"'] & C'[\lambda y.P'y] \arrow[r, two heads, "\geq n", "\eta^{-1}"'] & C[\lambda y.PQ] = M \arrow[d, "\eta!"', "n"] \\
 & & & & C[P] = M'
\end{tikzcd}$$

The more general statement of the lemma follows by repeated application of the above.

$\square$

Theorem B.7 (Preservation of η-expansions of x after $\eta!$). If $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M$ and $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} M'$, then $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M'$. **Back to Lemma 3.1**.

Proof: By compression of $\eta!$-reduction we may assume that the reduction $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} M'$ is at most ω steps long. By Theorem B.2, this reduction sequence can be sorted. Two possible situations can arise:

Case $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} M'$ is finite. We illustrate the proof for a sequence of length 3.

$$\begin{tikzcd}
x \arrow[r, two heads, "0", "\eta^{-1}"'] \arrow[dr, dotted, two heads, "0", "\eta^{-1}"'] & . \arrow[r, two heads, "1", "\eta^{-1}"'] & . \arrow[r, two heads, "2", "\eta^{-1}"'] & . \arrow[r, two heads, "\eta^{-1}"'] & M_0 \arrow[d, two heads, "\eta!"', "0"] & (= M) \\
 & N_1 \arrow[r, dotted, two heads, "1", "\eta^{-1}"'] \arrow[dr, dotted, two heads, "1", "\eta^{-1}"'] & . \arrow[r, dotted, two heads, "2", "\eta^{-1}"'] & . \arrow[r, dotted, two heads, "\eta^{-1}"'] & M_1 \arrow[d, two heads, "\eta!"', "1"] & \\
 & & N_2 \arrow[r, dotted, two heads, "2", "\eta^{-1}"'] \arrow[dr, dotted, two heads, "2", "\eta^{-1}"'] & . \arrow[r, dotted, two heads, "\eta^{-1}"'] & M_2 \arrow[d, two heads, "\eta!"', "3"] & \\
 & & & N_3 \arrow[r, dotted, "\eta^{-1}"'] & M_3 & (= M')
\end{tikzcd}$$

By Lemma B.6 we have $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M_1$. By Lemma B.2, there exists N_1 such that $x \xrightarrow{0}\!\!\twoheadrightarrow_{\eta^{-1}} N_1 \xrightarrow{>0}\!\!\twoheadrightarrow_{\eta^{-1}} M_1$. By Lemma B.6, we have that $N_1 \xrightarrow{>0}\!\!\twoheadrightarrow_{\eta^{-1}} M_2$. Hence we get $x \xrightarrow{0}\!\!\twoheadrightarrow_{\eta^{-1}} N_1 \xrightarrow{>0}\!\!\twoheadrightarrow_{\eta^{-1}} M_2$. By Lemma B.2, there exists N_2 such that $x \xrightarrow{0}\!\!\twoheadrightarrow_{\eta^{-1}} N_1 \xrightarrow{1}\!\!\twoheadrightarrow_{\eta^{-1}} N_2 \xrightarrow{>1}\!\!\twoheadrightarrow_{\eta^{-1}} M_2$. Again by Lemma B.6, we have that $N_2 \xrightarrow{>1}\!\!\twoheadrightarrow_{\eta^{-1}} M_3$. Once more by Lemma B.2, there exists N_3 such that $x \xrightarrow{0}\!\!\twoheadrightarrow_{\eta^{-1}} N_1 \xrightarrow{1}\!\!\twoheadrightarrow_{\eta^{-1}} N_2 \xrightarrow{2}\!\!\twoheadrightarrow_{\eta^{-1}} N_3 \xrightarrow{>2}\!\!\twoheadrightarrow_{\eta^{-1}} M_3$.

Case $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} M'$ has length ω. By repeated application of the previous argument, we can construct the diagonal sequence as shown in the following diagram:

$$
\begin{tikzcd}
x \arrow[r, ->>, "0", "\eta^{-1}"'] \arrow[dr, dotted, ->>, "0", "\eta^{-1}"'] & \cdot \arrow[r, ->>, "1", "\eta^{-1}"'] & \cdot \arrow[r, ->>, "2", "\eta^{-1}"'] & \cdot \arrow[r, ->>, "\eta^{-1}"'] & M_0 \arrow[d, ->>, "0", "\eta!"'] & (= M) \\
& N_1 \arrow[r, dotted, ->>, "1", "\eta^{-1}"'] \arrow[dr, dotted, ->>, "1", "\eta^{-1}"'] & \cdot \arrow[r, dotted, ->>, "2", "\eta^{-1}"'] & \cdot \arrow[r, dotted, ->>, "\eta^{-1}"'] & M_1 \arrow[d, ->>, "1", "\eta!"'] & \\
& & N_2 \arrow[r, dotted, ->>, "2", "\eta^{-1}"'] \arrow[drr, dotted, ->>, "\eta^{-1}"'] & \cdot \arrow[r, dotted, ->>, "\eta^{-1}"'] & M_2 \arrow[d, dotted, ->>, "\eta!"'] & \\
& & & & N_\omega = M_\omega & (= M')
\end{tikzcd}
$$

By construction, the diagonal sequence is strongly convergent and has a limit, say N_ω. It is easy to see that the limits M_ω and N_ω are the same, because for all k we have N_{k+1} and M_{k+1} have the same truncation to depth k. $\square$

Appendix C. Preservation of η-expansions of x after β

For the proof of local commutation, we need to prove preservation of $\eta!$-redexes under β-reduction. The proof follows the same pattern as the proof of preservation of $\eta!$-redexes under $\eta!$-reduction (Lemma 3.1).

Lemma C.1. If $x \xrightarrow{<n}\!\!\twoheadrightarrow_{\eta^{-1}} N \xrightarrow{\geq n}\!\!\twoheadrightarrow_{\eta^{-1}} M$ and $M \xrightarrow{\geq n}_\beta M'$ then $N \xrightarrow{\geq n}\!\!\twoheadrightarrow_{\eta^{-1}} M'$.

Proof: Assume $x \xrightarrow{<n}\!\!\twoheadrightarrow_{\eta^{-1}} N \xrightarrow{\geq n}\!\!\twoheadrightarrow_{\eta^{-1}} M$ and $M \xrightarrow{\geq n}_\beta M'$. We will show $N \xrightarrow{\geq n}\!\!\twoheadrightarrow_{\eta^{-1}} M'$. Since η^{-1} does not change the depth of any term, once the β-redex is created, its depth remains fixed. There are only two ways in which η-expansions can create a β-redex:

1 The application of the β-redex is created before its abstraction in the η-expansion. This happens as follows:

$$
\begin{tikzcd}
x \arrow[r, ->>, "<n", "\eta^{-1}"'] & N \arrow[r, ->>, "\geq n", "\eta^{-1}"'] & C'[P'Q'] \arrow[r, "\geq n", "\eta^{-1}"'] \arrow[drr, dotted, ->>, "\geq n", "\eta^{-1}"'] & C'[(\lambda y.P'y)Q'] \arrow[r, ->>, "\geq n", "\eta^{-1}"'] & C[(\lambda y.P)Q] = M \arrow[d, "\geq n", "\beta"'] \\
& & & & C[P[y := Q]] = M'
\end{tikzcd}
$$

Since $P'y \longrightarrow\!\!\twoheadrightarrow_{\eta^{-1}} P$ and $Q' \longrightarrow\!\!\twoheadrightarrow_{\eta^{-1}} Q$ and $y \notin FV(P')$, we have that

$$P'Q' = (P'y)[y := Q'] \longrightarrow\!\!\twoheadrightarrow_{\eta^{-1}} P[y := Q]$$

2 The abstraction in the β-redex gets created before its application in the η-expansion. This happens as follows:

$$
\begin{tikzcd}
x \arrow[r, ->>, "<n", "\eta^{-1}"'] & N \arrow[r, ->>, "\geq n", "\eta^{-1}"'] & C'[\lambda y.P'] \arrow[r, "\geq n", "\eta^{-1}"'] \arrow[drr, dotted, ->>, "\geq n", "\eta^{-1}"'] & C'[\lambda z.(\lambda y.P')z] \arrow[r, ->>, "\geq n", "\eta^{-1}"'] & C[\lambda z.(\lambda y.P)Q] = M \arrow[d, "\geq n", "\beta"'] \\
& & & & C[\lambda z.P[y := Q]] = M'
\end{tikzcd}
$$

Since $P' \twoheadrightarrow_{\eta^{-1}} P$ and $z \twoheadrightarrow_{\eta^{-1}} Q$, we have that

$$\lambda y.P' =_\alpha \lambda z.P'[y := z] \twoheadrightarrow_{\eta^{-1}} \lambda z.P[y := Q]$$

$\square$

Theorem C.2 (Preservation of η-expansion of x by β). If $x \twoheadrightarrow_{\eta^{-1}} M$ and $M \twoheadrightarrow_\beta M'$, then $x \twoheadrightarrow_{\eta^{-1}} M'$. **Back to Lemma 4.1**.

Proof: By strong convergence, we can assume that the β-reduction sequence is of the form $M_0 \xrightarrow{\geq 0}\!\!\twoheadrightarrow_\beta M_1 \xrightarrow{\geq 1}\!\!\twoheadrightarrow_\beta M_2 \ldots$. Now we can proceed similarly as in the proof of Theorem B.7 while exploiting Lemmas B.2 and B.8 instead. $\square$

Appendix D. Strip lemma for one step $\eta!$ over infinitely many β's

The full strip lemma for β over $\eta!$ is harder than the strip lemma for $\eta!$ over β (see Lemma 4.2). The difficulty lies in the fact that, due to overlap, the residuals of an $\eta!$ redex are not always immediately $\eta!$ redexes themselves. We illustrate this with an example. Consider $M = (\lambda x.zX)Q$, where $x \twoheadrightarrow_{\eta^{-1}} \lambda y_1y_2y_3.xy_1y_2y_3 = X$ and Q is some arbitrary term. What are the residuals of the $\eta!$-redex $\lambda x.zX$ in M after contracting the β-redex $(\lambda x.zX)Q$? We have that

$$\begin{array}{rl} M = (\lambda x.zX)Q & \longrightarrow_\beta z(\lambda y_1y_2y_3.Qy_1y_2y_3) \\ & \longrightarrow_{\eta!} z(\lambda y_1y_2.Qy_1y_2) \\ & \longrightarrow_{\eta!} z(\lambda y_1.Qy_1) \\ & \longrightarrow_{\eta!} zQ \end{array}$$

Only the first of these consecutive $\eta!$-redexes is readily present in $\lambda y_1y_2y_3.Qy_1y_2y_3$. From the next two redexes only their λ's are present in $\lambda y_1y_2y_3.Qy_1y_2y_3$. These lambda's are $\eta!$-redexes "in waiting". The residuals in $\lambda y_1y_2y_3.Qy_1y_2y_3$ of the original $\eta!$-redex $\lambda x.zX$ will be the three abstractions $\lambda y_3.Qy_1y_2y_3$, $\lambda y_2y_3.Qy_1y_2y_3$ and $\lambda y_1y_2y_3.Qy_1y_2y_3$ in spite of the fact that not all of them are $\eta!$-redexes. We make this precise with underlining. To track the residuals of $\lambda x.zX$, we will not only underline the λ of $\lambda x.zX$ but also all the λ's in X, i.e. $(\underline{\lambda}x.z\underline{X})Q$ where $\underline{X} = \underline{\lambda}y_1\underline{\lambda}y_2\underline{\lambda}y_3.xy_1y_2y_3$.
To simplify matters a bit, we will not do this in full generality. Instead we will do this only with $\eta!$-redexes of the form $\lambda x.MN$ where N is an η-expansion of x which is in $\beta\bot$-normal form, because such expansions have a straightforward format:

Lemma D.1. If $x \twoheadrightarrow_{\eta^{-1}} X$ and X is in $\beta\bot$-normal form, then X is of the form $\lambda y_1 \ldots y_n.xY_1 \ldots Y_n$, where $x \neq y_i$, $y_i \twoheadrightarrow_{\eta^{-1}} Y_i$ and Y_i is in $\beta\bot$-normal form for each $1 \leq i \leq n$.

Proof: The reduction steps in $x \twoheadrightarrow_{\eta!} X$ can be sorted by depth with Lemma B.2, so that we may assume without loss of generality that $x \twoheadrightarrow_{\eta!} X$ is of the form $x \xrightarrow{0}\!\!\twoheadrightarrow_{\eta^{-1}} X_1 \xrightarrow{>0}\!\!\twoheadrightarrow_{\eta^{-1}} X$. Because X is a $\beta\bot$-normal form, X_1 must be of the form $\lambda y_1 \ldots y_n.xy_1 \ldots y_n$ as can be shown with a proof by induction on n: if the expansions steps at depth 0 would be executed at other positions than the sequence of positions that leads to $\lambda y_1 \ldots y_n.xy_1 \ldots y_n$ a β-redex would be introduced and all further terms in the sequence would contain a β-redex, contradicting the normal form of the final term X.
Now, because the deeper reductions in $X_1 \xrightarrow{>0}\!\!\twoheadrightarrow_{\eta^{-1}} X$ do not alter the left spine of X_1, it

follows that also X must be of the shape $\lambda y_1 \ldots y_n.xY_1 \ldots Y_n$ where $y_i \twoheadrightarrow_{\eta^{-1}} Y_i$ and Y_i is in $\beta\bot$-normal form for each $1 \leq i \leq n$.

□

In the proof of the restricted strip lemma for one step $\eta!$ over β we will employ the underlining technique of (Barendregt, 1992) to track the residuals of $\eta!$-redexes of the particular form $\lambda x.PX$ where X is an η-expansion of x in $\beta\bot$-normal form. To introduce this technique in the infinitary setting, we extend the set $\Lambda_\bot^\infty$ to $\underline{\Lambda_\bot^\infty}$ which will contain underlined terms of the following form only:

$$\underline{\lambda} y_1 \ldots \underline{\lambda} y_n.M\underline{Y_1} \ldots \underline{Y_n}$$

where $Y_i \in \Lambda_\bot^\infty$ is in $\beta\bot$-normal form, Y_i is an η-expansion of y_i and $\underline{Y_i}$ is obtained by underlining all λs in Y_i. for all $1 \leq i \leq n$.

D.1 Definition [Family of sets $\underline{\mathcal{E}_x}$ for $x \in \mathcal{V}$]: We define a family of sets $\underline{\mathcal{E}_x}$ on $x \in \mathcal{V}$ by simultaneous induction:

$$X ::= x \mid \underline{\lambda} x_1 \ldots \underline{\lambda} x_n.xX_1 \ldots X_n$$

where $X_i \in \underline{\mathcal{E}_{x_i}}$ for all $1 \leq i \leq n$.

D.2 Definition [Set $\underline{\Lambda_\bot}$ of underlined finite lambda terms with $\bot$]: We define the set $\underline{\Lambda_\bot}$ of underlined finite λ-terms by induction:

$$M ::= \bot \mid x \mid (\lambda x M) \mid (MM) \mid \underline{\lambda} x_1 \ldots \underline{\lambda} x_n.MX_1 \ldots X_n$$

where $x \in \mathcal{V}$, $x_i \notin FV(M)$ and $X_i \in \underline{\mathcal{E}_{x_i}}$ for all $1 \leq i \leq n$.

The metric d on Λ can be easily extended to terms in $\underline{\Lambda_\bot}$ and in each of the $\underline{\mathcal{E}_x}$.

D.3 Definition [Sets of underlined finite and infinite terms]: 1 Let $x \in \mathcal{V}$. The set $\underline{\mathcal{E}_x^\infty}$ is the *metric completion* of the set $\underline{\mathcal{E}_x}$ with respect to the metric d.

2 The set $\underline{\Lambda_\bot^\infty}$ is the *metric completion* of the set of underlined finite lambda terms $\underline{\Lambda_\bot}$ with respect to the metric d.

Now we are ready to define underlined $\eta!$- and β-reduction.

D.4 Definition: 1 Let $\longrightarrow_{\underline{\eta!}}$ be the smallest binary relation on $\underline{\Lambda_\bot^\infty}$ containing the rule

$$\frac{X \in \underline{\mathcal{E}_x^\infty} \qquad x \notin FV(M)}{\underline{\lambda} x.MX \to M} \ (\underline{\eta!})$$

and closed under contexts.

2 Let $\longrightarrow_{\underline{\beta}}$ be the smallest binary relation on $\underline{\Lambda_\bot^\infty}$ containing the following rule:

$$(\underline{\lambda} x.P)Q \to P[x := Q] \quad (\underline{\beta})$$

and closed under contexts.

The definition of $\longrightarrow_{\underline{\beta}}$ is correct in the sense that $\underline{\Lambda_\bot^\infty}$ is closed under underlined β-reduction: one sees easily that $X[x := Q] \in \underline{\Lambda_\bot^\infty}$ holds for any $X \in \underline{\mathcal{E}_x^\infty}$ and any $Q \in \underline{\Lambda_\bot^\infty}$.
We will frequently use situations where $X_i \in \underline{\mathcal{E}_x^\infty}$ and $x_i \notin FV(M)$ for all $1 \leq i \leq n$, in which case we have the reductions:

$$\underline{\lambda} x_1 \ldots \underline{\lambda} x_n.MX_1 \ldots X_n \twoheadrightarrow_{\underline{\eta!}} M$$

and

$$(\underline{\lambda} x_1 \underline{\lambda} x_2 \ldots \underline{\lambda} x_n.MX_1X_2\ldots X_n)Q \to_{\underline{\beta}} \underline{\lambda} x_2 \ldots \underline{\lambda} x_n.MX_1[x_1 := Q]X_2 \ldots X_n$$

We will denote the union of $\longrightarrow_\beta$ and $\longrightarrow_{\underline{\beta}}$ by $\longrightarrow_{\beta\underline{\beta}}$.
As in the finitary setting we need mechanisms to remove the underlining:

D.5 Definition [Removing underlinings]: Let $M \in \underline{\Lambda_\perp^\infty}$.

1 We define $|M| \in \Lambda_\perp^\infty$ as the result of removing all the underlinings in M.

2 We define $\varphi(M) \in \Lambda_\perp^\infty$ as the result of contracting all $\underline{\eta!}$-redexes from M by co-recursion as follows.

$$\begin{array}{rl} \varphi(x) & = x \\ \varphi(PQ) & = \varphi(P)\varphi(Q) \\ \varphi(\lambda x.P) & = \lambda x.\varphi(P) \\ \varphi(\underline{\lambda} x_1 \ldots \underline{\lambda} x_n.MX_1\ldots X_n) & = \varphi(M) \end{array}$$

Note that $\varphi(M)$ is in $\underline{\eta!}$-normal form for all $M \in \underline{\Lambda_\perp^\infty}$.
For example, $\varphi((\underline{\lambda} x.zX)\mathsf{I}) = z\mathsf{I}$ where $X = \underline{\lambda} y_1 \underline{\lambda} y_2 \underline{\lambda} y_3.xy_1y_2y_3$.

Lemma D.2.

1 If $X \in \underline{\mathcal{E}_x^\infty}$ then $x \twoheadrightarrow_{\eta^{-1}} |X|$.

2 Let X be a $\beta\perp$-normal form. If $x \twoheadrightarrow_{\eta^{-1}} X$ then $\underline{X} \in \underline{\mathcal{E}_x^\infty}$ where $\underline{X}$ is the result of underlining all abstractions in X.

Proof: 1 Suppose $X \in \underline{\mathcal{E}_x^\infty}$. It is not difficult to show that $X = \underline{\lambda} y_1 \ldots \underline{\lambda} y_n.xY_1\ldots Y_n$ and $Y_i \in \underline{\mathcal{E}_{y_i}^\infty}$ for all $1 \le i \le n$ using Definitions C.1 and C.3. We consider the η^{-1}-reduction sequence: $x \twoheadrightarrow_{\eta^{-1}} \lambda y_1 \ldots \lambda y_n.xy_1\ldots y_n$. We repeat a similar argument for each Y_i with $1 \le i \le n$ as we did for X. This process can be repeated ad infinitum to obtain an η^{-1}-strongly converging reduction sequence from x to $|X|$.

2 Suppose $x \twoheadrightarrow_{\eta^{-1}} X$. We construct a Cauchy sequence $M_1, M_2, \ldots$ of terms in $\underline{\mathcal{E}_x}$ whose limit is $\underline{X}$ using Lemma C.1. By construction the limit $\underline{X}$ is an element of $\underline{\mathcal{E}_x^\infty}$. The first term M_1 in this sequence is x which belongs to $\underline{\mathcal{E}_x}$. By Lemma C.1, we have that $X = \lambda y_1 \ldots \lambda y_n.xY_1\ldots Y_n$ and $y_i \twoheadrightarrow_{\eta^{-1}} Y_i$ for each $1 \le i \le n$. The second term M_2 of the sequence is $\underline{\lambda} y_1 \ldots \underline{\lambda} y_n.xy_1\ldots y_n$ which belongs to $\underline{\mathcal{E}_x}$. We repeat this process to construct all the terms in the sequence. The limit of this sequence is $\underline{X}$ and it belongs to $\underline{\mathcal{E}_x^\infty}$ by Definition C.3.

□

Lemma D.3. If $M \twoheadrightarrow_{\underline{\eta!}} N$, then $|M| \twoheadrightarrow_{\eta!} |N|$.

Proof: This is proved by induction on the length of the reduction sequence. We only prove it for a reduction sequence of length 1. Suppose $\underline{\lambda} x.MX \longrightarrow_{\underline{\eta!}} M$. Then, $X \in \underline{\mathcal{E}_x^\infty}$ and $x \notin FV(M)$. By Lemma C.2(i), we have that $x \twoheadrightarrow_{\eta^{-1}} |X|$ and hence, $\lambda x.MX \longrightarrow_{\eta!} M$. □

The next lemma is a straightforward consequence of the definition of φ.

Lemma D.4. Let $X \in \underline{\mathcal{E}_x^\infty}$. Then $\varphi(X) = x$.

Lemma D.5. Let $M \in \underline{\Lambda_\perp^\infty}$. Then, there exists a reduction of length at most ω such that $M \twoheadrightarrow_{\underline{\eta!}} \varphi(M)$.

Proof: Contraction of the $\underline{\eta!}$-redexes using a depth-first left-most strategy gives a reduction $M \twoheadrightarrow_{\underline{\eta!}} \varphi(M)$ of length at most ω. $\square$

Lemma D.6 (φ on substitutions). Let $M, N \in \underline{\Lambda^{\infty}_{\perp}}$. Then, $\varphi(M[x := N]) = \varphi(M)[x := \varphi(N)]$.

Proof: We prove that $(\varphi(M[x := N]))^n = (\varphi(M)[x := \varphi(N)])^n$ for all n by induction on (n, m) where n is the depth of the truncation and m is the number of abstractions and applications in M at depth n. $\square$

Lemma D.7 (φ on one step β). Let $M \in \underline{\Lambda^{\infty}_{\perp}}$.

1 If $M \xrightarrow{n}_{\beta} N$ then $\varphi(M) \xrightarrow{n}_{\beta} \varphi(N)$.
2 If $M \longrightarrow_{\underline{\beta}} N$ then $\varphi(M) = \varphi(N)$.

Proof: In both cases, we proceed by induction on the pair (n, m) where n is the depth of the redex in M and m is the number of abstractions and applications in M at depth n.

1 Suppose $n = 0$. Then, $M = (\lambda x.P)Q \xrightarrow{0}_{\beta} P[x := Q] = N$ and we apply Lemma C.6. The case $n > 0$ follows by induction hypothesis.

2 The case $n > 0$ follows by induction hypothesis. Suppose $n = 0$. Since the only occurrence of x_1 in $(\underline{\lambda}x_2 \ldots \underline{\lambda}x_n.M_0X_1X_2 \ldots X_n)$ is in X_1, we have that:

$$\begin{array}{rl} M = & (\underline{\lambda}x_1 \ldots \underline{\lambda}x_n.M_0X_1 \ldots X_n)Q \\ \longrightarrow_{\underline{\beta}} & (\underline{\lambda}x_2 \ldots \underline{\lambda}x_n.M_0X_1[x_1 := Q]X_2 \ldots X_n) \\ = & N \end{array}$$

Since $X_1 = \underline{\lambda}y_1 \ldots y_k.x_1Y_1 \ldots Y_k \in \underline{\mathcal{E}^{\infty}_{x_i}}$, we have that:

$$\begin{array}{rcll} \varphi(M) & = & (\underline{\lambda}x_1 \ldots \underline{\lambda}x_n.M_0X_1 \ldots X_n)Q & \text{by definition of } \varphi \\ & = & \varphi(M_0)\varphi(Q) & \text{by definition of } \varphi \\ & = & \varphi(M_0)x_1[x_1 := \varphi(Q)] & \\ & = & \varphi(N)\varphi(X_1)[x_1 := \varphi(Q)] & \text{by Lemma C.4} \\ & = & \varphi(N)\varphi(X_1[x_1 := Q]) & \text{by Lemma C.6} \end{array}$$

$\square$

The function φ does not preserve truncations, i.e. $\varphi(M^n) \neq \varphi(M)^n$. For example, take $M = \underline{\lambda}x.y(\underline{\lambda}z.xz)$. We will define a notion of quasi-truncation which is preserved by φ. The quasi-truncation of a term at depth n truncates the term at depth n except for the η-expansions X in an $\underline{\eta!}$-redex.

D.6 Definition [Quasi-truncation]: We define quasi-truncation of M at depth n by induction on the lexicographically ordered pair (n, m) where m is the number of abstractions and applications at depth n:

$$\begin{array}{lcl} [\perp]^n & = & \perp \\ [M]^0 & = & \perp \\ [x]^{n+1} & = & x \\ [\lambda x.M]^{n+1} & = & \lambda x.[M]^{n+1} \\ [MN]^{n+1} & = & [M]^{n+1}[N]^n \\ [\underline{\lambda}x_1 \ldots \underline{\lambda}x_k.MX_1 \ldots X_k]^n & = & \underline{\lambda}x_1 \ldots \underline{\lambda}x_k.[M]^nX_1X_2 \ldots X_k \end{array}$$

For example, take $M = \underline{\lambda} x.y(y(y\ldots))(\underline{\lambda} z.xz)$. Then $M^1 = \underline{\lambda} x.y\perp\perp$ and $[M]^1 = \underline{\lambda} x.y\perp(\underline{\lambda} z.xz)$. Note that $([M]^n)^n = M^n$ for all $M \in \underline{\Lambda}^\infty_\perp$. The function φ preserves quasi-truncations:

Lemma D.8 (Preservation of quasi-truncations). $\varphi([M]^n) = \varphi(M)^n$.

Proof: This is proved by induction on (n,m) where n is the depth of the truncation and m is the number of abstractions and applications in M at depth n. □

Lemma D.9 (φ on many β-steps). Let $M \in \underline{\Lambda}^\infty_\perp$. If $M \twoheadrightarrow_{\beta\underline{\beta}} N$ has length at most ω, then $\varphi(M) \twoheadrightarrow_\beta \varphi(N)$.

Proof: We prove it by induction on the length of $M \twoheadrightarrow_{\beta\underline{\beta}} N$. The finite case follows from Lemma C.7. We prove the case when the length is ω. The following diagram can be constructed by repeated applications of Lemma C.7. Since φ preserves the depth of the contracted redex, we have that the bottom sequence is strongly convergent and the limit exists which is P.

$$\begin{tikzcd}
M = M_0 \arrow[r, "n_0", "\beta\underline{\beta}"'] & M_1 \arrow[r, "n_1", "\beta\underline{\beta}"'] & M_2 \arrow[r, "n_2", "\beta\underline{\beta}"'] & \ldots & M_\omega = N \\
\varphi(M_0) \arrow[r, dotted, "n_0", "{=,\beta}"'] & \varphi(M_1) \arrow[r, dotted, "n_1", "{=,\beta}"'] & \varphi(M_2) \arrow[r, dotted, "n_2", "{=,\beta}"'] & \ldots & P \stackrel{?}{=} \varphi(M_\omega)
\end{tikzcd}$$

It remains to prove that $P = \varphi(M_\omega)$. By strong convergence, there exists n_0 such that for all $n \geq n_0$, $(M_n)^k = (M_\omega)^k$ and $(\varphi(M_n))^k = P^k$. Now $[M_n]^k = [M_\omega]^k$ because$[M_n]^k$ and $[M_\omega]^k$ are obtained from $(M_n)^k$ and $(M_\omega)^k$ by adding terms of the form $X \in \underline{\mathcal{E}}^\infty_x$ which are in β-normal form. We have

$$\begin{aligned}
P^k &= (\varphi(M_n))^k \text{ by strong convergence} \\
&= \varphi([M_n]^k) \text{ by Lemma C.8} \\
&= \varphi([M_\omega]^k) \text{ because } [M_n]^k = [M_\omega]^k \\
&= (\varphi(M_\omega))^k \text{ by Lemma C.8}
\end{aligned}$$

□

Back to Lemma 4.3.

Proof: Let X be an η-expansion of x such that X is a $\beta\perp$-normal form. Suppose $M = C[\lambda x.M_0X]$. In order to track the residuals of this η!-redex, we consider the term $M' = C[\underline{\lambda} x.M_0\underline{X}]$ where $\underline{X}$ is the result of underlining all abstractions in X. Then $\underline{X} \in \underline{\mathcal{E}}^\infty_x$ by Lemma C.2(ii). The reduction $M \twoheadrightarrow_\beta N$ is lifted to $M' \twoheadrightarrow_{\beta\underline{\beta}} N'$. Using Lemmas C.3, C.5 and C.9, we obtain the following diagram:

$$\begin{tikzcd}
M = C[\lambda x.M_0X] \arrow[rr, twoheadrightarrow, "\beta"] \arrow[dd, "\eta!"'] & & N \arrow[dd, dotted, twoheadrightarrow, "\eta!"] & \\
& M' = C[(\underline{\lambda} x.M_0\underline{X}] \arrow[ul, "||"] \arrow[dl, "\varphi"] \arrow[rr, twoheadrightarrow, "\beta\underline{\beta}"] & & N' \arrow[ul, "||"'] \arrow[dl, "\varphi"] \\
P \arrow[rr, dotted, twoheadrightarrow, "\beta"'] & & Q &
\end{tikzcd}$$

□

Appendix E. Commutation properties of β and η^{-1}

In this section we will study some precise commutation properties of β and η^{-1}. We need these properties to prove that η-expansions of head normal forms again have a head normal form. As a consequence $\eta!$-reduction preserves $\bot_{out}$-redexes, which plays a crucial role in the proof of the commutation property for $\eta!$ and $\bot_{out}$ in Section 5.

E.1. *Strip Lemmas for one step β_0 over η^{-1}*

In this subsection we concentrate on the strip lemmas for one step β-reductions that takes place at depth 0 over η-expansion. There is a slight complication, because η-expansions can create β-redexes as shown by the next example.

$$
\begin{tikzcd}
(\lambda x.x^\omega)\mathsf{I} \arrow[r, "\eta^{-1}"'] \arrow[d, "\beta"'] & (\lambda y.(\lambda x.x^\omega)y)\mathsf{I} \arrow[r, "\eta^{-1}"'] & (\lambda y.(\lambda z.(\lambda x.x^\omega)z)y)\mathsf{I} \arrow[d, dotted, twoheadrightarrow, "\beta"'] \\
\mathsf{I}^\omega & & (\lambda y.y^\omega)\mathsf{I} \arrow[ll, dotted, "\beta"']
\end{tikzcd}
$$

In the above η^{-1}-reduction sequence, we have created two extra β-redexes which should be contracted to get a common reduct. These extra β-redexes are of a special nature, for which we will introduce the notion of β_0-reduction.

E.1 Definition [β_0-reduction]: We will introduce the notation $M \longrightarrow_{\beta_0} N$ for the situation where M is of the form $C[(\lambda x.P)Q]$ and N is of the form $C[P[x := Q]]$, while the hole $[\ ]$ in $C[\ ]$ occurs at depth 0 and the variable x occurs at depth 1 and exactly once in P.

Note that $\longrightarrow_{\beta_0}$ is a restricted form $\xrightarrow{0}_\beta$.
Examples of β_0-redexes are: $(\lambda x.yx)\mathsf{I}, (\lambda x.y(\lambda z.xz))\mathsf{I}$ and $(\lambda xy.y(x\mathsf{K}))\mathsf{I}$. Non-examples are $(\lambda x.xy)\mathsf{I}$ and $(\lambda x.yxx)\mathsf{I}$, as in the former the variable x does not occur at depth 1 and in the latter the variable x occurs twice.
The next diagram shows the usefulness of this restricted form of β-reduction in the context of a strip lemma of one-step β over η^{-1}-reduction. Consider the η^{-1}-reduction sequence $(\lambda x.x^\omega)\mathsf{I} \twoheadrightarrow_{\eta^{-1}} (\lambda y_1y_2.(\lambda x.x^\omega)(\lambda z.y_1z)y_2)\mathsf{I} = N$. Then,

$$
\begin{tikzcd}
(\lambda x.x^\omega)\mathsf{I} \arrow[r, twoheadrightarrow, "0", "\eta^{-1}"'] \arrow[dd, "\beta"] & (\lambda y_1y_2.(\lambda x.x^\omega)y_1y_2)\mathsf{I} \arrow[r, "1", "\eta^{-1}"'] \arrow[d, "\beta_0"] & (\lambda y_1y_2.(\lambda x.x^\omega)(\lambda z.y_1z)y_2)\mathsf{I} \arrow[d, "\beta_0"] \\
 & \lambda y_2.(\lambda x.x^\omega)\mathsf{I}y_2 \arrow[r, "1", "\eta^{-1}"'] \arrow[d, "\beta"] & \lambda y_2.(\lambda x.x^\omega)1y_2 \arrow[d, "\beta"] \\
\mathsf{I}^\omega \arrow[r, "0", "\eta^{-1}"'] & (\lambda y_2.\mathsf{I}^\omega y_2) \arrow[r, twoheadrightarrow, "1", "\eta^{-1}"'] & (\lambda y_2.1^\omega y_2)
\end{tikzcd}
$$

Lemma E.1 (Local Commutation for one step β_0 and one step η^{-1}). Given $M_0 \longrightarrow_{\eta^{-1}} M_1$ and $M_0 \longrightarrow_{\beta_0} M_2$, there exists M_3 such that either one of the following diagrams hold:

$$\begin{tikzcd}
M_0 \arrow[r, "m", "\eta^{-1}"'] \arrow[d, "\beta_0"'] \arrow[dr, phantom, "(1)"] & M_1 \arrow[d, dotted, "\beta_0"'] \\
M_2 \arrow[r, dotted, "m", "\eta^{-1}"'] & M_3
\end{tikzcd}
\qquad
\begin{tikzcd}
M_0 \arrow[r, "0", "\eta^{-1}"'] \arrow[dd, "\beta_0"'] & M_1 \arrow[d, dotted, "\beta_0"'] \\
\arrow[r, phantom, "(2)"] & N \arrow[d, dotted, "\beta_0"'] \\
M_2 \arrow[r, dotted, dash] & M_3
\end{tikzcd}$$

Proof: A term M_0 in $\Lambda_\perp^\infty$ can contain a β_0-redex $(\lambda y.P)Q$ at depth 0 and an η^{-1}-redex N at depth m in exactly one of the following situations:

1 The β_0-redex $(\lambda y.P)Q$ and the η^{-1}-redex N are not nested, i.e. $M_0 = C[(\lambda y.P)Q, N]$. This results in an instance of Diagram (1).

2 The β_0-redex is inside the η^{-1}-redex, that is M_0 is of the form $C_1[N]$, where $N \equiv C_2[(\lambda y.P)Q]$. This case results in an instance of Diagram (1) too.

3 The η^{-1}-redex is part of the body of the abstraction $\lambda y.P$ of the β_0-redex, i.e. $P \longrightarrow_{\eta^{-1}} P'$. Since η^{-1} does not affect the depth of the variable y, $(\lambda y.P')Q$ remains a β_0-redex and we have

$$\begin{tikzcd}
C[(\lambda y.P)Q] \arrow[r, "m", "\eta^{-1}"'] \arrow[d, "\beta_0"'] & C[(\lambda y.P')Q] \arrow[d, dotted, "\beta_0"'] \\
C[P[y := Q]] \arrow[r, dotted, "m", "\eta^{-1}"'] & C[P'[y := Q]]
\end{tikzcd}$$

which is an instance of Diagram (1).

4 The η^{-1}-redex is part of the argument Q of the β_0-redex. This results in the following:

$$\begin{tikzcd}
C[(\lambda y.P)Q] \arrow[r, "m", "\eta^{-1}"'] \arrow[d, "\beta_0"'] & C[(\lambda y.P)Q'] \arrow[d, dotted, "\beta_0"'] \\
P[y := Q] \arrow[r, dotted, "m", "\eta^{-1}"'] & P[y := Q']
\end{tikzcd}$$

which corresponds to Diagram (1). Since the variable y occurs only once in P, we need only one η^{-1}-step from $P[y := Q]$ to $P[y := Q']$. And because the depth of this variable is 1, the depth of that η^{-1}-redex in $P[y := Q]$ is m.

5 Only if $m = 0$, the η^{-1}-redex coincides with the abstraction $\lambda y.P$ of the β_0-redex, that is M_0 is of the form $C[NQ]$ where $N \equiv \lambda y.P$.

$$\begin{tikzcd}
C[(\lambda y.P)Q] \arrow[r, "0", "\eta^{-1}"'] \arrow[dd, "\beta_0"'] & C[(\lambda x.(\lambda y.P)x)Q] \arrow[d, dotted, "\beta_0"'] \\
 & C[(\lambda y.P)Q] \arrow[d, dotted, "\beta_0"'] \\
C[P[y := Q]] \arrow[r, dotted, dash] & C[P[y := Q]]
\end{tikzcd}$$

The above is an instance of Diagram (2).

□

Lemma E.2 (Strip Lemma for $\longrightarrow_{\beta_0}$ over $\twoheadrightarrow_{\eta^{-1}}$ at $m > 0$).

$$\begin{tikzcd}
M \arrow[r, ->>, "{>0}", "\eta^{-1}"'] \arrow[d, "\beta_0"'] & P \arrow[d, dotted, "\beta_0"'] \\
N \arrow[r, dotted, ->>, "{>0}", "\eta^{-1}"'] & Q
\end{tikzcd}$$

Proof: By Lemma 3.4 (Compression Lemma), we can assume that the η^{-1}-reduction sequence has length at most ω. If the length is finite, then the result follows by repeated application of Diagram (1) of Lemma D.1. Diagram (2) does not apply as the η-expansions are performed at depth greater than 0. When the length is ω we construct the diagram:

$$\begin{tikzcd}
M_0 \arrow[r, "n_0", "\eta^{-1}"'] \arrow[d, "\beta_0"] & M_1 \arrow[r, "n_1", "\eta^{-1}"'] \arrow[d, dotted, "\beta_0"] & M_2 \arrow[r, "n_2", "\eta^{-1}"'] \arrow[d, dotted, "\beta_0"] & M_3 \arrow[d, dotted, "\beta_0"] & \dots & M_\omega \arrow[d, dotted, "\beta_0"] \\
N_0 \arrow[r, dotted, "n_0", "\eta^{-1}"'] & N_1 \arrow[r, dotted, "n_1", "\eta^{-1}"'] & N_2 \arrow[r, dotted, "n_2", "\eta^{-1}"'] & N_3 & \dots & ?
\end{tikzcd}$$

Using Diagram (1) of Lemma D.1, we can complete all the subdiagrams except for the limit case. Since the η-expansions are performed at depth greater than 0, all M_k with $k \geq 0$ are of the form $C_k[(\lambda x.P_k)Q_k]$, where all the $C_k[\,]$ have the hole at the same position at depth 0, and all P_k have exactly one occurrence of x at depth 1. The limit term is of the form $C_\omega[(\lambda x.P_\omega)Q_\omega]$. The hole of C_ω occurs also at depth 0 and x occurs only once in P_ω and at depth 1 because η-expansions do not introduce variables and the existing variables remain at the same depth. Hence, the residual remains a β_0-redex in the limit. Contracting this redex in the limit M_ω reduces to $C_\omega[P_\omega[x := Q_\omega]]$ which is equal to the limit N_ω of the bottom sequence. □

E.2. *Strip Lemma for one step β at depth 0 over η^{-1} reduction*

We will now prove the strip lemma of $\xrightarrow{0}_\beta$ with $\longrightarrow_{\eta^{-1}}$ using the results of the previous subsection.

Lemma E.3 (Local Commutation of $\xrightarrow{0}_\beta$ and $\longrightarrow_{\eta^{-1}}$). Given $M_0 \xrightarrow{m}_{\eta^{-1}} M_1$ and $M_0 \xrightarrow{0}_\beta M_2$, there exists M_3 such that one of the following diagrams hold:

$$\begin{tikzcd}
M_0 \arrow[r, "m", "\eta^{-1}"'] \arrow[d, "0"', "\beta"] \arrow[dr, phantom, "(1)"] & M_1 \arrow[d, dotted, "0"', "\beta"] \\
M_2 \arrow[r, dotted, "m", "\eta^{-1}"'] & M_3
\end{tikzcd}
\qquad
\begin{tikzcd}
M_0 \arrow[r, "m>0", "\eta^{-1}"'] \arrow[d, "0"', "\beta"] \arrow[dr, phantom, "(2)"] & M_1 \arrow[d, dotted, "0"', "\beta"] \\
M_2 \arrow[r, dotted, ->>, "\geq(m-1)", "\eta^{-1}"'] & M_3
\end{tikzcd}
\qquad
\begin{tikzcd}
M_0 \arrow[r, "0", "\eta^{-1}"'] \arrow[dd, "0"', "\beta"] \arrow[ddr, phantom, "(3)"] & M_1 \arrow[d, dotted, "\beta_0"] \\
 & N \arrow[d, dotted, "0"', "\beta"] \\
M_2 \arrow[r, dotted, no head] & M_3
\end{tikzcd}$$

Note the extra information in Diagram (3): the first step of constructed down reduction in Diagram (3) is a β_0-reduction.

Proof: A term M_0 can contain a β-redex $(\lambda y.P)Q$ at depth 0 and an η^{-1}-redex N at depth m. A case analysis leads to the following exhaustive list of possible positions of β-redex and the η^{-1}-redex relative to each other:

1 The β-redex $(\lambda y.P)Q$ and the η^{-1}-redex N are not nested, i.e. $M_0 = C[(\lambda y.P)Q, N]$. This case leads to Diagram (1).

2 The β-redex is inside the η^{-1}-redex, that is M_0 is of the form $C_1[N]$, where $N \equiv C_2[(\lambda y.P)Q]$. This case leads to Diagram (1).

3 The η^{-1}-redex is part of the body of the abstraction $\lambda y.P$ of the β-redex. This case leads to Diagram (1).

4 The η^{-1}-redex is part of the argument Q of the β-redex. This case can only happen if $m > 0$ and it results in an instance of Diagram (2).

$$\begin{tikzcd}
C[(\lambda y.P)Q] \arrow[r, "m", "\eta^{-1}"'] \arrow[d, "\beta"', "0"] & C[(\lambda y.P)Q'] \arrow[d, dotted, "\beta"', "0"] \\
C[P[y := Q]] \arrow[r, dotted, twoheadrightarrow, "\geq m-1", "\eta^{-1}"'] & C[P[y := Q']]
\end{tikzcd}$$

If the variable y occurs infinite times in P, then we need ω-steps to get $P[y := Q']$ from $P[y := Q]$. If there is some occurrence of y at depth 0, then there will be some η^{-1}-redex in $P[y := Q]$ at depth $m - 1$.

5 The η^{-1}-redex coincides with the abstraction $\lambda y.P$ of the β-redex, that is M_0 is of the form $C[NQ]$ where $N \equiv \lambda y.P$. If this happens, M must be 0. It leads to the following instance of Diagram (3):

$$\begin{tikzcd}
C[(\lambda y.P)Q] \arrow[r, "0", "\eta^{-1}"'] \arrow[dd, "0"', "\beta"] & C[(\lambda x.(\lambda y.P)x)Q] \arrow[d, dotted, "\beta_0"] \\
 & C[(\lambda y.P)Q] \arrow[d, dotted, "0"', "\beta"] \\
C[P[y := Q]] \arrow[r, dotted, no head] & C[P[y := Q]]
\end{tikzcd}$$

Note that in the first step on the right vertical line, the contracted outermost β-redex that got created by the η-expansion is a β_0-redex. Again note the informative role of β_0 in the formulation of the lemma.

$\square$

The previous local commutation lemma generalises to a finite strip lemma of $\xrightarrow{0}_\beta$ over $\twoheadrightarrow_{\eta^{-1}}$.

Lemma E.4 (Finite Strip Lemma of $\xrightarrow{0}_\beta$ at depth 0 over $\twoheadrightarrow_{\eta^{-1}}$ at depth 0). Given a one step reduction $M \xrightarrow{0}_\beta P$ and a finite reduction $M \overset{0}{\twoheadrightarrow}_{\eta^{-1}} N$, then there exists Q and

Q_0 such that:

$$\begin{tikzcd}
M \arrow[r, two heads, "0"] \arrow[r, two heads, "\eta^{-1}"'] \arrow[dd, "\beta"] \arrow[dd, "0"'] & N \arrow[d, dotted, two heads, "\beta_0"] \\
 & Q_0 \arrow[d, dotted, "\beta"] \arrow[d, dotted, "0"'] \\
P \arrow[r, dotted, two heads, "0"] \arrow[r, dotted, two heads, "\eta^{-1}"'] & Q
\end{tikzcd}$$

Proof: By induction on the finite length of the η^{-1}-reduction sequence. We show the induction step.

$$\begin{tikzcd}
M \arrow[r, two heads, "0"] \arrow[r, two heads, "\eta^{-1}"'] \arrow[ddd, "\beta"] \arrow[ddd, "0"'] & N \arrow[r, dotted, "0"] \arrow[r, dotted, "\eta^{-1}"'] \arrow[d, dotted, two heads, "\beta_0"] \arrow[dr, phantom, "\text{(Lem. D.1)}"] & N' \arrow[d, dotted, two heads, "\beta_0"] \\
 \arrow[r, phantom, "\text{(IH)}"] & Q_0 \arrow[r, dotted, "0"] \arrow[r, dotted, "\eta^{-1}"'] \arrow[dd, dotted, "\beta"] \arrow[dd, dotted, "0"'] \arrow[ddr, phantom, "\text{(Lem. D.3) (1),(3)}"] & R_0 \arrow[d, dotted, "{=,\beta_0}"] \\
 & & Q_0' \arrow[d, dotted, "\beta"] \arrow[d, dotted, "0"'] \\
P \arrow[r, dotted, two heads, "0"] \arrow[r, dotted, two heads, "\eta^{-1}"'] & Q \arrow[r, dotted, "0"] \arrow[r, dotted, "{=,\eta^{-1}}"'] & Q'
\end{tikzcd}$$

The top right square follows from a repeated application of Lemma D.1. In the bottom right square we apply Diagram (1) and (3) of Lemma D.3. Note that Diagram (2) does not apply because β and η^{-1} are performed at the same depth. $\square$

Next we prove the strip lemma for one step β reduction over many step η^{-1}.

Lemma E.5 (Strip Lemma for $\xrightarrow{0}_\beta$ over $\twoheadrightarrow\!\!\!\!\rightarrow_{\eta^{-1}}$ at depth greater than 0). If $M \xrightarrow{0}_\beta P$ and $M \xrightarrow{>0}\!\!\!\!\twoheadrightarrow_{\eta^{-1}} N$, then there exists Q such that:

$$\begin{tikzcd}
M \arrow[r, "{>0}"] \arrow[r, "\eta^{-1}"'] \arrow[d, "\beta"] \arrow[d, "0"'] & N \arrow[d, dotted, "\beta"] \arrow[d, dotted, "0"'] \\
P \arrow[r, dotted, "\geq 0"] \arrow[r, dotted, "\eta^{-1}"'] & Q
\end{tikzcd}$$

Proof: Similar to Lemma D.2 using Lemma D.3 Diagrams (1) and (2). $\square$

Lemma E.6 (Strip Lemma for β at depth 0 over η^{-1}). Given a one step reduction $M \xrightarrow{0}_{\beta} P$ and a strongly converging reduction $M \twoheadrightarrow_{\eta^{-1}} N$, then there exists Q such that:

$$\begin{tikzcd}
M \arrow[rr, "\eta^{-1}"', ->>] \arrow[d, "0"', "\beta"] & & N \arrow[d, dotted, ->>, "0"', "\beta"] \\
P \arrow[rr, dotted, ->>, "\eta^{-1}"'] & & Q
\end{tikzcd}$$

Proof: By Lemma B.2, we can assume that the η^{-1}-reduction sequence is of the form $M \xrightarrow{0}\!\!\!\twoheadrightarrow_{\eta^{-1}} M_1 \xrightarrow{>0}\!\!\!\twoheadrightarrow_{\eta^{-1}} N$. The proof is sketched in the following diagram.

$$\begin{tikzcd}
M \arrow[r, ->>, "0", "\eta^{-1}"'] \arrow[dd, "0"', "\beta\ (\text{Lem. D.4})"] & M_1 \arrow[r, ->>, ">0", "\eta^{-1}"'] \arrow[d, dotted, ->>, "\beta_0"] \arrow[dr, phantom, "(\text{Lem. D.2})"] & N \arrow[d, dotted, ->>, "\beta_0"] \\
 & \cdot \arrow[r, dotted, ->>, ">0", "\eta^{-1}"'] \arrow[d, dotted, "0"', "\beta"] \arrow[dr, phantom, "(\text{Lem. D.5})"] & \cdot \arrow[d, dotted, "0"', "\beta"] \\
P \arrow[r, dotted, ->>, "0", "\eta^{-1}"'] & P_1 \arrow[r, dotted, ->>, "\geq 0", "\eta^{-1}"'] & Q
\end{tikzcd}$$

□

E.3. *Commutation properties for restricted β reduction and η-expansion*

In this subsection we will consider a particular instance of β-reduction in order to deal with the β-redexes created by η-expansions from truncated head normal forms. For example,

$$\begin{array}{ll}
\lambda x.x\bot\bot & \longrightarrow_{\eta^{-1}} \lambda y.(\lambda x.x\bot\bot)y \\
\lambda x.x\bot\bot & \longrightarrow_{\eta^{-1}} \lambda x.(\lambda y.x\bot y)\bot
\end{array}$$

In the first example, we see that the argument of the β-redex that we have created is a variable, while in the second example the argument is $\bot$.
In fact we will define two instances of β-reduction, called respectively β_v-reduction, and β_V-reduction in order to deal with β-redexes created by η-expansions starting from a head normal form.

E.2 Definition [β_v and β_V-reductions]: Let $C[\ ]$ be a context with the hole at depth 0.

1 We define β_v-reduction by $C[(\lambda x.P)Q] \longrightarrow_{\beta_v} C[P[x := Q]]$ if either $Q \equiv y$ for some variable y or $Q \equiv \bot$.

2 We define β_V-reduction by $C[(\lambda x.P)Q] \longrightarrow_{\beta_V} C[P[x := Q]]$ if Q is an η-expansion of either some variable y or $\bot$.

Note that $M \longrightarrow_{\beta_v} N$ implies $M \longrightarrow_{\beta_V} N$ implies $M \xrightarrow{0}_{\beta} N$ for all $M, N \in \Lambda^{\infty}_{\bot}$.
For example, $(\lambda x.x^{\omega})\bot$ and $(\lambda x.x^{\omega})y$ are β_v-redexes and also β_V-redexes while the terms $(\lambda x.x^{\omega})(\lambda z.\bot z)$ and $(\lambda x.x^{\omega})(\lambda z.yz)$ are β_V-redexes but they are not β_v-redexes.
We defined β_V because β_v and η^{-1} do not commute if the η^{-1}-step is performed at depth greater than 0. The following diagram can be completed because the right vertical line is a

β_V-step.

$$\begin{tikzcd}
(\lambda x.x\bot)y \arrow[rr, "1", "\eta^{-1}"'] \arrow[d, "\beta_v"] & & (\lambda x.x\bot)(\lambda z.yz) \arrow[d, dotted, "\beta_V"] \\
y\bot \arrow[rr, dotted, "0", "\eta^{-1}"'] & & (\lambda z.yz)\bot
\end{tikzcd}$$

Lemma E.7 (Local Commutation of β_v and η^{-1} at depth 0). If $M_0 \longrightarrow_{\eta^{-1}} M_1$ and $M_0 \longrightarrow_{\beta_v} M_2$, then there exists an M_3 such that one of the following diagrams holds:

$$\begin{tikzcd}
M_0 \arrow[r, "0", "\eta^{-1}"'] \arrow[d, "\beta_v"] \arrow[dr, phantom, "(1)"] & M_1 \arrow[d, dotted, "\beta_v"] \\
M_2 \arrow[r, dotted, "0", "\eta^{-1}"'] & M_3
\end{tikzcd}
\qquad
\begin{tikzcd}
M_0 \arrow[r, "0", "\eta^{-1}"'] \arrow[dd, "\beta_v"] \arrow[ddr, phantom, "(2)"] & M_1 \arrow[d, dotted, "\beta_v"] \\
 & N \arrow[d, dotted, "\beta_v"] \\
M_2 \arrow[r, dotted, dash] & M_3
\end{tikzcd}$$

Proof: Suppose M_0 can do both a β_v-redex $(\lambda y.P)Q$ at depth 0 and η^{-1}-redex N at depth 0. The only possible situations in which this can happen are:

1 The β_v-redex $(\lambda y.P)Q$ and the η^{-1}-redex N are not nested, i.e. $M_0 = C[(\lambda y.P)Q, N]$. This case leads to Diagram (1).

2 The β_v-redex is inside the η^{-1}-redex, that is M_0 is of the form $C_1[N]$, where $N \equiv C_2[(\lambda y.P)Q]$. This case leads to Diagram (1).

3 The η^{-1}-redex is part of the body of the abstraction $\lambda y.P$ of the β_v-redex. This case leads to Diagram (1).

4 The η^{-1}-redex can not be part of the argument Q of the β_v-redex, because the η^{-1}-redex occurs at depth 0.

5 The η^{-1}-redex coincides with the abstraction $\lambda y.P$ of the β_v-redex, that is M_0 is of the form $C[NQ]$ where $N \equiv \lambda y.P$.

$$\begin{tikzcd}
C[(\lambda y.P)Q] \arrow[r, "0", "\eta^{-1}"'] \arrow[dd, "\beta_v"] & C[(\lambda x.(\lambda y.P)x)Q] \arrow[d, dotted, "\beta_v"] \\
 & C[(\lambda y.P)Q] \arrow[d, dotted, "\beta_v"] \\
C[P[y := Q]] \arrow[r, dotted, dash] & C[P[y := Q]]
\end{tikzcd}$$

This case leads to Diagram (2). Note that $C[(\lambda x.(\lambda y.P)x)Q]$ contains a β_v-step and a β_0-step. The β_0-reduction contracts the outermost redex created by the η-expansion and the β_v-reduction contracts the innermost one.

□

Later we will need a sort of strip lemma of β_v over η^{-1} where the right vertical line is a

β_V-step. For example,

$$\begin{tikzcd}
(\lambda x.x^\omega)\bot \arrow[r, "\eta^{-1}"'] \arrow[d, "\beta_v"] & (\lambda x.x^\omega)(\lambda y.\bot y) \arrow[d, dotted, "\beta_V"] \\
\bot^\omega \arrow[r, dotted, twoheadrightarrow, "\eta^{-1}"'] & (\lambda y.\bot y)^\omega
\end{tikzcd}$$

For the bottom horizontal line, we will define a parallel reduction called η_v^{-1}-reduction which replaces some of the variables and $\bot$'s in a term by their η-expansions.

E.3 Definition [Parallel η_v^{-1}-reduction]: We define η_v^{-1}-reduction on $\Lambda_\bot^\infty$ as follows: $M \Longrightarrow_{\eta_v^{-1}} N$ if N is obtained from M by replacing each variable x by a term X such that $x \twoheadrightarrow_{\eta^{-1}} X$ and each $\bot$ by a term B such that $\bot \twoheadrightarrow_{\eta^{-1}} B$.

For example, $\lambda x.y\bot\bot x \Longrightarrow_{\eta_v^{-1}} \lambda x.yB_1B_2X$ where $X = \lambda z.xz$, $B_1 = \lambda x.\bot x$ and $B_2 = \lambda x.\bot X$. Note that the variable y has been replaced by itself, i.e. it has not changed.

Lemma E.8. If $M_0 \Longrightarrow_{\eta_v^{-1}} M_1$ and $M_1 \longrightarrow_{\beta_v} M_2$, then there exists a term M_3 such that:

$$\begin{tikzcd}
M_0 \arrow[r, Rightarrow, "\eta_v^{-1}"'] \arrow[d, "\beta_v"] & M_1 \arrow[d, dotted, "\beta_V"] \\
M_2 \arrow[r, dotted, Rightarrow, "\eta_v^{-1}"'] & M_3
\end{tikzcd}$$

Proof: Suppose $M_0 = C[(\lambda x.P)Q]$ and $(\lambda x.P)Q$ is a β_v-redex. Then Q is either a variable y or $\bot$. The following diagram can be completed

$$\begin{tikzcd}
C[(\lambda x.P)Q] \arrow[r, Rightarrow, "\eta_v^{-1}"'] \arrow[d, "\beta_v"] & C'[(\lambda x.P')Q'] \arrow[d, dotted, "\beta_V"] \\
C[P[x := Q]] \arrow[r, dotted, Rightarrow, "\eta_v^{-1}"'] & C'[P'[x := Q']]
\end{tikzcd}$$

Clearly $C[(\lambda x.P')Q'] \longrightarrow_{\beta_V} C'[P'[x := Q']]$. Since $Q \twoheadrightarrow_{\eta^{-1}} Q'$, either $y \twoheadrightarrow_{\eta^{-1}} Q'$ or $\bot \twoheadrightarrow_{\eta^{-1}} Q'$ and $(\lambda x.P')Q'$ is a β_V-redex.

Next we show that $C[P[x := Q]] \Longrightarrow_{\eta_v^{-1}} C'[P'[x := Q']]$. Note that P' is obtained from P by replacing some of the variables or $\bot$'s by their η-expansions. Suppose that x has been replaced in P by its η-expansion. There are now two options for Q:

1 If $Q = y$ then $P[x := Q] \equiv P[x := y]$ and $y = Q \twoheadrightarrow_{\eta^{-1}} Q'$. We replace y by $X[x := Q']$. We have that $y \twoheadrightarrow_{\eta^{-1}} X[x := Q']$ because $x \twoheadrightarrow_{\eta^{-1}} X$ and $y \twoheadrightarrow_{\eta^{-1}} Q'$.

2 If $Q = \bot$ then $P[x := Q] \equiv P[x := \bot]$ and $\bot = Q \twoheadrightarrow_{\eta^{-1}} Q'$. We replace all the $\bot$'s of $P[x := \bot]$ obtained from substituting x by $\bot$ by $X[x := Q']$. We have that $\bot \twoheadrightarrow_{\eta^{-1}} X[x := Q']$ because $x \twoheadrightarrow_{\eta^{-1}} X$ and $\bot \twoheadrightarrow_{\eta^{-1}} Q'$.

□

Appendix F. Preservation of Head Normalisation by $\eta!$ and η^{-1}

In this section we prove that the property "having a reduction to head normal form" is preserved both by $\eta!$ and η^{-1}. Both properties will be used in the proof of local commutation of $\eta!$ and $\bot_{out}$.

F.1. *Preservation of head normalisation by $\eta!$*

Lemma F.1. If $M \twoheadrightarrow_{\eta!} N$ and M is a head normal form, then N is a head normal form as well.

Proof: The reduction $M \twoheadrightarrow_{\eta!} N$ can be sorted by Lemma B.4, so that it starts with a finite number of reductions at depth 0 followed by deeper reductions. Clearly each of the depth 0 reductions preserves the head normal form. And the remaining deeper reductions can not alter the left spine of the resulting normal form. □

Theorem F.2 (Preservation of Head Normalization by $\eta!$). If $M \twoheadrightarrow_{\eta!} N$ and M has a head normal form, then so does N.

Proof: Suppose M has a β-reduces to a head normal form H. Then it has a finite β-reduction to head normal form as well by strong convergence. By repeated application of Lemma 4.2 we construct the diagram:

$$\begin{tikzcd} M \arrow[r, two heads, "\eta!"'] \arrow[d, two heads, "\beta"'] \arrow[dr, phantom, "\text{(Lem 4.2)}"] & N \arrow[d, two heads, dotted, "\beta"] \\ H \arrow[r, two heads, dotted, "\eta!"'] & H' \end{tikzcd}$$

By Lemma E.1, H' is in head normal form. □

Using the above theorem, we can prove the following result:

Theorem F.3 (Preservation of $\beta\bot$-normal forms by $\eta!$). If $M \twoheadrightarrow_{\eta!} N$ and M is a $\beta\bot$-normal form, then N is a $\beta\bot$-normal form.

Back to E.3.

Proof: Using Theorem 3.5, it is equivalent to prove that if $N \twoheadrightarrow_{\eta^{-1}} M$ and N is not a $\beta\bot$-normal form then M is not a $\beta\bot$-normal form. Suppose that $N \twoheadrightarrow_{\eta^{-1}} M$ and N is not a $\beta\bot$-normal form. We have two cases:

1 Suppose N is not a β-normal form. Then N has a subterm that is a β-redex. It is easy to show by induction on the length of the η^{-1}-reduction sequence that, if $N \twoheadrightarrow_{\eta^{-1}} M$ and N contains a β-redex then so does M.
2 Suppose N is not a $\bot$-normal form. Then, $N = C[P]$ for some P that has no head normal form. Then, there exists P_0 and C_0 such that $M = C_0[P_0]$ and $P \twoheadrightarrow_{\eta^{-1}} P_0$. By Theorem 3.5, $P_0 \twoheadrightarrow_{\eta!} P$. By Theorem E.2, P_0 does not have head normal form. But that implies that M contains a $\bot$-redex.

□

F.2. *Preservation of head normalisation by η^{-1}*

This is more involved than for $\eta!$ because of the obvious complication that when $M \longrightarrow_{\eta^{-1}} N$ and M is in head normal form, then N may not be in head normal form. Consider as an example of this the head normal form xPQ which η^{-1}-reduces to the term $(\lambda y.xPy)Q$. The latter term is not a head normal form itself. However, in this case as well in the general case, there is a further β-reduction to head normal form.

Lemma F.4 (Head normalization of η-expansions of a variable). Let x, y_1, …, y_n be all different variables. If $x \twoheadrightarrow_{\eta^{-1}} X$ then $X \twoheadrightarrow_{\beta_0} \lambda y_1 \dots y_n.xY_1 \dots Y_n$, where $y_i \twoheadrightarrow_{\eta^{-1}} Y_i$ and x does not occur free in Y_i for $1 \le i \le n$.

Proof: First we consider a special instance of the lemma. Suppose that $x \xrightarrow{0}\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X$. By induction on the length of this reduction, it follows that $X \twoheadrightarrow_{\beta_0} \lambda y_1 \dots y_n.xy_1 \dots y_n$. Because, if the length is zero, we are done, and if the length is non-zero, then $x \xrightarrow{0}\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X_1 \xrightarrow{0}_{\eta^{-1}} X$. Now, by induction hypothesis we get $X_1 \twoheadrightarrow_{\beta_0} H_1 \equiv \lambda y_1 \dots y_n.xy_1 \dots y_n$, so that, with by a repeated use of Lemma D.1, we obtain the diagram:

$$
\begin{tikzcd}
H \arrow[r, two heads, "0", "\eta^{-1}"'] & X_1 \arrow[r, "0", "\eta^{-1}"'] \arrow[d, dotted, two heads, "\beta_0"'] \arrow[dr, phantom, "\text{Lem. D.1}"] & X \arrow[d, dotted, two heads, "\beta_0"] \\
& H_1 \arrow[r, dotted, "0", "{=,\eta^{-1}}"'] & N \arrow[d, dotted, "{=,\beta_0}"] \\
& & H_2
\end{tikzcd}
$$

We distinguish four type of positions where an η-expansion at depth 0 can be performed in H_1:

1 at the position of the subterm $xy_1 \dots y_n$. Then N is the head normal form $\lambda y_1 \dots y_n z.xy_1 \dots y_n z$ and $H_2 = N$.

2 between two applications. Then $N \equiv \lambda y_1 \dots y_n.(\lambda z.xy_1 \dots y_i z)y_{i+1} \dots y_n$, so that $N \longrightarrow_{\beta_0} H_1$ and $H_2 = H_1$.

3 before an abstraction. Then $N \equiv \lambda y_1 \dots y_i z.(\lambda y_{i+1} \dots y_n.xy_1 \dots y_n)z$, so that $N \longrightarrow_{\beta_0} H_1$ and $H_2 = H_1$.

We are now ready for the general case. Assume $x \twoheadrightarrow_{\eta^{-1}} X$. By Lemma B.2, we can assume that this η^{-1}-reduction is of the form $x \xrightarrow{0}\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X_1 \xrightarrow{>0}\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X$. By the above, there exists some H_1 such that $X_1 \twoheadrightarrow_{\beta_0} H_1 = \lambda y_1 \dots y_n.xy_1 \dots y_n$. Then with a repeated use of Lemma D.2 we obtain the diagram:

$$
\begin{tikzcd}
x \arrow[r, two heads, "0", "\eta^{-1}"'] & X_1 \arrow[r, two heads, ">0", "\eta^{-1}"'] \arrow[d, dotted, two heads, "\beta_0"'] \arrow[dr, phantom, "\text{Lem. D.2}"] & X \arrow[d, dotted, two heads, "\beta_0"] \\
& H_1 \arrow[r, dotted, two heads, ">0", "\eta^{-1}"'] & H_2
\end{tikzcd}
$$

Since all the redexes in the bottom η^{-1}-reduction occur at depth greater than 0, H_2 is a head normal form of the form $\lambda y_1 \dots y_n.xY_1 \dots Y_n$ where $y_i \twoheadrightarrow_{\eta^{-1}} Y_i$ for all $1 \le i \le n$. $\square$

The previous lemma has an important consequence: we do not need the $\perp$-rule to obtain the β-normal form of an η-expansion of a variable.
Back to Theorem 6.1.

Proof: Suppose $x \twoheadrightarrow_{\eta^{-1}} X$. Then by Lemma E.4 X has a finite β-reduction to a head normal form $\lambda y_1 \dots y_n.xY_1 \dots Y_n$, where $y_i \twoheadrightarrow_{\eta^{-1}} Y_i$ for each $1 \le i \le n$. We can now repeat Lemma E.4 and β-reduce all the Y_i's to head normal forms $\lambda z_1 \dots z_{n_i}.y_i Z_1 \dots Z_{n_i}$. And we can continue the same process on the Z_j. In every step of this process we reveal a new layer of the β-normal form of X. In this way, we construct a strongly converging β-reduction from

X to its β-normal form $\mathsf{nf}_\beta(X)$. Hence, $X \twoheadrightarrow\!\!\!\!\twoheadrightarrow_\beta \mathsf{nf}_\beta(X)$. By Theorem B.9, we have that $x \twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} \mathsf{nf}_\beta(X)$. $\square$

Lemma F.5 (Head normalization of applications of η-expansions of x). If $x \twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X$, then $XN_1 \dots N_k$ is head normalising for any $N_1, \dots, N_k \in \Lambda^\infty_\perp$, and x occurs free as head variable in $XN_1 \dots N_k$.

Proof: Suppose $x \twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X$. By Lemma E.4, $X \twoheadrightarrow_\beta \lambda y_1 \dots y_m.xY_1 \dots Y_m$ where $y_i \neq x$ and $y_i \twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} Y_i$ for all $1 \le i \le m$. We have two cases:

1 Case $m \le k$. Then,

$$\begin{aligned} XN_1 \dots N_k &\twoheadrightarrow_\beta (\lambda y_1 \dots y_m.xY_1 \dots Y_m)N_1 \dots N_k \\ &\twoheadrightarrow_\beta xY_1^* \dots Y_m^* N_{m+1} \dots N_k \end{aligned}$$

where $Y_i^* = Y_i[y_i := N_i]$ for all $1 \le i \le m$.

2 Case $m > k$. Then,

$$\begin{aligned} XN_1 \dots N_k &\twoheadrightarrow_\beta (\lambda y_1 \dots y_m.xY_1 \dots Y_m)N_1 \dots N_k \\ &\twoheadrightarrow_\beta \lambda y_{k+1} \dots y_m.xY_1^* \dots Y_k^* Y_{k+1} \dots Y_m \end{aligned}$$

where $Y_i^* = Y_i[y_i := N_i]$ for all $1 \le i \le k$.

$\square$

The η-expansions of a variable only create β_0-redexes but the η-expansions of an arbitrary head normal form may also create β_v-redexes. For example, if $H = \lambda x.xP$ then, we have several cases depending on where the η-expansion in H is performed:

1 In the subterm xP, i.e. $N = \lambda xz.xPz$. Then N is in head normal form.
2 Between applications, i.e. $N = \lambda x.(\lambda z.xz)P$. So we have $N \longrightarrow_{\beta_0} H$.
3 Before the abstraction, i.e. $N = \lambda z.(\lambda x.xP)z$, in which case $N \longrightarrow_{\beta_v} H$.

The combination of β_0 and β_v (or β_V) does not have nice commuting properties with respect to η-expansions of head normal forms. In order to prove head normalisation of η-expansions of an arbitrary head normal form H, we will consider the truncation of H at depth 1. Because then η-expansions can create only β_V-redexes in such truncations.

Lemma F.6 (Head normalization of η-expansions of truncated hnf). Let $H \equiv \lambda x_1 \dots x_m.x\perp \dots \perp$. If $H \xrightarrow{0}\!\!\!\!\twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} M$ then there exists H' such that $M \twoheadrightarrow_{\beta_v} H'$ and $H' \equiv \lambda x_1 \dots x_m y_1 \dots y_n.x\perp \dots \perp y_1 \dots y_n$ where $y_i \neq x$ for all $1 \le i \le m$.

Proof: This is proved by induction on the length of the reduction. We prove the successor case. Suppose $H \xrightarrow{0}\!\!\!\!\twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} M_1 \xrightarrow{0}_{\eta^{-1}} M_2$. By induction hypothesis, there exists H_1 in head normal form such that $M_1 \twoheadrightarrow_{\beta_v} H_1$. By a repeated use of Lemma D.7 we obtain the following diagram:

$$\begin{tikzcd}
H \arrow[r, two heads, "0", "\eta^{-1}"'] & M_1 \arrow[r, "0", "\eta^{-1}"'] \arrow[d, dotted, two heads, "\beta_v"'] \arrow[dr, phantom, "\text{Lem. D.7}"] & M_2 \arrow[d, dotted, two heads, "\beta_v"] \\
& H_1 \arrow[r, dotted, "0", "\eta^{-1}"'] & N \arrow[d, dotted, "{=,\beta_v}"] \\
& & H_2
\end{tikzcd}$$

By induction hypothesis, $H_1 \equiv \lambda x_1 \dots x_m y_1 \dots y_n.x\bot \dots \bot y_1 \dots y_n$ where $x \neq y_i$ for $1 \leq i \leq n$. We distinguish 4 type of places where the η-expansion in H_1 can take place:

1 at the subterm $x\bot \dots \bot y_1 \dots y_n$, i.e.

$$N \equiv \lambda x_1 \dots x_n y_1 \dots y_n z.x\bot \dots \bot y_1 \dots y_n z.$$

Then, N is in head normal form and $H_2 = N$.

2 between two applications, i.e.

$$N \equiv \lambda x_1 \dots x_n y_1 \dots y_n.(\lambda z.x\bot \dots \bot z)\bot \dots \bot y_1 \dots y_n.$$

Then, $N \longrightarrow_{\beta_v} H_1$ and $H_2 = H_1$.

3 before one of the λy_i, e.g.

$$N \equiv \lambda x_1 \dots x_n y_1 \dots y_i z.(\lambda y_{i+1} \dots y_n.x\bot \dots \bot y_1 \dots y_n)z.$$

Then, $N \longrightarrow_{\beta_v} H_1$ and $H_2 = H_1$.

4 before one of the λx_i, e.g. $N \equiv \lambda x_1 \dots x_i z.(\lambda x_{i+1} \dots x_m y_1 \dots y_n.x\bot \dots \bot y_1 \dots y_n)z$. Then, $N \longrightarrow_{\beta_v} H_1$ and $H_2 = H_1$.

□

Lemma F.7 (Parallel η-expansions). Let $N \in \Lambda_\bot^\infty$ be the truncation of some term at depth 1. If $N \xrightarrow{0}\!\!\twoheadrightarrow_{\eta^{-1}} M_0 \xrightarrow{>0}\!\!\twoheadrightarrow_{\eta^{-1}} M_1$ then $M_0 \Longrightarrow_{\eta_v^{-1}} M_1$.

Proof: Assume that $N = M^1$ for some M in $\Lambda_\bot^\infty$ and suppose $N \xrightarrow{0}\!\!\twoheadrightarrow_{\eta^{-1}} M_0$. Then both N and M_0 have only subterms at depth 1 that are either a variable or $\bot$. Suppose further that $M_0 \xrightarrow{>0}\!\!\twoheadrightarrow_{\eta^{-1}} M_1$. Since these η-expansion steps are performed at depth 1 or deeper, we find that each subterm at depth 1 in M_1 is an η-expansion of a subterm at the same position in M_0. Hence, $M_0 \Longrightarrow_{\eta_v^{-1}} M_1$. □

Lemma F.8 (Approximation for η^{-1}). If $M \twoheadrightarrow_{\eta^{-1}} N$ then there is a P such that $M^1 \twoheadrightarrow_{\eta^{-1}} P$ where $N^1 \preceq P \preceq N$.

Proof: By Lemma 3.4, we can assume that the η^{-1}-reduction sequence has at most length ω and by Lemma B.2 we can assume it is sorted by increasing order of depth. Suppose the reduction sequence is finite, i.e.

$$M = M_0 \xrightarrow{n_0}_{\eta^{-1}} M_1 \xrightarrow{n_1}_{\eta^{-1}} M_2 \xrightarrow{n_2}_{\eta^{-1}} \dots \xrightarrow{n_m}_{\eta^{-1}} M_m$$

We construct a reduction sequence from M^1 of the form:

$$M^1 = P_0 \xrightarrow{n_0}_{=,\eta^{-1}} P_1 \xrightarrow{n_1}_{=,\eta^{-1}} P_2 \xrightarrow{n_2}_{=,\eta^{-1}} \dots \xrightarrow{n_m}_{=,\eta^{-1}} P_m$$

such that $(M_i)^1 \preceq P_i \preceq M_i$ for all $0 \leq i \leq m$ by induction on m.
The case $m = 0$ is trivial. Next consider the successor case $m = k + 1$. So, suppose $M = M_0 \twoheadrightarrow_{\eta^{-1}} M_k = C[N] \xrightarrow{n_k}_{\eta^{-1}} M_{k+1} = C[\lambda x.Nx]$.
We have two possibilities depending on the depth of the η^{-1}-step:

1 Suppose the depth n_k of the η^{-1}-step is 0, i.e. $M = M_0 \xrightarrow{0}\!\!\twoheadrightarrow_{\eta^{-1}} M_k = C[N] \xrightarrow{0}_{\eta^{-1}} C[\lambda x.Nx] = M_{k+1}$. By induction hypothesis, $(M_k)^1 \preceq P_k \preceq M_k$. Since $(M_k)^1 \preceq P_k$ and the hole in C occurs at depth 0, we have that there exist N_1 and C_1 such that $P_k = C_1[N_1]$

where $C^1 \preceq C_1$ and $N^1 \preceq N_1$. Since $P_k \preceq M_k$, we also have that $C_1 \preceq C$ and $N_1 \preceq N$. By setting $P_{k+1} = C_1[\lambda x.N_1 x]$, we have that:

$$P_k = C_1[N_1] \xrightarrow{0}_{\eta^{-1}} C_1[\lambda x.N_1 x] = P_{k+1}$$

where $(M_{k+1})^1 = C^1[\lambda x.N^1\bot] \preceq C_1[\lambda x.N_1 x] = P_{k+1} \preceq C[\lambda x.Nx] = M_{k+1}$.

2 Suppose the depth n_k of the η^{-1}-step is greater than 0. Then $M = M_0 \xrightarrow{\leq n_k}\!\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} M_k = C[N] \xrightarrow{n_k}_{\eta^{-1}} M_{k+1} = C[\lambda x.Nx]$. Since $P_k \preceq M_k$ (by induction hypothesis), P_k is obtained from M_k by replacing some of its subterms by $\bot$. We have two possibilities:

(a) A subterm of M_k containing N is replaced by $\bot$ in P_k, i.e. $M_k = C[N] = C'[C''[N]]$ and $P_k = C_1'[\bot]$ for some C_1' such that $C_1' \preceq C'$. We set $P_{k+1} = P_k = C_1'[\bot]$. Note that we also have that $(M^{k+1})^1 = (M_k)^1$ because the position of the hole in C occurs at depth n_k greater than 0. Hence, $(M_{k+1})^1 = (M_k)^1 \preceq P_k = P_{k+1} = C_1'[\bot] \preceq C'[C''[\lambda x.Nx]] = C[\lambda x.Mx] = M_{k+1}$.

(b) Otherwise, $M_k = C[N] = C'[C''[N]]$ and $P_k = C_1[N] = C_1'[C_1''[N_1]]$ where $C_1' \preceq C'$, $C_1'' \preceq C''$, $N_1 \preceq N$ and the holes of C' and C_1' occur at depth 1. By setting $P_{k+1} = C_1[\lambda x.N_1 x]$, we have that:

$$P_k = C_1[N_1] \xrightarrow{n_k}_{\eta^{-1}} C_1[\lambda x.N_1 x] = P_{k+1}$$

By induction hypothesis, $(M_k)^1 \preceq P_k$ and $C'^1[\bot] = (M_k)^1 \preceq P_k = C_1'[C_1''[N_1]]$. Hence $C'^1 \preceq C_1'$. Hence, $(M_{k+1})^1 = (M_k)^1 \preceq C'^1[\bot] \preceq C_1'[C_1''[\lambda x.N_1 x]] = P_{k+1} \preceq C[\lambda x.Nx] = M_{k+1}$.

Finally consider the limit case. Suppose we have a strongly convergent reduction of length ω:

$$M = M_0 \xrightarrow{n_0}_{\eta^{-1}} M_1 \xrightarrow{n_1}_{\eta^{-1}} M_2 \xrightarrow{n_2}_{\eta^{-1}} \dots M_\omega$$

By induction, we can construct the infinite reduction that performs the η-expansions at the same depth:

$$M^1 = P_0 \xrightarrow{n_0}_{=,\eta^{-1}} P_1 \xrightarrow{n_1}_{=,\eta^{-1}} P_2 \xrightarrow{n_2}_{=,\eta^{-1}} \dots P_\omega$$

The above sequence is strongly convergent and hence, it has a limit P_ω.
We first prove that $(M_\omega)^1 \preceq P_\omega$. There exists n_0 such that for all $n \geq n_0$, $(M_n)^1 = (M_\omega)^1$ and $(P_n)^1 = (P_\omega)^1$. By induction, $(M_n)^1 \preceq P_n$. Hence, $(M_\omega)^1 = (M_n)^1 \preceq (P_n)^1 = (P_\omega)^1 \preceq P_\omega$.
We prove that $P_\omega \preceq M_\omega$ by showing that $(P_\omega)^k \preceq (M_\omega)^k$ for all k. For any k, there exists n_0 such that for all $n \geq n_0$, $(M_n)^k = (M_\omega)^k$ and $(P_n)^k = (P_\omega)^k$. By induction, $P_n \preceq M_n$. Hence, $(P_\omega)^k = (P_n)^k \preceq (M_n)^k = (M_\omega)^k$. □

Lemma F.9 (Head normalization of η-expansions of hnfs). If H is a head normal form and $H \twoheadrightarrow_{\eta^{-1}} M$, then M is head normalising.

Proof: Let $H = \lambda x_1 \dots x_m.yN_1 \dots N_k$. We consider the truncation $H^1 = \lambda x_1 \dots x_m.y\bot \dots \bot$ of H at depth 1. If $H \twoheadrightarrow_{\eta^{-1}} M$ then there exists $M_1 \preceq M$ such that $H^1 \twoheadrightarrow_{\eta^{-1}} M_1$ by Lemma E.8. We will show that M_1 is head normalising. By Lemma B.2, we can assume that the η^{-1}-reduction is of the form $H^1 \xrightarrow{0}\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} M_0 \xrightarrow{>0}\!\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} M_1$. By Lemma E.7, we have that

$M_0 \Longrightarrow_{\eta_v^{-1}} M_1$. By Lemma E.6, we have that $M_0 \twoheadrightarrow_{\beta_v} H_0$ and H_0 is in head normal form.

$$\begin{tikzcd}
H^1 \arrow[r, two heads, "0", "\eta^{-1}"'] & M_0 \arrow[rr, Rightarrow, "\eta_v^{-1}"'] \arrow[d, dotted, two heads, "\beta_v"', "\text{Lem. E.6}"' near start, shift right=0] & & M_1 \arrow[d, dotted, two heads, "\beta_V"'] \\
& H_0 \arrow[rr, dotted, Rightarrow, "\eta_v^{-1}"'] \arrow[urr, phantom, "\text{Lem. D.8}"] & & N \arrow[d, dotted, two heads, "\beta_0"'] \\
& & & H_2
\end{tikzcd}$$

Suppose $H_0 = \lambda y_1 \dots y_n.zP_1 \dots P_l$. Then $N = \lambda y_1 \dots y_n.ZQ_1 \dots Q_l$ where $z \twoheadrightarrow_{\eta^{-1}} Z$ and $P_i \twoheadrightarrow_{\eta^{-1}} Q_i$ for $1 \leq i \leq l$. By Lemma E.5, we have that N is head normalising. Hence M_1 is also head normalising. By monotonicity of $\mathsf{nf}_{\beta\perp}$, we have that $\mathsf{nf}_{\beta\perp}(M_1) \preceq \mathsf{nf}_{\beta\perp}(M)$ and hence, if M_1 is head normalising, so is M. □

Theorem F.10 (Preservation of head normalization by η^{-1}). Let $M \twoheadrightarrow_{\eta^{-1}} N$. If M has a head normal form, so does N.

Proof: Suppose M β-reduces to a head normal form H. By Lemma 2.8, $M \overset{0}{\twoheadrightarrow}_\beta H$. Then,

$$\begin{tikzcd}
M \arrow[r, two heads, "\eta^{-1}"'] \arrow[d, two heads, "\beta"', "0"] \arrow[dr, phantom, "\text{(Lem D.6)}"] & N \arrow[d, dotted, two heads, "\beta"', "0"] \\
H \arrow[r, dotted, two heads, "\eta^{-1}"'] & Q
\end{tikzcd}$$

By Lemma E.9, Q has a head normal form. □

As a consequence of the above theorem and Theorem 3.5 we have the following:

Corollary F.11 (Preservation of $\perp$-redexes by $\eta!$). Let $M \twoheadrightarrow_{\eta!} N$. If M does not have a head normal form, neither does N.

The fact that the η-expansions of a variable do not contain subterms without head normal form does not necessarily follow from Theorem **??**. Actually, in order to prove that fact, we will need the following theorem:

Theorem F.12 (Preservation of subterm head normalization by η^{-1}). Let $M \twoheadrightarrow_{\eta^{-1}} N$. If all subterms of M are head normalising then all subterms of N are head normalising too.

Proof: We prove it by induction on the length of the reduction sequence. First we consider the one step case. Suppose $M = C[P] \longrightarrow_{\eta^{-1}} C[\lambda x.Px] = N$ and all subterms of M are head normalising. Let Q be a subterm of N at position q. We do a case analysis:

1 Q is a subterm of $\lambda x.Px$.

(a) $Q = x$. This case is trivial.

(b) $Q = Px$. Since P is a subterm of M, we have that P is head normalising, i.e. $P \twoheadrightarrow_\beta \lambda y_1 \dots y_k.zP_1 \dots P_n$. Hence $Q = Px \twoheadrightarrow_\beta \lambda y_2 \dots y_k.(zP_1 \dots P_n)[y_1 := x]$ is also head normalising.

(c) $Q = \lambda x.Px$. This case is similar to the previous case.
(d) Q is a subterm of P in N, then so it is in M as well, and therefore head normalising.

2 Q is not an subterm of $\lambda x.Px$. Then at the same position q we find a possible different subterm Q' in M. By assumption Q' is head normalising.

(a) P is a subterm of Q' then $Q' \longrightarrow_{\eta^{-1}} Q$ and hence by Theorem E.10, we see that Q is head normalising.
(b) P is not a subterm of Q' then $Q = Q'$, and so Q is head normalising.

Hence we have found that all subterms of N are head normalising.
For any finite reduction $M \twoheadrightarrow_{\eta^{-1}} N$ we have that, if all subterms of M are head normalising, then so are all of N. By η^{-1}-compression Lemma 3.4, the remaining situation we have to consider is a reduction $M \twoheadrightarrow\!\!\!\!\rightarrow_{\eta^{-1}} N$ of length ω. So consider a subterm Q of N at some depth n. By strong convergence there is $M \twoheadrightarrow_{\eta^{-1}} C[Q'] \twoheadrightarrow\!\!\!\!\rightarrow_{\eta^{-1}} N$ so that Q' occurs at depth n in $C[Q']$ and $Q' \twoheadrightarrow\!\!\!\!\rightarrow_{\eta^{-1}} Q$. Assuming that all subterms of M are head normalising, it follows from induction hypothesis that all subterms of $C[Q']$ are head normalising, in particular Q'. Again, by Theorem E.10, we find that Q is head normalising. ☐

Corollary F.13 (Preservation of $\eta!$-redexes by $\perp$). If $x \twoheadrightarrow\!\!\!\!\rightarrow_{\eta^{-1}} N$ then N does not contain any subterm without head normal form. Hence, if $\lambda x.MN$ is an $\eta!$-redex and $\lambda x.MN \twoheadrightarrow\!\!\!\!\rightarrow_{\perp} \lambda x.M'N'$ then $N = N'$ and $\lambda x.M'N'$ is an $\eta!$-redex.

Appendix G. Omitted Proofs for the Commutation of $\eta!$ and $\perp_{out}$

Back to Lemma 5.1.

Proof: Suppose M_0 can do both a $\perp_{out}$-reduction and an $\eta!$-reduction. Out of the potentially five relative positions of these two redexes only three are actually possible:

1 The $\perp_{out}$-redex U and the $\eta!$-redex $\lambda x.MN$ are not nested, i.e. $M_0 = C[U, \lambda x.MN]$. We have to show that if U is the outermost redex of $M_0 = C[U, \lambda x.MN]$ then U is also outermost in $M_1 = C[U, M]$. Using Theorem E.2, U cannot be a subterm of a term without head normal form. This results in the following diagram.

$$\begin{tikzcd}
M_0 \arrow[r, "m", "\eta!"'] \arrow[d, "\perp_{out}"', "n"] & M_1 \arrow[d, dotted, "\perp_{out}"', "n"] \\
M_2 \arrow[r, dotted, "m", "\eta!"'] & M_3
\end{tikzcd}$$

2 The $\perp_{out}$-redex is inside the first term of the $\eta!$-redex, that is M_0 is of the form $C_1[\lambda x.C_2[U]N]$. Using Theorem E.2, similarly to the previous case, we can show that U is outermost in $M_1 = C_1[C_2[U]]$. This results in the same diagram as the previous case.

3 The $\eta!$-redex is inside the $\perp_{out}$-redex. Corollary E.11 and Theorem E.2 ensure that the contracted term is still a $\perp_{out}$-redex. This results in the following diagram.

$$\begin{tikzcd}
C[U] \arrow[r, "m", "\eta!"'] \arrow[d, "\perp_{out}"', "n"] & C[U'] \arrow[d, dotted, "\perp_{out}"', "n"] \\
C[\perp] \arrow[r, dotted, no head] & C[\perp]
\end{tikzcd}$$

4 The $\perp_{out}$-redex is inside the expanded variable term of the $\eta!$-redex, that is M_0 is of the form $C_1[\lambda x.MC_2[U]]$. This option is impossible by Corollary E.13.

5 The $\perp_{out}$-redex is the body of the $\eta!$-redex, that is M_0 is of the form $C_1[\lambda x.U]$, where $U \equiv MN$. This possibility is excluded because U would not be an outermost $\perp$-redex.

$\square$

Back to Lemma 5.2.

Proof: By Lemma 3.4 (Compression Lemma), we can assume that the sequence has length ω.

$$\begin{tikzcd}
M_0 \arrow[r, "n_0", "\eta!"'] \arrow[d, "m"', "\perp_{out}"] & M_1 \arrow[r, "n_1", "\eta!"'] \arrow[d, dotted, "m"', "\perp_{out}"] & M_2 \arrow[r, "n_2", "\eta!"'] \arrow[d, dotted, "m"', "\perp_{out}"] & M_3 \arrow[d, dotted, "m"', "\perp_{out}"] & \ldots & M_\omega \arrow[d, dotted, "m"', "\perp_{out}"] \\
N_0 \arrow[r, dotted, "n_0", "{=,\eta!}"'] & N_1 \arrow[r, dotted, "n_1", "{=,\eta!}"'] & N_2 \arrow[r, dotted, "n_2", "{=,\eta!}"'] & N_3 & \ldots & ?
\end{tikzcd}$$

Using Lemma 5.1, we can complete all the subdiagrams except for the limit case. All the M_k are of the form $C_k[U_k]$ with the hole at the same position at depth m. The subterm U_0 is an outermost $\perp$-redex in M_0. All other U_k are terms without head normal form by Corollaries E.11 and E.13. The limit term is of the form $C_\omega[U_\omega]$ and the hole of C_ω occurs also at depth m. By Corollary E.11, U_ω does not have head normal form and by Theorem E.2, it cannot be a subterm of a term without head normal form and, hence, it is a $\perp_{out}$-redex. Contracting this redex in the limit M_ω reduces to $C_\omega[\perp]$ which is equal to the limit N_ω of the bottom sequence. $\square$

Back to Lemma 5.3.

Proof: By (Compression Lemma 3.4 we can assume that the length of the sequence is ω.

$$\begin{tikzcd}
M_0 \arrow[r, "n_0", "\perp_{out}"'] \arrow[d, "m"', "\eta!"] & M_1 \arrow[r, "n_1", "\perp_{out}"'] \arrow[d, dotted, "m"', "{=,\eta!}"] & M_2 \arrow[r, "n_2", "\perp_{out}"'] \arrow[d, dotted, "m"', "{=,\eta!}"] & M_3 \arrow[d, dotted, "m"', "{=,\eta!}"] & \ldots & M_\omega \arrow[d, dotted, "m"', "{=,\eta!}"] \\
N_0 \arrow[r, dotted, "n_0", "\perp_{out}"'] & N_1 \arrow[r, dotted, "n_1", "\perp_{out}"'] & N_2 \arrow[r, dotted, "n_2", "\perp_{out}"'] & N_3 & \ldots & ?
\end{tikzcd}$$

Using Lemma 5.1, we can complete all the subdiagrams except for the limit case. Either the vertical $\eta!$-reduction got cancelled out in one of the applications of Local Commutation or not. If it gets cancelled out, then from that moment on all vertical reductions are reductions of length 0, implying that M_ω is equal to the limit N_ω. Or the vertical $\eta!$-reduction did not get cancelled out, implying that its residual is present in M_k for all $k \geq 0$. That is all M_k with $k \geq 0$ are of the form $C_k[\lambda x.S_kT_k]$, where all the $C_k[\ ]$ have the hole at the same position at depth m, and all N_k with $k \geq 0$ are of the form $C_k[S_k]$. The limit term is of the form $C_\omega[\lambda x.S_\omega T_\omega]$ and the hole of C_ω occurs also at depth m. By Corollary E.13, $\lambda x.S_\omega T_\omega$ is an $\eta!$-redex. Contracting this redex in the limit M_ω reduces to $C_\omega[S_\omega]$ which is equal to the limit N_ω of the bottom sequence. $\square$

Appendix H. Omitted Proof on $\infty\eta$-Böhm tree as a Lambda Model

Back to Lemma 8.1.

Proof: First a remark: note that, because $(\lambda x.M)^\rho = \lambda x.M^{\rho(x:=x)}$, we have $[\![\lambda x.M]\!]_\rho = \infty\eta\mathsf{BT}(\lambda x.M^{\rho(x:=x)})$.

1 $[\![x]\!]_\rho = \infty\eta\mathsf{BT}(x^\rho) = \infty\eta\mathsf{BT}(\rho(x)) = \rho(x)$, as $\rho(x) \in \mathfrak{B}_{\infty\eta}$.

2 We have that

$$\begin{aligned}
[\![MN]\!]_\rho &= \infty\eta\mathsf{BT}((MN)^\rho) \\
&= \infty\eta\mathsf{BT}(M^\rho N^\rho) \\
&= \infty\eta\mathsf{BT}(\infty\eta\mathsf{BT}(M^\rho)\infty\eta\mathsf{BT}(N^\rho)) \text{ by Theorem 6.3} \\
&= \infty\eta\mathsf{BT}([\![M]\!]_\rho[\![N]\!]_\rho) \\
&= [\![M]\!]_\rho \bullet [\![N]\!]_\rho
\end{aligned}$$

3 For arbitrary P in $\mathfrak{B}_{\infty\eta}$ we have that

$$\begin{aligned}
[\![\lambda x.M]\!]_\rho \bullet P &= \infty\eta\mathsf{BT}([\![\lambda x.M]\!]_\rho P) \\
&= \infty\eta\mathsf{BT}(\infty\eta\mathsf{BT}(\lambda x.M^{\rho(x:=x)})P) \\
&= \infty\eta\mathsf{BT}((\lambda x.M^{\rho(x:=x)})P) \text{ by Theorem 6.3} \\
&= \infty\eta\mathsf{BT}(M^{\rho(x:=x)(x:=P)}) \text{ by } \beta\text{-reduction and Theorem 6.3} \\
&= [\![M]\!]_{\rho(x:=P)}
\end{aligned}$$

4 $\rho \mid FV(M) = \rho' \mid FV(M)$ implies $M^\rho = M^{\rho'}$, hence $\infty\eta\mathsf{BT}(M^\rho) = \infty\eta\mathsf{BT}(M^{\rho'})$, and so $[\![M]\!]_\rho = [\![M]\!]_{\rho'}$.

5 Similar to the proof of 18.3.10.i in (Barendregt, 1984):

$$\begin{aligned}
\forall P \in\ & \mathfrak{B}_{\infty\eta} [\![M]\!]_{\rho(x:=P)} = [\![N]\!]_{\rho(x:=P)} \\
\Rightarrow\ & [\![M]\!]_{\rho(x:=x)} = [\![N]\!]_{\rho(x:=x)} \\
\Rightarrow\ & \infty\eta\mathsf{BT}(M^{\rho(x:=x)}) = \infty\eta\mathsf{BT}(N^{\rho(x:=x)}) \\
\Rightarrow\ & \infty\eta\mathsf{BT}(\lambda x.\infty\eta\mathsf{BT}(M^{\rho(x:=x)})) = \infty\eta\mathsf{BT}(\lambda x.\infty\eta\mathsf{BT}(N^{\rho(x:=x)})) \text{ by Theorem 6.3} \\
\Rightarrow\ & \infty\eta\mathsf{BT}(\lambda x.M^{\rho(x:=x)}) = \infty\eta\mathsf{BT}(\lambda x.N^{\rho(x:=x)}) \text{ by Theorem 6.3} \\
\Rightarrow\ & \infty\eta\mathsf{BT}((\lambda x.M)^\rho) = \infty\eta\mathsf{BT}((\lambda x.N)^\rho) \\
\Rightarrow\ & [\![\lambda x.M]\!]_\rho = [\![\lambda x.N]\!]_\rho
\end{aligned}$$

6 Since $\lambda xy.xy \longrightarrow_{\eta!} \lambda x.x$, we have that $\infty\eta\mathsf{BT}((\lambda xy.xy)^\rho) = \infty\eta\mathsf{BT}((\lambda x.x)^\rho)$ and therefore $[\![\lambda xy.xy]\!]_\rho = [\![\lambda x.x]\!]_\rho$.

$\square$