WITNESS ALGEBRA AND ANYON BRAIDING

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ABSTRACT. Topological quantum computation employs twodimensional quasiparticles called anyons. The generally accepted mathematical basis for the theory of anyons is the framework of modular tensor categories. That framework involves a substantial amount of category theory and is, as a result, considered rather difficult to understand. Is the complexity of the present framework necessary? The computations of associativity and braiding matrices can be based on a much simpler framework, which looks less like category theory and more like familiar algebra. We introduce that framework here.

1. Introduction

Topological quantum computation employs two-dimensional quasiparticles called anyons [10, 5]. The generally accepted mathematical basis for the theory of anyons is the framework of modular tensor categories. That framework, as presented in [15] or [13] or [2] involves a substantial amount of category theory and is, as a result, considered rather difficult to understand. For example, Trebst et al. [14, page 385] write "In general terms, we can describe anyons by a mathematical framework called tensor category theory. ... Here we will not delve into this difficult mathematical subject "Similarly, Bonderson [4, page 13] writes "In mathematical terminology, anyon models are known as unitary braided tensor categories, but we will avoid descending too far into the abstract depths of category theory"

Is the complexity of the current framework necessary? We do not think so. Our opinion is based on the following.

(1) Our own experience, admittedly modest. In [2], after describing modular tensor categories, we presented a simplification based on Yoneda's Lemma. Then we exhibited some computations for the particular case of Fibonacci anyons, including the computation of the braiding operators that are central to proposed uses of anyons for quantum computation. It turned out that a good part of the axiomatics of tensor categories was not needed for those computations.

- (2) Only isomorphisms, rather than category theory's more general morphisms, are used in the anyon theory.
- (3) Physicists tend to avoid category theory.

The computations in [2] can be based on a much simpler framework, which looks less like category theory and more like familiar algebra. We introduce that framework here.

The main idea of the proposed framework is to work with ordinary algebraic structures, like rings, amplified with a notion of witnesses for equations. The rules of equational logic are then accompanied by constructions of new witnesses from old. For example, where equational logic says that, from an equation a = b, one can infer b = a, our framework will say that, from a witness for a = b, one can construct a witness for b = a. The other rules of equational logic are treated similarly; see Section 2 for details. We call this new framework witness algebra.

Traditional equational logic can be viewed as the special case of witness algebra where all witnesses are trivial and therefore do not need to be mentioned. In our work with anyons, the witnesses for the associativity and commutativity of multiplication will be highly nontrivial. Other witnesses, for example those for the associativity and commutativity of addition, will not be entirely trivial but will amount to minor bookkeeping information. See Section 4 for details.

Category-Theoretic Remark 1. Readers who care only about the witness algebra framework and not the traditional category-theoretic one can skip this and subsequent category-theoretic remarks. These will serve to connect our framework to the traditional one but will not be essential for the development of witness algebra itself.

Our work, having emerged from category theory, can be translated back into category-theoretic terminology. The result of that translation would be a theory of groupoids (categories in which all morphisms are isomorphisms) enhanced with additional structure. For example, in our braid semirings (defined in Section 3), the enhancement would be two monoidal structures, one of which (corresponding to addition) is symmetric while the other (corresponding to multiplication) is braided, plus a distributive law connecting the two, plus suitable coherence axioms.

Somewhat ironically, easing the category-theoretical complexity of the anyon theory required some category-theoretical investigation. The associativity, commutativity and distributivity isomorphisms should satisfy appropriate coherence conditions. What are these conditions? Much of the work has been done in the literature but there was a gap which we filled in the companion paper [3]. The situation is explained in detail in Section 3.

Related Work. Gödel suggested in unpublished work [6, Section V] that modal logic could profitably be amplified by introducing justifications. This idea was independently rediscovered and extensively studied by Artemov; see for example [1]. The role of justifications in modal logic is similar to the role of witnesses in our witness algebra. But, instead of modal logic, we work with elementary algebra. Another difference is that Artemov's justifications can be nested; that is, " ξ is a justification for φ " is itself a formula that can admit justifications. We use witnesses only for equations, and " ξ is a witness for a = b" is not itself an equation, so it cannot be witnessed.

2. WITNESSED EQUATIONAL LOGIC

2.1. **Preliminaries.** To establish terminology, we recall some definitions from universal algebra. A *signature* is a collection of operation symbols, each having a specified number of argument places (arity). An *algebra* A of signature Σ consists of a set |A|, the *universe* of A, together with interpretations of the symbols in Σ on |A|; an r-ary symbol is interpreted as an r-ary operation. The interpretations of nullary operation symbols are viewed as elements of |A|; that is, nullary operation symbols are individual constants.

It is often convenient to use the same symbol for an algebra and its universe; in such cases it should be clear from the context whether the symbol denotes the algebra or its universe.

2.2. Witness Frames. We begin the discussion of witness algebras with the special case where the signature Σ is empty. In this case, algebras of signature Σ are merely sets with no additional structure. Nevertheless, the key concept of witnessing appears even in this context and is easier to explain without the additional baggage of a nontrivial signature. So we begin with this special case; we use the name "witness frame" for witness algebras for the empty signature.¹

The idea here is that we have a set S, each of whose elements s might be viewed in a variety of ways. Witnesses will indicate whether and how two views represent the same element s. As a fairly typical

¹Since algebras for the empty signature are sets, it would seem reasonable to call witness algebras for the empty signature "witness sets". Unfortunately, that sounds too much like a set of witnesses. We have chosen the terminology "witness frame" to suggest that these are basic systems to which additional algebraic structure (nonempty signatures) could be attached.

example (similar to what we shall later use for anyon computations), the elements of S might be some vector spaces, and a view of a vector space might be that space equipped with a particular basis. A witness should then indicate how two vector spaces with specified bases (i.e., two views) are actually the same vector space (though the bases might be quite different). Such a witness could be an invertible matrix transforming the one basis into the other, i.e., expressing each vector of the latter basis as a linear combination of the former basis.

Thus, a witness frame really involves two sets: the set S and the set of views of elements of S. To formalize these notions, it is convenient to take the views as the basic entities; elements of S can then be identified with equivalence classes under the equivalence relation of "some witness shows that the two views represent the same element of S." (Of course, the formal definition will need to ensure, among other things, that this is an equivalence relation.)

This discussion leads to the somewhat unusual situation that the entities of primary interest and importance, the elements of S, are not the ones taken to be basic in the formalization. We shall emphasize the importance of the equivalence classes, the elements of S, by the following terminology and notation.

The views, which are formally the basic entities, will be called the raw elements of the witness frame, and the symbol \equiv will be used to mean that two of them are the same view, not merely representatives of the same element of S. The equivalence relation of "some witness says they're the same" will be denoted by \equiv , and the equivalence classes, the elements of S, will be called the true elements of the witness frame. In the vector space example above, $v \equiv w$ would mean that the views v and w are the same vector space with the same basis, whereas $v \equiv w$ would mean that they are the same vector space with possibly different bases.

Remark 2. Our use of = to denote an equivalence relation coarser than what one might consider complete equality, \equiv , is unusual but not unprecedented. For example, in combinatorial group theory, when discussing groups and their presentations, one sometimes writes v=w to mean that the words v and w represent the same group element, while $v \equiv w$ means that they are identical as words.

We emphasize that our unusual use of the equality symbol applies only to raw elements. In other contexts, the equality symbol retains its customary meaning. In particular, if ξ and η are witnesses, then $\xi = \eta$ means that they are the same witness.

After these explanations of the underlying intention, we are ready for the definition of witness frames.

Definition 3. A witness frame consists of a set A of raw elements and, for all elements $a, b \in A$, pairwise disjoint (possibly empty) sets W(a, b) of witnesses for equality between a and b. If $\xi \in W(a, b)$, we write $\xi \vdash a = b$ and we say that ξ witnesses that a = b. We call ξ a witness if it is in one of the sets W(a, b). The system of raw elements and witnesses is required to have the following structure.

- (W1) For each $a \in A$, there is a specified witness $1_a \vdash a = a$.
- (W2) For each witness $\xi \vdash a = b$, there is a specified $\xi^{-1} \vdash b = a$.
- (W3) For each pair of witnesses $\xi \vdash a = b$ and $\eta \vdash b = c$, there is a specified witness $\xi * \eta \vdash a = c$.

These specifications are subject to the following axioms.

- (W4) If $\xi \vdash a = b$ then $1_a * \xi = \xi * 1_b = \xi$.
- (W5) If $\xi \vdash a = b$ then $\xi * \xi^{-1} = 1_a$ and $\xi^{-1} * \xi = 1_b$.
- (W6) If $\xi \vdash a = b$, $\eta \vdash b = c$, and $\zeta \vdash c = d$, then $(\xi * \eta) * \zeta = \xi * (\eta * \zeta)$.

If we ignore the witnesses and pay attention only to the equations, then requirements (W1), (W2), and (W3) correspond to the usual axioms and rules of equational logic (in the case of the empty signature), saying that equality is reflexive, symmetric, and transitive. Thus, in witness frames these three requirements ensure that the relation = introduced in the following definition is an equivalence relation.

Definition 4. In any witness frame, with the notation of the preceding definition, we define, for $a, b \in A$,

$$a = b \iff (\exists \xi) \xi \vdash a = b.$$

The elements of A are called raw elements of the witness frame, and equality between them is symbolized by \equiv . The equivalence classes with respect to the relation = just defined are called true elements of the witness frame.

We have built into the definition of witness frames that the sets W(a,b) for the various pairs a,b are disjoint. In other words, a witness ξ witnesses only a single equation. This convention is very convenient for theoretical purposes. It implies in particular that the inverse ξ^{-1} of any witness ξ is uniquely defined and that the composition $\xi * \eta$ of any two witnesses ξ, η can be defined in at most one way. In some concrete situations, on the other hand, it is tempting to re-use the same witness for several equations. Indeed, this temptation arose in our vector space example above; one and the same matrix can serve as

the transformation matrix between many pairs of bases. Fortunately, there is an easy solution for this problem, namely to "mark" each witness with the equation that we want it to witness. That is, if the same ξ is in both W(a,b) and W(c,d), we replace it by $\langle a,\xi,b\rangle$ as an element of W(a,b), and we replace it by $\langle c,\xi,d\rangle$ as an element of W(c,d). More generally, we adopt the following convention, intended to give us the best of both worlds — authorization to re-use witnesses when describing concrete examples while maintaining disjointness of the sets of witnesses for official purposes.

Convention 5. If a witness frame is described in a way that allows the sets W(a,b) to overlap, then it is to be understood that the actual, official witnesses for an equation a=b are not the described ξ 's but rather the marked versions $\langle a, \xi, b \rangle$.

Our definition of witness frames includes requirements (W4), (W5), and (W6), which say that certain combinations of witnesses are equal. In each case, the required equality makes sense because the two sides are witnesses of the same equation. The intention behind these requirements is that the specifications in (W1), (W2), and (W3) should not be made randomly or arbitrarily but in some coherent way. For example, the witness $\xi * \eta$ in (W3) should not be just any witness for a = c but rather one that combines, in a sensible way, the information in the witnesses ξ and η (and the transitive law of equality).

Category-Theoretic Remark 6. Category theorists will recognize witness frames as just a notational variant of the familiar notion of groupoid, a category in which all morphisms are isomorphisms. Our raw elements are the objects of the groupoid, our witnesses are the morphisms, and $\xi \vdash a = b$ amounts to $\xi : a \to b$. Our (W1), (W3), (W4), and (W6) amount to the definition of a category, while (W2) and (W5) provide the inverses that make all the morphisms isomorphisms. The true elements are the connected components of the groupoid.

Notice that we compose witnesses in what is often called diagrammatic order. That is, the composition of $\xi \vdash a = b$ with $\eta \vdash b = c$ is written as $\xi * \eta$, not as $\eta * \xi$ or as $\eta \circ \xi$.

We emphasize that these groupoids are quite different from the abelian categories used in [13] and [2]. It is easy to check that the only way an abelian category can be a groupoid is to be trivial, i.e., to be equivalent to the category consisting of just a single object and its identity morphism.

The following lemma records some basic properties of the operations on witnesses that are involved in witness frames.

Lemma 7.

- (1) The operation * on witnesses admits cancellation. That is, if $\xi * \eta$ and $\xi * \zeta$ are defined and equal, then $\eta = \zeta$. Similarly if $\eta * \xi = \zeta * \xi$.
- (2) Inversion is involutive. That is, all witnesses ξ satisfy $(\xi^{-1})^{-1} = \xi$.

Proof. For part (1), assume $\xi * \eta = \xi * \zeta$, where $\xi \vdash a = b$ and $\eta, \zeta \vdash b = c$ so that the * operation is defined for these witnesses. Then compute

$$\eta = 1_b * \eta = (\xi^{-1} * \xi) * \eta = \xi^{-1} * (\xi * \eta) = \xi^{-1} * (\xi * \zeta) = (\xi^{-1} * \xi) * \zeta = 1_b * \zeta = \zeta.$$

The proof under the hypothesis $\eta * \xi = \zeta * \xi$ is symmetrical.

For part (2), notice that, if $\xi \vdash a = b$ then $(\xi^{-1})^{-1} * \xi^{-1} = 1_a = \xi * \xi^{-1}$, and use part (1) to cancel ξ^{-1} .

Remark 8. The definition of witness frames requires that the raw elements, and therefore also the true elements, constitute a set rather than a proper class. Everything we do, however, would work just as well if we used classes instead. So, for example,we could deal with a witness frame whose true elements are all of the vector spaces, not just some. Except for this remark, we shall ignore the set-class distinction in this paper.

2.3. Witness Algebras. In the preceding subsection, we dealt with the case of the empty signature, where algebras are just sets. In keeping with that special case, although our witness frames had considerable structure, as described in (W1)–(W6), the true elements formed just a set with no additional structure. In the present subsection, we deal with the case of general signatures Σ . A witness Σ -algebra will have enough additional structure to make the true elements into a Σ -algebra in the usual sense.

Definition 9. Let Σ be a signature. A witness Σ -algebra is a witness frame A (with notation as above) together with actions of the operation symbols from Σ on raw elements and on witnesses, as follows. Let $f \in \Sigma$ be r-ary. Then:

- (1) $f: A^r \to A$ is an r-ary operation on the raw elements.
- (2) If $\xi_i \vdash a_i = b_i$ for i = 1, ..., r then $f(\xi_1, ..., \xi_r)$ is defined and $f(\xi_1, ..., \xi_r) \vdash f(a_1, ..., a_r) = f(b_1, ..., b_r)$.
- (3) If $\xi_i \vdash a_i = b_i$ and $\eta_i \vdash b_i = c_i$ for i = 1, ..., r then $f(\xi_1, ..., \xi_r) * f(\eta_1, ..., \eta_r) = f(\xi_1 * \eta_1, ..., \xi_r * \eta_r).$

To avoid a proliferation of notation, we use f for all three of a symbol in Σ , its action as an operation on A, and its action on witnesses (as long as the context prevents ambiguity).

When the arity r is zero, a 0-ary operation on A is, of course, a function from the one-element set A^0 into A. It is convenient (and customary in algebra and logic) to identify such a function with its unique value. Thus, 0-ary operations amount to constants.

If we pay attention only to witnessed equality and not to the specific witnesses, then clause (2) in Definition 9 says that the algebraic laws of equality concerning function symbols are obeyed: we can substitute equals for equals. This clause and the clauses in the earlier definition of witness frames give all the usual laws of equational logic.

In the definition of witness algebras, clause (1) makes the raw elements into a Σ -algebra, which we call the raw algebra. Clause (2) makes the witnessed equality relation =, which we already know is an equivalence relation, into a congruence relation. As a result, the quotient, the set of true elements, becomes a Σ -algebra as well. We call it the true algebra (or the true Σ -algebra) of the witness algebra.

The purpose of clause (3) is, as with some of the clauses in the definition of witness frames earlier, to require that the witnesses $f(\xi_1, \ldots, \xi_r)$ in clause (2) should not be just arbitrarily chosen witnesses for the relevant equations $f(a_1, \ldots, a_r) = f(b_1, \ldots, b_r)$ but should be chosen in some reasonable way based on ξ_1, \ldots, ξ_r .

Example 10. Any Σ -algebra A can be converted trivially into a witness algebra whose raw and true algebras are both the given A. Just take all the sets W(a,b) to be singletons. (We could even choose to take just a single, trivial witness, say 0, to witness exactly those equalities a=b where a and b are the same raw element. Convention 5 would make this choice "legal" by replacing the single witness 0 with different witnesses for different equations.) In this way, we can regard ordinary algebras as a special case of witness algebras.

Example 11. More generally, consider a Σ -algebra A, a congruence relation \sim on it, and the quotient algebra A/\sim . Then, using only one witness 0 (until Convention 5 turns it into many witnesses), we can produce a witness algebra whose raw algebra is A and whose true algebra is A/\sim . It suffices to define $0 \vdash a = b$ to hold if and only if $a \sim b$.

As already mentioned in connection with some of the clauses in our definitions, our witnesses behave similarly to deductions in equational logic. To emphasize the similarity, we can use a notation where a

witness is displayed above the equation that it witnesses, resembling a deduction of that equation. Then we have, for all $a \in A$,

$$1_a$$
$$a = a,$$

corresponding to the axiom a = a of equational logic. For any

$$\xi \\
a = b,$$

we can depict ξ^{-1} as

$$\frac{x}{a=b}$$

$$b=a,$$

so that it looks like ξ followed by an application of the inference rule "from a = b infer b = a." Similarly, for any

$$\xi$$
 and η $b = c,$

we can depict $\xi * \eta$ as

$$\begin{array}{ccc}
\xi & & \eta \\
a = b & & b = c
\end{array}$$

so that it looks like ξ and η followed by an application of the inference rule "from a = b and b = c infer a = c." Finally, for an n-ary function symbol f, we can depict $f(\xi_1, \ldots, \xi_r)$ as

$$\frac{\xi_1}{a_1 = b_1} \frac{\xi_r}{\dots a_r = b_r}$$

$$f(a_1, \dots, a_r) = f(b_1, \dots, b_r)$$

The similarity between witnesses and deductions suggests the following variation on the theme of Example 11.

Example 12. Suppose A is the Σ -algebra presented by some generators and relations, and suppose B is the quotient obtained by imposing some additional relations. Then we can almost obtain a witness algebra by taking A as the raw algebra, taking the witnesses for any equation a = b to be the formal deductions of this equation from the relations defining B, and taking witnesses $1_a, \xi^{-1}, \xi * \eta$, and $f(\bar{\xi})$ to be compositions of deductions as depicted above. The reason for "almost" in the preceding sentence is that the axioms for witness algebras require certain identifications between deductions. For example, two consecutive

uses of symmetry as in

$$\begin{array}{c}
\xi \\
a = b \\
\hline
b = a \\
\hline
a = b
\end{array}$$

should have no effect (see part (2) of Lemma 7); this deduction should be identified with

$$\xi \\
a = b.$$

A special case of this example occurs when A is the algebra in a certain variety presented by certain generators and relations, and B is presented by the same generators and relations in a smaller variety, given by more identities. For example, A might be the group with a certain presentation while B is the abelian group with the same presentation (the abelianization of A). The witnesses would then be deductions (modulo identifications required by the axioms) from the commutative law.

Category-Theoretic Remark 13. Since witness algebras amount to groupoids, our definition of witness Σ -algebras makes the operations $f \in \Sigma$ act on both the objects (raw elements) and morphisms (witnesses). These operations constitute functors from powers of the groupoid to itself. In general, the definition of "functor" also requires preservation of identity morphisms, so it would seem that we need to require

$$f(1_{a_1},\ldots,1_{a_r})=1_{f(a_1,\ldots,a_r)}.$$

In the case of groupoids, though, this preservation of identity elements follows from compatibility with composition. This last obsrevation will be useful even apart from the category-theoretic point of view, so we formulate it as the following proposition.

Proposition 14. If $f \in \Sigma$ is r-ary then $f(1_{a_1}, \ldots, 1_{a_r}) = 1_{f(a_1, \ldots, a_r)}$. In particular, if f is a nullary operation symbol, then the corresponding witness is 1_f .

Proof. To simplify the notation, we assume for now that f is binary, and we write a and b instead of a_1 and a_2 . Note that both $f(1_a, 1_b)$ and $1_{f(a,b)}$ witness the equation f(a,b) = f(a,b). Furthermore, according to requirement (3) in the definition of witness algebras, we have

$$f(1_a, 1_b) * f(1_a, 1_b) = f(1_a * 1_a, 1_b * 1_b) = f(1_a, 1_b) = f(1_a, 1_b) * 1_{f(a,b)}.$$

The desired conclusion now follows by cancellation (part (1) of Lemma 7).

The argument for r=2 generalizes easily to larger arities r and to r=1. For r=0, the argument still works and in fact becomes simpler, but it requires some notational caution, as follows. If f is 0-ary then, remembering that 0-ary operations on a set amount to elements, we have a raw element f, and we also have a witness \tilde{f} (also called f if no confusion results, but here confusion would result) for the equation f=f. Then we have, from the definition of witness algebra,

$$\tilde{f} * \tilde{f} = \tilde{f} = \tilde{f} * 1_f,$$

and cancellation (part (1) of Lemma 7) gives us $\tilde{f} = 1_f$.

3. Braid Semirings

In this section, we introduce the particular algebraic theory underlying the application of witness algebra to anyons. The basic idea is quite simple, but various complications arise in the details.

We shall arrange our witness algebras so that the associated true algebras are commutative semirings with unit. Here "semiring" is defined like "ring" except that additive inverses are not required to exist. Some authors use "rig" to mean "semiring" (the idea being that removing the letter "n" from "ring" corresponds to removing negatives from the definition). Because multiplication will be commutative in our semirings, we adopt the following convention for brevity.

Convention 15. "Rig" means commutative semiring with unit.

Thus, the signature for our algebras consists of two binary operations + and \times and two constants 0 and 1. A rig is an algebra for this signature in which

- both + and \times are associative and commutative,
- 0 and 1 are identity elements for + and \times , respectively, and
- \bullet × distributes over +.

Distributivity here means not only that multiplication distributes over sums of two elements², $A \times (B + C) = (A \times B) + (A \times C)$ (which implies distribution over any larger number of summands) but also distribution over zero summands, i.e., $A \times 0 = 0$. (This equation would be redundant in the presence of additive inverses, but it needs to be assumed in the axiomatization of rigs.)

²We begin here to use capital letters for elements of our raw algebras, for two reasons. First, in the application to anyons, these raw elements will be tuples of vector spaces, for which capital letters are more natural than lower-case letters. Second, we shall need to import some diagrams from [3]. That paper was written from the point of view of category theory, so raw elements were objects of categories, denoted, as usual, by capital letters.

The raw algebras admit an even simpler description: They are algebras for the same signature $\{+, \times, 0, 1\}$ but subject to no equations. This means that, for the true algebras to satisfy the equations that define rigs, we need to introduce witnesses for all those equations. That is, our witness algebras must have (at least) the following witnesses specified as part of the structure, for all raw elements A, B, C:

$$\begin{array}{lll} \textbf{associative} + & \alpha_{A,B,C}^+ \vdash (A+B) + C = A + (B+C) \\ \textbf{unit} + & \lambda_A^+ \vdash 0 + A = A \quad \text{and} \quad \rho_A^+ \vdash A + 0 = A \\ \textbf{commutative} + & \gamma_{A,B}^+ \vdash A + B = B + A \\ \textbf{associative} \times & \alpha_{A,B,C}^\times \vdash (A \times B) \times C = A \times (B \times C) \\ \textbf{unit} \times & \lambda_A^\times \vdash 1 \times A = A \quad \text{and} \quad \rho_A^\times \vdash A \times 1 = A \\ \textbf{commutative} \times & \gamma_{A,B}^\times \vdash A \times B = B \times A \\ \textbf{distributive} & 2 & \delta_{A,B,C} \vdash A \times (B+C) = (A \times B) + (A \times C) \\ \textbf{distributive} & 0 & \varepsilon_A \vdash A \times 0 = 0 \\ \end{array}$$

We shall refer to these witnesses as the rig witnesses.

So far, our description of the desired witness algebras can be easily summarized: Algebras for the signature $\{+, \times, 0, 1\}$ with enough specified witnesses to make the true algebra a rig. More is needed, though, for this structure to make good sense and (more importantly) to be useful for anyon theory. We need to specify, or at least constrain, how the rig witnesses listed here interact with each other and with other witnesses that might be present. Let us begin with two examples before presenting the general situation.

Example 16. Suppose we have witnesses $\xi \vdash A = A'$ and $\eta \vdash B = B'$. Then the fact that A and B commute under addition has, in addition to the witness $\gamma_{A,B}^+$ above, another witness that works via the commutativity of A' and B', namely $(\xi + \eta) * \gamma_{A',B'}^+ * (\eta + \xi)^{-1}$. If we think of ξ and η as giving us alternative ways to view A and B, then this second witness for A + B = B + A is just an alternative way to view the original witness $\gamma_{A,B}^+$. So it is reasonable to require that these two witnesses be equal³. Equivalently,

$$\gamma_{A,B}^{+} * (\eta + \xi) = (\xi + \eta) * \gamma_{A',B'}^{+}.$$

 $^{^3}$ Recall that equality of witnesses means genuine identity, not existence of some meta-witness.

This requirement can be summarized as "The witnesses γ^+ respect witnessed equalities." In Section 3.1, we will impose analogous requirements on all of the other rig witnesses. The justifications for these requirements are the same as for γ^+ here.

Category-Theoretic Remark 17. In category theory, the requirement in Example 16 is expressed by saying that γ^+ is a natural transformation. The analogous requirements for the other rig witnesses will say that all of them are natural transformations.

Example 18. We have two witnesses for A+0=A, namely ρ_A^+ and $\gamma_{A,0}^+ * \lambda_A^+$. It seems reasonable to require that they coincide.

The requirements in Example 16 and Example 18 are qualitatively different. In the former, we were concerned with how a rig witness (γ^+) interacts with arbitrary other witnesses $(\xi$ and η); in the latter, we are concerned with how rig witnesses interact with each other. In the former, there is an obvious generalization from the γ^+ considered there to all the other rig witnesses. In the latter, there is no obvious generalization, and indeed there is considerable freedom in choosing what requirements of this sort should be imposed.

The rest of this section is devoted to presenting the requirements that we impose on the witnessess in our braid rigs (also called braid semirings). We present these requirements in three parts, subsections 3.1, 3.3, and 3.4, with an intermediate subsection 3.2 explaining how some of the requirements were chosen.

The requirements in subsection 3.1 are analogous to what we saw in Example 16. This subsection is simply the evident generalization of the example from γ^+ to all rig witnesses. Subsections 3.3 and 3.4 present requirements analogous to that in Example 18. As indicated above, we have some freedom in choosing requirements of this sort. Subsection 3.2 discusses how we chose to exercise that freedom in making the decisions in the next two subsections.

Much of the material in subsections 3.2, 3.3, and 3.4 has already appeared, up to notational and terminological differences, in our earlier paper [3]. For the reader's convenience, we import it here, with the necessary modifications, rather than contenting ourselves with a list of the changes.

Braid rigs will be defined as witness algebras for the rig signature $\{+, \times, 0, 1\}$, equipped with all the rig witnesses listed above and satisfying all the requirements imposed in the following subsections.

3.1. Rig Witnesses Respect Witnessed Equalities. Suppose we have witnesses

$$\xi \vdash A = A', \quad \eta \vdash B = B', \text{ and } \zeta \vdash C = C'.$$

Then we require that all rig witnesses involving any of A, B, C and the corresponding rig witnesses involving A', B', C' match via the given ξ, η, ζ . In detail, these requirements are as follows.

$$\alpha_{A,B,C}^{+} * (\xi + (\eta + \zeta)) = ((\xi + \eta) + \zeta) * \alpha_{A',B',C'}^{+}$$

$$\lambda_{A}^{+} * \xi = (1_{0} + \xi) * \lambda_{A'}^{+}$$

$$\rho_{A}^{+} * \xi = (\xi + 1_{0}) * \rho_{A'}^{+}$$

$$\gamma_{A,B}^{+} * (\eta + \xi) = (\xi + \eta) * \gamma_{A',B'}^{+}$$

$$\alpha_{A,B,C}^{\times} * (\xi \times (\eta \times \zeta)) = ((\xi \times \eta) \times \zeta) * \alpha_{A',B',C'}^{\times}$$

$$\lambda_{A}^{\times} * \xi = (1_{1} \times \xi) * \lambda_{A'}^{\times}$$

$$\rho_{A}^{\times} * \xi = (\xi \times 1_{1}) \times \rho_{A'}^{\times}$$

$$\gamma_{A,B}^{\times} * (\eta \times \xi) = (\xi \times \eta) * \gamma_{A',B'}^{\times}$$

$$\delta_{A,B,C} * ((\xi \times \eta) + (\xi \times \zeta)) = (\xi \times (\eta + \zeta)) * \delta_{A',B',C'}$$

$$\varepsilon_{A} = (\xi \times 1_{0}) * \varepsilon_{A'}$$

Notice that the fourth of these equations is what we had in Example 16. All ten of the equations have the same general form: The first factor on the left and the second on the right are from one of our ten types of rig witnesses, with unprimed subscripts on the left and primed on the right. The other factors are built from (some of) ξ , η , ζ . How they are built matches the specifications of the rig witness. (In the last of these equations, in strict analogy to the previous ones, the left side would have been $\varepsilon_A * 1_0$; we have performed the trivial simplification to ε_A .)

Category-Theoretic Remark 19. In the language of category theory, these ten equations merely say that the rig witnesses constitute ten natural transformations.

3.2. Coherence Conditions. We turn next to identities, known as coherence conditions, between various compositions of rig witnesses. We have seen one coherence condition, $\gamma_{A,0}^+ * \lambda_A^+ = \rho_A^+$, in Example 18, but there is no evident way to determine all of the coherence conditions that should be imposed on our rig witnesses. In fact, as we shall see, the choice of coherence conditions is influenced by the intended application. For example, since addition alone and multiplication alone are subject to the same requirements in the definition of rig (associativity,

unit, and commutativity), it would seem natural to impose the same coherence conditions on the purely additive and purely multiplicative rig witnesses. That approach, however, turns out to be completely inappropriate for the study of non-abelian anyons. (See the discussion of symmetry versus braiding later in the present subsection.) Accordingly, we devote this subsection to explaining how we selected suitable coherence conditions to be satisfied by braid rigs; the conditions themselves will be presented in the next two subsections.

There are two mathematical constraints on our selection of coherence conditions, plus a practical consideration that also influenced our choices. The first and most important mathematical constraint is that our coherence conditions should be satisfied in the witness algebras arising in anyon models, the witness algebras that we propose as a simplification of modular tensor categories. Axioms that fail in the intended examples are useless. So we must not make our coherence conditions too strong.

The second mathematical constraint is that our coherence conditions should not be too weak; they should entail all the information needed in our computations of specific examples. For example, our coherence conditions should support the computations, as in [2], of the associativity and braiding matrices for Fibonacci anyons.

For practical purposes, we stay close to the coherence conditions already available in the literature for structures resembling some of our rig witnesses. Let us briefly summarize the relevant literature.

If we consider either addition by itself or multiplication by itself, then the definition of rigs requires that we have a commutative monoid. An analogous structure has been studied in category theory, namely symmetric monoidal categories, and suitable coherence conditions were found by Mac Lane [12] and simplified by Kelly [9]. Here is an example to clarify what "suitable" means.

Example 20. In ordinary (not witness) algebra, the associative law for addition, (A+B)+C=A+(B+C), implies that one can safely omit parentheses in sums of any number of terms. For example, one can deduce ((A+B)+C)+D=A+(B+(C+D)), and similarly for more summands and for other arrangements of the parentheses. In fact, for the specific case of ((A+B)+C)+D=A+(B+(C+D)), two deductions are available. One goes via (A+B)+(C+D), and the other goes via (A+(B+C))+D and A+((B+C)+D).

In witness algebra, the corresponding facts are as follows. Given witnesses $\alpha_{A,B,C}^+$ as above, for all raw elements A,B,C, we can construct, by composing them, witnesses for ((A+B)+C)+D=A+(B+(C+D))

and similarly for more summands and for other arrangements of the parentheses. In fact, for the specific case of ((A+B)+C)+D=A+(B+(C+D)), two such compositions are available, corresponding to the two deductions in ordinary algebra. A typical coherence condition would require that these two compositions coincide.

If we use associativity to rearrange parentheses in sums with more than four summands, there will, in general, be many deductions for a single equation in ordinary algebra, and therefore many witnesses, composites of witnesses of type α^+ , for the same equation in witness algebra. We would like all these witnesses for the same equation to coincide. So, a priori, we would impose infinitely many coherence conditions, with more and more variables. Fortunately, Mac Lane showed in [12] that the coherence condition for associativity with four summands implies all the other coherence conditions for associativity. Moreover, he found a small number of coherence conditions for associativity, unit, and commutativity that imply that, whenever two reasonable compositions of these witnesses witness the same equation, they coincide. ("Reasonable" requires careful formulation, to avoid, for example, expecting the special case $\gamma_{A,A}^+ \vdash A + A = A + A$ of commutativity to coincide with 1_{A+A} .) We shall adopt Mac Lane's coherence conditions, as simplified by Kelly [9], for the additive structure of our braid semirings, i.e., for the rig witnesses of the forms $\alpha^+, \lambda^+, \rho^+, \gamma^+$.

Although it seems natural and simple to treat multiplication the same way, we shall not adopt these same conditions for the multiplicative structure. The reason lies in the behavior of anyons that we intend to model. Here is a rough explanation of the situation; a more detailed (and thus more accurate) explanation can be found in [2, 13]. Think of a product $A \times B$ as representing an anyon (or anyon system) of type A located next to one of type B. The commutativity witness $\gamma_{A,B}^{\times} \vdash A \times B = B \times A$ represents interchanging the locations of A and B. Anyons inhabit a two-dimensional space, and so there are two different ways to move A from, say, the left of B to the right of B: A could pass in front of B or behind it. If we (arbitrarily) take $\gamma_{A,B}^{\times}$ to represent the transposition that moves A in front of B, then $(\gamma_{B,A}^{\times})^{-1} \vdash A + B = B + A$ represents the transposition moving A behind B, and we do not want these to always coincide. In other words, we do not want the so-called symmetry condition

$$\gamma_{A,B}^{\times} * \gamma_{B,A}^{\times} = 1_{A+B}$$

to hold in general. Indeed, the left side of the symmetry condition represents moving A all the way around B, back to its original location.

The non-triviality of such *braiding* operations is the key to the usefulness of anyons in quantum computation. But the symmetry condition is among the coherence conditions of Mac Lane and Kelly. Our multiplicative structure should therefore be subject only to some weaker system of coherence conditions, allowing non-trivial braiding.

Fortunately, Joyal and Street [7, 8] have given a system of coherence conditions that accomplishes exactly what we need. Their notion of "braided monoidal category" is like "symmetric monoidal category" except that the symmetry condition is omitted (and another condition, deducible using symmetry but not otherwise, is added). The coherence conditions for braided monoidal categories are included among the axioms for modular tensor categories; see [2, 13]. We shall adopt the Joyal–Street coherence conditions for the multiplicative structure of our braid semirings.

Beyond the additive and multiplicative structures, whose coherence conditions we borrow from Mac Lane, Kelly, Joyal, and Street, we also have the distributivity witnesses which connect the additive and multiplicative structures.

The available literature concerning coherence conditions for distributivity is the paper [11] of Laplaza. He introduces and justifies a system of such coherence conditions for the situation where both the additive and multiplicative structures are symmetric monoidal structures. In our situation, however, only the additive structure is symmetric; the multiplicative structure is merely braided. As a result, we must modify Laplaza's coherence conditions to work properly with braided multiplication. We have carried out this modification in [3] and proved some theorems there that indicate its appropriateness. In the present paper, we shall only record the coherence conditions that we found and some remarks about them, referring to [3] for details.

Convention 21. Rather than writing these conditions as equations, we shall exhibit them as diagrams, in accordance with the following conventions. Each diagram will be a directed graph, with vertices labeled by raw elements and with directed edges labeled by witnesses. If ξ labels an edge from a vertex A to a vertex B then $\xi \vdash A = B$. Consider a path in the underlying undirected graph (obtained by forgetting the orientations of the edges); note that the edges in such a path may be directed forward or backward along the path. Associate to this path the witness for A = B obtained, by composing (by *), in order along the path, the labels of those edges that are directed forward along the path and the inverses of the labels of the edges directed backward along the path. In our diagrams, the underlying undirected graph

will always be just a cycle, so for each pair of vertices A, B, there will be exactly two paths from A to B, associated with two witnesses for A = B. The diagram is to be interpreted as the equation saying that these two witnesses are equal. It is easy to check that, if we had chosen two other vertices A' and B' instead of A and B, then the resulting equation between two witnesses for A' = B' would be equivalent to the equation described here between witnesses for A = B. (The proof uses clauses (W4), (W5), and (W6) of the definition of witness frames.)

For another equivalent way to interpret the diagram, consider any vertex A in the cycle and consider a "path" that goes from A, once around the cycle (in either direction), ending back at A. (We put "path" in quotation marks because, strictly speaking, a path should not have a repeated vertex.) Associated with this "path" is a witness ξ for A = A. The equations that interpret the diagram as above are equivalent not only to each other but also to the equation $\xi = 1_A$.

3.3. Coherence for Addition and Multiplication. Our coherence conditions for the additive structure, $\{\alpha^+, \lambda^+, \rho^+, \gamma^+\}$, are Kelly's simplification [9] of Mac Lane's coherence conditions [12] for symmetric monoidal categories. In the diagram form explained in Convention 21, they are the following Figures 1–4, in whose captions we have given names for the conditions. (The first figure, the pentagon, is the previously discussed case of two witnesses for moving the parentheses, in a sum of four terms, from the extreme left to the extreme right.)

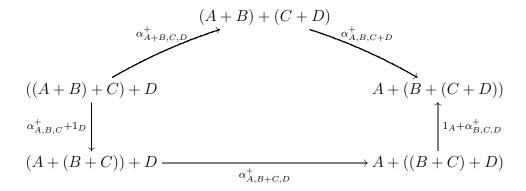


Figure 1. Additive Pentagon Condition

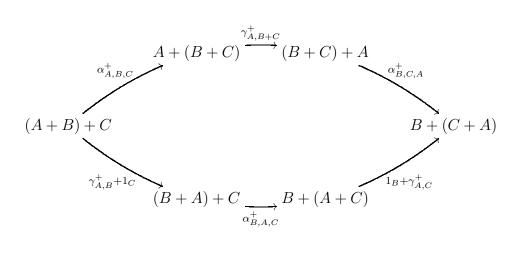


FIGURE 2. Additive Hexagon Condition

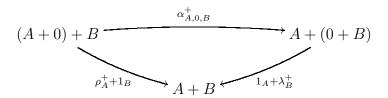


FIGURE 3. Additive Unit Associativity

$$A + B \xrightarrow{\gamma_{A,B}^+} B + A$$

$$\xrightarrow{\gamma_{B,A}^+}$$

FIGURE 4. Additive Symmetry

Our coherence conditions for the multiplicative structure, $\{\alpha^{\times}, \lambda^{\times}, \rho^{\times}, \gamma^{\times}\}$, are those given by Joyal and Street [7, 8] for braided monoidal categories, namely the following Figures 5–8. Note that they differ from the additive ones by the absence of symmetry and the presence of a second hexagon condition. This second hexagon condition is like the first but with every $\gamma_{X,Y}^{\times}$ replaced with $\gamma_{Y,X}^{\times}$ and the direction of the associated edge reversed. In view of our Convention 21 about reading diagrams as equations, this replacement is equivalent to replacing every $\gamma_{X,Y}^{\times}$ with $\gamma_{Y,X}^{\times}^{-1}$. Thus, the two hexagon conditions are equivalent in the presence of symmetry, so we

needed only one of them in the additive situation. But when symmetry is unavailable, the two hexagon conditions must both be assumed.

In the names of the mutiplicative hexagon conditions, "in front of" and "behind" refer to the way two anyons are interchanged by the commutativity witnesses γ^{\times} . This corresponds to the customary picture of braided commutativity in terms of geometric braids (the same picture that gave the name "braided" to this weakening of symmetry).

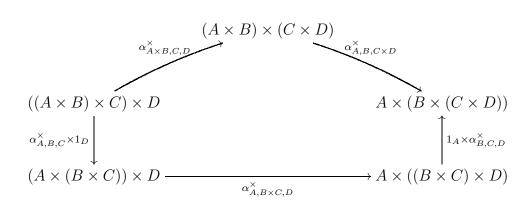


Figure 5. Multiplicative Pentagon condition

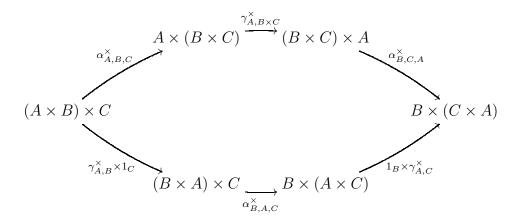


FIGURE 6. Multiplicative Hexagon: Moving one factor in front of two

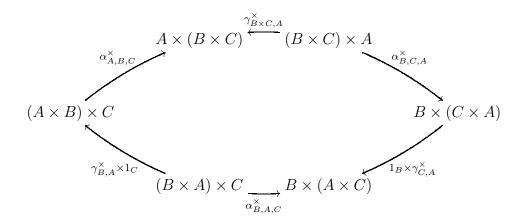


FIGURE 7. Multiplicative Hexagon: Moving one factor behind two

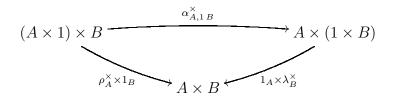


FIGURE 8. Multiplicative Unit Associativity

Category-Theoretic Remark 22. The content of this subsection is that our groupoid is equipped with two monoidal structures, a symmetric one written with + and a braided one written with \times . These two structures will be connected by distributivity, whose coherence conditions constitute the next subsection.

3.4. Coherence for Distributivity. Our coherence requirements for the distributivity witnesses δ and ε are given by Figures 9–18, taken from our paper [3]. We refer to this paper for motivation and additional information about these conditions, in particular their connections with Laplaza's coherence conditions for the case where both + and \times are symmetric.

Notation 23. We shall sometimes use the usual conventions from algebra that XY means $X \times Y$ and that, for example, X + YZ means X + (YZ), not (X + Y)Z.

It will be convenient to have the following notation for the deviation from symmetry.

Notation 24. $\beta_{X,Y} = \gamma_{X,Y}^{\times} * \gamma_{Y,X}^{\times}$.

Thus, symmetry amounts to the requirement that $\beta_{X,Y} = 1_{X\times Y}$. In the general braided situation, $\beta_{X,Y} \vdash X \times Y = X \times Y$. Pictorially, if we imagine $\gamma_{X,Y}^{\times}$ as interchanging X with Y by moving X in front of Y, then $\beta_{X,Y}$ moves X all the way around Y back to its initial position, first passing in front of Y and then returning behind Y.

We now present our coherence conditions for distributivity, along with some remarks intended to make them easier to understand.

Remark 25. We have required witnesses for the distributive law $A \times (B+C) = (AB) + (AC)$ but not for the analogous law $(B+C) \times A = (BA) + (CA)$. This is reasonable, since we have commutativity and can therefore deduce either of these distributive laws from the other. In terms of witnesses, we have

$$\gamma_{B+C,A}^{\times} * \delta_{A,B,C} * (\gamma_{B,A}^{\times} + \gamma_{C,A}^{\times})^{-1} \vdash (B+C)A = (BA) + (CA).$$

In fact, we have a second, equally good witness for the same equation:

$$(\gamma_{AB+C}^{\times})^{-1} * \delta_{A,B,C} * (\gamma_{AB}^{\times} + \gamma_{AC}^{\times}) \vdash (B+C)A = (BA) + (CA).$$

In terms of the braiding picture of products, the first of these witnesses moves A behind B, C, and B+C, and the second witness moves A in front of these other factors. There is no reason to prefer one of these witnesses to the other, so we shall impose a coherence condition saying that they are equal. Rather than writing out that equality, we simplify it a bit by "clearing fractions", i.e., by multiplying both sides by factors to cancel the inverses that occur in our two witnesses. Once that is done, each side of the desired coherence condition involves a composition of two γ^{\times} witnesses, a composition that fits our definition of β above. Thus, the desired coherence condition takes the simple form of the left diagram in Figure 9.

$$A \times (B+C) \xrightarrow{\delta_{A,B,C}} (A \times B) + (A \times C)$$

$$\downarrow^{\beta_{A,B+C}} \qquad \qquad \downarrow^{\beta_{A,B}+\beta_{A,C}}$$

$$A \times (B+C) \xrightarrow{\delta_{A,B,C}} (A \times B) + (A \times C)$$

$$A \times (B+C) \xrightarrow{\delta_{A,B,C}} (A \times B) + (A \times C)$$

$$A \times (B+C) \xrightarrow{\delta_{A,B,C}} (A \times B) + (A \times C)$$

FIGURE 9. Right Distributive

Remark 26. The right diagram in Figure 9 is essentially the analog of the left for a sum of no summands in place of the sum B + C of two summands. The precise analog would result from the left diagram by

changing the vertex labels on the left to $A \times 0$ and on the right to 0, changing the horizontal arrows to ε_A , changing the left vertical arrow $\beta_{A,0}$, and changing the right vertical arrow to 1_0 . (To see that this last 1_0 is correct, use Proposition 14 with f being the 0-ary operation 0.) Multiplying by the inverse of ε_A , we simplify the desired equality to $\beta_{A,0} = 1_{A\times 0}$, which is depicted on the right side of Figure 9.

Remark 27. It is worthwhile to keep in mind the two witnesses (equal by the left part of Figure 9) described above for (B+C)A = (BA)+(CA). They will occur twice in Figure 17, and that rather large figure becomes easier to understand if one realizes that, in the two places indicated by dashed lines⁴, what looks like a composition of three witnesses can be understood as just a witness for distributivity from the right rather than the left.

Notice also that we have an analogous pair of witnesses (equal by the right part of Figure 9) with no summands rather than two,

$$\gamma_{0,A}^{\times} * \varepsilon_A \vdash 0A = 0$$
 $(\gamma_{A,0}^{\times})^{-1} * \varepsilon_A \vdash 0A = 0.$

Remark 28. The next three coherence conditions, Figures 10 through 12, say that distribution respects additive manipulations — commutativity, associativity, and unit properties. That is, given $A \times S$ where S is a sum, it doesn't matter whether we perform additive manipulations within S and then apply distributivity or first apply distributivity and then perform the corresponding manipulations on the resulting sum.

In these and subsequent figures, we indicate in the caption the corresponding condition in [11].

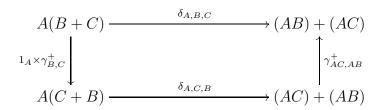


FIGURE 10. Distribution Respects Additive Commutativity (Laplaza Cond. I)

⁴Red in the pdf version of the paper

$$A(B + (C + D)) \xrightarrow{\delta_{A,B,C+D}} AB + A(C + D) \xrightarrow{1_{AB} + \delta_{A,C,D}} AB + (AC + AD)$$

$$\uparrow_{1_A \times \alpha_{B,C,D}^+} AB + A(C + D) \xrightarrow{\alpha_{AB,AC,AD}^+} AB + (AC + AD)$$

$$\uparrow_{A(B + C) + D} AB + A(C + D) \xrightarrow{\delta_{A,B,C+1}} AB + AC + AD$$

FIGURE 11. Distribution Respects Additive Associativity (Laplaza Cond. V)

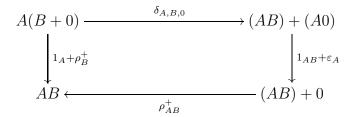


FIGURE 12. Distribution Respects 0 as neutral (Laplaza Cond. XXI)

Remark 29. Next are four coherence conditions saying that, when distributing a product of several factors across a sum, it doesn't matter whether one distributes the whole product at once or the individual factors one after the other. The case of a product of two factors distributing across a sum of two summands is the obvious one; it implies (in the presence of the other coherence conditions) the cases with more factors or summands. It is, however, also necessary to cover the cases where the number of factors or the number of summands is zero. So we get the four coherence conditions in Figures 13 through 16. In our names for the conditions, the numbers 2 or 0 refer first to the number of factors and second to the number of summands.

$$(AB)(C+D) \xrightarrow{\alpha_{A,B,C+D}^{\times}} A(B(C+D))$$

$$\downarrow 1_{A} \times \delta_{B,C,D}$$

$$A((BC)+(BD))$$

$$\downarrow \delta_{A,BC,BD}$$

$$((AB)C)+((AB)D) \xrightarrow{\alpha_{A,B,C}^{\times}+\alpha_{A,B,D}^{\times}} (A(BC))+(A(BD))$$

FIGURE 13. Sequential Distribution 2×2 (Laplaza Cond. VI)

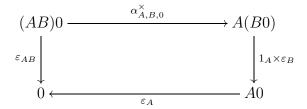


FIGURE 14. Sequential Distribution 2×0 (Laplaza Cond. XVIII)

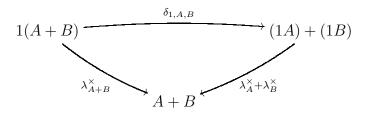


FIGURE 15. Sequential Distribution 0×2 (Laplaza Cond. XXIII)

$$1 \times 0 \xrightarrow{\varepsilon_1} 0$$

FIGURE 16. Sequential Distribution 0×0 (Laplaza Cond. XIV)

Remark 30. The remaining coherence conditions for distributivity, in Figures 17 and 18, concern a product of two sums, like (A+B)(C+D). Distributivity lets us expand this as a sum of four products, but there is a choice whether to apply distributivity first from the left, obtaining ((A+B)C) + ((A+B)D), or from the right, obtaining (A(C+D)) + (B(C+D)). One coherence condition (Figure 17) says that both choices produce the same final result, up to associativity and commutativity of addition. (Unfortunately, the associativity and commutativity make the diagram rather large. It gets even larger because a single witness for distributivity from the right looks like a witness for distributivity from the left flanked by two commutativity witnesses.) In addition, there are analogous but far simpler coherence conditions for the case where one or both of the factors is the sum of no terms rather than of two. Our labels for these conditions include numbers 2 or 0 indicating the number of summands in each factor.

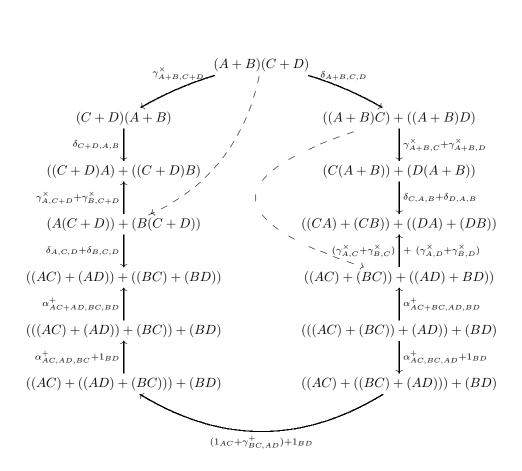


FIGURE 17. Expand 2×2 (Laplaza Cond. IX)

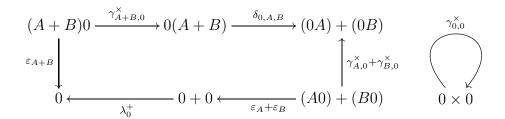


FIGURE 18. Expand 2×0 and 0×0 (Laplaza Conds. XII and X)

This completes our list of coherence conditions and allows us to define braid rigs.

Definition 31. A braid rig (or braid semiring) is a witness algebra for the signature $\{+, \times, 0, 1\}$ together with chosen witnesses of the forms $\alpha^+, \lambda^+, \rho^+, \gamma^+, \alpha^\times, \lambda^\times, \rho^\times, \gamma^\times, \delta, \varepsilon$ specified above and subject to the coherence conditions in subsections 3.1, 3.3, and 3.4.

We emphasize that the specifications of the chosen witnesses in braid rigs make the associated true algebras into rigs.

4. Unitary Fusion Semirings

In this section, we describe the particular braid rigs that are used in anyon models. We begin by describing the true rigs, as this description will be rather straightforward. Afterward, we shall describe the raw elements and witnesses.

- 4.1. **True Algebra.** The true rigs associated to our unitary fusion rigs will be rigs in which
 - (1) there is a finite set $\{x_0, x_1, \ldots, x_q\}$ of additive generators,
 - (2) each element is a finite sum of generators in a unique way (up to order and parentheses), and
 - (3) one of the generators is the multiplicative unit element 1.

The first two requirements in this list say that, as far as the additive structure of the rig is concerned, it is the free commutative monoid on the finite set $\{x_0, x_1, \ldots, x_q\}$ of generators. Note that the finite sums in requirement (2) include the empty sum 0. In connection with requirement (3), we adopt the convention that the generators are numbered so that $x_0 = 1$.

We also adopt the standard convention that nx, for a natural number n and a rig element x, means the sum of n copies of x.

Apart from $x_0 = 1$ and our intention to produce a rig, no requirements are imposed here on the multiplicative structure. Of course, the

distributive law for rigs and requirement (2) together imply that the multiplicative structure is completely determined by the products of the generators. Thus, in the true rig associated to any unitary fusion rig, we shall have a system of equations of the form

$$x_i x_j = \sum_k N_{ij}^k x_k,$$

where the N_{ij}^k are natural numbers. These equations, which in anyon theory are usually called *fusion rules*, suffice to determine the whole true rig. They define the products of the generators, and we extend the definition to arbitrary elements, i.e., sums of generators, by distributivity.

The coefficients N_{ij}^k in the fusion rules, called *fusion coefficients*, are subject to several constraints, because the multiplication operation is required to be associative and commutative with x_0 as the unit element. Specifically, to ensure that $x_0x_i = x_ix_0 = x_i$, we must have

$$(1) N_{i0}^k = N_{0i}^k = \delta_{ik} \text{for all } i, k,$$

where δ is the Kronecker delta. To ensure that $x_i x_j = x_j x_i$, we must have

(2)
$$N_{ij}^k = N_{ii}^k \quad \text{for all } i, j, k.$$

To ensure the associativity equation $(x_ix_j)x_k = x_i(x_jx_k)$, we must have

(3)
$$\sum_{r} N_{ij}^{r} N_{rk}^{l} = \sum_{s} N_{jk}^{s} N_{is}^{l} \quad \text{for all } i, j, k, l.$$

(The two sides of this equation are simply the coefficients of x_l in the two sides of the associativity equation.) It is well known that the associativity, commutativity, and unit laws, which we have ensured for the generators by means of these constraints on the fusion coefficients, imply the corresponding laws for all elements, because products of arbitrary elements were defined from the products of generators via distributivity.

To summarize this description of the true rigs associated to unitary fusion rigs: Such a true rig is completely specified by a positive integer q and a system of fusion coefficients subject to the constraints (1), (2), and (3). The additive structure is a free monoid on $\{x_0(=1), x_1, \ldots, x_q\}$, and the multiplicative structure is given by the fusion rules and distributivity.

4.2. Witness Frame. We now turn from the true rigs to the full structure of unitary fusion rigs, i.e., to the raw elements and witnesses. Of

course, these must be defined in a way that produces true algebras of the sort described above.

For the rest of this section, we work with a fixed set of fusion rules, and we use the notation N_{ij}^k as above for the fusion coefficients. In particular, we have a fixed value for q, the number of generators other than 1.

The raw elements are, by definition, all of the (q+1)-tuples of finite-dimensional complex Hilbert spaces

$$\mathbf{A} = (A_0, A_1, \dots, A_q),$$

each with a specified orthonormal basis, such that the elements of each A_i are the formal linear combinations (over \mathbb{C}) of basis elements. When we speak of basis elements, we always mean elements of the specified bases.

The witnesses ξ for the equality of two raw elements, $\xi \vdash \mathbf{A} = \mathbf{B}$, are all of the (q+1)-tuples of Hilbert-space isomorphisms (i.e., unitary transformations) between the corresponding components of \mathbf{A} and \mathbf{B} ,

$$\xi = (\xi_0, \xi_1, \dots, \xi_q)$$
 where $\xi_i : A_i \to B_i$.

It is **not** required that the unitary transformations ξ_i respect the specified bases. When they do, i.e., when each ξ_i is induced by a bijection between the specified bases of A_i and of B_i , we call ξ a basic witness.

The reflexivity witnesses are just (q + 1)-tuples of identity maps. Composition and inversion of witnesses are done componentwise.

4.3. **Addition.** Raw elements are added by forming the component-wise direct sum of the Hilbert spaces. In more detail, the i^{th} component of $\mathbf{A} + \mathbf{B}$ has as its specified basis the disjoint union of the specified bases of the components A_i and B_i of \mathbf{A} and \mathbf{B} , respectively.

Before proceeding further, we need to add some details about the additive structure just defined. There are several ways to formally define direct sums of vector spaces. The different ways produce isomorphic results, so it usually doesn't matter which way one chooses. In our situation, though, our witnesses are themselves (tuples of) isomorphisms, and when we need to manipulate these witnesses, it will not do to say that things are well-defined up to isomorphism. We therefore describe a few ways to formalize direct sums and, afterward, indicate a notation that will be convenient for the rest of our work.

(1) The most common construction of the direct sum $A \oplus B$ of two vector spaces A and B is the set of ordered pairs (a,b) with $a \in A$ and $b \in B$. Addition and scalar multiplication are defined componentwise.

Theoretically, this works well, but it becomes awkward in some situations that we shall have to consider. Notice that, in direct sums of three vector spaces we shall have elements of the form ((a,b),c) in $(A \oplus B) \oplus C$ and elements of the form (a,(b,c)) in $A \oplus (B \oplus C)$. Mathematicians frequently ignore the distinction, because of the obvious isomorphism, but in our situation, the obvious isomorphism $\alpha_{A,B,C}^+$ is part of the structure we are defining, so it cannot simply be swept under the rug. Of course, with more than three summands, we would have an even greater proliferation of parenthesis patterns, making it more difficult to see the structures involved.

One could also introduce sums of the form $A \oplus B \oplus C$ (without parentheses) as a vector space of ordered triples (a,b,c), and, depending on one's set-theoretic conventions, such a triple might or might not be considered the same as one (but not both) of ((a,b),c) and (a,(b,c)). Similarly for direct sums of more vector spaces. When we consider multiplication of our raw elements, we shall need to deal with direct sums of many vector spaces at a time, with no natural parenthesization of the summands, nor even a really natural ordering. Representing elements of the direct sum by tuples (or by tuples of tuples of . . .) becomes increasingly arbitrary and awkward.

(2) Another way to construct direct sums like $A \oplus B$ is to begin with the specified bases for A and for B and to take the disjoint union of these bases as a basis for $A \oplus B$; the other elements of $A \oplus B$ are then formal linear combinations of these basis elements.

An immediate difficulty here concerns the notion of disjoint union: What if the two bases are not disjoint? (As an extreme example, we might have A = B with the same specified basis.) Fortunately, there is an easy solution, namely to tag the elements of our bases, so that the disjoint union consists of elements (a,0) and (b,1) with a and b in the bases of A and of B, respectively. The tag notation can be extended to non-basis vectors from A and B. Given a vector $x \in A$, the corresponding vector in $A \oplus B$ is obtained by expanding x as a linear combination of basis vectors and replacing each of those basis vectors a by (a,0). We call the resulting vector (x,0). Similarly, if $y \in B$, the corresponding vector in $A \oplus B$ is called (y,1).

This approach works well for a direct sum of many spaces, even if these are given as an indexed family without any particular ordering. We can take the basis elements of $\bigoplus_{i\in I} A_i$ to be tagged elements of the specified bases of the A_i 's, i.e., ordered pairs (a,i) with $i\in I$ and $a\in A_i$. This observation will be useful because such naturally indexed but not naturally ordered direct sums will occur in our discussion of multiplication of raw elements. Incidentally, note that, if we imposed some arbitrary ordering on the indices and then used the traditional

approach (1) above, then what we have written as (a, i) here would be an |I|-tuple with one component equal to a and all the other components equal to 0; the location of the a would encode i (in effect, i is written in unary notation).

- (3) There is a way to attach tags to vectors, as in (2), without the need for specified bases. In (2), we did this as syntactic sugar, writing (x,t), when x is a (possibly non-basis) vector and t is a tag, for something constructed out of tagged basis vectors. Without resorting to syntactic sugar, we can achieve the same goal as follows. Select some 1-dimensional Hibert spaces (copies of \mathbb{C}), one space \mathbb{C}_t for each tag t that we might want to use, and let $|t\rangle$ be a fixed unit vector in \mathbb{C}_t (the copy in that space of $1 \in \mathbb{C}$). Then define the direct sum $A \oplus B$ to consist of formal sums of vectors from $A \otimes C_0$ and $B \otimes C_1$. In general, $\bigoplus_{i \in I} A_i$ consists of formal sums of vectors from the spaces $A_i \otimes \mathbb{C}_i$. Because \mathbb{C}_i is one-dimensional and spanned by $|i\rangle$, every vector in $A_i \otimes \mathbb{C}_i$ has the form $a \otimes |i\rangle$ for a unique $a \in A_i$. A fairly common simplification of the notation for tensor products would write $a \otimes |i\rangle$ as just $a|i\rangle$, which brings us back to almost the same notation as in (2).
- (4) Having introduced witness algebra as a generalization of universal algebra, we mention another viewpoint that we hope will appeal to universal algebraists. In each of the preceding three approaches, a vector a from a summand A_j appears in $\bigoplus_{i\in I} A_i$ as a with additional information that indicates the value of j. In (1), the additional information is the location of the component a amid many other components (and parentheses) in a tuple (of tuples ...); in (2) it is the tag j in (a, j); and in (3) it is the tag j in $a|j\rangle$. We can view any such arrangement of tags as an operation (in the sense of universal algebra) applied to a. Nesting of tags becomes composition of operations. This more abstract point of view allows considerably more freedom in tagging. We have not yet had need for this freedom, but it may prove useful in further studies.

Convention 32. For the purposes of this paper, we shall use the tag notation as in (2), including the use of tags with non-basis vectors. It will do no harm if the reader views (a,t) as syntactic sugar for the $a|t\rangle$ of (3), nor will it do harm if the reader views the tagging operation (-,t) as in (4).

In view of the definition of the addition operation on raw witnesses, it is clear that the raw element 0 = (0, 0, ..., 0) consisting of zero-dimensional Hilbert spaces serves as the additive identity element.

Addition of witnesses is defined componentwise in the obvious way. That is, if $\xi \vdash \mathbf{A} = \mathbf{A}'$ and $\eta \vdash \mathbf{B} = \mathbf{B}'$, then $\xi + \eta$ has as its i^{th}

component the isomorphism $A_i + B_i \rightarrow A'_i + B'_i$ given by

$$(a,0) \mapsto (\xi_i(a),0)$$
 and $(b,1) \mapsto (\eta_i(b),1)$.

This completes our description of how addition works in our unitary fusion rig. Notice that this structure depends on our fixed fusion rules only through q, the number of non-1 generators. The fusion coefficients $N_{i,j}^k$ will affect only the multiplicative structure.

Notice also the following property of witness addition, which will be generalized later and will be useful in the verification of several of the requirements for witness algebras.

Tag Invariance (preliminary form): When the sum of witnesses acts on a vector, it leaves the tags unchanged and merely applies the summand witnesses in the unique reasonable way.

Let us check that addition as defined here for raw elements produces the desired additive structure in the true algebra. Two raw elements are equal in the true algebra if and only if they are componentwise isomorphic, which means just that corresponding components have the same dimension. The elements of the true algebra thus correspond bijectively to (q+1)-tuples of dimensions, natural numbers (n_0, n_1, \ldots, n_q) , and thus to the formal sums

$$\sum_{i=0}^{q} n_i x_i = n_0 1 + \sum_{i=1}^{q} n_i x_i$$

that we want as elements of the true algebra. This correspondence can be formalized by defining, for each i, the raw element \mathbf{x}_i to have \mathbb{C} (with specified basis $\{1\}$) in component i and zero in all other components. Then every raw element is equal (i.e., componentwise isomorphic) to a unique sum of these \mathbf{x}_i 's. The equivalence classes in the true algebra of the \mathbf{x}_i 's serve as the additive generators of the true algebra, and addition of raw elements corresponds to the formal addition used in our earlier description of the true rig.

Remark 33. Addition essentially works on (specified) bases; the Hilbert space structure (linear structure and inner product) just comes along for the ride. By this we mean two things. First, the specified bases in $\mathbf{A} + \mathbf{B}$ are built purely set-theoretically (no linear combinations involved) from the specified bases in \mathbf{A} and \mathbf{B} . Second, if ξ and η happen to be basic witnesses (recall that this means they respect specified bases), then $\xi + \eta$ is also basic. Everything we have done so far would continue to work if raw elements were (q+1)-tuples of finite sets rather than Hilbert spaces.

4.4. **Multiplication.** The product $\mathbf{A} \times \mathbf{B}$ of raw elements $\mathbf{A} = (A_0, A_1, \dots, A_q)$ and $\mathbf{B} = (B_0, B_1, \dots, B_q)$ has in its k^{th} component the direct sum of N_{ij}^k copies of the tensor product $A_i \otimes B_j$ for all i and j.

Quite generally, when forming the tensor product of two Hilbert spaces A and B with specified bases, we let the specified basis of $A \otimes B$ consist of ordered pairs (a, b) where a and b range over the specified bases of A and of B, respectively.

In our present situation, when forming a direct sum of many such tensor products, we must adjoin tags to identify the various components of the sum. In accordance with Convention 32, we write a typical basis element of the k^{th} component of $\mathbf{A} \times \mathbf{B}$ in the form

$$(a_i, b_j, i, j, t)$$

where a_i and b_j are basis elements from the components A_i of \mathbf{A} and B_j of \mathbf{B} , and where $1 \leq t \leq N_{ij}^k$. Here the tags i and j serve to identify the tensor product $A_i \otimes B_j$ in which (a_i, b_j) is a basis element, and the last tag t serves to tell which of the N_{ij}^k copies of this tensor product our basis element is in. It will often be convenient in calculations (though not technically required) to add a subscript indicating which component of $\mathbf{A} \times \mathbf{B}$ this basis element is in; thus, instead of (a_i, b_j, i, j, t) , we may write $(a_i, b_j, i, j, t)_k$.

Recall that we defined \mathbf{x}_i to be the raw element consisting of \mathbb{C} in component i and 0 in all q other components. Let us calculate the product of two of these raw elements, say $\mathbf{x}_r \times \mathbf{x}_s$. For any k, the k^{th} component of this product, as described above, is the direct sum of numerous tensor products, but, because of the many 0 components in \mathbf{x}_r and \mathbf{x}_s , many of these tensor products will be 0. Indeed, the only non-zero summands $(x_r)_i \otimes (x_s)_j$ occur when i = r and j = s, and those summands are $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$. So the direct sum of these tensor products will be the direct sum of N_{rs}^k one-dimensional spaces. Since this happens for every k, we see that the product $\mathbf{x}_r \times \mathbf{x}_s$ has the same dimensions, in all components, as $\sum_k N_{rs}^k \mathbf{x}_k$. It follows that the true elements x_k represented by the raw elements \mathbf{x}_k satisfy the given fusion rules. Thus, our definition of mutiplication of raw elements produces the correct true algebra.

In particular, we can define the element 1 of our unitary fusion rig to be \mathbf{x}_0 , and this serves as a multiplicative identity element in the true algebra.

We define the multiplication of witnesses so that the principle of Tag Invariance applies to them, just as for addition. That is, if $\xi \vdash \mathbf{A} = \mathbf{A}'$ and $\eta \vdash \mathbf{B} = \mathbf{B}'$ then $\xi \times \eta$ is the (q+1)-tuple whose k^{th} component sends (a_i, b_j, i, j, t) to

$$(\xi_i(a_i), \eta_j(b_j), i, j, t).$$

We repeat the principle of Tag Invariance, now in its final form, including both addition and multiplication.

Tag Invariance: When the sum or product of witnesses acts on a vector, it leaves the tags unchanged and merely applies the summand and factor witnesses in the unique reasonable way.

This completes the definition of a witness algebra. To make it into a braid semiring, we must specify the associativity, commutativity, and unit witnesses for both addition and multiplication; specify the distributivity witnesses; verify that these rig witnesses respect witnessed equalities (Section 3.1); and verify the coherence conditions (Sections 3.3 and 3.4).

4.5. Additive Rig Witnesses. In this subsection, we specify witnesses for the associative, commutative, and identity laws of addition.

For the associative law, $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$, let us consider what happens in one component, say the k^{th} , and let us omit, for brevity, the subscripts k.

We begin by observing that the standard basis vectors for (A+B)+C have three possible forms. Basis elements from A+B look like (a,0) or (b,1), and they provide basis elements ((a,0),0) and ((b,1),0) in (A+B)+C. In addition, C provides basis vectors (c,1). Similarly, we see that the standard basis vectors for A+(B+C) are of the three forms (a,0), ((b,0),1), and ((c,1),1). Now we can define the associativity isomorphism $\alpha^+:(A+B)+C\to A+(B+C)$ in the obvious way:

$$((a,0),0) \mapsto (a,0)$$

 $((b,1),0) \mapsto ((b,0),1)$
 $(c,1) \mapsto ((c,1),1).$

Restoring the subscripts that we omitted for brevity earlier, we should write this α^+ as $(\alpha_{\mathbf{A},\mathbf{B},\mathbf{C}}^+)_k$. It is the k^{th} component of the associativity witness $\alpha_{\mathbf{A},\mathbf{B},\mathbf{C}}^+$ for $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

Before proceeding to other witnesses for addition, we point out an important property, which will also hold for many — but not all — of the rig witnesses to be introduced later. It concerns what happens to tags (the 0's and 1's in our present situation) and the generic elements of specified bases (the a,b,c in our present situation).

Tag Manipulation: All rig witnesses, with the exception of the multiplicative associativity and commutativity witnesses α^{\times} and γ^{\times} , merely manipulate tags, leaving vectors from the given Hilbert spaces unchanged. These manipulations of the tags do not depend on the particular vectors from the given Hilbert spaces.

The exceptional witnesses α^{\times} and γ^{\times} are what makes unitary fusion rigs interesting and useful for quantum computation.

The two principles of Tag Invariance (for sums of witnesses) and Tag Manipulation (for α^+) together imply that α^+ respects witnessed equality in the sense explained in Section 3.1. That is, if $\xi \vdash A = A'$, $\eta \vdash B = B'$, and $\zeta \vdash C = C'$ then

$$\alpha_{A,B,C}^+ * (\xi + (\eta + \zeta)) = ((\xi + \eta) + \zeta) * \alpha_{A',B',C'}^+$$

Rather than doing a detailed calculation to verify this, we just notice that manipulations of tags that ignore the input vectors (as in the α^+ 's) and manipulations of the input vectors that ignore the tags (as in both versions of $\xi + \eta + \zeta$) do not interfere with each other and can therefore be carried out in either order.

The rest of the additive structure is easier than what we have already done with associativity. For commutativity, the obvious witness $\gamma_{\mathbf{A},\mathbf{B}}^+$ for $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ has in its k^{th} component,

$$(a,0) \mapsto (a,1)$$
$$(b,1) \mapsto (b,0).$$

The raw zero element $\mathbf{0}$ is the (q+1)-tuple of zero-dimensional Hilbert spaces. The required witnesses for $\mathbf{A} + \mathbf{0} = \mathbf{A}$ and $\mathbf{0} + \mathbf{A} = \mathbf{A}$ are given on basis vectors and therefore, by linearity, on all vectors, by (note that there are no basis vectors in $\mathbf{0}$)

$$(a,0) \mapsto a$$
 and $(a,1) \mapsto a$,

respectively.

These witnesses also satisfy the Tag Manipulation principle, and it follows, just as in the case of associativity, that they respect witnessed equalities. Furthermore, it is easy to check the coherence conditions for the additive structure, Figures 1–4 in Section 3.3.

4.6. Multiplicative Rig Witnesses, Part 1. In this subsection, we handle only the multiplicative identity witnesses. Associativity and commutativity are more complicated and will be treated later.

The raw mutiplicative identity element 1 is the (q + 1)-tuple with a one-dimensional space in component 0 and 0-dimensional spaces in all

q of the other components. In the 0th component, the 1-dimensional Hilbert space is \mathbb{C} with basis $\{1\}$.

The witness for $1 \times \mathbf{A} = \mathbf{A}$ has in its k^{th} component the isomorphism given on basis elements (and therefore by linearity on all elements) by

$$(1, a, 0, k, 1) \mapsto a$$
.

To see that this makes sense, notice that the k^{th} component of $1 \times \mathbf{A}$ has, according to the definition of \times , basis elements of the form (u, a, i, j, t) with $1 \leq t \leq N_{i,j}^k$, with u a basis element of 1_i , and with a a basis element of A_j . But for all $i \neq 0$, we have that 1_i is zero-dimensional, and so it has no basis vectors. So we get elements (u, a, i, j, t) only when i = 0, and then u is the number $1 \in \mathbb{C}$. But then $N_{i,j}^k = N_{0,j}^k = \delta_{j,k}$ is zero for all $j \neq k$, so there are no values of t available. So we get elements (u, a, i, j, t) only when j = k. And then $N_{0,k}^k = 1$, so the only available value for t is 1. So all our basis elements for the k^{th} component of $1 \times \mathbf{A}$ are of the form (1, a, 0, k, 1). These are in one-to-one correspondence with the basis elements a of A_k (since j = k), and that provides our witness $\lambda_{\mathbf{A}}^{\times}$ for $1 \times \mathbf{A} = \mathbf{A}$.

Similarly, we define the required witness $\rho_{\mathbf{A}}^{\times}$ for $\mathbf{A} \times 1 = \mathbf{A}$ to have, in its k^{th} component, the isomorphism

$$(a, 1, k, 0, 1) \mapsto a$$
.

As before, we still have the Tag Manipulation principle and therefore these witnesses respect witnessed equality.

4.7. **Distributivity.** We must specify witnesses $\delta_{\mathbf{A},\mathbf{B},\mathbf{C}}$ for $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$. Standard basis elements for components of B + C have two possible forms, namely (b,0) and (c,1) with b and c in the standard bases for the corresponding components of \mathbf{B} and \mathbf{C} . Therefore standard basis elements for the k^{th} component of $\mathbf{A} \times (\mathbf{B} + \mathbf{C})$ have the possible forms (a,(b,0),i,j,t) and (a,(c,1),i,j,t), with $1 \le t \le N_{i,j}^k$. In the k^{th} component of $(\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$, we have elements of the two forms ((a,b,i,j,t),0) and ((a,c,i,j,t),1). So it is clear how to set up the desired isomorphism in accordance with the Tag Manipulation principle:

$$(a, (b, 0), i, j, t) \mapsto ((a, b, i, j, t), 0)$$

 $(a, (c, 1), i, j, t) \mapsto ((a, c, i, j, t), 1).$

Similarly, for $(\mathbf{B} + \mathbf{C}) \times \mathbf{A} = (\mathbf{B} \times \mathbf{A}) + (\mathbf{C} \times \mathbf{A})$, we have the witness in accordance with Tag Manipulation

$$((b,0),a,i,j,t) \mapsto ((b,a,i,j,t),0) ((c,1),a,i,j,t) \mapsto ((c,a,i,j,t),1).$$

Recall that, when we defined braid semirings, only the left distributivity witnesses were taken as primitive parts of the structure; the right distributivity witnesses were defined in terms of the left ones and the commutativity witnesses. Here, in contrast, we have specified both left and right distributivity witnesses via Tag Manipulation. Thus, when we define the multiplicative commutativity witnesses γ^{\times} , we shall need to ensure that they cohere with what we have done here, i.e., that the commutativity witnesses commute appropriately with the left and right distributivity witnesses.

With distributivity, we should also include the case where the addition has no summands, i.e., $\mathbf{A} \times 0 = 0$ and $0 \times \mathbf{A} = 0$. In both cases, the raw elements are identical; they have empty bases in all components. So the desired witnesses can (indeed must) be taken to be identity isomorphisms (i.e., reflexivity witnesses).

Because of Tag Manipulation, we obtain, by the same argument as before, that our distributivity witnesses respect witnessed equality.

4.8. **Coherence.** At this point, we have defined all the rig witnesses for a unitary fusion rig, except for the associativity and commutativity of multiplication, α^{\times} and γ^{\times} . All the witnesses defined so far satisfy the Tag Manipulation principle. Because of this principle and the Tag Invariance principle for the operations on witnesses, we know that the rig witnesses defined so far respect witnessed equality as required in Section 3.1.

In fact, the Tag Manipulation principle also gives us many of the coherence conditions required in Sections 3.3 and 3.4. Specifically, Figures 1, 2, 3, 4, 10, 11, 12, and 15 involve no α^{\times} or γ^{\times} , so all the rig witnesses in these figures are already defined, and it is routine to check that these coherence conditions hold.

The same goes for Figure 17 if we use the dashed arrows for right distributivity rather than the solid arrows that express right distributivity in terms of left distributivity and γ^{\times} . So, provided we ensure that our Tag Manipulation definition of right distributivity witnesses agrees with the definition via γ^{\times} , we shall have verified Figure 17.

Several more of the coherence conditions are satisfied simply because there is only one isomorphism from a zero-dimensional Hilbert space to itself. Thus, any two witnesses of 0 = 0 coincide. This gives us the coherence conditions in the second part of Figure 9, and in Figures 14, 16, and (both parts of) 18.

What remains to be checked, after we discuss α^{\times} and γ^{\times} ?

First, we must make sure that our Tag Manipulation definition for right distributivity witnesses agrees with what we obtain from the left distributivity witnesses by conjugation with commutativity witnesses of either form, as displayed in Remark 25. That will ensure not only that our definitions for right distributivity witnesses are coherent but also that the coherence condition in the first part of Figure 9 holds. Indeed, that part of Figure 9 was exactly the statement that the two formulas in Remark 25 agree.

Second, we must verify the coherence conditions for the multiplicative structure in Figures 5, 6, 7, and 8 (the Joyal-Street conditions for braided monoidal structure).

Finally, we must make sure that distributivity and multiplicative associativity cohere as required in Figure 13.

4.9. Associativity of Multiplication. We turn to the problem of defining associativity witnesses

$$\alpha_{\mathbf{A},\mathbf{B},\mathbf{C}}^{\times} \vdash (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C}).$$

With our usual notational conventions, the left side of the associativity equation, $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$, has, in its l^{th} component, basis elements of the form

$$((a,b,i,j,t)_r,c,r,k,u)$$

with i, j, and k ranging from 0 to q; with a, b, and c basis elements of the Hilbert spaces A_i , B_j , and C_k , respectively; and with $1 \le t \le N_{ij}^r$ and $1 \le u \le N_{rk}^l$. Similarly, the l-component of the right side, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ has basis elements of the form

$$(a, (b, c, j, k, v)_s, i, s, w)$$

with i, j, k and a, b, c as above, $1 \le v \le N_{jk}^s$, and $1 \le w \le N_{is}^l$. Notice that both forms involve the same basis elements a, b, c and the same indices i, j, k, l for the elements and the final result, but that the configurations of tags are quite different. Fortunately, equation (3) above ensures that, for any fixed i, j, k, l, the number of tag configurations is the same in both cases.

If we try to define the associativity witness $\alpha_{A,B,C}^{\times}$ in the spirit of the Tag Manipulation principle, then we should set up a bijection between the two sorts of basis elements, $((a,b,i,j,t)_r,c,r,k,u)$ and $(a,(b,c,j,k,v)_s,i,s,w)$, that leaves a,b,c and therefore also i,j,k and l unchanged, but sets up a bijection between the possible triples of

tags (r, t, u) and (s, v, w). As noted above, equation (3) ensures that a bijection exists, for each fixed i, j, k, l, but it is not evident how or even whether we can choose such bijections coherently. It turns out that such a choice of bijections is not possible in general. We devote the next subsection to showing why it is impossible in a specific, rather simple example.

4.10. **Fibonacci Example and Tag Manipulation.** We consider the fusion rules for the Fibonacci anyon model. This model has q = 1, so there are only two generators, x_0 and x_1 . (They are often called 1 and τ respectively, but for the time being we retain the x_0, x_1 notation for consistency with previous sections.) The fusion rules say that

$$x_1 \times x_1 = x_1 + x_0$$
 (i.e., $\tau^2 = \tau + 1$)

and, as usual, x_0 is the multiplicative identity

$$x_0 \times x_1 = x_1 \times x_0 = x_1$$
 and $x_0 \times x_0 = x_0$.

Thus, the non-zero fusion coefficients are

$$N_{11}^1 = N_{11}^0 = N_{01}^1 = N_{10}^1 = N_{00}^0 = 1.$$

Since all the fusion coefficients are 0 or 1, we can considerably simplify the notation in the previous subsection: If t, u, v, w exist at all, they must be equal to 1, so it is not necessary to mention them. And the conditions for their existence are precisely that the corresponding fusion coefficients must be 1 rather than 0.

Thus, instead of seeking a bijection between triples of tags $(r, t, u) \leftrightarrow (s, v, w)$ as above, we seek bijections $r \leftrightarrow s$ such that appropriate t, u exist for r if and only if appropriate v, w exist for s.

Now let us consider some specific cases for i, j, k, l.

Case 1: i = 0

Recalling equation (1), we find that existence of t requires r=j, and existence of w requires s=l. Furthermore, given these equations, the existence of u and the existence of v require the same thing, namely $N_{jk}^l=1$. (If either of j,k is 0, then l equals the other one; if j=k=1 then l can be 0 or 1.) In any case, we need to chose a bijection between $\{j\}$ and $\{l\}$; There's only one bijection between two singletons, so that's what we choose.

Case 2:
$$k = 0$$

This is symmetrical to Case 1. Existence of v and u requires s = j and r = l, respectively, and then the existence conditions for w and t give the same requirement $N_{ij}^l = 1$. So we need to choose a bijection between $\{l\}$ and $\{j\}$; there's only one bijection, so we choose it.

Case 3:
$$i = k = 1$$
 but $j = 0$

Then existence of t and of v requires r = s = 1. No additional requirements arise from existence of u and v; either value of l is possible. But in either case, we again need to choose a bijection between $\{1\}$ and $\{1\}$; we choose the only bijection there is.

Case 4:
$$i = j = k = 1$$
 but $l = 0$

Existence of u and v requires r = s = 1. No additional requirements arise from existence of t and v. We again need to choose a bijection between $\{1\}$ and $\{1\}$; we choose the only bijection there is.

Case 5:
$$i = j = k = l = 1$$

Now both values of r and both values of s are available; no constraints arise from existence of any of t, u, v, w. So we need to choose a bijection between $\{0, 1\}$ and $\{0, 1\}$.

So we have two reasonable attempts to define associativity witnesses for multiplication according to the Tag Manipulation principle in the Fibonacci model, namely to use the identity bijection in Case 5 or to use the "switching" bijection $0 \leftrightarrow 1$. In the other four cases, we use the only bijection that there is.

Let us see what happens if we use the identity bijection in Case 5. And let us look at the simplest non-trivial case of the pentagon condition, namely the case where $\mathbf{A} = \mathbf{B} = \mathbf{C} = \mathbf{D} = (0, \mathbb{C}) = \mathbf{x}_1$. So all four factors have the one-dimensional space \mathbb{C} with standard basis element 1 in their 1-component, and they have the 0-dimensional space with empty basis in the 0-component.

Before beginning the computations, let us simplify the notation a bit. We have written the basis elements for a product $X \times Y$ as (x, y, i, j, t), where i and j indicate which components of X and Y the basis elements x and y come from, and t is an index ranging from 1 to a suitable fusion coefficient N_{ij}^k . In our situation, we can omit t, because the only value it ever has is 1. We can also omit i and j provided we know, in some other way, which components of X and Y our x and y come from. If X, Yare any of A, B, C, D then we know that basis elements come from the 1 component, because the bases for the 0-components are empty. If Xand Y are products of several factors, then x and y would themselves be compound expressions, and we would know which components they are in if, as mentioned earlier, we append subscripts indicating the components. We adopt, for the present computation, this convention: Tag compound expressions to indicate which component they lie in, and then omit i, j, t from the standard notation. Thus, if (x, y, i, j, 1)is a basis element of the k^{th} component of some product, we shall write it as $(x,y)_k$.

With this notation, the basis elements for the 1-component of $((\mathbf{A} \times \mathbf{B}) \times \mathbf{C}) \times \mathbf{D}$ have three possible forms

$$(((a,b)_1,c)_0,d)_1$$
 $(((a,b)_0,c)_1,d)_1$ $(((a,b)_1,c)_1,d)_1$

where a, b, c, d are basis elements. Actually, it is unnecessary to write a, b, c, d here, since these basis elements are all simply 1. All the real information is in the subscripts. Nevertheless we continue to write a, b, c, d to match previous notation.

Let us trace what happens to these three sorts of basis elements in $((\mathbf{A} \times \mathbf{B}) \times \mathbf{C}) \times \mathbf{D}$ along the two paths to $\mathbf{A} \times (\mathbf{B} \times (\mathbf{C} \times \mathbf{D}))$ in the multiplicative pentagon condition, Figure 5.

We begin by following the longer of the two paths, around the bottom of the figure. The first witness on that path, for the equation $((\mathbf{A} \times \mathbf{B}) \times \mathbf{C}) \times \mathbf{D} = (\mathbf{A} \times (\mathbf{B} \times \mathbf{C})) \times \mathbf{D}$, leaves D alone and applies $\alpha_{\mathbf{A},\mathbf{B},\mathbf{C}}^{\times}$ to the other three factors. For the first of our three possible forms, we are in Case 4, so we get $((a, (b, c)_1)_0, d)_1$. The second and third cases are in Case 5, so our decision to use the identity bijection in this case leads to $((a, (b, c)_0)_1), d)_1$ for the second form and $((a, (b, c)_1)_1, d)_1$ for the third. Summarizing, our three forms have become, after this first step along the long side of the pentagon,

$$((a, (b, c)_1)_0, d)_1$$
 $((a, (b, c)_0)_1), d)_1$ $((a, (b, c)_1)_1, d)_1$.

The next step along this path is the witness for $(\mathbf{A} \times (\mathbf{B} \times \mathbf{C})) \times \mathbf{D} = \mathbf{A} \times ((\mathbf{B} \times \mathbf{C}) \times \mathbf{D})$, namely $\alpha_{A,B \times C,D}^{\times}$. This time the second form is in Case 3, so we use the unique available bijection and obtain $(a,((b,c)_0,d)_1)_1$. The first and third forms are in Case 5. Because we're using the identity bijection in this case, we get $(a,((b,c)_1,d)_0)_1$ for the first form and $(a,((b,c)_1,d)_1)_1$ for the third. The summary now reads

$$(a, ((b,c)_1,d)_0)_1$$
 $(a, ((b,c)_0,d)_1)_1$ $(a, ((b,c)_1,d)_1)_1$.

The last step on this path of the pentagon is the witness for $\mathbf{A} \times ((\mathbf{B} \times \mathbf{C}) \times \mathbf{D}) = \mathbf{A} \times (\mathbf{B} \times (\mathbf{C} \times \mathbf{D}))$, which leaves \mathbf{A} alone but applies $\alpha_{\mathbf{B},\mathbf{C},\mathbf{D}}^{\times}$ to the rest. For the first form, we have Case 4 and we obtain $(a,(b,(c,d)_1)_0)_1$. The second and third forms are in Case 5, and our decision to use the identity bijection produces $(a,(b,(c,d)_0)_1)_1$ for the second form and $(a,(b,(c,d)_1)_1)_1$ for the third. So the summary, for the entire long path in the pentagon, is

$$(a, (b, (c, d)_1)_0)_1$$
 $(a, (b, (c, d)_0)_1)_1$ $(a, (b, (c, d)_1)_1)_1$.

This completes the calculation for the long path; we return to the original three forms and calculate what happens to them along the short path, around the top of the pentagon, still using the identity bijection in Case 5.

The first step is the witness for $((\mathbf{A} \times \mathbf{B}) \times \mathbf{C}) \times \mathbf{D} = (\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})$, namely $\alpha_{\mathbf{A} \times \mathbf{B}, \mathbf{C}, \mathbf{D}}^{\times}$. The second form is in Case 1, so we get $((a, b)_0, (c, d)_1)_1$. The first and third forms are in Case 5, so our choice of the identity bijection produces $((a, b)_1, (c, d)_0)_1$ for the first form and $((a, b)_1, (c, d)_1)_1$ for the third. The current summary is therefore

$$((a,b)_1,(c,d)_0)_1$$
 $((a,b)_0,(c,d)_1)_1$ $((a,b)_1,(c,d)_1)_1.$

The remaining step on the short path, the associativity witness for $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = \mathbf{A} \times (\mathbf{B} \times (\mathbf{C} \times \mathbf{D}))$ is $\alpha_{\mathbf{A}, \mathbf{B}, \mathbf{C} \times \mathbf{D}}^{\times}$. The first form is in Case 2, so we get $(a, (b, (c, d)_0)_1)_1$. The second and third forms are in Case 5, so our choice of the identity bijection yields $(a, (b, (c, d)_1)_0)_1$ for the second form and $(a, (b, (c, d)_1)_1)_1$ for the third. The final summary for the short path is

$$(a, (b, (c, d)_0)_1)_1$$
 $(a, (b, (c, d)_1)_0)_1$ $(a, (b, (c, d)_1)_1)_1$.

This is not the same as the summary for the long path. The third form yielded the same result for both ways around the pentagon, but the results for the first and second forms have been interchanged. This means that our choice of the identity bijection in Case 5 cannot be correct; it fails to satisfy the pentagon condition.

Would the alternative choice in Case 5, the switching bijection, fare better? We could repeat the entire computation above using the switching bijection in place of the identity, but there is a more efficient method. Instead of repeating the calculation, we merely keep track of the changes. Every time Case 5 occurred in the preceding calcuation, we must now switch its two outcomes. Around the long path, the bottom of the pentagon, we have three switches, which when composed just interchange the first and second forms in the final summary, producing

$$(a, (b, (c, d)_0)_1)_1$$
 $(a, (b, (c, d)_1)_0)_1$ $(a, (b, (c, d)_1)_1)_1$.

Around the short side, we get two switches, which when composed give a 3-cycle permutation of the final summary, producing

$$(a, (b, (c, d)_1)_0)_1$$
 $(a, (b, (c, d)_1)_1)_1$ $(a, (b, (c, d)_0)_1)_1$.

The long and short paths still don't agree; in fact none of the three forms produce the same results for both paths. So the "switching" choice in Case 5 doesn't work either; it violates the pentagon condition.

4.11. Associativity and Commutativity of Multiplication. The preceding computations show that we cannot insist upon the Tag Manipulation principle for the associativity witnesses for multiplication.

Nor can we expect the Tag Manipulation principle to hold for the commutativity witnesses for multiplication. Indeed, for the same Fibonacci example used above, once we compute the associativity isomorphisms α^{\times} in Section 5, we shall find that they and the hexagon conditions require the commutativity isomorphisms γ^{\times} to violate the Tag Manipulation principle.

This situation is the source of the complexity of finding anyon models with prescribed fusion rules. Given the fusion rules, we have produced all of the structure of a unitary fusion rig except for α^{\times} and γ^{\times} straightforwardly, according to the Tag Invariance and Tag Manipulation principles.

The task of producing suitable α^{\times} and γ^{\times} looks daunting, first because we must define suitable $\alpha^{\times}_{\mathbf{A},\mathbf{B},\mathbf{C}}$ and $\gamma^{\times}_{\mathbf{A},\mathbf{B}}$ for all of the infinitely many raw elements $\mathbf{A},\mathbf{B},\mathbf{C}$ and second because we must satisfy several coherence conditions. Recall that we found, in Section 4.8, that many coherence conditions will automatically hold, but several, listed at the end of that subsection, remain as constraints on α^{\times} and γ^{\times} .

Fortunately, some of the coherence conditions work in our favor, reducing the number of instances $\alpha_{\mathbf{A},\mathbf{B},\mathbf{C}}^{\times}$ and $\gamma_{\mathbf{A},\mathbf{B}}^{\times}$ that we need in order to determine all the other instances. Specifically, we can obtain the general witnesses $\alpha_{\mathbf{A},\mathbf{B},\mathbf{C}}^{\times}$ and $\gamma_{\mathbf{A},\mathbf{B}}^{\times}$ from the special case where \mathbf{A} , \mathbf{B} , and \mathbf{C} are among the q+1 elements \mathbf{x}_i , as follows.

Let us write (adopting Laplaza's notation in [11]) $\delta^{\#}$ for the right distributivity witness

$$\delta_{\mathbf{A} \mathbf{B} \mathbf{C}}^{\#} \vdash (\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{B}\mathbf{A} + \mathbf{C}\mathbf{A}$$

as defined earlier using the Tag Manipulation principle. Then we use $\delta^{\#}$ and the left distributivity witness δ to reduce commutativity witnesses with sums as subscripts, by setting

(4)
$$\gamma_{\mathbf{B}+\mathbf{C},\mathbf{A}}^{\times} = \delta_{\mathbf{A},\mathbf{B},\mathbf{C}}^{\#} * (\gamma_{\mathbf{B},\mathbf{A}}^{\times} + \gamma_{\mathbf{C},\mathbf{A}}^{\times}) * (\delta_{\mathbf{A},\mathbf{B},\mathbf{C}})^{-1}$$
$$\gamma_{\mathbf{A},\mathbf{B}+\mathbf{C}}^{\times} = \delta_{\mathbf{A},\mathbf{B},\mathbf{C}} * (\gamma_{\mathbf{A},\mathbf{B}}^{\times} + \gamma_{\mathbf{A},\mathbf{C}}^{\times}) * (\delta_{\mathbf{A},\mathbf{B},\mathbf{C}}^{\#})^{-1}$$

Because every raw element is equal (=, not necessarily \equiv) to a sum of \mathbf{x}_i 's and because witnesses must respect witnessed equalities, these formulas (4) allow us to represent all the infinitely many commutativity witnesses $\gamma_{\mathbf{A},\mathbf{B}}^{\times}$ that we need to define in terms of just finitely many (at most $(q+1)^2$) of them, namely those where the subscripts are \mathbf{x}_i 's.

There is another useful way to view the equations (4). In the preceding paragraph, we worked on the basis that both δ and $\delta^{\#}$ are given (by Tag Manipulation), and these equations serve to reduce the task of defining γ^{\times} . Let us now return to the viewpoint of Section 3, namely

that $\delta^{\#}$ is not primitive but rather defined from δ and γ^{\times} . Specifically, we had, in Remark 25, two formulas providing such definitions, and the coherence condition in the first part of Figure 9 said precisely that these two definitions agree. Notice now that those two definitions for $\delta^{\#}$ are equivalent to equations (4). This has several pleasant consequences.

First, equations (4) are not arbitrary, nor are they produced merely by the desire to simplify our task of defining γ^{\times} . They are forced upon us if we want to have both a prescribed $\delta^{\#}$ (from Tag Manipulation) and the definitions of $\delta^{\#}$ in Remark 25. In effect, we need no longer worry about the two viewpoints espoused in the two preceding paragraphs; the two agree.

Second, since the two equations in (4) give us both of the proposed definitions of $\delta^{\#}$ in Remark 25, we get that those two definitions agree. That is, we get that the coherence condition in the first part of Figure 9 holds. (Recall that we already had the other part of Figure 9, because it involves witnesses for 0 = 0.)

The task of defining γ^{\times} can be reduced a bit more. When one of the subscripts is 1 (i.e., \mathbf{x}_0), the commutativity witness is determined by Proposition 2.1 of [8], a consequence of the braided monoidal coherence conditions (Figures 5–8):

$$\gamma_{A,1}^\times = \rho_A^\times * (\lambda_A^\times)^{-1} \quad \text{and} \quad \gamma_{1,A}^\times = \lambda_A^\times * (\rho_A^\times)^{-1}.$$

Thus, to define γ^{\times} , it suffices to define $\gamma^{\times}_{\mathbf{x}_i,\mathbf{x}_j}$ with $i,j=1,2,\ldots,q$.

Similar simplifications are possible for the multiplicative associativity witnesses α^{\times} . Whenever one of the subscripts of α^{\times} is a sum, we can express that witness in terms of α^{\times} with the individual summands as subscripts together with suitable distributivity witnesses δ . Figure 13 gives this information when the sum is in the third subscript:

$$\alpha_{A,B,C+D}^{\times} = \delta_{AB,C,D} * (\alpha_{A,B,C}^{\times} + \alpha_{A,B,D}^{\times}) * (\delta_{A,BC,BD})^{-1} * (1_A \times \delta_{B,C,D})^{-1}.$$

Analogous information with the sum in the first or second subscript was deduced from our coherence conditions in [3]. We reproduce here Figures 20 and 24 from that paper:

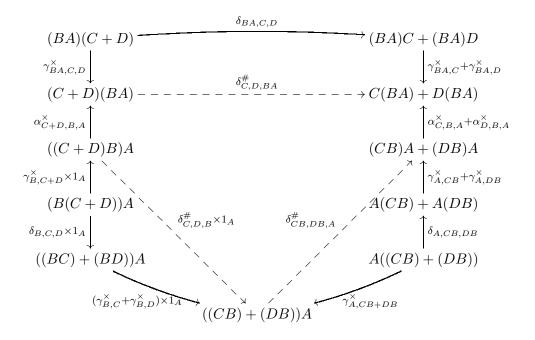


FIGURE 19. Laplaza Cond. VII

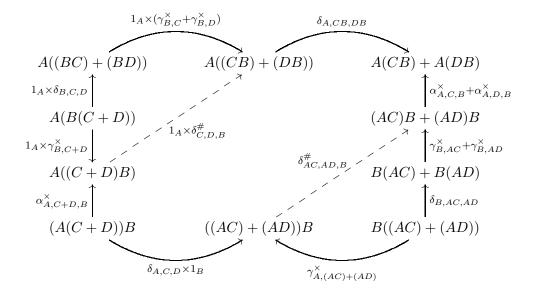


FIGURE 20. Laplaza Cond. VIII

(As indicated in the captions, these results are two of Laplaza's coherence conditions in [11].) As before, the dashed arrows represent right

distributivity witnesses $\delta^{\#}$. The inner parts of these two figures, between the dashed lines, give the desired simplifications of α^{\times} whenever one of the subscripts is a sum. So we need only define α^{\times} when its subscripts are among the \mathbf{x}_i 's.

We can also eliminate the case where one of the subscripts is $\mathbf{x}_0 = 1$. Figure 8 handles the case where the second subscript is 1:

$$\alpha_{A,1,B}^{\times} = (\rho_A^{\times} \times 1_B) * (1_A \times \lambda_B^{\times})^{-1}.$$

The cases where the first or third subscript of α^{\times} is 1 are handled by Proposition 1.1 of [8] (a consequence of the coherence conditions for braided monoidal structure, Figures 5–8)):

$$\alpha_{1,A,B}^{\times} = (\lambda_A^{\times} \times 1_B) * (\lambda_{A \times B}^{\times})^{-1}$$

$$\alpha_{A,B,1}^{\times} = (\rho_{A \times B}^{\times}) * (1_A \times \rho_B^{\times})^{-1}.$$

So α^{\times} needs to be defined only when the subscripts are among the \mathbf{x}_i 's for $1 \leq i \leq q$.

4.12. **Summary.** We have explcitly defined most of the structure of a unitary fusion rig (based on given fusion rules). Specifically, we have defined the raw elements, the witnesses, the operations on witnesses that provide a witness frame, the binary operations + and + and nullary operations 0 and 1 on raw elements and on witnesses, and all of the rig witnesses except α^{\times} and γ^{\times} . Even these two exceptional cases have been reduced to the cases where the subscripts are among the generators other than 1, by using the formulas above for the cases where a subscript is a sum or is 1. (The degenerate sum 0 of no summands is also covered, because any α^{\times} or γ^{\times} with a 0 subscript is a witness for 0=0 and is therefore uniquely determined.) Furthermore, these definitions and reductions ensure that all of the coherence conditions except possibly Figures 5, 6, and 7 are satisfied.

This completes our description of unitary fusion rigs in general. Specific unitary fusion rigs, for specific fusion rules, are determined by witnesses $\alpha_{\mathbf{x}_i,\mathbf{x}_j,\mathbf{x}_k}^{\times}$ and $\gamma_{\mathbf{x}_i,\mathbf{x}_j}^{\times}$ with i,j,k ranging from 1 to q. Since we are dealing with finite tuples of finite-dimensional Hilbert spaces, the data needed to specify a unitary fusion rig are just finitely many matrices of complex numbers. (The matrices for α^{\times} and γ^{\times} are often called F-matrices and braiding matrices.) Unitary fusion rigs correspond to such systems of matrices, subject to the requirements that the pentagon condition (Figure 5) and the two hexagon conditions (Figures 6 and 7) must be satisfied.

For a given set of fusion rules, these coherence conditions may have several solutions, or just one, or none. Accordingly, there may be several unitary fusion rigs, or just one, or none for those fusion rules.

In the next section, we review the calculation, done in detail in [2], for one particular fusion rule. The results for several other fusion rules are given in Bonderson's thesis [4], along with indications of how to do some such computations more efficiently on a computer.

5. Fibonacci Anyons

In this section, we show how a particular anyon model, Fibonacci anyons, looks in our witness algebra framework. In particular, we show how the new framework accommodates the computations in [2] of the associativity and braiding operators of this model.

The Fibonacci anyon model, already discussed in Section 4.10, has only two types, i.e., two generators of the true algebra, the multiplicative identity 1 (corresponding physically to the vacuum) and the anyon type τ . The fusion rules are $\tau \times \tau = \tau + 1$ and the rules saying that 1 is a two-sided identity element.

Applying the framework of unitary fusion rigs from the preceding section, we have a raw algebra consisting of pairs of Hilbert spaces, equipped with specified bases. We prefer to write these pairs as (H_1, H_τ) rather than (H_0, H_1) because the types, rather than their indices, are more informative subscripts. Witnesses are pairs of unitary transformations. Addition is componentwise direct sum, and multiplication is given by the sum-of-tensor-products formula as above.

We intend to set up the computation of associativity and commutativity witnesses in the unitary fusion semiring for Fibonacci anyons. As noted earlier, these witnesses in any fusion semiring are completely determined by a rather small number of them, namely those with types other than 1 as subscripts. In the case of Fibonacci anyons, this means that the fusion semiring structure will be completely determined if we can compute two witnesses, $\alpha_{\tau,\tau,\tau}^{\times}$ and $\gamma_{\tau,\tau}^{\times}$. These witnesses are constrained by the pentagon and hexagon conditions from the definition of braid semirings, Figures 5–7.

Before proceeding to set up these computations, we simplify the notation for the relevant basis elements, not only for the sake of simplicity but also to match the notation that we used in [2].

The raw element 1 (also known as \mathbf{x}_0) is the pair of Hilbert spaces $(\mathbb{C},0)$, and the specified basis of the first component consists of the complex number 1. We do not attempt to simplify this.

The raw element τ (also known as \mathbf{x}_1) is $(0, \mathbb{C})$, and the specified basis of the second component again consists of the complex number 1. It will be convenient to give this basis element a different name, so that the name tells us where the basis vector came from. We give it the name τ . Thus, τ denotes the unique basis vector in the raw element τ , just as, in the preceding paragraph, 1 was the unique basis vector in the raw element 1.

We shall rename some more complicated basis elements, in iterated products of τ 's, but we shall do so in a way that always allows us to read off, from a basis element, the raw element that it is in, and indeed the specific component of that raw element that it is in.

In fact, a suitable system of simplified names for basis elements in iterated products of τ , can be produced by a remarkably simple scheme. Begin with the basis vectors for the trivial products 1 and τ as described above. For non-trivial products $x \times y$, we have, from the preceding section, the official notation $(a, b, i, j, 1)_k$, where k tells us which component of $x \times y$ this basis vector is in, where i and j tell us which components of the factors x and y the basis vectors a and b are taken from, and where the final 1 tells us nothing because Fibonacci anyons have all $N_{i,j}^k \leq 1$. Instead of this official notation, we shall use

$$(a \cdot_k b),$$

and we shall usually write 1 or τ under the dot, instead of 0 or 1, respectively. This shorter notation works because the additional information i, j in the official notation and the sources x, y can all be read off from a and b.

Example 34. The simplest non-trivial product, $\tau \times \tau$, is a pair of 1-dimensional Hilbert spaces. The specified basis vector in the 1-component (the first Hilbert space in the pair) is officially $(\tau, \tau, 1, 1, 1)_0$, and our simplified notation for it is (τ, τ) .

Similarly, the specified basis vector in the τ -component (the second Hilbert space in the pair) is officially $(\tau, \tau, 1, 1, 1)_1$, and our simplified notation for it is $(\tau \cdot \tau)$.

In $(\tau \times \tau) \times \tau$, the first component is a one-dimensional Hilbert space with specified basis vector $((\tau \cdot \tau)_{1} \cdot \tau)$. The second component is two-dimensional with specified basis vectors $((\tau \cdot \tau)_{\tau} \cdot \tau)$ and $((\tau \cdot \tau)_{\tau} \cdot \tau)$.

Recalling that multiplication in the raw algebra is not generally asociative, we also consider $\tau \times (\tau \times \tau)$. Its components have the same dimensions as in the preceding paragraph, but the specified bases are potentially different. The first component has the basis vector $(\tau \cdot (\tau \cdot \tau))$. The second component has the basis vectors $(\tau \cdot (\tau \cdot \tau))$ and $(\tau \cdot (\tau \cdot \tau))$.

Note that the parentheses in the simplified notation are in the same places as the parentheses in the raw element $(\tau \times \tau) \times \tau$ or $\tau \times (\tau \times \tau)$.

Continuing as in the example, we have simplified notations for all the basis vectors in iterated products of τ 's.

We are now in a position to discuss the associativity witness $\alpha_{\tau,\tau,\tau}^{\times}$. It consists of two unitary transformations, one between the one-dimensional first components of $(\tau \times \tau) \times \tau$ and $\tau \times (\tau \times \tau)$, and one between their two-dimensional second components.

For purposes of calculation, we want to exhibit these unitary transformations as unitary matrices. Exhibiting these matrices on the page requires choosing an ordering of the vectors within each of the specified bases for the two-dimensional spaces. We arbitrarily order them in the order in which they were mentioned above. Now the two components of $\alpha_{\tau,\tau,\tau}^{\times}$ are a unitary 1×1 matrix (i.e., a complex number of absolute value 1) and a 2×2 unitary matrix, say

$$(p)$$
 and $\begin{pmatrix} q & r \\ s & t \end{pmatrix}$.

In detail, this means that

$$\begin{split} &\alpha_{\tau,\tau,\tau}^{\times}((\tau \cdot \tau)_{1} \cdot \tau) = p(\tau_{1} \cdot (\tau \cdot \tau)) \\ &\alpha_{\tau,\tau,\tau}^{\times}((\tau_{1} \cdot \tau)_{\tau} \cdot \tau) = q(\tau_{\tau} \cdot (\tau_{1} \cdot \tau)) + r(\tau_{\tau} \cdot (\tau_{\tau} \cdot \tau)) \\ &\alpha_{\tau,\tau,\tau}^{\times}((\tau \cdot \tau)_{\tau} \cdot \tau) = s(\tau_{\tau} \cdot (\tau_{1} \cdot \tau)) + t(\tau_{\tau} \cdot (\tau_{\tau} \cdot \tau)). \end{split}$$

These equations match those in [2, Section 5.4] except that we have here written the associativity witness $\alpha_{\tau,\tau,\tau}^{\times}$ explicitly whereas in [2] the corresponding isomorphism was used to identify the corresponding components of $(\tau \times \tau) \times \tau$ and $\tau \times (\tau \times \tau)$ (see footnote 8 of [2]).

The commutativity witnesses are similar but simpler. Since both components of $\tau \times \tau$ are one-dimensional, $\gamma_{\tau,\tau}^{\times}$ is just a pair of 1×1 unitary matrices, i.e., a pair of complex numbers a,b of absolute value 1. In detail,

$$\gamma_{\tau,\tau}^{\times}(\tau_{1},\tau) = a(\tau_{1},\tau)$$
$$\gamma_{\tau,\tau}^{\times}(\tau_{2},\tau) = b(\tau_{2},\tau).$$

Again, these equations match those in [2, Section 5.5].

With this notation in place, and remembering that symmetry and transitivity operations on witnesses are given by inversion and composition of unitary transformations, it is easy to write out the pentagon and hexagon conditions as equations for the matrix entries p, q, r, s, t, a, b. Since the computations are done in detail in [2], we do not repeat them here but merely record that the pentagon condition reads

$$\begin{pmatrix} rs & q & rt \\ q & 0 & r \\ st & s & t^2 \end{pmatrix} = \begin{pmatrix} p^2q & prs & prt \\ prs & q^2 + rst & qr + rt^2 \\ pst & qs + st^2 & rs + t^3 \end{pmatrix}$$

for the τ component and

$$\begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} q^2 + prs & qr + ptr \\ qs + pts & rs + pt^2 \end{pmatrix}$$

for the 1 component.

It follows from these equations (in fact, from just the τ component — the 1 component is redundant) and the unitarity of the witness matrices that

$$p = 1,$$
 $q = -t = \frac{-1 + \sqrt{5}}{2},$ $r = \sqrt{q}e^{i\theta},$ $s = \sqrt{q}e^{-i\theta}$

for some real θ . If we modify the standard basis vectors by suitable phase factors, we can simplify the result by getting rid of θ , so $r = s = \sqrt{q}$.

The hexagon condition reads, if we take into account that p=1,

$$\begin{pmatrix} q^2 + brs & (q+bt)r \\ (q+bt)s & rs+bt^2 \end{pmatrix} = \begin{pmatrix} b^4q & b^3r \\ b^3s & b^2t \end{pmatrix}$$

for the τ component and

$$a = b^2$$

for the 1 component.

These equations and unitarity yield that, except for possible complex conjugation of both a and b,

$$b = \frac{-1 + \sqrt{q^2 - 4}}{2} = e^{3\pi i/5}, \qquad a = b^2 = e^{6\pi i/5}.$$

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