

# NILPOTENT TYPES AND FRACTURE SQUARES IN HOMOTOPY TYPE THEORY

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ABSTRACT. We develop the basic theory of nilpotent types and their localizations away from sets of numbers in Homotopy Type Theory. For this, general results about the classifying spaces of fibrations with fiber an Eilenberg–Mac Lane space are proven. We also construct fracture squares for localizations away from sets of numbers. All of our proofs are constructive.

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## 1. INTRODUCTION

Nilpotent spaces play a very important role in the homotopy theory of spaces. Many results that hold for simply connected spaces can be generalized to nilpotent spaces. For example, any cohomology isomorphism between nilpotent spaces is a weak equivalence. Nilpotent spaces also have a rich theory of localizations away from sets of numbers, including fracture squares that reconstruct a space out of some of its localizations.

Nilpotent spaces are characterized by the fact that the maps in their Postnikov tower factor as finite composites of principal fibrations. Nevertheless, they are usually defined in terms of some algebraic structure on the actions of their fundamental group on their homotopy groups. Both characterizations are useful, and one of the basic theorems in the theory is the fact that the characterizations are equivalent.

In this paper we develop the basic theory of nilpotent spaces in Homotopy Type Theory. In Section 2 we prove the equivalence between the two characterizing properties of nilpotency (Theorem 2.60). In order to do this, we study the relationship between unpointed Eilenberg–Mac Lane spaces and doubly-pointed Eilenberg–Mac Lane spaces, and prove that the type of unpointed  $n$ -dimensional Eilenberg–Mac Lane spaces is equivalent to the type of doubly-pointed  $(n + 1)$ -dimensional Eilenberg–Mac Lane spaces (Theorem 2.25). In Section 3 we follow the suggestion in (Shulman; 2014), and we prove that cohomology isomorphisms between nilpotent types induce isomorphisms in all homotopy groups. In Section 4 we study the localization of nilpotent types and its effect on homotopy groups. The main result of this section is that localization of a nilpotent type localizes its homotopy groups in the expected way (Theorem 4.19). Finally, in Section 5 we construct fracture squares for localizations away from sets of numbers. In particular, we construct a fracture square for simply connected types that avoids assuming Whitehead’s principle (Theorem 5.4). This construction is very different from the classical one that can be found in, e.g., (May and Ponto; 2012). For nilpotent types we only construct a square in the case the type is truncated. The last two sections build on the results of (Rijke et al.; 2017) and (Christensen et al.; 2018).

The proofs of the main results in this paper are different from the classical proofs (see the treatment of nilpotent spaces and their localizations in (May and Ponto; 2012), (Bousfield and Kan; 1972), (Hilton et al.; 1975)). As we work in the type theory described in (Univalent Foundations Program; 2013), avoiding the use of Whitehead’s theorem (Univalent Foundations Program; 2013, Section 8.8), we get constructive proofs that work in any  $\infty$ -topos (Kapulkin and Lumsdaine; 2018), (Lumsdaine and Shulman; n.d.), (Shulman; 2019), assuming the initiality conjecture for Homotopy Type Theory. What makes the theory of nilpotent types work in this setting is Theorem 2.25, which gives a hands-on characterization of  $K(A, n)$ -bundles, and a simple definition of the action  $\pi_1(X) \curvearrowright A$  associated to a pointed  $K(A, n)$ -bundle  $Y \rightarrow X$ . This is inspired by (Shulman; 2014), which discusses the advantages of working directly with the classifying spaces of  $K(A, n)$ -bundles.

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**1.1. Conventions and Background.** We work in Homotopy Type Theory as described in (Univalent Foundations Program; 2013). That is, intensional Martin–Löf dependent type theory together with the univalence axiom and higher inductive types.

We use the existence and universal property of Eilenberg–Mac Lane spaces (Licata and Finster; 2014), and the existence of localizations at families of maps (Rijke et al.; 2017), as well as many of their properties in the special case of localizations at self-maps of the circle (Christensen et al.; 2018).

## 2. NILPOTENT TYPES

In this section we develop the basic theory of nilpotent types. The main result is Theorem 2.60, which characterizes nilpotent spaces in terms of certain factorizations of the maps in their Postnikov towers. Although the main result is known classically, most of the proofs we give are quite different. In particular, we give a description of fibrations

with fiber an Eilenberg–Mac Lane space (Corollary 2.28), which allows us to prove a generalization of the classification of nilpotent types (Theorem 2.59).

In Section 2.1 we show that the theory of groups and group actions can be seen as a particular case of the theory of spaces and fibrations. In Section 2.2 we study  $K(A, n)$ -bundles, by studying their classifying space. We prove a general stabilization result about these classifying spaces (Theorem 2.25) which allows us to give a workable description of  $K(A, n)$ -bundles. In Section 2.3 we discuss principal  $K(A, n)$ -bundles, and give a characterization in terms of group actions. Finally, in Section 2.4 and Section 2.5, we introduce nilpotent groups and types and characterize them in terms of group actions.

**2.1. Homotopical interpretation of group theory.** We start by setting up some basic notation and results about group theory and its interpretation as a particular case of the theory of Eilenberg–Mac Lane spaces. We will rely on several results in (Buchholtz et al.; 2018).

Throughout the section,  $G$  and  $H$  will denote arbitrary groups. As is usual in Homotopy Type Theory, we assume that the underlying type of a group is a set. The type of maps between groups  $G$  and  $H$  will be denoted by  $G \rightarrow_{\text{Grp}} H$ .

The type  $K(G, n)$  will denote the Eilenberg–Mac Lane space of type  $G, n$  as defined in (Licata and Finster; 2014). If  $n > 1$ , when we write  $K(G, n)$  we will be assuming that  $G$  is abelian. When regarded as a pointed type, the pointing of  $K(G, n)$  will be denoted by  $*$  :  $K(G, n)$ , and the inclusion of the point by  $\iota : \mathbf{1} \rightarrow K(G, n)$ . When regarded as a doubly-pointed type, we will use the point  $*$  :  $K(G, n)$  twice. The first building block in the homotopical interpretation of group theory is the following result.

**Theorem 2.1** (see (Buchholtz et al.; 2018, Theorem 5.1)). *The map  $\Omega^n : \mathcal{U}_\bullet \rightarrow \mathcal{U}_\bullet$  restricts to an equivalence of categories between pointed, connected, 1-truncated types and groups, when  $n = 1$ ; and between pointed,  $(n - 1)$ -connected,  $n$ -truncated types and abelian groups, when  $n \geq 2$ .*

**Definition 2.2.** *Let  $G$  and  $H$  be groups and let  $\text{Aut}(H)$  be the group of group automorphisms of  $H$ . An **action** of  $G$  on  $H$  is a group morphism  $G \rightarrow_{\text{Grp}} \text{Aut}(H)$ . The type of actions of  $G$  on  $H$  is denoted by  $G \curvearrowright H$ .*

**Definition 2.3.** *The **trivial action**, denoted by  $\mathbf{1} : G \curvearrowright H$ , is the trivial group homomorphism  $G \rightarrow_{\text{Grp}} \text{Aut}(H)$  given by  $g \mapsto \text{id}_H$ . We say that an action is trivial if it is equal to the trivial action.*

**Definition 2.4.** *Given groups  $G, H, H'$ , and actions  $\alpha : G \curvearrowright H, \alpha' : G \curvearrowright H'$ , a **map of actions**  $\phi : \alpha \rightarrow_{\text{act}} \alpha'$  is given by a group morphism  $\phi : H \rightarrow_{\text{Grp}} H'$  such that for every  $g : G$  and every  $h : H$ , we have  $\phi(\alpha(g, h)) = \alpha'(g, \phi(h))$ . We say that such a map is a **subaction** if its underlying group morphism is a monomorphism. Similarly, a map of actions is **surjective** if its underlying group morphism is an epimorphism.*

Notice that for a group morphism  $\phi : H \rightarrow_{\text{Grp}} H'$ , being a map of actions is a mere property, since respecting the action is specified by a family of equalities in a set.

The following lemma follows from Theorem 2.1.

**Lemma 2.5.** *Let  $X$  be a pointed, connected type. Then, using the identity morphism  $\pi_1(X) \rightarrow_{\text{Grp}} \pi_1(X)$  in the induction principle of  $K(\pi_1(X), 1)$  (Licata and Finster; 2014, Section 3.1) we get a pointed map  $K(\pi_1(X), 1) \rightarrow_\bullet \|X\|_1$ . This map is an equivalence.  $\square$*

**Definition 2.6.** Given a natural number  $n \geq 1$ , and a group  $G$  (assumed to be abelian if  $n > 1$ ), define the **type of pointed Eilenberg–Mac Lane spaces** of type  $G$ ,  $n$  as follows:

$$\mathbf{EM}_\bullet(G, n) := \sum_{(K: \mathcal{U}_\bullet)} \left\| K =_{\mathcal{U}_\bullet} K(G, n) \right\|_{-1}.$$

The type  $\mathbf{EM}_\bullet(G, n)$  classifies group actions on  $G$ , in the following sense.

**Lemma 2.7.** Given  $n \geq 1$  and  $K, K' : \mathbf{EM}_\bullet(G, n)$ , the map  $(K =_{\mathcal{U}_\bullet} K') \rightarrow (\Omega^n K =_{\mathbf{Grp}} \Omega^n K')$  given by the functoriality of  $\Omega$  is an equivalence. This equivalence provides us with an equivalence  $\mathbf{EM}_\bullet(G, n) \simeq K(\mathbf{Aut}(G), 1)$  since  $\mathbf{EM}_\bullet(G, n)$  is pointed and connected, and its loop space is  $K(G, n) =_{\mathcal{U}_\bullet} K(G, n)$ .

*Proof.* The first statement follows from Theorem 2.1. The second statement follows from the first one.  $\square$

**Lemma 2.8.** Given a pointed, connected type  $X$ , an integer  $n \geq 1$ , and a group  $G$ , we have a map

$$(X \rightarrow_\bullet \mathbf{EM}_\bullet(G, n)) \rightarrow (\pi_1(X) \curvearrowright G)$$

given by functoriality of  $\pi_1$  and the fact that  $\mathbf{EM}_\bullet(G, n) \simeq K(\mathbf{Aut}(G), 1)$ . This map is an equivalence. Moreover, an action  $\alpha : \pi_1(X) \curvearrowright G$  is trivial precisely when its corresponding map  $\alpha : X \rightarrow \mathbf{EM}_\bullet(G, n)$  is null-homotopic.

*Proof.* Using the fact that  $\|X\|_1 \simeq K(\pi_1(X), 1)$ , together with Lemma 2.7, we get the following chain of equivalences:

$$\begin{aligned} (X \rightarrow_\bullet \mathbf{EM}_\bullet(G, n)) &\simeq (\|X\|_1 \rightarrow_\bullet \mathbf{EM}_\bullet(G, n)) \\ &\simeq (\pi_1(X) \rightarrow_{\mathbf{Grp}} \mathbf{Aut}(G)) \\ &\equiv (\pi_1(X) \curvearrowright G), \end{aligned}$$

where the second equivalence follows from Theorem 2.1.

The second claim follows from the fact that the constant map  $X \rightarrow \mathbf{EM}_\bullet(G, n)$  induces the trivial action.  $\square$

Whenever  $X$  is pointed and connected, and we are given an action  $\alpha : \pi_1(X) \curvearrowright G$ , we denote its associated map of type  $X \rightarrow_\bullet \mathbf{EM}_\bullet(G, n)$  by  $\hat{\alpha}$ . When there is no risk of confusion, we will abuse notation and denote both with the same symbol.

This characterization is functorial, in the following sense.

**Lemma 2.9.** Given a pointed, connected type  $X$ , an integer  $n \geq 1$ , groups  $H, H'$ , and actions  $\alpha : \pi_1(X) \curvearrowright H$  and  $\alpha' : \pi_1(X) \curvearrowright H'$ , there is an equivalence between the type of maps of actions  $\alpha \rightarrow_{\mathbf{act}} \alpha'$ , and the type of fiberwise pointed maps  $\prod_{(x: X)} \hat{\alpha}(x) \rightarrow_\bullet \hat{\alpha}'(x)$ .

*Proof.* The maps  $\hat{\alpha} : X \rightarrow_\bullet \mathbf{EM}_\bullet(H, n)$  and  $\hat{\alpha}' : X \rightarrow_\bullet \mathbf{EM}_\bullet(H', n)$  factor through the truncation  $X \rightarrow \|X\|_1$ , since both  $\mathbf{EM}_\bullet(H, n)$  and  $\mathbf{EM}_\bullet(H', n)$  are 1-truncated. Let us refer to these extensions as  $\beta$  and  $\beta'$  respectively. Moreover, for every  $x : X$ , the type  $\hat{\alpha}(x) \rightarrow_\bullet \hat{\alpha}'(x)$  is 1-truncated, by Theorem 2.1, so  $\prod_{(x: X)} \hat{\alpha}(x) \rightarrow_\bullet \hat{\alpha}'(x)$  is equivalent to  $\prod_{(x: \|X\|_1)} \beta(x) \rightarrow_\bullet \beta'(x)$ . Now, by Lemma 2.5, it is enough to provide an equivalence

$$\left( \prod_{(x: X')} \hat{\alpha}(x) \rightarrow_\bullet \hat{\alpha}'(x) \right) \simeq (\alpha \rightarrow_{\mathbf{act}} \alpha')$$

where we have implicitly used the equivalence  $X \simeq X' \equiv K(\pi_1(X), 1)$  of the lemma.

This reduction lets us use the induction principle of  $K(\pi_1(X), 1)$  (Licata and Finster; 2014, Section 3.1). In this case, the principle tells us that the type  $\prod_{(x:X')} \hat{\alpha}(x) \rightarrow_{\bullet} \hat{\alpha}'(x)$  is equivalent to the type of maps  $\phi : \hat{\alpha}(\ast) \rightarrow_{\bullet} \hat{\alpha}'(\ast)$  together with a proof that for every  $l : \ast = \ast$ , we have  $\text{transport}^{\hat{\alpha}(-) \rightarrow_{\bullet} \hat{\alpha}'(-)}(l, \phi) = \phi$ . Now, by path-induction, this last type is equivalent to the type of proofs that the following square is pointed commutative:

$$\begin{array}{ccc} \hat{\alpha}(\ast) & \xrightarrow{\phi} & \hat{\alpha}'(\ast) \\ \text{transport}^{\hat{\alpha}}(l, -) \downarrow & & \downarrow \text{transport}^{\hat{\alpha}'}(l, -) \\ \hat{\alpha}(\ast) & \xrightarrow{\phi} & \hat{\alpha}'(\ast). \end{array}$$

Recall that the map  $\phi : \hat{\alpha}(\ast) \rightarrow_{\bullet} \hat{\alpha}'(\ast)$  corresponds to a group morphism  $\psi : H \rightarrow_{\text{Grp}} H'$ . Then, Theorem 2.1 implies that the above square is pointed commutative exactly when the square of group morphisms

$$\begin{array}{ccc} H & \xrightarrow{\psi} & H' \\ \alpha(|l|) \downarrow & & \downarrow \alpha'(|l|) \\ H & \xrightarrow{\psi} & H' \end{array}$$

is commutative. So the type  $\prod_{(x:X')} \hat{\alpha}(x) \rightarrow_{\bullet} \hat{\alpha}'(x)$  is equivalent to the type of group morphisms  $H \rightarrow_{\text{Grp}} H'$  that respect the actions, as needed.  $\square$

**Definition 2.10.** A map  $i : H \rightarrow_{\text{Grp}} H'$  is a **normal inclusion** if it is injective and whenever we have  $h : H$ ,  $h' : H'$ , the type  $\text{fib}_i(h'i(h)h'^{-1})$  is inhabited. We denote the type of normal inclusions of  $H$  into  $H'$  by  $H \triangleleft H'$ .

Notice that being a normal inclusion is a mere property of a group morphism.

*Example 2.11.* Any inclusion with codomain an abelian group is a normal inclusion.

Using quotients it is easy to construct the cokernel, or quotient, of a normal inclusion. We will not give the details here, since the construction is analogous to the set theoretic construction.

**Definition 2.12.** Given a normal inclusion  $i : H \rightarrow_{\text{Grp}} H'$  we denote its **cokernel**, or **quotient**, by  $\text{coker}(i) : \text{Grp}$ , or  $H'/H : \text{Grp}$ .

**Definition 2.13.** Given an action  $\alpha' : G \curvearrowright H'$ , a **normal subaction** is given by an action  $\alpha : G \curvearrowright H$ , and a map of actions  $\phi : \alpha \rightarrow_{\text{act}} \alpha'$  such that its corresponding group morphism  $H \rightarrow_{\text{Grp}} H'$  is a normal inclusion. The type  $\alpha \triangleleft_{\text{act}} \alpha'$  denotes the type of maps of actions  $\alpha \rightarrow_{\text{act}} \alpha'$ , together with a proof that the underlying morphism is a normal inclusion.

In the context of the above definition, the usual set theoretic construction provides, for each normal subaction  $\phi : \alpha \rightarrow_{\text{act}} \alpha'$ , a quotient action  $\alpha'/\alpha : G \curvearrowright H'/H$ .

**Definition 2.14.** For a group morphism  $G \xrightarrow{q} H$ , let the **kernel** of  $q$  be the group morphism  $I \rightarrow_{\text{Grp}} G$  where  $I := \left( \sum_{(l:G)} q(l) = e \right)$  with the group structure inherited from  $G$ , and the morphism  $I \rightarrow_{\text{Grp}} G$  given by the first projection.

There is an induced action  $G \curvearrowright I$  defined as follows. Given  $g : G$  and  $(l, r) : I$ , let  $g \cdot (l, r) := (g^{-1}lg, \phi_r)$ , where  $\phi_r$  is the straightforward proof that  $g^{-1}lg$  gets mapped to  $e : H$  by  $q$ . It is easy to check that this action is a normal subaction of the action  $G \curvearrowright G$  given by conjugation.

**Definition 2.15.** A sequence of group morphisms  $I \rightarrow G \rightarrow H$  is **short exact** if  $G \rightarrow H$  is surjective and  $I \rightarrow G$  is equivalent to the kernel of  $G \rightarrow H$ . A short exact sequence of groups  $I \rightarrow G \rightarrow H$  is a **central extension** if the action  $G \curvearrowright I$  given in Definition 2.14 is trivial.

Note that being short exact and being a central extension are both mere properties of a sequence  $I \rightarrow G \rightarrow H$ .

As is standard in group theory, whenever we have normal inclusions  $H \triangleleft H' \triangleleft G$  such that the composite inclusion  $H \rightarrow_{\text{Grp}} G$  is normal, we can form a short exact sequence of groups  $H'/H \rightarrow G/H \rightarrow G/H'$ .

**Definition 2.16.** Let  $H$  be a group. Given a sequence of group morphisms

$$\mathbf{1} \equiv H_0 \rightarrow H_1 \rightarrow \cdots \rightarrow H_k \equiv H,$$

we say that it is a **central series** if all the morphisms are normal inclusions, all the composites  $H_i \rightarrow H$  are normal inclusions, and for all  $0 \leq i < k$ , the short exact sequence

$$H_{i+1}/H_i \rightarrow H/H_i \rightarrow H/H_{i+1}$$

is a central extension.

Note that being a central series is a mere property.

**Definition 2.17.** The type of **nilpotent structures** for a group  $H$  is defined to be the type of central series for  $H$ .

**Definition 2.18.** Given an action  $\alpha : G \curvearrowright H$  of a group on another group, a **nilpotent structure** on this action consists of a sequence of maps of actions

$$\mathbf{1} \equiv \alpha_0 \rightarrow_{\text{act}} \alpha_1 \rightarrow_{\text{act}} \cdots \rightarrow_{\text{act}} \alpha_k \equiv \alpha,$$

such that the underlying sequence of group morphisms is a central series for  $H$ , and such that, for every  $i$ , the map  $\alpha_i \rightarrow_{\text{act}} \alpha_{i+1}$  and the composite  $\alpha_i \rightarrow_{\text{act}} \alpha$  are normal subactions, and the quotient action  $\alpha_{i+1}/\alpha_i : G \curvearrowright H_{i+1}/H_i$  is trivial.

Notice that the type of nilpotent structures on a group  $G$  is equivalent to the type of nilpotent structures for the action  $G \curvearrowright G$  given by conjugation.

**Lemma 2.19.** The type of nilpotent structures on a group  $G$  is equivalent to the type of finite sequences of group epimorphisms

$$G \equiv G'_0 \twoheadrightarrow G'_1 \twoheadrightarrow \cdots \twoheadrightarrow G'_k \equiv \mathbf{1}$$

such that for every  $0 \leq i < k$ , the short exact sequence

$$K_i \rightarrow G'_i \twoheadrightarrow G'_{i+1},$$

where  $K_i$  is the kernel, is a central extension. The equivalence is given by mapping a nilpotent structure for  $G$

$$\mathbf{1} \equiv G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_k \equiv G$$

to the sequence of group epimorphisms given by setting  $G'_i := G/G_i$ , and defining the morphism  $G'_i \rightarrow_{\text{Grp}} G'_{i+1}$  to be the epimorphism  $G/G_i \twoheadrightarrow G/G_{i+1}$ .

*Proof.* It is clear that the short exact sequence

$$G_{i+1}/G_i \rightarrow G/G_i \rightarrow G/G_{i+1}.$$

is isomorphic to the short exact sequence

$$K_i \rightarrow G'_i \rightarrow G'_{i+1},$$

and thus this last short exact sequence is a central extension, as needed.

Going the other way, assume given

$$G \equiv G'_0 \twoheadrightarrow G'_1 \twoheadrightarrow \cdots \twoheadrightarrow G'_k \equiv \mathbf{1}$$

such that the short exact sequence

$$K_i \rightarrow G'_i \rightarrow G'_{i+1}$$

is a central extension for every  $i$ . Let  $G_i$  be the kernel of the map  $G \rightarrow G'_i$  obtained by composing the maps in the sequence. By functoriality of kernels, we have induced maps

$$1 \equiv G_0 \rightarrow G_1 \rightarrow \cdots \rightarrow G_k \equiv G,$$

which give rise to a filtration of  $G$ . We will prove that this filtration is a nilpotent structure for  $G$ . Since  $G_i$  is normal in  $G$ , it must be normal in  $G_{i+1}$ , so the filtration consists of normal inclusions. It remains to show that the short exact sequences

$$G_{i+1}/G_i \rightarrow G/G_i \rightarrow G/G_{i+1}$$

are central extensions. By construction, this sequence is isomorphic to

$$K_i \rightarrow G'_i \rightarrow G'_{i+1}$$

which is a central extension by hypothesis.

The fact that the constructions form an equivalence is clear.  $\square$

**2.2.  $K(-, n)$ -bundles.** The goal of this section is to give a hands-on description of fibrations with fiber an Eilenberg–Mac Lane space (Corollary 2.28) and their associated actions that will allow us to work with principal fibrations and nilpotent types. This description follows from Theorem 2.25, which, inspired by (Shulman; 2014), studies the classifying spaces of  $K(A, n)$ -bundles, and establishes the equivalence between the type of unpointed  $n$ -dimensional Eilenberg–Mac Lane spaces and the type of doubly-pointed  $(n + 1)$ -dimensional Eilenberg–Mac Lane spaces.

The main reason to introduce doubly-pointed Eilenberg–Mac Lane spaces is the following. In order to associate an action of  $\pi_1(X)$  on a group  $G$  to a  $K(G, n)$ -bundle over  $X$  in a natural way, we must construct a map from the space classifying  $K(G, n)$ -bundles to the space classifying  $\pi_1(X)$ -actions on  $G$ . To state this more concretely, we give the following definition.

**Definition 2.20.** *Given a natural number  $n \geq 1$ , and a group  $G$  (assumed to be abelian if  $n > 1$ ), define the **type of unpointed Eilenberg–Mac Lane spaces** of type  $G$ ,  $n$  as follows:*

$$\mathrm{EM}(G, n) := \sum_{K: \mathcal{U}} \|K =_U K(G, n)\|_{-1}.$$

We show in Lemma 2.22 that the type of Definition 2.20 classifies  $K(G, n)$ -bundles, so we need to construct a map  $\mathrm{EM}(G, n) \rightarrow \mathrm{EM}_\bullet(G, n)$ , since  $\mathrm{EM}_\bullet(G, n)$  classifies  $\pi_1(X)$ -actions on  $G$  (Lemma 2.7). In (Shulman; 2014), this is done by showing that the map  $\mathrm{EM}_\bullet(G, n) \rightarrow \mathrm{EM}(G, n)$  that forgets the pointing admits a retraction. The argument

uses the truncated Whitehead theorem, and thus the corresponding retraction is hard to work with. In this section we give a more explicit characterization of the retraction by showing that the map  $\mathbf{EM}_\bullet(G, n) \rightarrow \mathbf{EM}(G, n)$  is equivalent to the map  $\mathbf{EM}_\bullet(G, n+1) \rightarrow \mathbf{EM}_{\bullet\bullet}(G, n+1)$  that repeats the pointing, and as retraction we use the map  $\mathbf{EM}_{\bullet\bullet}(G, n+1) \rightarrow \mathbf{EM}_\bullet(G, n+1)$  that forgets one of the points. Here,  $\mathbf{EM}_{\bullet\bullet}(G, n) := \sum_{K:\mathcal{U}_{\bullet\bullet}} \|K = \mathcal{U}_{\bullet\bullet} K(G, n)\|_{-1}$  is the space of doubly-pointed Eilenberg–Mac Lane spaces of type  $G, n$ .

In this section,  $A$  and  $B$  will denote abelian groups, and  $G$  an arbitrary group. As usual, when writing  $K(G, n)$ , we will assume that  $G$  is abelian, if  $n > 1$ .

**Definition 2.21.** *Given  $n \geq 1$ , a  $K(G, n)$ -bundle is a map such that all of its fibers are merely equivalent to  $K(G, n)$ . Formally, for a map  $f : Y \rightarrow X$ , we let*

$$\text{is-}K(G, n)\text{-bundle}(f) := \prod_{x:X} \|\text{fib}_f(x) \simeq K(G, n)\|_{-1}.$$

A **pointed  $K(G, n)$ -bundle** is a pointed map that is also a  $K(G, n)$ -bundle, together with a proof that its fiber over the base point is equivalent to  $K(G, n)$  as a pointed type. Formally, for a pointed map  $f : Y \rightarrow_\bullet X$ , with  $x_0 : X$  the basepoint of  $X$ , we let

$$\text{is-}K(G, n)\text{-bundle}_\bullet(f) := \text{is-}K(G, n)\text{-bundle}(f) \times (\text{fib}_f(x_0) \simeq_\bullet K(G, n)).$$

Equivalently, a pointed  $K(G, n)$ -bundle is a pointed map  $f : Y \rightarrow_\bullet X$  whose underlying (unpointed) map is a  $K(G, n)$ -bundle, together with a pointed map  $K(A, n) \rightarrow_\bullet Y$ , and a proof that the following square commutes as a square of pointed maps, and is a pullback:

$$\begin{array}{ccc} K(A, n) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & X. \end{array}$$

We will write  $K(-, n)$ -bundle when we mean a  $K(G, n)$ -bundle for some group  $G$ .

The forgetful map  $F : \mathbf{EM}_\bullet(G, n) \rightarrow \mathbf{EM}(G, n)$  classifies  $K(G, n)$ -bundles and pointed  $K(G, n)$ -bundles in the following sense.

**Lemma 2.22.** *Let  $X : \mathcal{U}$ . There is an equivalence between maps  $X \rightarrow \mathbf{EM}(G, n)$  and  $K(G, n)$ -bundles with codomain  $X$ . The equivalence is given, on the one hand, by pulling back  $F$  along the map  $X \rightarrow \mathbf{EM}(G, n)$  to get a map  $Y \rightarrow X$ . And on the other hand, by sending  $f : Y \rightarrow X$  to the type family  $\text{fib}_f(-) : X \rightarrow \mathbf{EM}(G, n)$ .*

*In particular, the map*

$$\begin{aligned} Y &\rightarrow \sum_{x:X} \text{fib}_f(x) \\ y &\mapsto (f(y), (y, \text{refl})) \end{aligned}$$

*is an equivalence over  $X$ . This equivalence is natural in  $Y$ , in that, given  $K(G, n)$ -bundles  $f : Y \rightarrow X$  and  $f' : Y' \rightarrow X$ , a map  $g : Y \rightarrow Y'$ , and a homotopy  $h : f' \circ g \sim f$ , we get an induced fiberwise map*

$$\begin{aligned} \prod_{x:X} \text{fib}_f(x) &\xrightarrow{g^{\text{fib}}} \prod_{x:X} \text{fib}_{f'}(x) \\ (y, p) &\mapsto (g(y), h(y) \cdot p) \end{aligned}$$

and the homotopy that is constantly reflexivity makes the following square commute

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y' \\ \sim \downarrow & & \downarrow \sim \\ \sum_{x:X} \text{fib}_f(x) & \xrightarrow{(\text{id}, g_{\text{fib}})} & \sum_{x:X} \text{fib}_{f'}(x). \end{array}$$

*Proof.* This is a specialization of (Univalent Foundations Program; 2013, Theorem 4.8.3), where instead of the full universe  $\mathcal{U}$ , we have the subuniverse of types equivalent to  $K(G, n)$ , namely  $\text{EM}(G, n)$ .  $\square$

The same results in Lemma 2.22 hold when working with pointed  $K(G, n)$ -bundles: in this case we see  $F$  as a pointed map equipping the domain and codomain with the point given by  $K(G, n)$ . In that case, we have an equivalence between pointed maps  $X \rightarrow \bullet \text{EM}(G, n)$  and pointed  $K(G, n)$ -bundles with codomain  $X$ , defined again by pulling back  $F$  along the pointed map  $X \rightarrow \bullet \text{EM}(G, n)$ .

Notice that given a map  $X \rightarrow K(A, n+1)$ , its fiber  $Y \rightarrow X$  is a  $K(A, n)$ -bundle. But, as we will see in Proposition 2.38, not every  $K(A, n)$ -bundle arises in this way.

**Definition 2.23.** A (pointed)  $K(A, n)$ -bundle  $f : Y \rightarrow X$  is a **(pointed) principal fibration** if it is the fiber of a (pointed) map  $f' : X \rightarrow K(A, n+1)$ . Formally we define

$$\text{isPrincipal}(f) := \sum_{\substack{f' : X \rightarrow K(A, n+1) \\ e : \text{fib}_{f'}(*) \simeq Y}} f \circ e \sim i,$$

where  $i : \text{fib}_{f'}(*) \rightarrow X$  is the projection from the fiber of  $f'$  to  $X$ . In the case of pointed principal fibrations,  $f'$ ,  $e$ , and the homotopy  $f \circ e \sim i$  are taken to be pointed.

The goal of this section is to characterize principal fibrations and, more generally, maps that can be factored as a finite composite of principal fibrations.

We start with a stabilization theorem for Eilenberg–Mac Lane spaces of abelian groups that follows directly from the main construction in (Licata and Finster; 2014).

**Lemma 2.24.** Given  $n \geq 1$  and  $(K, k) : \text{EM}_\bullet(A, n)$ , the following composite is a pointed equivalence:

$$K \rightarrow \Omega \Sigma K \rightarrow \Omega \|\Sigma K\|_{n+1}.$$

The first map is the Freudenthal map, and the second one is the functorial action of  $\Omega$  on the  $(n+1)$ -truncation unit. We are equipping suspensions with a point using the point constructor  $\mathbf{N}$ .

*Proof.* To see that it is a pointed map notice that the base point  $k : K$  gets mapped to  $\text{refl}_{\mathbf{N}}$  by the Freudenthal map. The second map then sends this point to  $\text{refl}_{|\mathbf{N}|}$ , by the functoriality of  $\Omega$  on paths.

Since being an equivalence is a mere proposition, it is enough to prove the statement when  $K \equiv K(A, n)$ . Although this is proven in (Licata and Finster; 2014, Theorem 5.4), it is not stated in exactly this way, so we give some details. In the proof, the cases  $n = 1$  and  $n > 1$  are considered separately. When  $n = 1$ , (Licata and Finster; 2014, Theorem 4.3) is used, and the Freudenthal map appears as `decode'`. When  $n > 1$ , (Licata and Finster; 2014, Lemma 5.3) is used, and in that case the Freudenthal map appears explicitly in (Licata and Finster; 2014, Corollary 5.2).  $\square$

The stabilization theorem can be used to construct an equivalence  $\mathbf{EM}_\bullet(A, n) \simeq \mathbf{EM}_\bullet(A, n+1)$ , as in (Buchholtz et al.; 2018, Theorem 6.7). The next theorem (Theorem 2.25) is a generalization of this fact. We start with the crucial construction.

Consider the maps  $\sigma : \mathbf{EM}_\bullet(A, n) \rightleftarrows \mathbf{EM}_\bullet(A, n+1) : \rho$  given by

$$\begin{aligned}\sigma(K, k) &::= \left( \|\Sigma K\|_{n+1}, |\mathbf{N}| \right), \\ \rho(K, k) &::= (\Omega(K, k), \text{refl}_k).\end{aligned}$$

Here  $\mathbf{N}$  and  $\mathbf{S}$  are the point constructors of the suspension. Notice that we are omitting the proofs that the maps land in the correct types, which follows from Lemma 2.24.

Consider also the “non-pointed versions” of  $\sigma$  and  $\rho$ , namely the maps  $\mathbf{s} : \mathbf{EM}(A, n) \rightleftarrows \mathbf{EM}_{\bullet\bullet}(A, n+1) : \mathbf{r}$  given by

$$\begin{aligned}\mathbf{s}(K) &::= \left( \|\Sigma K\|_{n+1}, |\mathbf{N}|, |\mathbf{S}| \right), \\ \mathbf{r}(K', p, q) &::= (p =_{K'} q).\end{aligned}$$

We have the following commutative square:

$$\begin{array}{ccc} \mathbf{EM}_\bullet(A, n) & \xrightarrow{\sigma} & \mathbf{EM}_\bullet(A, n+1) \\ F \downarrow & & \downarrow R \\ \mathbf{EM}(A, n) & \xrightarrow{\mathbf{s}} & \mathbf{EM}_{\bullet\bullet}(A, n+1), \end{array}$$

where  $R : \mathbf{EM}_\bullet(A, n+1) \rightarrow \mathbf{EM}_{\bullet\bullet}(A, n+1)$  is the map that repeats the pointing. To see that this square commutes notice that given  $(K, k) : \mathbf{EM}_\bullet(A, n)$ , the carriers of  $\mathbf{s}(F(K, k))$  and  $R(\sigma(K, k))$  are definitionally equal. To see that the (double) pointings coincide, notice that  $\mathbf{s}(F(K, k))$  is pointed with  $|\mathbf{N}|$  and  $|\mathbf{S}|$ , whereas  $R(\sigma(K, k))$  is pointed with  $|\mathbf{N}|$  and  $|\mathbf{N}|$ . But since we have  $k : K$ , we get  $\text{ap}_{|-|}(\text{merid}(k)) : |\mathbf{N}| = |\mathbf{S}|$ , as required.

Similarly, we have the following square that commutes on the nose:

$$\begin{array}{ccc} \mathbf{EM}_\bullet(A, n) & \xleftarrow{\rho} & \mathbf{EM}_\bullet(A, n+1) \\ F \downarrow & & \downarrow R \\ \mathbf{EM}(A, n) & \xleftarrow{\mathbf{r}} & \mathbf{EM}_{\bullet\bullet}(A, n+1). \end{array}$$

**Theorem 2.25.** *For any  $n \geq 1$ , the pair  $(\sigma, \mathbf{s})$  gives an equivalence between the map  $F : \mathbf{EM}_\bullet(A, n) \rightarrow \mathbf{EM}(A, n)$  and the map  $R : \mathbf{EM}_\bullet(A, n+1) \rightarrow \mathbf{EM}_{\bullet\bullet}(A, n+1)$ . Its inverse is the pair  $(\rho, \mathbf{r})$ .*

*Proof.* Our goal is to show that the pairs  $\sigma$  and  $\rho$ , and  $\mathbf{s}$  and  $\mathbf{r}$  form equivalences. The arguments in this proof are easier to understand by informally thinking of  $F$ ,  $R$ ,  $\sigma$ ,  $\rho$ ,  $\mathbf{s}$  and  $\mathbf{r}$  as  $(\infty, 1)$ -functors between  $(\infty, 1)$ -categories. From this perspective, the proof strategy is to show that  $\sigma$  and  $\rho$ , and  $\mathbf{s}$  and  $\mathbf{r}$  form “adjoint equivalences”. Start by noticing that we have maps (which we interpret as unit and counit maps for  $\sigma$  and  $\rho$ )

$$\begin{aligned}\mu : (K, k) \rightarrow_\bullet \left( \Omega(\|\Sigma K\|_{n+1}, |\mathbf{N}|), \text{refl}_\mathbf{N} \right) &\equiv \rho(\sigma(K, k)), \\ \epsilon : \sigma(\rho(K, k)) \equiv \left( \|\Sigma \Omega(K, k)\|_{n+1}, |\mathbf{N}| \right) \rightarrow_\bullet (K, k),\end{aligned}$$

defined using the map of Lemma 2.24 for  $\mu$  and by truncation-induction and suspension-induction, mapping  $|\mathbf{N}|$ ,  $|\mathbf{S}|$ , and  $\mathbf{ap}_{|-|}(\mathbf{merid}(e : k = k))$  to  $k$ ,  $k$ , and  $e$ , respectively. Similarly, we have maps (which we interpret as unit and counit maps for  $\mathbf{s}$  and  $\mathbf{r}$ )

$$\begin{aligned} u : K &\rightarrow \left( |\mathbf{N}| = \|\Sigma K\|_{n+1} \quad |\mathbf{S}| \right) \equiv \mathbf{r}(\mathbf{s}(K)), \\ c : \mathbf{s}(\mathbf{r}(K', p, q)) &\equiv \left( \|\Sigma(p =_{K'} q)\|_{n+1}, |\mathbf{N}|, |\mathbf{S}| \right) \rightarrow_{\bullet\bullet} (K', p, q), \end{aligned}$$

with  $u$  defined by  $u(x) := \mathbf{ap}_{|-|}(\mathbf{merid}(x))$ , and  $c$  defined by truncation-induction and suspension-induction, mapping  $\mathbf{N}$  to  $p$ ,  $\mathbf{S}$  to  $q$ , and an equality  $e : p = q$  to itself. It is then enough to show that  $\mu$ ,  $\epsilon$ ,  $u$ , and  $c$  are equivalences, since, together with univalence, this establishes the result.

Lemma 2.24 directly implies that  $\mu$  is an equivalence. The proof that  $\epsilon$  is an equivalence requires a bit more work. Assume given an  $n$ -connected and  $(n + 1)$ -truncated type  $K$ , and  $k : K$ . Since the domain and codomain of the counit are connected, it is enough to show that  $\Omega\epsilon$  is an equivalence (using (Univalent Foundations Program; 2013, Theorem 8.8.2)). To do this, we must prove and use one of the ‘‘triangle identities’’. Concretely, we want to show that the following triangle commutes:

$$\begin{array}{ccc} \Omega(K, k) & \xrightarrow{\mu_{\Omega(K, k)}} & \Omega\left(\|\Sigma\Omega(K, k)\|_{n+1}\right) \\ & \searrow & \downarrow \Omega\epsilon_{(K, k)} \\ & & \Omega(K, k). \end{array}$$

The top map is homotopic to the map that sends  $l : k = k$  to  $\mathbf{ap}_{|-|}(\mathbf{merid}(l)) \cdot \mathbf{ap}_{|-|}(\mathbf{merid}(\mathbf{refl}_k))$ . And since  $\Omega\epsilon$  respects composition, we see that  $\mathbf{ap}_{|-|}(\mathbf{merid}(l)) \cdot \mathbf{ap}_{|-|}(\mathbf{merid}(\mathbf{refl}_k))$  gets mapped to  $l \cdot \mathbf{refl}_k = l$ , by the computation rules of truncation-induction and suspension-induction. This establishes the commutativity of the triangle. To use the ‘‘triangle identity’’, just notice that the top map is an equivalence and thus the right vertical map must be an equivalence, as needed.

To prove that  $u$  and  $c$  are equivalences, we will use the fact that  $F$  and  $R$  respect the units and counits. By that we mean that, for  $(K, k) : \mathbf{EM}_{\bullet}(A, n)$ , we have a commutative square

$$\begin{array}{ccc} F(K, k) & \xrightarrow{F\mu} & F(\rho(\sigma(K, k))) \\ \parallel & & \downarrow \sim \\ F(K, k) & \xrightarrow{u} & \mathbf{r}(\mathbf{s}(F(K, k))) \end{array}$$

and for  $(K, k) : \mathbf{EM}_{\bullet}(A, n + 1)$  we have a commutative square

$$\begin{array}{ccc} R(\sigma(\rho(K, k))) & \xrightarrow{R\epsilon} & R(K, k) \\ \sim \downarrow & & \parallel \\ \mathbf{s}(\mathbf{r}(R(K, k))) & \xrightarrow{c} & R(K, k). \end{array}$$

To construct the first square we let the right vertical equivalence

$$F(\rho(\sigma(K, k))) \equiv \left( |\mathbf{N}| =_{\|\Sigma K\|_{n+1}} |\mathbf{N}| \right) \simeq \left( |\mathbf{N}| =_{\|\Sigma K\|_{n+1}} |\mathbf{S}| \right) \equiv \mathbf{r}(\mathbf{s}(F(K, k)))$$

be given by concatenation with  $\mathbf{ap}_{|-|}(\mathbf{merid}(k))$ . It is then clear that the square commutes, since the top map is homotopic to  $x \mapsto \mathbf{ap}_{|-|}(\mathbf{merid}(x)) \cdot \mathbf{ap}_{|-|}(\mathbf{merid}(k)^{-1})$  by the functoriality of  $\mathbf{ap}$ .

The equivalence for the second square

$$R(\sigma(\rho(K, k))) \equiv \left( \|\Sigma(k = k)\|_{n+1}, |\mathbf{N}|, |\mathbf{N}| \right) \simeq \left( \|\Sigma(k = k)\|_{n+1}, |\mathbf{N}|, |\mathbf{S}| \right) \equiv \mathbf{s}(\mathbf{r}(R(K, k)))$$

is given by the identity on the carrier, and by equating  $|\mathbf{N}|$  and  $|\mathbf{S}|$  using  $\mathbf{ap}_{|-|}(\mathbf{merid}(\mathbf{refl}_k))$  for the second pointing. To see that the square commutes start by noticing that the composites agree on the carriers definitionally. The top right composite is pointed using  $\mathbf{refl}_k$  for both points. The left bottom composite is pointed using  $\mathbf{refl}_k$  for the first point and a path homotopic to  $\mathbf{refl}_k$  for the second point, by the computation rule of truncation-induction and suspension-induction. So the square in fact commutes.

Finally, the commutativity of the first square implies that  $u$  is an equivalence whenever we have  $k : K$ . But since being an equivalence is a mere proposition and any  $K : \mathbf{EM}(A, n)$  is merely inhabited,  $u$  is always an equivalence. Similarly, the commutativity of the second square implies that  $c$  is also an equivalence.  $\square$

It should be possible to generalize the above argument to encompass bundles with fiber any higher group in a stable range, as in (Buchholtz et al.; 2018, Section 6).

Since  $K(A, n)$ -bundles over  $X$  are classified by maps  $X \rightarrow \mathbf{EM}_\bullet(A, n)$ , we can use the above properties for each  $x : X$  to get an alternative description of  $K(-, n)$ -bundles and of fiberwise maps between them.

Recall the map  $\mathbf{s} : \mathbf{EM}(A, n) \rightarrow \mathbf{EM}_{\bullet\bullet}(A, n + 1)$  given by suspension followed by  $(n + 1)$ -truncation.

**Definition 2.26.** *Given a (pointed)  $K(A, n)$ -bundle  $f : Y \rightarrow X$ , its **classifying map** is defined to be the corresponding map  $\lceil f \rceil := \mathbf{s} \circ \mathbf{fib}_f(-) : X \rightarrow \mathbf{EM}_{\bullet\bullet}(A, n + 1)$ .*

**Corollary 2.27.** *The type of (pointed)  $K(A, n)$ -bundles over  $X$  is equivalent to the type of (pointed) maps  $X \rightarrow \mathbf{EM}_{\bullet\bullet}(A, n + 1)$ . This equivalence is given by mapping a  $K(A, n)$ -bundle  $f : Y \rightarrow X$  to its classifying map  $\lceil f \rceil : X \rightarrow \mathbf{EM}_{\bullet\bullet}(A, n + 1)$ .  $\square$*

The above result says that a (pointed)  $K(A, n)$ -bundle over  $X$  is determined by the following data:

- a (pointed) map  $\overline{f} : X \rightarrow \mathbf{EM}(A, n + 1)$ ;
- two (pointed) sections  $s_1, s_2 : \prod_{(x:X)} \overline{f}(x)$ .

More concretely, we have the following.

**Corollary 2.28.** *Let  $X : \mathcal{U}$ . Any  $K(A, n)$ -bundle  $f : Y \rightarrow X$  is equivalent to the projection map  $\mathbf{pr}_1 : \sum_{x:X} s_1(x) =_{\overline{f}(x)} s_2(x) \rightarrow X$ , where  $(\overline{f}, s_1, s_2) \equiv \lceil f \rceil$  as in Corollary 2.27, and the fiberwise equivalence  $Y \rightarrow \sum_{x:X} s_1(x) =_{\overline{f}(x)} s_2(x)$  is given by the composite*

$$Y \rightarrow \sum_{x:X} \mathbf{fib}_f(x) \xrightarrow{(\mathbf{id}, u)} \sum_{x:X} |\mathbf{N}| =_{\|\mathbf{fib}_f(x)\|_{n+1}} |\mathbf{S}|.$$

*Proof.* This follows from the equivalences of Theorem 2.25 and Lemma 2.22.  $\square$

As was hinted in its proof, the equivalence of Theorem 2.25 is not just an equivalence of types. Although more general properties can be proven, it will be enough for us to know the following.

**Proposition 2.29.** *Given  $K : \mathbf{EM}(A, n)$ ,  $K' : \mathbf{EM}(B, n)$ , and a map  $\phi : K \rightarrow K'$ , consider the map  $\phi' := \|\Sigma\phi\|_{n+1} : \mathbf{s}(K) \equiv \|\Sigma K\|_{n+1} \rightarrow \|\Sigma K'\|_{n+1} \equiv \mathbf{s}(K')$  defined by truncation-induction and suspension-induction, by mapping  $\mathbf{N}, \mathbf{S} : \Sigma K$  to  $\mathbf{N}, \mathbf{S} : \Sigma K'$  respectively, and  $\mathbf{merid}(k)$  to  $\mathbf{merid}(\phi(k))$ . Then, the computation rule of suspension-induction gives a homotopy making the following square commute*

$$\begin{array}{ccc} K & \xrightarrow{\phi} & K' \\ u \downarrow & \mathbf{ap}_{\phi'} & \downarrow u \\ \mathbf{r}(\mathbf{s}(K)) & \longrightarrow & \mathbf{r}(\mathbf{s}(K')). \end{array}$$

*Proof.* Given  $k : K$ , we have on the one hand  $u(\phi(k)) \equiv \mathbf{ap}_{|-|}(\mathbf{merid}(\phi(k)))$ . On the other hand  $\mathbf{ap}_{\phi'}(u(k)) \equiv \mathbf{ap}_{\phi'}(\mathbf{ap}_{|-|}(\mathbf{merid}(k))) = \mathbf{ap}_{|-|}(\mathbf{merid}(\phi(k)))$ , where the equality is given by the computation rule of suspension-induction.  $\square$

Proposition 2.29 implies that the description in Corollary 2.28 is functorial. To see this, assume given  $K(-, n)$ -bundles  $f : Y \rightarrow X$ ,  $f' : Y' \rightarrow X$  and a map  $g : Y \rightarrow Y'$ , such that  $f' \circ g \sim f$ . We have  $t_g : \prod_{x:X} \bar{f}(x) \rightarrow \bar{f}'(x)$  given by  $t_g(x) \equiv (g_{\mathbf{fib}}(x))'$ , as in Proposition 2.29. For every  $x : X$  the map  $t_g(x)$  is doubly-pointed definitionally, since it is defined by suspension-induction. Then Proposition 2.29 implies the following.

**Corollary 2.30.** *Assume given  $K(-, n)$ -bundles  $f : Y \rightarrow X$ ,  $f' : Y' \rightarrow X$  and a map  $g : Y \rightarrow Y'$ , such that  $f' \circ g \sim f$ . Then, the computation rule of suspension-induction gives a homotopy making the following square commute:*

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y' \\ \sim \downarrow & & \downarrow \sim \\ \left( \sum_{(x:X)} s_1(x) =_{\bar{f}(x)} s_2(x) \right) & \xrightarrow{(\mathbf{id}, \lambda x. \mathbf{ap}_{t_g(x)})} & \left( \sum_{(x:X)} s'_1(x) =_{\bar{f}'(x)} s'_2(x) \right). \end{array}$$

Here the vertical maps are given by the equivalence of Corollary 2.28.  $\square$

Let  $F_{\bullet} : \mathbf{EM}_{\bullet\bullet}(A, n+1) \rightarrow \mathbf{EM}_{\bullet}(A, n+1)$  be the map that forgets the second point.

**Definition 2.31.** *Given a pointed  $K(A, n)$ -bundle  $f : Y \rightarrow X$  we define the **associated action** of  $X$  on  $A$  as the pointed map  $F_{\bullet} \circ [f] : X \rightarrow_{\bullet} \mathbf{EM}_{\bullet}(A, n+1)$ .*

*We say that a pointed  $K(A, n)$ -bundle  $f$  **lives over** an action  $\phi : X \rightarrow_{\bullet} \mathbf{EM}_{\bullet}(A, n+1)$  if  $\phi$  is equal to the action associated to  $f$ .*

*Example 2.32.* As is familiar from Homotopy Theory, for any pointed type  $X$  and  $n \geq 1$ , we have an action  $\pi_1(X) \curvearrowright \pi_n(X)$ . The action of  $\pi_1(X)$  on itself is defined to be the action by conjugation. The action of  $\pi_1(X)$  on  $\pi_n(X)$  is defined to be the action associated to the fibration  $\|X\|_n \rightarrow \|X\|_{n-1}$ , using Definition 2.31 and Lemma 2.8.

*Remark 2.33.* In the same vein as Corollary 2.28, the associated action of a pointed  $K(A, n)$ -bundle over  $X$  is nothing but:

- the pointed map  $\overline{f} : X \rightarrow_{\bullet} \mathbf{EM}(A, n + 1)$ ;
- the first of its pointed sections  $s_1 : \prod_{x:X}^{\bullet} \overline{f}(x)$ .

where  $\prod^{\bullet}$  is the pointed pi-type. An element of the pointed pi-type  $\prod_{x:X}^{\bullet} \overline{f}(x)$  consists of an element  $s$  of the usual, unpointed pi-type, together with a path from  $s(x_0)$  to the base point of  $\overline{f}(x_0)$ .

**Lemma 2.34.** *Given pointed  $K(-, n)$ -bundles  $f : Y \rightarrow_{\bullet} X$  and  $f' : Y' \rightarrow_{\bullet} X$ , and a pointed map  $g : Y \rightarrow_{\bullet} Y'$  such that  $f = f' \circ g$  as pointed maps, we have an induced map of actions  $(\overline{f}, s_1) \rightarrow_{\text{act}} (\overline{f}', s'_1)$ .*

*Proof.* Using the description of Corollary 2.30, we see that we have in particular a map of actions  $(\overline{f}, s_1) \rightarrow_{\text{act}} (\overline{f}', s'_1)$ .  $\square$

**2.3. Principal fibrations.** In this section, following (Shulman; 2014), we prove that principal fibrations are precisely the fibrations that live over the trivial action. We will need the following general lemma.

**Lemma 2.35.** *Assume given a pair of maps  $r : E \rightleftarrows B : s$  such that  $q : r \circ s \sim \text{id}_B$ . Then, for every  $b : B$ , the following square is a pullback*

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{b} & B \\ (s(b), q(b)) \downarrow & & \downarrow s \\ \text{fib}_r(b) & \longrightarrow & E. \end{array}$$

*Proof.* This can be proven by writing the pullback using sigma types, or by using the two pullback lemma (Avigad et al.; 2015, Lemma 4.1.11) applied to the following diagram:

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & B \\ \downarrow & & \downarrow s \\ \text{fib}_r(b) & \longrightarrow & E \\ \downarrow & \xrightarrow{b} & \downarrow r \\ \mathbf{1} & \longrightarrow & B \end{array}$$

$\square$

**Lemma 2.36.** *The map  $R$  has a retraction, given by  $F_{\bullet} : \mathbf{EM}_{\bullet\bullet}(A, n + 1) \rightarrow \mathbf{EM}_{\bullet}(A, n + 1)$ , the map that forgets the second point.*  $\square$

From Theorem 2.25, it follows that  $F$  has a retraction.

The following proposition is a strengthening of the main theorem in (Shulman; 2014). Instead of proving a logical equivalence, we provide an equivalence of types. For this proposition, it is convenient to define the notion of unpointed associated action. Similarly to Definition 2.31, given a  $K(A, n)$ -bundle  $f : Y \rightarrow X$ , we define the **unpointed associated action** of  $X$  on  $A$  as the (unpointed) map  $F_{\bullet} \circ [f] : X \rightarrow \mathbf{EM}_{\bullet}(A, n + 1)$ .

**Proposition 2.37** (cf. (Shulman; 2014)). *Let  $f : Y \rightarrow X$  be a  $K(A, n)$ -bundle. Then the following types are equivalent:*

- (1) *The type of homotopies between the unpointed associated action of  $X$  on  $A$  and the constant map at the pointing of  $\mathbf{EM}_{\bullet}(A, n + 1)$ .*
- (2) *The type of principal fibration structures on  $f$ .*

*Proof.* Observe that we have a fiber sequence

$$K(A, n+1) \xrightarrow{i} \mathbf{EM}_{\bullet\bullet}(A, n+1) \xrightarrow{F_{\bullet}} \mathbf{EM}_{\bullet}(A, n+1),$$

and thus, the first type in the statement is equivalent to the type of factorizations of  $[f]$  through  $i$ . Concretely, this is the type  $\sum_{(f': X \rightarrow K(A, n+1))} i \circ f' \sim [f]$ .

Now, by Lemma 2.22 and Theorem 2.25, this last type is equivalent to the type of maps  $f' : X \rightarrow K(A, n+1)$  together with a proof that  $i \circ f'$  classifies  $f$ . That is, a pointed map  $\psi : Y \rightarrow \mathbf{EM}_{\bullet}(A, n+1)$  together with a homotopy, making the following square commute and a pullback:

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & \mathbf{EM}_{\bullet}(A, n+1) \\ f \downarrow & & \downarrow R \\ X & \xrightarrow{f'} K(A, n+1) \xrightarrow{i} & \mathbf{EM}_{\bullet\bullet}(A, n+1). \end{array}$$

By Lemma 2.35, and the universal property of pullbacks, this is equivalent to having maps  $f' : X \rightarrow K(A, n+1)$ ,  $\psi : Y \rightarrow \mathbf{EM}_{\bullet}(A, n+1)$ , and  $\phi : Y \rightarrow \mathbf{1}$ , homotopies  $H$  and  $J$ , filling the left square and the top triangle respectively in the following diagram

$$\begin{array}{ccccc} & & \psi & & \\ & & \curvearrowright & & \\ Y & \xrightarrow{\phi} & \mathbf{1} & \xrightarrow{\quad} & \mathbf{EM}_{\bullet}(A, n+1) \\ f \downarrow & & \downarrow & & \downarrow R \\ X & \xrightarrow{f'} & K(A, n+1) & \xrightarrow{i} & \mathbf{EM}_{\bullet\bullet}(A, n+1). \end{array}$$

and a proof that the pasting of the two squares is a pullback.

The type of maps  $Y \rightarrow \mathbf{1}$  is contractible, with center of contraction the canonical map  $Y \rightarrow \mathbf{1}$ . Observe also that the pair  $(J, \psi)$  inhabits a contractible type, since  $J$  is witnessing the fact that  $\psi$  is the composite of  $\phi$  with the inclusion of the base point  $\mathbf{1} \rightarrow \mathbf{EM}_{\bullet}(A, n+1)$ . So this last type is equivalent to the type of maps  $f' : X \rightarrow K(A, n+1)$  together with a homotopy  $H$  making the left square commute, and a proof that the composite square is a pullback. Finally, by the two pullback lemma (Avigad et al.; 2015, Lemma 4.1.11), this last type is equivalent to the type of proofs that  $f$  is pointed principal, as required.  $\square$

A similar argument, using pointed maps, pointed homotopies, and Lemma 2.8, proves an analogous result for pointed  $K(A, n)$ -bundles between connected types.

**Proposition 2.38.** *Let  $f : Y \rightarrow X$  be a pointed  $K(A, n)$ -bundle between connected types. Then the following types are equivalent:*

- (1) *The type of trivializations of the associated action of  $X$  on  $A$ .*
- (2) *The type of principal fibration structures on  $f$ .*  $\square$

**Corollary 2.39.** *The type of principal fibration structures on a pointed  $K(A, n)$ -bundle between connected types is a mere proposition.*

*Proof.* This follows from Proposition 2.38, noting that Lemma 2.8 implies that the first of the two equivalent types in the statement of Proposition 2.38 is a mere proposition.  $\square$

**2.4. Nilpotent groups.** In this section we give a homotopical characterization of nilpotent groups that is familiar from classical homotopy theory. Before doing this, recall, from (Buchholtz et al.; 2018, Section 4.5), that short exact sequences of groups of the form  $I \rightarrow G \rightarrow H$  correspond to pointed fiber sequences of Eilenberg–Mac Lane spaces of the form  $K(I, 1) \rightarrow K(G, 1) \rightarrow K(H, 1)$ .

**Definition 2.40.** *Given a short exact sequence of groups  $I \rightarrow G \rightarrow H$ , where  $I$  is abelian, we define an action  $H \curvearrowright I$  by applying Definition 2.31 together with Lemma 2.8 to the fiber sequence  $K(I, 1) \rightarrow K(G, 1) \rightarrow K(H, 1)$  corresponding to the short exact sequence.*

**Lemma 2.41.** *Given a short exact sequence of groups  $I \rightarrow G \xrightarrow{q} H$  with  $I$  abelian, the action  $G \curvearrowright I$  given in Definition 2.14 factors as  $q$  composed with the action  $H \curvearrowright I$  given in Definition 2.40.*

*Proof.* We have equivalences  $G \simeq \Omega K(G, 1)$ , and  $H \simeq \Omega K(H, 1)$ , such that multiplication maps to composition of loops. For readability, we will avoid using the equivalences explicitly. Assume given a pointed map  $f : K(G, 1) \rightarrow K(H, 1)$  that represents  $q$ . Let us recall the setting of Definition 2.14. Let  $I \equiv \text{fib}_{\Omega f}(\text{refl}_*)$ , and fix  $g : G$ . We then have a map  $I \rightarrow I$  given by  $(l, r) \mapsto (g^{-1} \cdot l \cdot g, \phi_r)$ , where  $\phi_r$  is the straightforward proof that  $g^{-1} \cdot l \cdot g$  maps to  $\text{refl}_*$  under  $\Omega f$ . Our goal is to show that using the action from Definition 2.40,  $\Omega f(g)$  acts as the above map. To do this, let us recall the construction from Definition 2.40. We start by considering the composite in Definition 2.31:

$$K(H, 1) \rightarrow \text{EM}(I, 1) \xrightarrow{\simeq} \text{EM}_{\bullet\bullet}(I, 2) \rightarrow \text{EM}_{\bullet}(I, 2).$$

Together with  $f$ , this gives us a map  $K(G, 1) \rightarrow \text{EM}_{\bullet}(I, 2)$  that sends  $*$  to  $\|\Sigma \text{fib}_f(*)\|_2$ . By looping this map, we get a map  $\alpha : G \rightarrow (\|\Sigma \text{fib}_f(*)\|_2 =_{\mathcal{U}_{\bullet}} \|\Sigma \text{fib}_f(*)\|_2)$ . The action  $G \curvearrowright I$  is obtained by using Lemma 2.7, and recalling that we have

$$\Omega^2 \|\Sigma \text{fib}_f(*)\|_2 \simeq \Omega \|\Omega \Sigma \text{fib}_f(*)\|_1 \simeq \Omega \text{fib}_f(*) \simeq \text{fib}_{\Omega f}(\text{refl}_*) \equiv I$$

This gets us the map  $\beta : G \rightarrow (I \rightarrow I)$  which we want to prove is homotopic to  $(l, r) \mapsto (g^{-1} \cdot l \cdot g, \phi_r)$ .

Now, to prove this, we generalize and use path induction. Instead of considering a loop  $g : \Omega K(G, 1)$ , we take a point  $x : K(G, 1)$  and a path  $p : * = x$ . Generalizing the construction from Definition 2.40 (using  $\text{ap}_f$  instead of  $\Omega(f)$ ) we get a map  $(* = x) \rightarrow (\text{fib}_{\text{ap}_f}(\text{refl}_*) \rightarrow \text{fib}_{\text{ap}_f}(\text{refl}_{f(x)}))$ , and it is now enough to show that this map is homotopic to  $p \mapsto ((c, r) \mapsto (p^{-1} \cdot c \cdot p, \psi_r))$ . But this is easily proven by path induction, since we only have to verify it in the case where  $x \equiv *$  and  $p \equiv \text{refl}_*$ . In that case, on the one hand, the map  $(c, r) \mapsto (p^{-1} \cdot c \cdot p, \psi_r)$  is just the identity on  $I$ . On the other hand,  $\alpha$ , being the looping of a pointed map, maps  $\text{refl}_*$  to  $\text{refl}_{\|\Sigma \text{fib}_f(*)\|_2}$ . So  $\beta$  maps  $\text{refl}_*$  to  $\text{id}_I$ , since for  $K, K' : \text{EM}_{\bullet}(A, 2)$ , the equivalence  $(K =_{\mathcal{U}_{\bullet}} K') \simeq (\Omega^2 K =_{\text{Grp}} \Omega^2 K')$  is given by  $\text{ap}_{\Omega^2}$ .  $\square$

**Corollary 2.42.** *Using the correspondence between short exact sequences and pointed fiber sequences of Eilenberg–Mac Lane spaces, a short exact sequence is a **central extension** if and only if its corresponding fiber sequence is a **principal fibration**.*

The statement of the corollary is only a logical equivalence and a priori not an equivalence of types. But notice that being a central extension is a mere proposition, and that Corollary 2.39 implies that, in this case, being principal is a mere proposition too. So the statement is actually an equivalence of types.

*Proof.* Fix a short exact sequence  $I \rightarrow G \xrightarrow{q} H$ . On the one hand, if it corresponds to a principal fibration, then  $I$  is abelian and the action of Definition 2.40 is trivial, so the action of Definition 2.14 must be too, by Lemma 2.41. On the other hand, since the fact that an element  $h : H$  acts trivially as a map  $I \rightarrow I$  is a mere proposition, and since  $q$  is surjective, we can assume that  $h = q(g)$  for some  $g : G$ . But if we assume that the action of Definition 2.14 is trivial, then  $g$  acts trivially, and, by Lemma 2.41,  $h$  must act trivially too.  $\square$

**Proposition 2.43.** *Given a group  $G$ , the following types are equivalent:*

- (1) *The type of nilpotent structures on  $G$ .*
- (2) *The type of factorizations of the map  $K(G, 1) \rightarrow \mathbf{1}$  as a finite composite of pointed principal fibrations involving pointed, connected 1-types.*

*Proof.* By the correspondence between central extensions and principal fibrations (Corollary 2.42) the statement is equivalent to Lemma 2.19.  $\square$

**2.5. Nilpotent types.** This section characterizes nilpotent types in terms of the maps in their Postnikov towers, as is familiar from classical homotopy theory.

**Definition 2.44.** *A **nilpotent structure** on a pointed, connected type  $X$  is given by a nilpotent structure for  $\pi_1(X)$  and, for each  $n > 1$ , a nilpotent structure for the action  $\pi_1(X) \curvearrowright \pi_n(X)$ .*

Notice that a connected type is merely pointed, and thus it makes sense to ask whether a connected type merely has a nilpotent structure.

**Definition 2.45.** *A **nilpotent type** is a connected type such that it merely has a nilpotent structure.*

*Example 2.46.* All simply connected types are nilpotent.

*Example 2.47.* The truncation of a nilpotent type is nilpotent.

We can prove many facts about truncated nilpotent types by inducting over the nilpotency degree.

**Definition 2.48.** *Given  $n \geq 1$  and a pointed, connected,  $n$ -truncated type  $Y$  together with a nilpotent structure, its **nilpotency degree** is the sum, over  $i < n$ , of the lengths of the factorizations of the maps  $\|Y\|_{i+1} \rightarrow \|Y\|_i$ , given by the nilpotent structure.*

Our next goal is to prove the key property of a nilpotent type, namely that all of the maps in its Postnikov tower factor as finite composites of principal fibrations. In order to do this, we need to transfer  $K(-, n)$ -bundles along maps of actions.

Given a map of  $\pi_1(X)$ -actions  $m : \phi \rightarrow \psi$ , and a  $K(A, n)$ -bundle over  $\phi$ , we can construct a bundle over  $\psi$  as follows.

**Definition 2.49.** *Let  $f : Y \rightarrow X$  be a bundle classified by  $\bar{f} : X \rightarrow \mathbf{EM}(A, n + 1)$  and  $s_1, s_2 : \prod_{(x:X)} f(x)$ . Given an action  $\psi \equiv (\bar{f}', s'_1)$  and a map of actions*

$$m : \prod_{(x:X)} (\bar{f}(x), s_1(x)) \rightarrow \bullet (\bar{f}'(x), s'_1(x)),$$

*the **bundle transferred along  $m$** , denoted by  $m_*(f)$ , is the bundle classified by  $\bar{f}'$ ,  $s'_1$ , and  $s'_2 := \lambda x.m(x)(s_2(x))$ .*

Notice that if  $f$  is a pointed bundle and  $\psi$  is a pointed action, then the transferred bundle is canonically pointed.

**Definition 2.50.** Let  $f : Y \rightarrow X$  be a (pointed)  $K(-, n)$ -bundle. We define a (pointed) fiberwise map  $f \rightarrow m_*(f)$ . Under the identification of Corollary 2.28, this is the map

$$\left( \sum_{(x:X)} s_1(x) = s_2(x) \right) \rightarrow \left( \sum_{(x:X)} s'_1(x) = s'_2(x) \right)$$

given by mapping  $(x, p)$  to  $(x, r^{-1} \cdot \mathbf{ap}_{m(x)}(p))$ , where  $r : m(s_1(x)) = s'_1(x)$  is the proof that  $m$  is pointed.

The transferred bundle construction has a nice interpretation in terms of cohomology with local coefficients.

*Remark 2.51.* Recall from (Cavallo; 2015) that  $K(A, n)$  represents the  **$n$ -th reduced cohomology group with coefficients in  $A$** . That is, for a pointed type  $X$ , one defines

$$\tilde{H}^n(X; A) := \left\| X \rightarrow_{\bullet} K(A, n) \right\|_0.$$

More generally, the map  $F_{\bullet} : \mathbf{EM}_{\bullet\bullet}(A, n+1) \rightarrow \mathbf{EM}_{\bullet}(A, n+1)$  represents the  **$n$ -th reduced cohomology group with local coefficients**. In this setting, a **local system** is given by a pointed map  $c : X \rightarrow_{\bullet} \mathbf{EM}_{\bullet}(A, n+1)$  (which, by Lemma 2.8, corresponds to an action of  $\pi_1(X)$  on  $A$ , when  $X$  is connected), and the  $n$ -th reduced cohomology group of  $X$  with coefficients in  $c$  is defined by

$$\tilde{H}^n(X; c) := \left\| \prod_{x:X}^{\bullet} c(x) \right\|_0,$$

where  $\prod^{\bullet}$  is the pointed Pi-type.

Under the above interpretation, the transferred bundle construction gives a map  $m : \tilde{H}^*(X, \phi) \rightarrow \tilde{H}^*(X, \psi)$  for every map of actions  $m : \phi \rightarrow \psi$ . This is the functoriality of cohomology with local coefficients with respect to the coefficients variable.

**Definition 2.52.** Given composable maps  $Y \xrightarrow{f} Y' \xrightarrow{g} Y''$  and  $y' : Y'$ , we define a fiber sequence

$$\mathbf{fib}_f(y') \rightarrow \mathbf{fib}_{g \circ f}(g(y')) \rightarrow \mathbf{fib}_g(g(y')),$$

where we are considering the fiber over  $(y', \text{refl}) : \mathbf{fib}_g(g(y'))$ . To construct this fiber sequence, we take the fibers of all the maps in the bottom right square of the following diagram:

$$\begin{array}{ccccc} & & \mathbf{fib}_{g \circ f}(g(y')) & \rightarrow & \mathbf{fib}_g(g(y')) \\ & & \downarrow & & \downarrow \\ \mathbf{fib}_f(y') & \longrightarrow & Y & \xrightarrow{f} & Y' \\ & & \downarrow g \circ f & & \downarrow g \\ \mathbf{1} & \longrightarrow & Y'' & \xlongequal{\quad} & Y'' \end{array}$$

and then use the commutativity of limits to get an equivalence between the fiber of the map  $\mathbf{fib}_{g \circ f}(g(y')) \rightarrow \mathbf{fib}_g(g(y'))$  and  $\mathbf{fib}_f(y')$ .

**Lemma 2.53.** For every  $n \geq 1$ , pointed  $K(-, n)$ -bundles are closed under composition.

*Proof.* Assume given composable pointed maps  $f : Y \rightarrow Y'$ ,  $g : Y' \rightarrow Y''$  such that  $f$  is a pointed  $K(A, n)$ -bundle and  $g$  is a pointed  $K(B, n)$ -bundle. Let  $y : Y$ ,  $y' : Y'$ ,  $y'' : Y''$  be the pointings. The fiber sequence of Definition 2.52, in this case, is equivalent to a fiber sequence of the following form:

$$K(A, n) \rightarrow \mathbf{fib}_{g \circ f}(y'') \rightarrow K(B, n).$$

In particular  $\mathbf{fib}_{g \circ f}(y'')$  is a pointed,  $(n - 1)$ -connected,  $n$ -truncated type. So it is an Eilenberg–Mac Lane space. By looking at the long exact sequence of homotopy groups, we see that  $g \circ f$  is in fact a  $K(C, n)$ -bundle, with  $C$  an extension of  $B$  by  $A$ .  $\square$

**Lemma 2.54.** *Let  $X$  be a pointed, connected type, and assume given pointed  $K(-, n)$ -bundles  $f : Y \rightarrow X$ ,  $f' : Y' \rightarrow X$ ,  $g : Y \rightarrow Y'$ , such that  $f = f' \circ g$  as pointed maps. Then the induced map of actions  $t_g : (\overline{f}, s_1) \rightarrow (\overline{f'}, s'_1)$  is surjective (Definition 2.4).*

*Proof.* Let  $x_0 : X$  and  $y'_0 : Y'$  be the base points. Notice that by the construction of the induced action (Lemma 2.34) the map  $t_g(x_0) : \overline{f}(x_0) \rightarrow \overline{f'}(x_0)$  is equivalent to the second map in the following fiber sequence:

$$\mathbf{fib}_g(y'_0) \rightarrow \mathbf{fib}_f(x_0) \rightarrow \mathbf{fib}_{f'}(x_0).$$

By hypothesis,  $\mathbf{fib}_g(y'_0) \simeq K(A, n)$ ,  $\mathbf{fib}_f(x_0) \simeq K(C, n)$ , and  $\mathbf{fib}_{f'}(x_0) \simeq K(B, n)$ . This fiber sequence corresponds to a short exact sequence of groups, so after looping  $n$  times,  $t_g(x_0) : \overline{f}(x_0) \rightarrow \overline{f'}(x_0)$  is a surjective map of groups.  $\square$

**Lemma 2.55.** *Assume given commutative triangles of pointed  $K(-, n)$ -bundles*

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y' \\ & \searrow f & \swarrow f' \\ & X & \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{g'} & Y'' \\ & \searrow f & \swarrow f'' \\ & X & \end{array}$$

and an equivalence of actions  $e : (\overline{f'}, s'_1) \simeq (\overline{f''}, s''_1)$ , such that  $e \circ t_g = t'_g$  as maps of actions. Then we have a pointed equivalence  $h : Y \simeq Y'$  such that  $h \circ g = g'$  and  $f'' \circ h = f'$  as pointed maps.

*Proof.* Notice that  $e(x)(s'_i(x)) = s''_i(x)$ , for  $i$  either 1 or 2. This is because on the one hand  $e(x) \circ t_g(x) = t'_g(x)$  as maps, and on the other hand,  $t_g$  and  $t'_g$  are doubly pointed, so we have equalities  $s'_i(x) = t_g(x)(s_i(x))$  and  $s''_i(x) = t'_g(x)(s_i(x))$ . Corollary 2.30 then implies that  $Y$  and  $Y'$  are equivalent, with the required compatibility.  $\square$

**Lemma 2.56.** *Let  $X, Y, Y'$  be pointed, connected types, and  $f : Y \rightarrow X$ ,  $f' : Y' \rightarrow X$  be pointed  $K(-, n)$ -bundles. Given a map  $g : Y \rightarrow Y'$  such that  $f = g \circ f'$ , if the action  $(\overline{f}, s_1) \rightarrow_{\text{act}} (\overline{f'}, s'_1)$  induced by  $g$  is surjective, then  $g$  is a  $K(-, n)$ -bundle, and its associated action is equivalent to  $f'$  composed with  $\ker \left( (\overline{f}, s_1) \rightarrow_{\text{act}} (\overline{f'}, s'_1) \right)$ .*

*Proof.* Using Definition 2.50, we replace  $g$  by an equivalent map

$$(\text{id}, \lambda x. \text{ap}_{t_g(x)}) : \left( \sum_{(x:X)} s_1(x) =_{\overline{f}(x)} s_2(x) \right) \rightarrow \left( \sum_{(x:X)} s'_1(x) =_{\overline{f'}(x)} s'_2(x) \right).$$

Now, in general, whenever we have a map  $\eta : U \rightarrow V$  between types, and points  $u_1, u_2 : U$ , we have an equivalence

$$(u_1 = u_2) \simeq \left( \sum_{(p:\eta(u_1)=\eta(u_2))} (u_1, \text{refl}_{\eta(u_1)}) =_{\mathbf{fib}_\eta(\eta(u_1))} \text{transport}^{\mathbf{fib}_\eta(-)}(p^{-1}, (u_2, \text{refl}_{\eta(u_2)})) \right)$$

such that a path  $z$  gets mapped to a pair with  $\mathbf{ap}_\eta(z)$  as its first coordinate. This implies that the fiber of  $g$  is equivalent to

$$(s_1(x), \mathbf{refl}_{s'_1(x)}) =_{\mathbf{fib}_{t_g(x)}(s'_1(x))} \mathbf{transport}^{\mathbf{fib}_{t_g(x)}(-)}(p^{-1}, (s_2(x), \mathbf{refl}_{s_2(x)})).$$

Since there is a fiber sequence  $\mathbf{fib}_{t_g(x)}(s'_1(x)) \rightarrow \bar{f}(x) \xrightarrow{t_g} \bar{f}'(x)$ , if the action induced by  $g$  is surjective,  $\mathbf{fib}_{t_g(x)}(s'_1(x))$  must be  $(n+1)$ -truncated and  $n$ -connected. This means that  $g$  is a  $K(-, n)$ -bundle, since its fibers are path spaces of an  $(n+1)$ -dimensional Eilenberg–Mac Lane space. This proves the first statement.

By the very description of  $(\mathbf{id}, \lambda x. \mathbf{ap}_{t_g(x)})$  as a  $K(-, n)$ -bundle, we see that its action is given by mapping  $(x, p)$  to the pointed type  $(\mathbf{fib}_{t_g(x)}(s'_1(x)), (s_1(x), \mathbf{refl}_{s'_1(x)}))$ , which is precisely  $\ker\left(\left(\bar{f}, s_1\right) \rightarrow_{\mathbf{act}} \left(\bar{f}', s'_1\right)\right)$  applied to  $x$ . To finish the proof, notice that under the equivalence  $Y' \simeq \left(\sum_{(x:X)} s'_1(x) =_{\bar{f}'(x)} s'_2(x)\right)$ ,  $f'$  corresponds to the projection  $(x, p) \mapsto p$ .  $\square$

**Proposition 2.57.** *Let  $f : Y \rightarrow X$  be a  $K(A, n)$ -bundle between pointed, connected types living over an action  $\phi : X \rightarrow \bullet \mathbf{EM}_\bullet(A, n+1)$ . Given a nilpotent structure on  $\phi$  there is a factorization of  $f$  as a finite composite of principal fibrations.*

*Proof.* Let  $\mathbf{1}$  denote the trivial  $\pi_1(X)$ -action on the trivial group. Given a filtration of normal subactions

$$\mathbf{1} \equiv \phi_0 \triangleleft \cdots \triangleleft \phi_k \equiv \phi$$

all of them normal in  $\phi$ , we can construct a sequence of maps of actions

$$\phi \simeq \phi'_0 \rightarrow \cdots \rightarrow \phi'_k \simeq \mathbf{1}$$

that induce surjective maps after looping  $n$  times. Analogously to Proposition 2.43, we get this by defining  $\phi'_i := \phi / \phi_i$ . We can then apply Definition 2.49 to the quotient map  $\phi \rightarrow \phi'_{k-1}$ , to get a factorization

$$\begin{array}{ccc} Y & \xrightarrow{f^{(1)}} & Y_{k-1} \\ & \searrow f & \swarrow f_{k-1} \\ & & X \end{array}$$

such that the associated action of  $f_{k-1}$  is  $\phi'_{k-1}$ . Moreover, by Lemma 2.56,  $f^{(1)}$  is a  $K(-, n)$ -bundle, and its associated action is  $\phi_{k-1} \circ f_{k-1}$ .

By hypothesis, the action associated to  $f_{k-1}$  is trivial, and thus  $f_{k-1}$  is a principal fibration by Proposition 2.38. Notice that composing each  $\phi_i$  with  $f_{k-1}$ , we get a filtration of the action  $\phi_{k-1} \circ f_{k-1}$

$$\mathbf{1} \equiv \phi_0 \circ f_k \triangleleft \cdots \triangleleft \phi_{k-1} \circ f_{k-1}.$$

We can now proceed inductively, by applying the same argument to  $f^{(1)}$ . In the end, we get a factorization of  $f$  as a sequence of principal fibrations

$$Y \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_{k-1} \rightarrow X,$$

as needed.  $\square$

**Proposition 2.58.** *Let  $f : Y \rightarrow X$  be a  $K(A, n)$ -bundle between pointed, connected types living over an action  $\phi : X \rightarrow \bullet \text{EM}_\bullet(A, n+1)$ . Given a factorization of  $f$  as a finite composite of principal fibrations, there is a nilpotent structure on  $\phi$ .*

*Proof.* Suppose we are given a factorization of  $f$

$$Y \equiv Y_0 \xrightarrow{g_1} Y_1 \xrightarrow{g_2} \dots \xrightarrow{g_{k-1}} Y_{k-1} \xrightarrow{g_k} Y_k \equiv X.$$

Let  $f_i : Y_i \rightarrow X$  denote the composite of the maps  $g_{i+1}$  to  $g_k$ , so that we have commuting triangles

$$\begin{array}{ccc} Y_i & \xrightarrow{g_{i+1}} & Y_{i+1} \\ & \searrow f_i & \swarrow f_{i+1} \\ & X & \end{array}$$

where  $f_k : Y_k \rightarrow X$  is the identity. By Lemma 2.53, the map  $f_i$  is a  $K(-, n)$ -bundle for every  $i$ .

For each  $i$ , consider the actions associated to the maps  $f_i$  and  $f_{i+1}$ , and call them  $\phi'_i$  and  $\phi'_{i+1}$ . From Lemma 2.34 we know that the map  $g_{i+1}$  induces a map of actions  $\phi'_i \rightarrow_{\text{act}} \phi'_{i+1}$ , and, from Lemma 2.54, we know this map is surjective. By composing these maps, we obtain, for each  $i$ , a surjective map  $\phi'_0 \rightarrow \phi'_i$ . Let  $\phi_i$  be the kernel of this map. This gives us a sequence of normal subactions

$$\phi_0 \triangleleft \phi_1 \triangleleft \dots \triangleleft \phi_k$$

such that  $\phi_k$  is equivalent to  $\phi'_0$ , the action associated to  $f$ . So this sequence in fact gives us a filtration of the action associated to  $f$ . Now, analogously to Proposition 2.43, the quotient  $\phi_{i+1}/\phi_i$  is equivalent to  $k_i$ , the kernel of the map  $\phi'_i \rightarrow_{\text{act}} \phi'_{i+1}$ . So it remains to show that  $k_i$  is a trivial action.

The action associated to  $g_i$  is surjective. Lemma 2.56 then implies that the action associated to  $g_i$  is equivalent to  $k_i \circ f_i$ . Moreover,  $g_i$  is principal, so its associated action is trivial, by Proposition 2.38. Since  $f_i$  is homotopy surjective, it follows that  $k_i$  is a trivial action, as needed.  $\square$

**Theorem 2.59.** *Let  $X$  and  $Y$  be pointed, connected types, and  $f : X \rightarrow Y$  a pointed  $K(-, n)$ -bundle. Then the type of nilpotent structures on  $(\overline{f}, s_1)$  is equivalent to the type of factorizations of  $f$  as a finite composite of principal fibrations.*

*Proof.* We constructed maps back and forth in Proposition 2.58 and Proposition 2.57. The fact that given a nilpotent structure on  $(\overline{f}, s_1)$  we obtain the same structure after applying Proposition 2.57 and then Proposition 2.58 is clear by construction.

The other composite follows from Lemma 2.55.  $\square$

As a direct corollary we get the main characterization of nilpotent types.

**Theorem 2.60.** *For a connected type  $X$  the following are equivalent:*

- *The type  $X$  is nilpotent.*
- *Each map  $\|X\|_{n+1} \rightarrow \|X\|_n$  in the Postnikov tower of  $X$  merely factors as a finite composite of principal fibrations.*  $\square$

## 3. COHOMOLOGY ISOMORPHISMS

In this section we prove that if a pointed map between nilpotent types induces a cohomology isomorphism with coefficients in every abelian group, then it induces an isomorphism in all homotopy groups. We use this to prove that the suspension of an  $\infty$ -connected map between connected types is  $\infty$ -connected. The main ideas in this section appear in (Shulman; 2014).

**Definition 3.1.** *A pointed map  $f : Y \rightarrow_{\bullet} X$  between pointed, connected types is a **cohomology isomorphism** if for every abelian group  $A$  we have that  $f^* : \tilde{H}^{\bullet}(X; A) \rightarrow \tilde{H}^{\bullet}(Y; A)$  is an isomorphism.*

Notice that we are using reduced cohomology. The main result of this section can also be stated and proved using non-reduced cohomology.

**Definition 3.2.** *A map  $f : Y \rightarrow X$  is  **$\infty$ -connected** if for all  $n : \mathbb{N}$  we have that  $\|f\|_n : \|Y\|_n \rightarrow \|X\|_n$  is an equivalence.*

It is straightforward to see that a pointed map between pointed, connected types is  $\infty$ -connected if and only if it induces an isomorphism in all homotopy groups.

**Lemma 3.3.** *A map  $f : X \rightarrow_{\bullet} Y$  between pointed, connected types is a cohomology isomorphism if and only if the map  $\|f\|_n : \|X\|_n \rightarrow_{\bullet} \|Y\|_n$  is a cohomology isomorphism for all  $n : \mathbb{N}$ . In particular, any  $\infty$ -connected, pointed map between pointed, connected types is a cohomology isomorphism.*

*Proof.* The second statement follows at once from the first one. The first one follows from the fact that cohomology isomorphisms are defined by mapping into truncated types.  $\square$

**Lemma 3.4.** *For any cohomology isomorphism  $f : Y \rightarrow_{\bullet} X$ , abelian group  $A$ , and  $n : \mathbb{N}$ , the precomposition map  $f^* : (X \rightarrow_{\bullet} K(A, n)) \rightarrow (Y \rightarrow_{\bullet} K(A, n))$  is an equivalence.*

*Proof.* Since  $K(A, n)$  is  $n$ -truncated, the domain and codomain of  $f^*$  are  $n$ -truncated, so we can use the truncated Whitehead theorem (Univalent Foundations Program; 2013, Theorem 8.8.3). It is thus enough to check that  $f^*$  induces a bijection after 0-truncating, and an isomorphism in  $\pi_k$  for every pointing  $g : X \rightarrow_{\bullet} K(A, n)$  and every  $k : \mathbb{N}$ .

The statement about  $\pi_0$  is clear, since after 0-truncating, we get the induced map on cohomology  $\tilde{H}^n(X; A) \rightarrow \tilde{H}^n(Y; A)$ , which is an isomorphism by hypothesis.

Given  $g : X \rightarrow_{\bullet} K(A, n)$ , notice that

$$\Omega(X \rightarrow_{\bullet} K(A, n), g) \simeq \Omega(X \rightarrow_{\bullet} K(A, n), (\lambda x. *))$$

since  $X \rightarrow_{\bullet} K(A, n)$  is a group object, so any loop at  $g$  can be multiplied pointwise by  $g^{-1}$  to get a loop at the constant map  $(\lambda x. *)$ . The same can be argued about  $Y \rightarrow_{\bullet} K(A, n)$ , using  $f^*(g)$  instead of  $g$ . Under this correspondence, given  $k > 0$ ,  $\|\Omega^k f^*\|_0$  corresponds to the induced map in cohomology  $\tilde{H}^{n-k}(X; A) \rightarrow \tilde{H}^{n-k}(Y; A)$ , which again is an isomorphism by hypothesis.  $\square$

**Lemma 3.5.** *For any cohomology isomorphism  $f : Y \rightarrow_{\bullet} X$  and truncated, pointed, nilpotent type  $Z$ , the precomposition map  $f^* : (X \rightarrow_{\bullet} Z) \rightarrow (Y \rightarrow_{\bullet} Z)$  is an equivalence.*

*Proof.* Since we have to prove a mere proposition, we can assume that we have a nilpotent structure for  $Z$ . We can then prove the result by induction on the nilpotency degree of

$Z$ . If  $Z$  is contractible, the statement is clear. So let us assume that we have a principal fibration  $K(A, n) \rightarrow Z \rightarrow Z' \rightarrow K(A, n+1)$ . By mapping out of  $f : Y \rightarrow X$ , we obtain a map of fiber sequences

$$\begin{array}{ccccc} (X \rightarrow_{\bullet} Z) & \hookrightarrow & (X \rightarrow_{\bullet} Z') & \longrightarrow & (X \rightarrow_{\bullet} K(A, n+1)) \\ \downarrow & & \downarrow & & \downarrow \\ (Y \rightarrow_{\bullet} Z) & \hookrightarrow & (Y \rightarrow_{\bullet} Z') & \longrightarrow & (Y \rightarrow_{\bullet} K(A, n+1)) \end{array}$$

By Lemma 3.4, the right vertical map is an equivalence, and by inductive hypothesis, the middle vertical map is an equivalence too. So we conclude that the left vertical map is an equivalence, as needed.  $\square$

**Theorem 3.6.** *A map between pointed, nilpotent types is a cohomology isomorphism if and only if it is  $\infty$ -connected.*

*Proof.* Any  $\infty$ -connected map is a cohomology isomorphism, by Lemma 3.3.

For the other implication, notice that, for all  $n : \mathbb{N}$ , the map  $\|f\|_n : \|X\|_n \rightarrow \|Y\|_n$  is a cohomology isomorphism between pointed, nilpotent,  $n$ -truncated types. This follows from Example 2.47 and Lemma 3.3. Using Lemma 3.5 and a Yoneda-type argument, we deduce that  $\|f\|_n$  is an equivalence for every  $n : \mathbb{N}$ , as required.  $\square$

As an interesting corollary, we get the following.

**Theorem 3.7.** *The suspension of an  $\infty$ -connected map between connected types is  $\infty$ -connected.*

*Proof.* Suppose given an  $\infty$ -connected map  $f : X \rightarrow Y$  between connected types. Since we are proving a mere proposition, we can assume that  $X$ ,  $Y$  and  $f$  are pointed. From Lemma 3.3, it follows that  $f$  is a cohomology isomorphism. Since cohomology isomorphisms are defined by mapping into Eilenberg–Mac Lane spaces, and these are closed under taking loop spaces, the suspension of a cohomology isomorphism is a cohomology isomorphism, by the adjunction between suspension and loop space. So  $\Sigma f : \Sigma X \rightarrow \Sigma Y$  is a cohomology isomorphism between simply connected types. Since any simply connected type is nilpotent, it follows that  $\Sigma f$  is  $\infty$ -connected, as needed.  $\square$

#### 4. LOCALIZATION

This section is about the localization of nilpotent types away from sets of numbers. Let  $S : \mathbb{N} \rightarrow \mathbb{N}$  be a function enumerating a set of natural numbers. A group  $G$  is **deg( $S$ )-local** if, for each  $n : \mathbb{N}$ , the power map  $S(n) : G \rightarrow G$  is a bijection. A morphism of groups  $G \rightarrow_{\text{Grp}} G'$  is an **algebraic localization** of  $G$  away from  $S$  if it is the initial morphism into a **deg( $S$ )-local** group. For a natural number  $m$  and a pointed type  $X$ , let  $m : \Omega X \rightarrow \Omega X$  denote the map that sends a loop  $l$  to  $l^m$ . In (Christensen et al.; 2018), the reflective subuniverse of **deg( $S$ )-local** types is defined. A type  $X$  is **deg( $S$ )-local** if, for each  $n : \mathbb{N}$ , the map  $S(n) : \Omega X \rightarrow \Omega X$  is an equivalence. In Homotopy Type Theory, this localization plays the role of localization away from the set  $S$  of classical homotopy theory.

The main goal of the section is to prove that, when applied to a nilpotent type, **deg( $S$ )-localization** localizes the homotopy groups of the type.

In Section 4.1 we recall the theory of reflective subuniverses and separated types. In Section 4.2 we describe the effect of localization on the homotopy groups of a nilpotent type.

**4.1. Reflective subuniverses and separated types.** We start with a few general results about reflective subuniverses. For details we refer the reader to (Rijke et al.; 2017) and (Christensen et al.; 2018). We will need the notion of separated type for a reflective subuniverse  $L$ .

**Definition 4.1** ((Christensen et al.; 2018, Definition 2.13)). *Let  $L$  be a reflective subuniverse and let  $X : \mathcal{U}$  be a type. We say that  $X$  is  **$L$ -separated** if its identity types are  $L$ -local types. We write  $L'$  for the subuniverse of  $L$ -separated types.*  $\square$

Separated types form a reflective subuniverse.

**Theorem 4.2** ((Christensen et al.; 2018, Theorem 2.26)). *For any reflective subuniverse  $L$ , the subuniverse of  $L$ -separated types is again reflective.*  $\square$

Given a reflective subuniverse  $L$  and a type  $X : \mathcal{U}$ , we denote the unit of the  $L'$ -localization of  $X$  by  $\eta' : X \rightarrow L'X$ .

Given any family of maps  $f : \prod_{i:I} A_i \rightarrow B_i$ , there is an associated reflective subuniverse  $L_f$  of  $f$ -local types (Rijke et al.; 2017, Theorem 2.16). As proven in (Christensen et al.; 2018, Lemma 2.15), the separated types with respect to the subuniverse of  $f$ -local types are precisely the subuniverse of  $\Sigma f$ -local types. Here  $\Sigma f$  denotes the family  $\prod_{i:I} \Sigma A_i \rightarrow \Sigma B_i$  given by suspending  $f_i$  for each  $i : I$ . Moreover, in the case of localization away from sets of numbers, we have the following:

**Proposition 4.3** (see (Christensen et al.; 2018, Theorem 4.11)). *If  $X$  is simply connected, then the natural map  $L_{\Sigma \deg(S)} X \rightarrow L_{\deg(S)} X$  is an equivalence.*

**Definition 4.4.** *Given a reflective subuniverse  $L$ , we say that a map  $f : X \rightarrow Y$  is an  **$L$ -equivalence** if  $Lf : LX \rightarrow LY$  is an equivalence.*

**Definition 4.5.** *We say that a reflective subuniverse  $L$  **preserves a fiber sequence**  $F \rightarrow E \xrightarrow{g} B$  if the induced map  $F \rightarrow \text{fib}_{Lg}(\eta b)$  is an  $L$ -localization. Here  $b : B$  is the base point of  $B$ . In this case we will also say that  $L$  **preserves the fiber** of  $g$ .*

We now prove some useful exactness properties of  $L'$ -localization.

**Lemma 4.6.** *Let  $P : L'X \rightarrow \mathcal{U}$  be a type family over  $L'X$ . Then the map*

$$f : \left( \sum_{(x:X)} P(\eta'(x)) \right) \rightarrow \left( \sum_{(y:L'X)} P(y) \right)$$

*given by  $(x, p) \mapsto (\eta'(x), p)$  is an  $L$ -equivalence.*

*Proof.* Using the characterization of  $L$ -equivalences of (Christensen et al.; 2018, Lemma 2.9), it is enough to prove that  $f$  induces an equivalence of mapping spaces whenever we map into an  $L$ -local type. Assume given an  $L$ -local type  $Z$  and notice that we have the following factorization of  $f^*$ :

$$\begin{aligned} \left( \sum_{(y:L'X)} P(y) \right) \rightarrow Z &\simeq \prod_{(y:L'X)} P(y) \rightarrow Z \\ &\simeq \prod_{(x:X)} P(\eta'x) \rightarrow Z \\ &\simeq \left( \sum_{(x:X)} P(\eta'x) \right) \rightarrow Z, \end{aligned}$$

where, in the second equivalence, we use (Christensen et al.; 2018, Proposition 2.22) together with the fact that  $P(y) \rightarrow Z$  is  $L$ -local, since  $Z$  is.  $\square$

**Proposition 4.7.** *Given a fiber sequence  $F \rightarrow E \xrightarrow{f} X$ , there is a map of fiber sequences*

$$\begin{array}{ccccc} F & \hookrightarrow & E & \longrightarrow & X \\ \eta \downarrow & & \downarrow & & \eta' \downarrow \\ LF & \hookrightarrow & E' & \longrightarrow & L'X \end{array}$$

such that the type  $E'$  is  $L'$ -local and the middle vertical map is an  $L$ -equivalence. Here  $F$  is the fiber over the base point  $x_0 : X$ , and  $LF$  is the fiber over  $\eta'(x_0)$ .

*Proof.* For the proof it is more convenient to work with type families. We start by constructing the map of fiber sequences. Assume given a type  $X$  and a type family  $P : X \rightarrow \mathcal{U}$ , with total space  $E \equiv \sum_{(x:X)} P(x)$ . By (Christensen et al.; 2018, Lemma 2.19), the composite  $L \circ P : X \rightarrow \mathcal{U}_L$  can be extended along the localization  $\eta' : X \rightarrow L'X$  as follows:

$$\begin{array}{ccc} X & \xrightarrow{P} & \mathcal{U} \\ \eta' \downarrow & & \downarrow L \\ L'X & \dashrightarrow^{P'} & \mathcal{U}_L. \end{array}$$

In particular, we have  $t : \prod_{(x:X)} L(P(x)) \simeq P'(\eta'x)$  and the localization units induce a map

$$\begin{aligned} \sum_{(x:X)} P(x) &\xrightarrow{l} \sum_{(y:L'X)} P'(y) \\ (x, p) &\longmapsto (\eta'x, t(\eta p)). \end{aligned}$$

Define  $E' \equiv \sum_{(y:L'X)} P'(y)$ . This gives us the map of fiber sequences.

Now,  $E'$  is  $L'$ -local, by (Christensen et al.; 2018, Lemma 2.21), so to conclude the proof we must show that  $l : E \rightarrow E'$  is an  $L$ -equivalence. To see this observe that the  $L$ -localization of  $l$  factors as the following chain of equivalences

$$\begin{aligned} L\left(\sum_{(x:X)} P(x)\right) &\simeq L\left(\sum_{(x:X)} L(P(x))\right) \\ &\simeq L\left(\sum_{(x:X)} P'(\eta'x)\right) \\ &\simeq L\left(\sum_{(y:L'X)} P'(y)\right) \end{aligned}$$

using (Rijke et al.; 2017, Theorem 1.24) in the first equivalence, the map  $t$  in the second equivalence and Lemma 4.6 in the third one.  $\square$

**Lemma 4.8.** *For  $S : \mathbb{N} \rightarrow \mathbb{N}$  and  $L$  any reflective subuniverse, the  $L'$ -localization of a  $\deg(S)$ -local type is  $\deg(S)$ -local.*

*Proof.* Let  $X$  be a  $\deg(S)$ -local type. Fix  $n : \mathbb{N}$  and let  $k \equiv S(n)$ . We must show that  $k : \Omega L'X \rightarrow \Omega L'X$  is an equivalence for each base point  $x' : L'X$ . Since being an equivalence is a mere proposition, and the localization unit  $X \rightarrow L'X$  is surjective

(Christensen et al.; 2018, Lemma 2.17), we can assume that  $x' = \eta(x)$  for some  $x : X$ . Consider the square

$$\begin{array}{ccc} L\Omega X & \xrightarrow{\sim} & \Omega L'X \\ Lk \downarrow & & \downarrow k \\ L\Omega X & \xrightarrow{\sim} & \Omega L'X. \end{array}$$

To show that the square commutes, it is enough to check it after precomposing with the unit  $\Omega X \rightarrow L\Omega X$ , and this is clear. Finally, the map on the left is an equivalence by hypothesis, and thus  $L'X$  is  $\mathbf{deg}(S)$ -local.  $\square$

**4.2. Localization of homotopy groups.** In this section we prove that  $\mathbf{deg}(S)$ -localization is well behaved when applied to nilpotent types. In particular, we prove that  $\mathbf{deg}(S)$ -localization localizes the homotopy groups of a nilpotent type algebraically.

**Lemma 4.9.** *The total space of a fibration with  $\mathbf{deg}(S)$ -local fibers and simply connected  $\mathbf{deg}(S)$ -local base is  $\mathbf{deg}(S)$ -local.*

*Proof.* Let  $P : B \rightarrow \mathcal{U}_{\mathbf{deg}(S)}$  be a fibration over a simply connected,  $\mathbf{deg}(S)$ -local type. We will prove that for every  $x : \sum_{(b:B)} P(b)$  and every  $k : \mathbb{N}$ , the map  $S(k) : \Omega \left( \sum_{(b:B)} P(b), x \right) \rightarrow \Omega \left( \sum_{(b:B)} P(b), x \right)$  is an equivalence. The point  $x : \sum_{(b:B)} P(b)$  corresponds to a point  $b : B$  together with a point  $f : P(b)$  in its fiber, and thus induces a fiber sequence

$$\Omega(P(b), f) \rightarrow \Omega \left( \sum_{(b:B)} P(b), x \right) \rightarrow \Omega(B, b),$$

where, as usual, the fiber is taken to be the fiber over  $\mathbf{refl}_b : \Omega(B, b)$ . By naturality, this fiber sequence maps to itself by the map  $S(k)$ . By hypothesis, this map of fiber sequences is an equivalence for the base, and thus it is an equivalence for the total space if and only if it is a fiberwise equivalence. Since the base is connected by hypothesis, it is enough to check that  $S(k)$  is an equivalence for the fiber over  $\mathbf{refl}_b : \Omega(B, b)$ . And this last fact holds by hypothesis.  $\square$

**Proposition 4.10.** *Let  $n \geq 1$  and let  $Y \rightarrow X$  be a pointed principal  $K(A, n)$ -bundle. In particular, we have a classifying map  $X \rightarrow K(A, n+1)$ , and fiber sequences  $K(A, n) \rightarrow Y \rightarrow X$  and  $Y \rightarrow X \rightarrow K(A, n+1)$ . Then,  $\mathbf{deg}(S)$ -localization preserves both fiber sequences. Moreover, the localized map  $L_{\mathbf{deg}(S)}Y \rightarrow L_{\mathbf{deg}(S)}X$  is a pointed principal  $K(A', n)$ -bundle, where  $A'$  is the algebraic localization of  $A$ .*

*Proof.* Notice that it is enough to show that  $\mathbf{deg}(S)$ -localization preserves the fiber sequence  $Y \rightarrow X \rightarrow K(A, n+1)$ . This is because, if  $A'$  is the algebraic localization of  $A$  away from  $S$ , looping the localization map  $K(A, n+1) \rightarrow K(A', n+1)$  gives us the localization map  $K(A, n) \rightarrow K(A', n)$  by (Christensen et al.; 2018, Corollary 4.13) and (Christensen et al.; 2018, Theorem 4.22). This implies that  $L_{\mathbf{deg}(S)}Y \rightarrow L_{\mathbf{deg}(S)}X$  is classified by a map  $L_{\mathbf{deg}(S)}X \rightarrow K(A', n+1)$ , and that  $\mathbf{deg}(S)$ -localization preserves the fibration  $K(A, n) \rightarrow Y \rightarrow X$ .

Consider the fiber sequence  $Y \rightarrow X \rightarrow K(A, n+1)$ . Using (Christensen et al.; 2018, Lemma 2.15) and applying Proposition 4.7 to this fiber sequence we get a map of fiber sequences

$$\begin{array}{ccccc}
Y & \longrightarrow & X & \longrightarrow & K(A, n+1) \\
\downarrow & & \downarrow & & \downarrow \\
L_{\deg(S)}Y & \longrightarrow & X' & \longrightarrow & L_{\Sigma \deg(S)}K(A, n+1),
\end{array}$$

where the first two vertical maps are  $\deg(S)$ -equivalences. Since  $K(A, n+1)$  is simply connected, the third vertical map is actually a  $\deg(S)$ -localization by (Christensen et al.; 2018, Theorem 4.11), so it is enough to show that  $X'$  is  $\deg(S)$ -local. To see this, use again the fact that  $K(A, n+1)$  is simply connected, and that  $\deg(S)$ -localization preserves simply connectedness (Christensen et al.; 2018, Corollary 4.14), to deduce that the bottom fiber sequence satisfies the hypothesis of Lemma 4.9. We thus see that  $L_{\deg(S)}Y \rightarrow L_{\deg(S)}X$  is the fiber of the localization of the classifying map  $X \rightarrow K(A, n+1)$ , as needed.  $\square$

Notice that the above proof can be used to prove a slight strengthening of (Christensen et al.; 2018, Theorem 4.16), namely, that  $\deg(S)$ -localization preserves fiber sequences with simply connected base.

As an application of Proposition 4.10, we give a concrete construction of the  $\deg(S)$ -localization of a pointed, truncated type with a nilpotent structure.

**Definition 4.11.** *We give an inductive construction of the  $\deg(S)$ -localization of a pointed, truncated type with a nilpotent structure. For the inductive step, this construction uses Proposition 4.10 and the  $\deg(S)$ -localization of Eilenberg–Mac Lane spaces of abelian groups, which is done in (Christensen et al.; 2018, Section 4.5). This construction is analogous to the classical one (e.g. (May and Ponto; 2012)).*

*Given a pointed, truncated type  $Y$  with a nilpotent structure, we induct over the nilpotency degree of  $Y$ . If  $Y$  is a point, then it is its own  $\deg(S)$ -localization. If  $Y$  is the fiber of a map  $Y' \rightarrow K(A, n+1)$ , then we localize the map, by applying the inductive hypothesis to  $Y'$  and the localization of Eilenberg–Mac Lane spaces to  $K(A, n+1)$ . By Proposition 4.10,  $\deg(S)$ -localization preserves the fiber of the localized map, so the localization of  $Y$  is just the fiber of the localized map.*

**Corollary 4.12.** *For  $n \geq 2$ ,  $\deg(S)$ -localization of a pointed,  $n$ -truncated, nilpotent type is  $n$ -truncated.*

*Proof.* Since being  $n$ -truncated is a mere proposition, we can assume that we have a nilpotent structure for the nilpotent type. The construction of Definition 4.11 finishes the proof.  $\square$

Although we haven't yet proven that the  $\deg(S)$ -localization of a nilpotent type is nilpotent, we can already prove the following fact.

**Corollary 4.13.** *For  $n \geq 2$ ,  $n$ -truncation and  $\deg(S)$ -localization commute, when restricted to pointed, nilpotent types.*

*Proof.* From (Christensen et al.; 2018, Lemma 4.17) we know that the  $n$ -truncation of a  $\deg(S)$ -local type is  $\deg(S)$ -local. It follows from (Christensen et al.; 2018, Lemma 2.11) that for any type  $X$  we have  $\|L_{\deg(S)}X\|_n \simeq \|L_{\deg(S)}\|X\|_n\|_n$ . The result follows from observing that Corollary 4.12 implies that, if  $X$  is nilpotent, we have  $\|L_{\deg(S)}\|X\|_n\|_n \simeq L_{\deg(S)}\|X\|_n$ .  $\square$

Our goal now is to show that  $\mathbf{deg}(S)$ -localization preserves composites of principal fibrations. To this end, we prove a slightly more general result.

**Lemma 4.14.** *Given pointed, composable maps  $Y \xrightarrow{f} Y' \xrightarrow{g} Y''$ , if  $f$  is a pointed principal  $K(A, n)$ -bundle, then the map  $\mathbf{fib}_f(y') \rightarrow \mathbf{fib}_{g \circ f}(g(y'))$  of Definition 2.52 is also a pointed principal  $K(A, n)$ -bundle.*

*Proof.* Let  $f' : Y' \rightarrow K(A, n + 1)$  classify  $f$ . By taking fibers of all the maps in the square

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & K(A, n + 1) \\ g \downarrow & & \downarrow \\ Y'' & \longrightarrow & \mathbf{1} \end{array}$$

we obtain a diagram equivalent to the following

$$\begin{array}{ccccc} \mathbf{fib}_{g \circ f}(g(y')) & \longrightarrow & \mathbf{fib}_g(g(y')) & \longrightarrow & K(A, n + 1) \\ \downarrow & & \downarrow & & \parallel \\ Y & \xrightarrow{f} & Y' & \xrightarrow{f'} & K(A, n + 1) \\ g \circ f \downarrow & & g \downarrow & & \downarrow \\ Y'' & \xlongequal{\quad} & Y'' & \longrightarrow & \mathbf{1}. \end{array}$$

Now notice that the induced fiber sequence of Definition 2.52 comes from extending the top fiber sequence to the left.  $\square$

**Proposition 4.15.** *Given pointed, composable maps  $Y \xrightarrow{f} Y' \xrightarrow{g} Y''$ , if  $\mathbf{deg}(S)$ -localization preserves the fiber of  $g$ , and  $f$  is a pointed principal  $K(-, n)$ -bundle, then  $\mathbf{deg}(S)$ -localization preserves the fiber of  $g \circ f$ .*

*Proof.* Consider the square in the beginning of the previous proof and localize it. Taking fibers of all the maps in the localized square we claim that we get a diagram equivalent to the following

$$\begin{array}{ccccc} L_{\mathbf{deg}(S)} \mathbf{fib}_g(g(y')) & \longrightarrow & K(A', n + 1) & & \\ \downarrow & & \downarrow & & \parallel \\ L_{\mathbf{deg}(S)} Y & \xrightarrow{L_{\mathbf{deg}(S)} f} & L_{\mathbf{deg}(S)} Y' & \longrightarrow & K(A', n + 1) \\ L_{\mathbf{deg}(S)}(g \circ f) \downarrow & & L_{\mathbf{deg}(S)} g \downarrow & & \downarrow \\ L_{\mathbf{deg}(S)} Y'' & \xlongequal{\quad} & L_{\mathbf{deg}(S)} Y'' & \longrightarrow & \mathbf{1}, \end{array}$$

where  $A'$  is the algebraic localization of  $A$  away from  $S$ . This is clear for the right vertical fiber sequence and the bottom fiber sequence. It is true for the middle vertical fiber sequence by hypothesis, and it is true for the middle horizontal fiber sequence by Proposition 4.10. Finally, notice that taking the fiber of the top right map we get  $L_{\mathbf{deg}(S)} \mathbf{fib}_{g \circ f}(g(y'))$  by Lemma 4.14 and Proposition 4.10, so, by commutativity of limits, we see that  $\mathbf{deg}(S)$ -localization preserves the fiber of  $g \circ f$ .  $\square$

**Proposition 4.16.**  *$\deg(S)$ -localization preserves the fiber of pointed maps that factor as composites of pointed principal fibrations.*

*Proof.* We do induction on the number of maps in the factorization. The base case can be taken to be the empty factorization, that is, the case when the map is an identity map between the same type. This case is immediate.

For the inductive step, use Proposition 4.15 together with Proposition 4.10.  $\square$

**Corollary 4.17.** *If  $X$  is a pointed, nilpotent type, then for every  $n : \mathbb{N}$ ,  $\deg(S)$ -localization preserves the fiber of  $\|X\|_{n+1} \rightarrow \|X\|_n$ .*

*Proof.*  $\deg(S)$ -localization preserving the fiber of a map is a mere proposition, so the result follows from Proposition 4.16.  $\square$

**Corollary 4.18.** *The  $\deg(S)$ -localization of a nilpotent type is nilpotent.*

*Proof.* Given  $X$  nilpotent, use Corollary 4.13 to conclude that the  $\deg(S)$ -localization of the truncation map  $\|X\|_{n+1} \rightarrow \|X\|_n$  is equivalent to the truncation map  $\|L_{\deg(S)}X\|_{n+1} \rightarrow \|L_{\deg(S)}X\|_n$ . Now Corollary 4.17 implies that this map, being equivalent to a  $\deg(S)$ -localization of a composite of principal fibrations, must be a composite of principal fibrations.  $\square$

**Theorem 4.19.** *The  $\deg(S)$ -localization of a pointed, nilpotent type  $X$  localizes all of the homotopy groups away from  $S$ .*

The proof is now essentially the same as the proof of (Christensen et al.; 2018, Theorem 4.25).

*Proof.* The fact that  $\deg(S)$ -localization localizes  $\pi_1(X)$  is a special case of (Christensen et al.; 2018, Theorem 4.15). Now, take  $n \geq 1$  and consider the fiber sequence

$$K(\pi_{n+1}(X), n+1) \hookrightarrow \|X\|_{n+1} \twoheadrightarrow \|X\|_n.$$

Notice that all the types in the fiber sequence are nilpotent. Applying  $\deg(S)$ -localization we obtain a map of fiber sequences

$$\begin{array}{ccccc} K(\pi_{n+1}(X), n+1) & \hookrightarrow & \|X\|_{n+1} & \twoheadrightarrow & \|X\|_n \\ \downarrow & & \downarrow & & \downarrow \\ L_{\deg(S)}K(\pi_{n+1}(X), n+1) & \hookrightarrow & \|L_{\deg(S)}X\|_{n+1} & \twoheadrightarrow & \|L_{\deg(S)}X\|_n \end{array}$$

by Corollary 4.17 and Corollary 4.13. Looking at the bottom fiber sequence, we see that  $\pi_n(L_{\deg(S)}X)$  is the algebraic localization of  $\pi_n(X)$ , by (Christensen et al.; 2018, Theorem 4.22), concluding the proof.  $\square$

## 5. FRACTURE SQUARES

In this section we construct two fracture squares. The first one holds for any simply connected type. Its construction is different from the classical construction that can be found in, e.g., (May and Ponto; 2012). A different construction is needed in Homotopy Type Theory if one wants to avoid the use of Whitehead's theorem. The second square applies to nilpotent types, but requires that they are truncated. Its construction is analogous to the classical construction.

In Section 5.1 we show how we can treat simply connected types as generalized abelian groups, and construct a fracture square for simply connected types with a simple and direct argument. In Section 5.2 we use Postnikov towers to construct a fracture square for truncated nilpotent types.

**5.1. A fracture theorem for simply connected types.** In this section we will prove one of the many fracture theorems. The classical version is as follows. We are given two subrings of the rational numbers  $R, S \subseteq \mathbb{Q}$  with  $R \cap S = \mathbb{Z}$ , and a pointed simply connected space  $X$ . The theorem then says that the following square given by localizations is a homotopy pullback

$$\begin{array}{ccc} X & \longrightarrow & X_S \\ \downarrow & & \downarrow \\ X_R & \longrightarrow & X_T \end{array}$$

where  $T = R \otimes S$  and the localizations correspond to inverting the units in the corresponding ring.

In the constructive setting we let  $R, S : \mathbb{N} \rightarrow \mathbb{N}$  denote denumerable (multi) sets of natural numbers, and we define  $T : \mathbb{N} \rightarrow \mathbb{N}$  by multiplying  $R$  and  $S$  pointwise. We also assume that for all  $n, m : \mathbb{N}$ ,  $R_n$  is coprime to  $S_m$ .

We start by constructing a candidate for the fracture square in Homotopy Type Theory. For this, we need the following.

**Lemma 5.1.** *The maps*

$$\begin{aligned} L_{\deg(R)}X &\rightarrow L_{\deg(S)}L_{\deg(R)}X, \\ L_{\deg(S)}X &\rightarrow L_{\deg(R)}L_{\deg(S)}X, \end{aligned}$$

and the composites

$$\begin{aligned} X &\rightarrow L_{\deg(R)}X \rightarrow L_{\deg(S)}L_{\deg(R)}X, \\ X &\rightarrow L_{\deg(S)}X \rightarrow L_{\deg(R)}L_{\deg(S)}X, \end{aligned}$$

are  $\deg(T)$ -localizations. □

*Proof.* This follows from Lemma 4.8 and the fact that for a simply connected type  $\deg(S)$ - and  $\Sigma \deg(S)$ -localization coincide (Proposition 4.3). □

This lets us write  $L_{\deg(T)}X$  instead of  $L_{\deg(S)}L_{\deg(R)}X$  and  $L_{\deg(R)}L_{\deg(S)}X$ . In particular we have a commutative square

$$\begin{array}{ccc} X & \longrightarrow & L_{\deg(S)}X \\ \downarrow & & \downarrow \\ L_{\deg(R)}X & \longrightarrow & L_{\deg(T)}X, \end{array}$$

such that all of its maps are localizations.

**Lemma 5.2.** *Let  $n, m : \mathbb{N}$  be coprime numbers. Then, for every pointed, simply connected type  $X$ , the square*

$$\begin{array}{ccc}
\Omega X & \xrightarrow{n} & \Omega X \\
m \downarrow & & \downarrow m \\
\Omega X & \xrightarrow{n} & \Omega X
\end{array}$$

is a pullback.

*Proof.* Let  $\alpha n + \beta m = 1$ . Since  $X$  is connected,  $\Omega$  reflects equivalences, and since it also preserves limits, the square in the statement is a pullback if and only if it is after looping. Let  $Y := \Omega^2 X$ . It is then enough to show that

$$Y \xrightarrow{\binom{m}{n}} Y \times Y \xrightarrow{\binom{n}{-m}} Y$$

is a fiber sequence, where we are considering the fiber over  $\text{refl}$ . Here we are using matrix notation, so that, for example, the map  $\binom{n}{-m} : Y \times Y \rightarrow Y$  is given by  $(l, l') \mapsto (l^n \cdot l'^{-m})$ . The reason why it is enough to consider the above fiber sequence is that the fiber of  $\binom{n}{-m}$  is  $\sum_{(l, l': Y)} l^n \cdot l'^{-m} = \text{refl}$ , which is equivalent to the pullback of  $Y \xrightarrow{n} Y \xleftarrow{m} Y$ , namely  $\sum_{(l, l': Y)} l^n = l'^m$ .

Notice that we have maps

$$\binom{n}{\beta} \quad \binom{-m}{\alpha}, \quad \binom{\alpha}{-\beta} \quad \binom{m}{n} : Y \times Y \rightarrow Y \times Y$$

and these are mutual inverses, using the fact that  $Y$  is homotopy commutative.

This means that we have an equivalence of maps

$$\begin{array}{ccc}
Y \times Y & \xrightarrow{\binom{n}{-m}} & Y \\
\binom{n}{\beta} \quad \binom{-m}{\alpha} \downarrow & & \downarrow \parallel \\
Y \times Y & \xrightarrow{\binom{1}{0}} & Y
\end{array}$$

But the fiber of the bottom map is clearly  $Y$ , so using the inverse of the left vertical map we get the fiber sequence that we needed.  $\square$

*Remark 5.3.* Define  $r, s, t, \rho, \sigma : \mathbb{N} \rightarrow \mathbb{N}$  by:

$$r_n := \prod_{i=0}^n R_i, \quad s_n := \prod_{i=0}^n S_i, \quad t_n := r_n \times s_n, \quad \rho_n := \prod_{i=0}^n r_i, \quad \sigma_n := \prod_{i=0}^n s_i.$$

Then if  $H$  is a loop space, the following diagram commutes:

$$\begin{array}{ccccccc}
 H & \xrightarrow{r_1} & H & \xrightarrow{r_2} & H & \xrightarrow{r_3} & H & \xrightarrow{r_4} & \dots \\
 \sigma_0 \downarrow & & \sigma_1 \downarrow & & \sigma_2 \downarrow & & \sigma_3 \downarrow & & \\
 H & \xrightarrow{t_1} & H & \xrightarrow{t_2} & H & \xrightarrow{t_3} & H & \xrightarrow{t_4} & \dots \\
 \rho_0 \uparrow & & \rho_1 \uparrow & & \rho_2 \uparrow & & \rho_3 \uparrow & & \\
 H & \xrightarrow{s_1} & H & \xrightarrow{s_2} & H & \xrightarrow{s_3} & H & \xrightarrow{s_4} & \dots
 \end{array} \tag{5.1}$$

**Theorem 5.4.** *If  $X$  is simply connected and for all  $n, m : \mathbb{N}$ ,  $R_n$  is coprime to  $S_m$ , then the square*

$$\begin{array}{ccc}
 X & \longrightarrow & L_{\deg(S)}X \\
 \downarrow & & \downarrow \\
 L_{\deg(R)}X & \longrightarrow & L_{\deg(T)}X,
 \end{array}$$

is a pullback.

*Proof.* Since we must prove a mere proposition and  $X$  is connected, we can assume that we have  $x : X$ . Since  $X$  is connected, and the loop functor from pointed connected types to types preserves limits and reflects equivalences, the square is a pullback if and only if it is after looping. Using the fact that  $X$  is simply connected, we see that applying loop space to the diagram, we get a diagram equivalent to

$$\begin{array}{ccc}
 \Omega X & \longrightarrow & L_{\deg(S)}\Omega X \\
 \downarrow & & \downarrow \\
 L_{\deg(R)}\Omega X & \longrightarrow & L_{\deg(T)}\Omega X
 \end{array}$$

by (Christensen et al.; 2018, Corollary 4.13).

By taking  $H$  to be  $\Omega X$  in Eq. (5.1) and taking the colimit of each of the rows we get the diagram:

$$L_{\deg(R)}\Omega X \rightarrow L_{\deg(T)}\Omega X \leftarrow L_{\deg(S)}\Omega X,$$

by (Christensen et al.; 2018, Theorem 4.20).

Observe that by hypothesis  $\rho_n$  and  $\sigma_n$  are coprime, so by Lemma 5.2, if we instead take the pullback in each column we get a diagram equivalent to:

$$\Omega X \rightarrow \Omega X \rightarrow \Omega X \rightarrow \Omega X \rightarrow \dots$$

Let us assume for a moment that the maps in this diagram are the identity. Then the colimit of this diagram is  $\Omega X$ , so the result follows from the commutativity of pullbacks and filtered colimits (Doorn et al.; 2018).

To prove that the maps are the identity observe that, by Lemma 5.2, in the  $n$ -th place we have a map of pullbacks:

$$\begin{array}{ccccc}
\Omega X & \overset{\text{-----}}{\longrightarrow} & \Omega X & & \\
\downarrow \sigma_n & \searrow \rho_n & \Omega X & \xrightarrow{r_{n+1}} & \Omega X & \swarrow \rho_{n+1} & \downarrow \sigma_{n+1} \\
& & \Omega X & \downarrow \sigma_n & \Omega X & \downarrow \sigma_{n+1} & \\
& & \Omega X & \xrightarrow{t_{n+1}} & \Omega X & & \\
& \swarrow \rho_n & & & & \swarrow \rho_{n+1} & \\
\Omega X & \xrightarrow{s_{n+1}} & \Omega X & & \Omega X & & \\
& & & & & & 
\end{array}$$

where the pullbacks are the left and right squares, and the dotted map is the map we want to show is the identity. Since this map is induced by the universal property of pullbacks, it is enough to check that the identity makes the cube commute, which is straightforward.  $\square$

**5.2. A fracture theorem for truncated nilpotent types.** We now prove an analogous fracture theorem, but for truncated nilpotent types. Getting rid of the truncation hypothesis remains an open problem.

We start with a fracture theorem for abelian groups.

**Proposition 5.5.** *Given an abelian group  $A$  and  $n \geq 1$ , if for all  $n, m : \mathbb{N}$ ,  $R_n$  is coprime to  $S_m$ , then the square*

$$\begin{array}{ccc}
K(A, n) & \longrightarrow & L_{\deg(S)}K(A, n) \\
\downarrow & & \downarrow \\
L_{\deg(R)}K(A, n) & \longrightarrow & L_{\deg(T)}K(A, n)
\end{array}$$

is a pullback.

*Proof.* Use Theorem 5.4, taking  $X$  to be  $K(A, n + 1)$ . Looping the pullback square, we get a pullback square. To conclude use the fact that, since  $A$  is abelian, we have  $\Omega(L_{\deg(R)}K(A, n + 1)) \simeq L_{\deg(R)}K(A, n)$  by (Christensen et al.; 2018, Lemma 4.13), and likewise for  $S$  and  $T$ .  $\square$

**Theorem 5.6.** *If  $X$  is truncated and nilpotent and for all  $n, m : \mathbb{N}$ ,  $R_n$  is coprime to  $S_m$ , then the square*

$$\begin{array}{ccc}
X & \longrightarrow & L_{\deg(S)}X \\
\downarrow & & \downarrow \\
L_{\deg(R)}X & \longrightarrow & L_{\deg(T)}X
\end{array}$$

is a pullback.

The following proof is classical, and appears in (May and Ponto; 2012).

*Proof.* Since we have to prove a mere proposition, we can assume given a nilpotent structure for  $X$ . We proceed by induction on the nilpotency degree of  $X$ . The base case is clear. For the inductive step, let  $K(A, n) \rightarrow X \rightarrow X'$  be a principal fibration, and consider its classifying map  $X' \rightarrow K(A, n + 1)$ . We can form the following diagram

$$\begin{array}{ccccc}
 L_{\deg(R)}X' & \longrightarrow & L_{\deg(R)}K(A, n + 1) & \longleftarrow & \mathbf{1} \\
 \downarrow & & \downarrow & & \downarrow \\
 L_{\deg(T)}X' & \longrightarrow & L_{\deg(T)}K(A, n + 1) & \longleftarrow & \mathbf{1} \\
 \uparrow & & \uparrow & & \uparrow \\
 L_{\deg(S)}X' & \longrightarrow & L_{\deg(S)}K(A, n + 1) & \longleftarrow & \mathbf{1}.
 \end{array}$$

Using Proposition 4.10, we see that taking the pullback of each row gives us the cospan  $L_{\deg(R)}X \rightarrow L_{\deg(T)}X \leftarrow L_{\deg(S)}X$ . On the other hand, by taking the pullback of each column, we get the cospan  $X' \rightarrow K(A, n) \leftarrow \mathbf{1}$ . So by the commutativity of limits, we are done.  $\square$

## 6. CONCLUSIONS

We have developed the theory of nilpotent types in Homotopy Type Theory, including some of their cohomological properties and the basic properties of their localizations away from sets of numbers, such as the effect of localization on their homotopy groups.

As part of this, we developed the theory of  $K(A, n)$ -bundles, and their induced group actions. We saw how Homotopy Type Theory helps in seeing this theory as a generalization of the theory of groups and group actions.

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