# **Dynamic Game Semantics**

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The present paper gives a mathematical, in particular, *syntax-independent*, formulation of *intensionality* and *dynamics* of computation in terms of games and strategies. Specifically, we give a game semantics for a higher-order programming language that distinguishes programs with the same value yet different algorithms (or intensionality), equipped with the *hiding operation* on strategies that precisely corresponds to the (small-step) operational semantics (or dynamics) of the language. Categorically, our games and strategies give rise to a *cartesian closed bicategory*, and our game semantics forms an instance of a generalization of the standard interpretation of functional programming languages in cartesian closed categories. This work is intended to be the first step towards a mathematical (both categorical and game-semantic) foundation of intensional and dynamic aspects of logic and computation; our approach should be applicable to a wide range of logics and computations.

### 1. Introduction

In (Girard et al., 1989), J.-Y. Girard mentions the dichotomy between the *static* and the *dynamic* viewpoints in logic and computation; the former identifies terms (i.e., proofs or programs) with their *denotations* (i.e., results of their computations in an ideal sense), while the latter focuses on their *senses* (i.e., algorithms or intensionality) and dynamics (i.e., proof-normalization or reduction). This distinction has been certainly reflected in the two mutually complementary semantics of programming languages: *denotational* and *operational* ones (Amadio and Curien, 1998; Winskel, 1993; Gunter, 1992). He points out that a *mathematical* formulation of the former has been well-developed, based on Scott's beautiful *domain theory* (Scott, 1976; Gierz et al., 2003; Abramsky and Jung, 1994), but it is not the case for the latter; the treatment of senses has been based on ad-hoc *syntactic manipulation*. He then emphasizes the importance of *mathematics of senses*:

The establishment of a truly operational semantics of algorithms is perhaps the most important problem in computer science (Girard et al., 1989).

The present work addresses this problem; specifically, it gives an interpretation  $\llbracket_{-} \rrbracket_{\mathcal{D}}$ of a programming language  $\mathcal{L}$  with a small-step operational semantics  $\rightarrow$  and a syntaxindependent operation  $\mathcal{H}$  that satisfy the following *dynamic correspondence property* (DCP):  $M_1 \rightarrow M_2$  if and only if (a.k.a. iff)  $\llbracket M_1 \rrbracket_{\mathcal{D}} \neq \llbracket M_2 \rrbracket_{\mathcal{D}}$  and  $\mathcal{H}(\llbracket M_1 \rrbracket_{\mathcal{D}}) = \llbracket M_2 \rrbracket_{\mathcal{D}}$ for any programs  $M_1$  and  $M_2$  of  $\mathcal{L}$ . Note that the 'only if' and 'if' directions correspond

respectively to certain soundness and completeness properties of the interpretation  $\mathcal{H}$  of  $\rightarrow$ . Note also that the interpretation  $\llbracket_{-} \rrbracket_{\mathcal{D}}$  is *finer* than the usual (sound) denotational semantics because  $M_1 \rightarrow M_2$  implies  $\llbracket M_1 \rrbracket_{\mathcal{D}} \neq \llbracket M_2 \rrbracket_{\mathcal{D}}$ . Thus, the interpretation  $\llbracket_{-} \rrbracket_{\mathcal{D}}$  and the operation  $\mathcal{H}$  capture *intensionality* and *dynamics* of computation, respectively.

Although our framework is intended to be a *general* approach, being applicable to a wide range of logics and computations, as the first step, we focus on a finite fragment of the programming language PCF (Scott, 1993; Plotkin, 1977) customized for our aim.

### 1.1. Game Semantics

Our approach is based on *game semantics* (Abramsky et al., 1997; Hyland, 1997), a particular kind of denotational semantics of logic and computation, in which formulas (or types) and proofs (or programs) are interpreted as *games* and *strategies*, respectively.

We employ game semantics for its conceptual naturality and mathematical precision, which has been demonstrated by various *full completeness* and *full abstraction* results (Curien, 2007) in the literature, leading to a conceptually and mathematically deeper understanding of logic and computation. Also, game semantics is very flexible: It has modeled a wide range of formal systems and programming languages by simply varying constraints on strategies (Abramsky and McCusker, 1999), which enables us to compare and relate various concepts *syntax-independently*. We hope that these advantages of game semantics are true also for intensionality and dynamics of logic and computation.

A game, roughly, is a certain kind of a rooted forest whose branches represent possible 'developments' or (valid) positions of a 'game in the usual sense' (such as chess, poker, etc.). Moves of a game are nodes of the game, where some moves are distinguished and called *initial*; only initial moves can be the first element (or occurrence) of a position of the game. Plays of a game are (finitely or infinitely) increasing sequences ( $\epsilon$ ,  $m_1$ ,  $m_1m_2$ ,...) of positions of the game, where  $\epsilon$  is the *empty sequence*. For our purpose, it suffices to focus on rather standard sequential (as opposed to concurrent (Abramsky and Melliès, 1999)), unpolarized (as opposed to polarized (Laurent, 2004)) games played by two participants, Player, representing a 'computational agent', and Opponent, representing an 'environment', in each of which Opponent always starts a play (i.e., unpolarized), and then they alternately and separately perform moves (i.e., sequential) allowed by the rules of the game. Strictly speaking, a position of each game is not just a finite sequence of moves: Each occurrence m of Opponent's or O- (resp. Player's or P-) non-initial move in a position is assigned or points to a previous occurrence m' of P- (resp. O-) move in the position, representing that m is performed specifically as a response to m'.

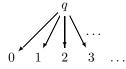
A strategy on a game, on the other hand, is what tells Player which move (together with a pointer) she should make at each of her turns in the game. Hence, a game semantics  $\llbracket_{-}\rrbracket_{\mathcal{G}}$  of a programming language  $\mathcal{L}$  interprets a type A of  $\mathcal{L}$  as a game  $\llbracket A \rrbracket_{\mathcal{G}}$  that specifies possible plays between Player and Opponent, and a term  $M : A^{\dagger}$  of  $\mathcal{L}$  as a strategy  $\llbracket M \rrbracket_{\mathcal{G}}$ on the game  $\llbracket A \rrbracket_{\mathcal{G}}$  that describes for Player how to play on  $\llbracket A \rrbracket_{\mathcal{G}}$ ; an execution of the term M is then modeled as a play of the game  $\llbracket A \rrbracket_{\mathcal{G}}$  in which Player follows  $\llbracket M \rrbracket_{\mathcal{G}}$ .

 $<sup>^{\</sup>dagger}$  For simplicity, here we focus on closed terms, i.e., ones with the  $empty\ context.$ 

### Dynamic Game Semantics

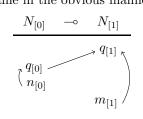
Let us consider a simple example. The simplest game is the *terminal game* T, which has no moves, and thus it has only the trivial position  $\epsilon$ .

As another example, consider the *natural number game* N, which is the following rooted tree (which is infinite in width):



in which a play starts with Opponent's question q ('What is your number?') and ends with Player's answer  $n \in \mathbb{N}$  ('My number is n!'), where  $\mathbb{N}$  is the set of all natural numbers, and n points to q (though this pointer is omitted in the above diagram). A strategy <u>10</u> on N, for instance, that corresponds to  $10 \in \mathbb{N}$  can be represented by the map  $q \mapsto 10$ equipped with a pointer from 10 to q (though it is the only choice). In the following, pointers of most strategies are obvious, and thus we often omit them.

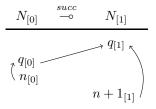
As yet another example, consider the game  $N \multimap N$  of *linear functions* (Girard, 1987) (also written informally  $N_{[0]} \multimap N_{[1]}$ ) on natural numbers, whose typical maximal position is  $q_{[1]}q_{[0]}n_{[0]}m_{[1]}$ , where  $n, m \in \mathbb{N}$ , and  $(\_)_{[i]}$  for i = 0, 1 are arbitrary, unspecified 'tags' to distinguish the two copies of N (in the rest of the paper, we employ a similar notation for three or more copies of a game in the obvious manner too), or diagrammatically<sup>‡</sup>:



which can be read as follows:

- 1 Opponent's question  $q_{[1]}$  for an output ('What is your output?');
- 2 Player's question  $q_{[0]}$  for an input ('Wait, what is your input?);
- 3 Opponent's answer, say,  $n_{[0]}$  to  $q_{[0]}$  ('OK, here is an input n.');
- 4 Player's answer, say,  $m_{[1]}$  to  $q_{[1]}$  ('Alright, the output is then m.').

A strategy *succ* on this game that corresponds to the (linear) successor function can be represented by the map  $q_{[1]} \mapsto q_{[0]}, q_{[1]}q_{[0]}n_{[0]} \mapsto n + 1_{[1]}$ , where n ranges over N, or diagrammatically:

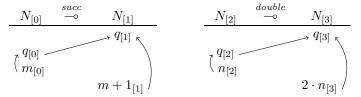


<sup>&</sup>lt;sup>‡</sup> The diagram is depicted as above only to clarify which component game each move belongs to; it should be read just as a finite sequence, namely,  $q_{[1]}q_{[0]}n_{[0]}m_{[1]}$ , equipped with the pointers represented by the arrows in the diagram.

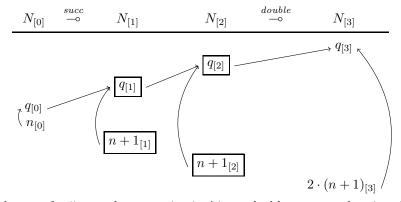
# 1.2. Static Game Semantics

Game semantics is often said to be an *intensional, dynamic* semantics for a category of games and strategies is usually not well-pointed, and plays of a game may be regarded as 'intensional, dynamic interactions' between the participants of the game. However, it has been employed as denotational semantics, and thus it is in particular *sound*: If two programs evaluate to the same value, then their denotations in conventional game semantics are identical. Consequently, conventional game semantics  $[\_]_{\mathcal{G}}$  is actually *extensional* and *static* in the sense that if there is a reduction  $M_1 \rightarrow M_2$  in syntax, then the equation  $[M_1]_{\mathcal{G}} = [[M_2]]_{\mathcal{G}}$  holds in the semantics (i.e., it does not capture the dynamics  $M_1 \rightarrow M_2$  or the intensional difference between  $M_1$  and  $M_2$ ). In other words, it is *not* intensional or dynamic in the sense that it does not satisfy DCPs.

Therefore, to establish mathematics of senses, we need to introduce a more dynamic, intensional refinement of games and strategies so that it satisfies DCPs for logical systems and programming languages. To get some insights to develop such games and strategies, let us see how conventional game semantics fails to be dynamic or intensional. The point in a word is that 'internal communication' between strategies for their composition is a priori 'hidden', and thus the resulting strategy is always in 'normal form'. For instance, the composition succ; double :  $N \multimap N$  of strategies  $succ : N \multimap N$  and double :  $N \multimap N$ , implementing the successor and the doubling (linear) functions, respectively,



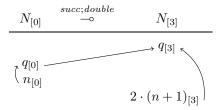
is formed as follows. First, by 'internal communication', we mean that Player plays the role of Opponent in the intermediate component games  $N_{[1]}$  and  $N_{[2]}$  just by 'copycatting' her last moves, resulting in the following play:



where each move for 'internal communication' is marked by a square box just for clarity, and the pointer from  $q_{[1]}$  to  $q_{[2]}$  is added because the move  $q_{[1]}$  is no longer initial. Importantly, it is assumed that Opponent plays on the game  $N_{[0]} \multimap N_{[3]}$ , 'seeing' only moves of  $N_{[0]}$  or  $N_{[3]}$ . The resulting play is to be read as follows:

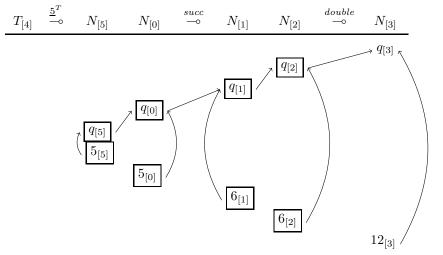
- Opponent's question  $q_{[3]}$  for an output in  $N_{[0]} \multimap N_{[3]}$  ('What is your output?'); 1
- Player's question  $q_{[2]}$  by *double* for an input in  $N_{[2]} \multimap N_{[3]}$  ('What is an input?'); 2
- $q_{[2]}$  triggers the question  $q_{[1]}$  for an output in  $N_{[0]} \multimap N_{[1]}$  ('What is an output?'); 3
- Player's question  $q_{[0]}$  by succ for an input in  $N_{[0]} \multimap N_{[1]}$  ('Wait, what is an input?'); 4
- Opponent's answer, say,  $n_{[0]}$  to  $q_{[0]}$  in  $N_{[0]} \multimap N_{[3]}$  ('Here is an input n.'); 5
- 6
- Player's answer  $n + 1_{[1]}$  to  $\overline{q_{[1]}}$  by *succ* in  $N_{[0]} \multimap N_{[1]}$  ('The output is then n + 1.');  $n + 1_{[1]}$  triggers the answer  $n + 1_{[2]}$  to  $\overline{q_{[2]}}$  in  $N_{[2]} \multimap N_{[3]}$  ('Here is the input 7 n+1.'):
- Player's answer  $2 \cdot (n+1)_{[3]}$  to  $q_{[3]}$  by double in  $N_{[2]} \rightarrow N_{[3]}$  ('The output is then 8  $2 \cdot (n+1)!'$ ).

Next, 'hiding' means to hide or delete every move with a square box from the play, resulting in the strategy for the (linear) function  $n \mapsto 2 \cdot (n+1)$  as expected:



Note that it is 'hiding' that makes the resulting play a valid one on the game  $N \rightarrow N$ .

Now, let us plug in the strategy  $\underline{5}^T : q_{[5]} \mapsto 5_{[5]}$  on the game  $T_{[4]} \multimap N_{[5]}$ , which coincides with N up to 'tags'. The composition  $\underline{5}^T$ ; succ; double :  $T \multimap N^{\S}$  is computed again by 'internal communication':



plus 'hiding':

§ Composition of strategies isassociative(Abramsky et al., 1997; Hyland, 1997; Abramsky and McCusker, 1999); thus, the order of applying composition does not matter.

$$\begin{array}{c} T_{[4]} & \stackrel{\underline{5}^{T}; succ; double}{\multimap} & N_{[3]} \\ & & & \\ & &$$

In syntax, on the other hand, assuming that there are a (ground) type  $\iota$  of natural numbers, a numeral <u>n</u> of type  $\iota$  for each  $n \in \mathbb{N}$ , and constants succ and double of type  $\iota$  for the successor and the doubling functions, respectively, equipped with the operational semantics succ  $\underline{n} \rightarrow \underline{n+1}$  and double  $\underline{n} \rightarrow \underline{2 \cdot n}$  for all  $n \in \mathbb{N}$  in an arbitrary functional programming language, the program  $p_1 \stackrel{\text{df.}}{\equiv} \lambda x.(\lambda y. \text{double } y)((\lambda z. \text{ succ } z) x)$  represents the syntactic composition succ; double. When it is applied to the numeral <u>5</u>, we have the following chain of reductions:

$$\begin{array}{l} \mathsf{p}_1 \, \underline{5} \to^* (\lambda \mathsf{x}. \, \mathsf{double} \, (\mathsf{succ} \, \mathsf{x})) \, \underline{5} \\ \to^* \, \mathsf{double} \, (\mathsf{succ} \, \underline{5}) \\ \to^* \, \mathsf{double} \, \underline{6} \\ \to^* \, \underline{12}. \end{array}$$

Therefore, it seems that reduction in syntax corresponds in game semantics to 'hiding internal communication'. As seen in the above example, however, this game-semantic normalization is a priori executed and thus invisible in conventional game semantics  $[-]_{\mathcal{G}}$ . As a result, the two programs  $p_1 \underline{5}$  and  $\underline{12}$  are interpreted by  $[-]_{\mathcal{G}}$  as the same strategy. Moreover, observe that moves with a square box describe *intensionality* or *step-by-step processes* to compute an output from an input, but they are invisible after 'hiding'. Thus, e.g., a program  $p_2 \stackrel{\text{df.}}{\equiv} \lambda x.(\lambda y. \operatorname{succ} y)(\lambda v.(\lambda z. \operatorname{succ} z)((\lambda w. \operatorname{double} w) v)x)$ , representing the same function as  $p_1$  yet a different algorithm double; succ; succ is modeled as:

 $[\![p_2]\!]_{\mathcal{G}} = [\![double; succ; succ]\!]_{\mathcal{G}} = [\![succ; double]\!]_{\mathcal{G}} = [\![p_1]\!]_{\mathcal{G}}.$ 

To sum up, we have observed the following:

- 1 (REDUCTION AS HIDING). Reduction in syntax corresponds in game semantics to 'hiding intermediate moves (i.e., moves with a square box)';
- 2 (A PRIORI NORMALIZATION). However, the 'hiding' process is *a priori* executed in conventional game semantics, and thus strategies are always in 'normal form';
- 3 (INTERMEDIATE MOVES AS INTENSIONALITY). Also, 'intermediate moves' constitute *intensionality* of computation; however, they are not captured in conventional game semantics again due to the a priori execution of the 'hiding' operation.

# 1.3. Dynamic Games and Strategies

From these observations, we have obtained a promising solution: to define a variant of games and strategies, in which 'intermediate moves' are not a priori 'hidden', representing intensionality of logic and computation, and the *hiding operations*  $\mathcal{H}$  on the games and strategies that 'hide intermediate moves' in a step-by-step fashion, interpreting dynamics of logic and computation. Let us call such a variant of games (resp. strategies) *dynamic games* (resp. *dynamic strategies*).

### Dynamic Game Semantics

In doing so, we shall develop mathematical structures that are conceptually natural and mathematically elegant. This effort is to inherit the natural, intuitive nature of conventional game semantics so that the resulting interpretation would be insightful, convincing and useful. Also, mathematics often leads to a 'correct' formulation: If a definition gives rise to neat mathematical structures, then it is likely to succeed in capturing the essence of concepts and phenomena of concern, and subsume various instances (n.b., recall that our aim is to establish *mathematics* of senses). In fact, dynamic games and strategies are a natural generalization of conventional games and strategies, and they satisfy beautiful algebraic laws; as a consequence, they form a *cartesian closed bicategory (CCB)* in the sense of (Ouaknine, 1997)  $\mathcal{LDG}$  (Definition 4.1), in which 0- (resp. 1-) cells are certain dynamic games (resp. dynamic strategies), and 2-cells are the *extensional equivalence* between 1-cells; the countably-infinite iteration of the hiding operations  $\mathcal{H}$  on dynamic games and strategies induces the 2-functor  $\mathcal{H}^{\omega} : \mathcal{LDG} \to \mathcal{LMG}$ , where the CCC  $\mathcal{LMG}$ of conventional games and strategies can be seen as an 'extensionally collapsed'  $\mathcal{LDG}$ .

### 1.4. Dynamic Game Semantics

We then give, as the main result of the present work, a game semantics  $[-]_{\mathcal{DG}}$  of finitary PCF (i.e., the simply-typed  $\lambda$ -calculus equipped with the boolean type) in  $\mathcal{LDG}$  that together with the hiding operation  $\mathcal{H}$  satisfies the DCP (Corollary 4.1), which we call **dynamic game semantics** as it captures dynamics and intensionality of computation better than conventional ones. We select finitary PCF as our target language since a simple language would be appropriate for the first work on dynamic game semantics.

Note that it does not make much sense to ask whether full abstraction holds for dynamic game semantics as its aim is rather to capture intensionality of computation.

Also, the semantics does not satisfy faithfulness: The semantic equation is of course finer than  $\beta$ -equivalence but also coarser than  $\alpha$ -equivalence, e.g., non- $\alpha$ -equivalent terms  $(\lambda x. \underline{0}) \underline{1}$  and  $(\lambda x. \underline{0}) \underline{2}$  are interpreted to be the same in dynamic game semantics, which is because the semantic equation captures algorithmic difference of terms, while  $\alpha$ -equivalence distinguishes how they are constructed even if their algorithms coincide (n.b., this point calls for (syntax-independent) mathematics of senses).

On the other hand, it makes sense to ask if full completeness holds for dynamic game semantics. In fact, we shall establish *dynamic full completeness* (Corollary 4.2).

#### 1.5. Our Contribution and Related Work

To the best of our knowledge, the present work is the first syntax-independent characterization of dynamics and intensionality of computation in the sense of DCPs.

The work closest in spirit is Girard's geometry of interaction (GoI) (Girard, 1989; Girard, 1990; Girard, 1995; Girard, 2003; Girard, 2011; Girard, 2013). However, GoI appears mathematically *ad-hoc* for it does not conform to the standard categorical semantics

<sup>&</sup>lt;sup>¶</sup> N.b., for the present work, it suffices to know that a CCB is a generalized CCC in the sense that the equational axioms of CCCs are required to hold only up to 2-cell isomorphisms.

of type theories (Lambek and Scott, 1988; Pitts, 2001; Crole, 1993; Jacobs, 1999); also, it does not capture the *step-by-step* process of reduction in the sense of DCPs. In contrast, dynamic game semantics refines the standard semantics and does satisfy a DCP.

Next, the idea of exhibiting 'intermediate moves' in the composition of strategies is *nothing new*; there are game-semantic approaches (Dimovski et al., 2005; Greenland, 2005; Ong, 2006) that give such moves an official status. However, because their aims are rather to develop a tool for program analysis and verification, they do not study in depth mathematical structures thereof, give an intensional game semantics that refines the standard categorical semantics or formulate a *step-by-step* 'hiding' process. Therefore, our contribution for this point is to study algebraic structures of games and strategies when we do not a priori 'hide intermediate moves' and refine the standard categorical semantics in such a way that satisfies DCPs.

Also, there are several approaches to model dynamics of computation by 2-categories (Seely, 1987; Hilken, 1996; Mellies, 2005). In these papers, however, the horizontal composition of 1-cells is the *normalizing* one, which is why the structure is 2-categories rather than bicategories.<sup>||</sup> In addition, the 2-cells of their 2-categories are rewriting, while the 2-cells of our bicategory are the external equivalence between 1-cells; note that 2-cells in a bicategory cannot interpret rewriting unless the horizontal composition is normalizing since associativity of non-normalizing composition with respect to such 2-cells does not hold.<sup>††</sup> Thus, although their motivations are similar to ours, our *bicategorical* approach seems novel, interpreting an application of terms by non-normalizing composition, the extensional equivalence of terms by 2-cells and rewriting by the hiding operation  $\mathcal{H}$ . Moreover, their frameworks are categorical, while we instantiate our categorical model by game semantics. Furthermore, neither of the previous work establishes a DCP.

Finally, note that the present work has some implications from theoretical as well as practical viewpoints. From the theoretical perspective, it enables us to study dynamics and intensionality of computation as purely *mathematical* (or *semantic*) concepts, just like any concepts in pure mathematics such as differentiation and integration in calculus, homotopy in topology, etc. Thus, we would be able to rigorously analyze the essence of these concepts, ignoring superfluous syntactic details. From the practical point, on the other hand, it might become a useful tool for language analysis and design, e.g., our variant of finitary PCF would not exist without the present work.

# 1.6. Structure of the paper

The rest of the present paper proceeds as follows. This introduction ends with fixing some notations. Then, Section 2 formulates our target programming language and its bicategorical semantics that satisfies the DCP so that it remains to establish its game-semantic instance. Next, Section 3 introduces dynamic games and strategies and studies their basic algebraic structures, and Section 4 gives dynamic game semantics of the language. Finally, Section 5 draws a conclusion and proposes some future work.

<sup>||</sup> N.b., the unit law on the nose does not hold if the composition is non-normalizing.

<sup>&</sup>lt;sup>††</sup> N.b., there is no rewriting between 1-cells (f;g);h and f;(g;h) if the composition is non-normalizing.

Notation. We use the following notations throughout the paper:

- We use bold letters s, t, u, v, etc. for sequences, in particular  $\epsilon$  for the *empty sequence*, and letters a, b, c, d, m, n, x, y, z, etc. for elements of sequences;
- Given  $k \in \mathbb{N}$ , we write  $\overline{k}$  for the finite set  $\{1, 2, \dots, k\} \subseteq \mathbb{N}$  (n.b.,  $\overline{0} = \emptyset$ );
- We often abbreviate a finite sequence  $\mathbf{s} = (x_1, x_2, \dots, x_{|\mathbf{s}|})$  as  $x_1 x_2 \dots x_{|\mathbf{s}|}$ , where  $|\mathbf{s}|$ denotes the *length* (i.e., the number of elements) of s, and write s(i), where  $i \in [s]$ , as another notation for  $x_i$ ;
- A *concatenation* of sequences is represented by the juxtaposition of them, but we often write as, tb, ucv for (a)s, t(b), u(c)v, etc., and also write s.t for st;
- We define  $s^n \stackrel{\text{df.}}{=} ss \cdots s$  for a sequence s and a natural number  $n \in \mathbb{N}$ ;
- We write  $\mathsf{Even}(s)$  (resp.  $\mathsf{Odd}(s)$ ) iff s is of even-length (resp. odd-length);
- We define  $S^{\mathsf{P}} \stackrel{\text{df.}}{=} \{ s \in S \mid \mathsf{P}(s) \}$  for a set S of sequences and  $\mathsf{P} \in \{\mathsf{Even}, \mathsf{Odd}\};$
- $-s \leq t$  means s is a *prefix* of t, i.e., t = s.u for some sequence u, and given a set S of sequences, we define  $\operatorname{Pref}(S) \stackrel{\mathrm{df.}}{=} \{ s \mid \exists t \in S. s \leq t \};$
- For a poset P and a subset  $S \subseteq P$ , Sup(S) denotes the supremum of S;
- $X^* \stackrel{\text{df.}}{=} \{ x_1 x_2 \dots x_n \mid n \in \mathbb{N}, \forall i \in \overline{n} . x_i \in X \} \text{ for each set } X;$  For a function  $f : A \to B$  and a subset  $S \subseteq A$ , we define  $f \upharpoonright S : S \to B$  to be the restriction of f to S, and  $f^*: A^* \to B^*$  by  $f^*(a_1a_2 \dots a_n) \stackrel{\text{df.}}{=} f(a_1)f(a_2) \dots f(a_n) \in$  $B^*$  for all  $a_1a_2\ldots a_n \in A^*$ ;
- Given sets  $X_1, X_2, \ldots, X_n$ , and  $i \in \overline{n}$ , we write  $\pi_i$  (or  $\pi_i^{(n)}$ ) for the *i*<sup>th</sup>-projection function  $X_1 \times X_2 \times \cdots \times X_n \to X_i$  that maps  $(x_1, x_2, \dots, x_n) \mapsto x_i$ ;
- --  $\simeq$  denote the Kleene equality, i.e.,  $x \simeq y \stackrel{\text{df.}}{\Leftrightarrow} (x \downarrow \land y \downarrow \land x = y) \lor (x \uparrow \land y \uparrow)$ , where we write  $x \downarrow$  if an element x is defined, and  $x \uparrow$  otherwise.

### 2. Dynamic Bicategorical Semantics

Let us first present a *categorical* description of how dynamic games and strategies capture dynamics and intensionality of logic and computation, and show that it is a *refinement* of the standard categorical semantics of type theories (Lambek and Scott, 1988; Pitts, 2001; Crole, 1993; Jacobs, 1999).

#### 2.1. Beta-Categories of Computation

The categorical structure for our interpretation of logic and computation is  $\beta$ -categories of computation (BoCs), a certain kind of bicategories whose 2-cells are the extensional equivalence between 1-cells, equipped with an evaluation satisfying certain axioms.

Let us first introduce a more general notion of  $\beta$ -categories, which are categories up to an equivalence relation on morphisms:

**Definition 2.1** ( $\beta$ -categories). A  $\beta$ -category is a pair  $\mathcal{C} = (\mathcal{C}, \simeq)$  that consists of:

- A class  $ob(\mathcal{C})$  of *objects*, where we usually write  $A \in \mathcal{C}$  for  $A \in ob(\mathcal{C})$ ;
- A class  $\mathcal{C}(A, B)$  of  $\beta$ -morphisms from A to B for each pair  $A, B \in \mathcal{C}$ , where we often write  $f: A \to B$  for  $f \in \mathcal{C}(A, B)$  if  $\mathcal{C}$  is obvious from the context;

- A (class) function  $\mathcal{C}(A, B) \times \mathcal{C}(B, C) \xrightarrow{;A,B,C} \mathcal{C}(A, C)$ , called the  $\beta$ -composition on  $\beta$ -morphisms from A to B and from B to C, for each triple  $A, B, C \in \mathcal{C}$ ;
- A  $\beta$ -morphism  $id_A \in \mathcal{C}(A, A)$ , called the  $\beta$ -identity on A, for each  $A \in \mathcal{C}$ ;
- An equivalence (class) relation  $\simeq_{A,B}$  on  $\mathcal{C}(A,B)$ , called the *equivalence* on  $\beta$ -morphisms from A to B, for each pair  $A, B \in \mathcal{C}$

where we also write  $\mathcal{C}(B,C) \times \mathcal{C}(A,B) \xrightarrow{\circ_{A,B,C}} \mathcal{C}(A,C)$  for the  $\beta$ -composition ;<sub>A,B,C</sub> and often omit the subscripts on ;<sub>A,B,C</sub>,  $\circ_{A,B,C}$  and  $\simeq_{A,B}$ , such that it satisfies:

$$(f;g); h \simeq f; (g;h)$$

$$f; id_B \simeq f$$

$$id_A; f \simeq f$$

$$f \simeq f' \land g \simeq g' \Rightarrow f; g \simeq f'; g'$$

for any  $A, B, C, D \in \mathcal{C}$ ,  $f, f' : A \to B$ ,  $g, g' : B \to C$  and  $h : C \to D$ . Moreover, it is *cartesian closed* iff:

— There is an object  $T \in C$ , called a *β*-terminal object, equipped with a *β*-morphism  $!_A : A \to T$ , called the *canonical β*-morphism on *A*, for each  $A \in C$  that satisfies:

$$!_A \simeq t$$
 for any  $t : A \to T$ ;

--- There is an object  $A \times B \in \mathcal{C}$  for each pair  $A, B \in \mathcal{C}$ , called a  $\beta$ -(binary) product of A and B, equipped with  $\beta$ -morphisms  $\pi_1^{A,B} : A \times B \to A$  and  $\pi_2^{A,B} : A \times B \to B$ , called the first and the second  $\beta$ -projections of  $A \times B$ , respectively, and an assignment  $\langle -, - \rangle$  of a  $\beta$ -morphism  $\langle a, b \rangle_{A,B}^C : C \to A \times B$ , called the  $\beta$ -pairing of a and b, to given  $C \in \mathcal{C}$ ,  $a : C \to A$  and  $b : C \to B$ , that satisfies:

$$\begin{split} \langle a, b \rangle_{A,B}^{C}; \pi_{1}^{A,B} &\simeq a \\ \langle a, b \rangle_{A,B}^{C}; \pi_{2}^{A,B} &\simeq b \\ \langle h; \pi_{1}^{A,B}, h; \pi_{2}^{A,B} \rangle_{A,B}^{C} &\simeq h \text{ for any } h: C \to A \times B \\ a &\simeq a' \wedge b \simeq b' \Rightarrow \langle a, b \rangle \simeq \langle a', b' \rangle \text{ for any } a': C \to A \text{ and } b': C \to B; \end{split}$$

$$\begin{split} \langle \pi_1^{A,B}; \Lambda_{A,B,C}(k), \pi_2^{A,B} \rangle_{C^B,B}^{A \times B}; ev_{B,C} \simeq k \\ \Lambda_{A,B,C}(\langle \pi_1^{A,B}; l, \pi_2^{A,B} \rangle_{C^B,B}^{A \times B}; ev_{B,C}) \simeq l \text{ for any } l : A \to C^B \\ k \simeq k' \Rightarrow \Lambda_{A,B,C}(k) \simeq \Lambda_{A,B,C}(k') \text{ for any } k' : A \times B \to C \end{split}$$

where we often omit the sub/superscripts on  $\pi_i^{A,B}$ ,  $\langle -, - \rangle_{A,B}^C$ ,  $ev_{B,C}$  and  $\Lambda_{A,B,C}$ .

That is, a (resp. cartesian closed)  $\beta$ -category  $\mathcal{C} = (\mathcal{C}, \simeq)$  is a (resp. cartesian closed) category  $up \ to \simeq$  (i.e., the equation = on morphisms is replaced with the equivalence relation  $\simeq$  on 1-cells), where the prefix ' $\beta$ -' signifies the compromise 'up to  $\simeq$ '. Alternatively,

regarding objects and  $\beta$ -morphisms of C as 0-cells and 1-cells, respectively, and defining 2-

cells by  $\mathcal{C}(A, B)(d, c) \stackrel{\text{df.}}{=} \begin{cases} \{\simeq\} & \text{if } d \simeq c; \\ \emptyset & \text{otherwise} \end{cases}$  for any  $A, B \in \mathcal{C}$  and  $d, c : A \to B$ , where  $\{\simeq\}$ 

is any singleton set, we may identify C with a (resp. cartesian closed (Ouaknine, 1997)) bicategory whose 2-cells are only the trivial one.

We are now ready to define  $\beta$ -categories of computation (BoCs):

**Definition 2.2 (BoCs).** A  $\beta$ -category of computation (BoC) is a  $\beta$ -category  $C = (C, \simeq)$  equipped with a (class) function  $\mathcal{E}$  on  $\beta$ -morphisms of C, called the *evaluation* (of computation), that satisfies:

- (SUBJECT REDUCTION).  $\mathcal{E}(f) : A \to B$  for all  $A, B \in \mathcal{C}$  and  $f : A \to B$ ;
- (TERMINATION).  $f \downarrow$  for all  $A, B \in \mathcal{C}$  and  $f : A \to B$ ;
- ( $\beta$ -IDENTITIES).  $id_A \in \mathcal{V}_{\mathcal{C}}(A, A)$  for all  $A \in \mathcal{C}$ ;
- -- (EVALUATION).  $f \simeq f' \Leftrightarrow \exists v \in \mathcal{V}_{\mathcal{C}}(A, B)$ .  $f \downarrow v \land f' \downarrow v$  for all  $A, B \in \mathcal{C}$  and  $f, f': A \to B$

where  $\mathcal{V}_{\mathcal{C}}(A, B) \stackrel{\text{df.}}{=} \{v \in \mathcal{C}(A, B) \mid \mathcal{E}(v) = v\}$ , whose elements are called **values** from A to B, and we write  $f \downarrow$ , or specifically  $f \downarrow \mathcal{E}^n(f)$ , if  $\mathcal{E}^n(f) \in \mathcal{V}_{\mathcal{C}}(A, B)$  for some  $n \in \mathbb{N}$ .<sup>‡‡</sup> It is **cartesian closed**, which we call a **cartesian closed BoC** (**CCBoC**), iff so is C as a  $\beta$ -category, all the canonical  $\beta$ -morphisms, the  $\beta$ -projections and the  $\beta$ -evaluations of  $\mathcal{C}$  are values, and all the  $\beta$ -pairing and the  $\beta$ -currying of  $\mathcal{C}$  preserve values.

**Convention.** Since the equivalence  $\simeq$  of a BoC C may be completely recovered from the evaluation  $\mathcal{E}$ , we usually specify the BoC by a pair  $\mathcal{C} = (\mathcal{C}, \mathcal{E})$ . If  $f \downarrow \mathcal{E}^n(f)$  for some  $n \in \mathbb{N}$ , then we call  $\mathcal{E}^n(f)$  the **value** of f and also write  $\mathcal{E}^{\omega}(f)$  for it.

The intuition behind Definition 2.2 is as follows. In a BoC  $\mathcal{C} = (\mathcal{C}, \mathcal{E})$ ,  $\beta$ -morphisms are (possibly *intensional* but not necessarily 'effective') *computations* with the domain and the codomain (objects) specified, and values are *extensional* computations such as functions (as graphs). The  $\beta$ -composition is 'non-normalizing composition' or *concatenation* of computations, and  $\beta$ -identities are *unit computations* (they are just like identity functions). The execution of a computation f is achieved by *evaluating* it into a unique value  $\mathcal{E}^{\omega}(f)$ , which corresponds to *dynamics* of computation.<sup>§§</sup> In addition, the equivalence relation  $\simeq$  witnesses the *extensional equivalence* between  $\beta$ -morphisms modulo  $\mathcal{E}^{\omega}$ . The four axioms then should make sense from this perspective. In this way, a BoC provides a 'universe' of dynamic, intensional computations.

It is easy to see that a BoC  $\mathcal{C} = (\mathcal{C}, \mathcal{E})$  induces the category  $\mathcal{V}_{\mathcal{C}}$  given by:

- Objects are those of C;
- Morphisms  $A \to B$  are elements in  $\mathcal{V}_{\mathcal{C}}(A, B)$ , i.e., values from A to B in  $\mathcal{C}$ ;

<sup>&</sup>lt;sup>‡‡</sup> Note that if  $\mathcal{E}^{n_1}(f), \mathcal{E}^{n_2}(f) \in \mathcal{V}_{\mathcal{C}}(A, B)$  for any  $n_1, n_2 \in \mathbb{N}$ , then clearly  $\mathcal{E}^{n_1}(f) = \mathcal{E}^{n_2}(f)$ , where  $\mathcal{E}^n$  denotes the *n*-times iteration of  $\mathcal{E}$  for all  $n \in \mathbb{N}$ .

<sup>&</sup>lt;sup>§§</sup> In the present work, every dynamic strategy (or  $\beta$ -morphism) becomes a value by a *finite* iteration of the hiding operation (or evaluation) due to the axiom on labeling functions (Definition 3.1), and thus the axiom Termination (Definition 2.2) makes sense. Of course, if we consider another, in particular finer, evaluation of computations (which is left as future work), then this point may no longer hold.

- The composition of morphisms  $u: A \to B$  and  $v: B \to C$  is  $\mathcal{E}^{\omega}(u; v): A \to C$ ;
- Identities are  $\beta$ -identities in  $\mathcal{C}$ .

Regarding the BoC C as the trivial bicategory as already specified above, and the category  $\mathcal{V}_{\mathcal{C}}$  as the trivial 2-category, the evaluation  $\mathcal{E}$  induces the 2-functor  $\mathcal{E}^{\omega} : \mathcal{C} \to \mathcal{V}_{\mathcal{C}}$  that maps  $A \mapsto A$  for 0-cells  $A, f \mapsto \mathcal{E}^{\omega}(f)$  for 1-cells f, and  $\simeq \mapsto =$  for 2-cells  $\simeq$ . Clearly,  $\mathcal{V}_{\mathcal{C}}$  is cartesian closed if so is  $\mathcal{C}$ , where canonical morphisms into a terminal object, projections, evaluations, pairing and currying of  $\mathcal{V}_{\mathcal{C}}$  are respectively the corresponding ' $\beta$ -ones' in  $\mathcal{C}$ .

The point here is that we may decompose the standard interpretation  $\llbracket_{-} \rrbracket_{\mathcal{S}}$  of functional programming languages in a CCC  $\mathcal{V}_{\mathcal{C}}$  (Lambek and Scott, 1988; Pitts, 2001; Crole, 1993; Jacobs, 1999) as a more intensional interpretation  $\llbracket_{-} \rrbracket_{\mathcal{D}}$  in a CCBoC  $\mathcal{C} = (\mathcal{C}, \mathcal{E})$  and the full evaluation  $\mathcal{E}^{\omega} : \mathcal{C} \to \mathcal{V}_{\mathcal{C}}$ , i.e.,  $\llbracket_{-} \rrbracket_{\mathcal{S}} = \mathcal{E}^{\omega}(\llbracket_{-} \rrbracket_{\mathcal{D}})$ , and talk about *intensional difference* between computations: Terms M and M' are interpreted to be *intensionally equal* if  $\llbracket M \rrbracket_{\mathcal{D}} = \llbracket M' \rrbracket_{\mathcal{D}}$  and *extensionally equal* if  $\llbracket M \rrbracket_{\mathcal{D}} \simeq \llbracket M' \rrbracket_{\mathcal{D}}$ . Also, the *one-step* evaluation  $\mathcal{E}$  is to capture the small-step operational semantics of the target language, i.e., to satisfy the DCP (see Definition 2.5 for the precise definition specialized to our target language).

# 2.2. Finitary PCF

Next, let us introduce our target programming language for dynamic game semantics.

First, recall that there is a one-to-one correspondence between *PCF Böhm trees* (i.e., terms of PCF in  $\eta$ -long normal form) (Amadio and Curien, 1998) and innocent, well-bracketed strategies (Hyland and Ong, 2000; Abramsky and McCusker, 1999; Curien, 2006); this highlight in the literature of game semantics is called *strong definability*. Naturally, we would like to exploit the strong definability result to establish the first instance of dynamic game semantics as the task would be easier than otherwise.

On the other hand, the higher-order functional programming language PCF (Scott, 1993; Plotkin, 1977) has the *natural number type* and the *fixed-point combinators*, which make PCF Böhm trees *infinitary* in width and depth, respectively. However, we would like to select, as the first target language for dynamic game semantics, the simplest one possible because then the idea and the mechanism would be most visible. For this reason, let us choose *finitary PCF*, i.e., the finite fragment of PCF that has only the *boolean type* as the ground type (or equivalently, the *simply-typed \lambda-calculus* (Church, 1940; Sørensen and Urzyczyn, 2006) equipped with the boolean type).

We then define a simple small-step operational semantics (or reduction strategy) of finitary PCF whose execution order is obvious from types and has an immediate counterpart in dynamic game semantics.

**Remark.** Note that an execution of *linear head reduction* (LHR) (Danos and Regnier, 2004) corresponds in a step-by-step fashion to an 'internal communication' between strategies (Danos et al., 1996). Hence, one may wonder if it would be better to employ LHR as the operational semantics of finitary PCF; however, note that:

- The correspondence is *not* between terms and strategies;
- LHR is executed by *linear substitution*, which makes the calculus very different from the usual  $\lambda$ -calculus with  $\beta$ -reduction.

By these two points, we have conjectured that it would require significantly more work than the present work to establish a game-semantic DCP with respect to LHR, and therefore we leave it as future work.

In the following, we give the precise definition of the resulting target programming language (viz., finitary PCF equipped with the small-step operational semantics).

Notation. We employ the following notations:

- Let  $\mathscr{V}$  be a countably infinite set of *variables*, written x, y, z, etc., for which we assume the *variable convention* (or *Barendreqt's convention* (Hankin, 1994)<sup>¶¶</sup>);
- We use sans-serif letters such as  $\Gamma$ , A and a for syntactic objects and  $\equiv$  for syntactic equality up to  $\alpha$ -equivalence, i.e., up to renaming of bound variables.

**Definition 2.3 (FPCF).** The *finitary PCF (FPCF)* is a functional programming language defined as follows:

- (TYPES). A **type** A is an expression generated by the grammar:

$$\mathsf{A} \stackrel{\mathrm{dr.}}{\equiv} o \mid \mathsf{A}_1 \Rightarrow \mathsf{A}_2$$

where *o* is the **boolean type** and  $A_1 \Rightarrow A_2$  is the **function type** from  $A_1$  to  $A_2$  ( $\Rightarrow$  is right associative). We write A, B, C, etc. for types. Note that each type A may be written uniquely of the form  $A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_k \Rightarrow o$ , where  $k \in \mathbb{N}$ .

— (RAW-TERMS). A *raw-term* M is an expression generated by the grammar:

$$\mathsf{M} \stackrel{\mathrm{df.}}{\equiv} \mathsf{x} \mid \mathsf{tt} \mid \mathsf{ff} \mid \mathsf{case}(\mathsf{M})[\mathsf{M}_1;\mathsf{M}_2] \mid \lambda \mathsf{x}^\mathsf{A}.\mathsf{M} \mid \mathsf{M}_1\mathsf{M}_2$$

where x ranges over variables, and A over types. We call tt, ff,  $\lambda x^A.M$  and  $M_1M_2$  respectively the *true constant*, the *false constant*, an *abstraction* and an *application*. We write M, P, Q, R, etc. for raw-terms and often omit A in an abstraction  $\lambda x^A$ ; an application is always left-associative, e.g.,  $M_1M_2M_3$  may be written informally  $(M_1M_2)M_3$ . The set  $\mathscr{FV}(M) \subseteq \mathscr{V}$  of all *free variables* occurring in a raw-term M is defined by the following induction on M:

-- (CONTEXTS). A **context** is a finite sequence  $x_1 : A_1, x_2 : A_2, \ldots, x_k : A_k$  of (variable : type)-pairs with  $x_i \neq x_i$  if  $i \neq j$ , where  $i, j \in \overline{k}$ . We write  $\Gamma$ ,  $\Delta$ ,  $\Theta$ , etc. for contexts.

 $<sup>\</sup>P$  I.e., we assume that in any term of concern every bound variable is chosen to be different from any free variable occurring in that mathematical context.

- (TERMS). A *term* is an expression of the form  $\Gamma \vdash M : B$ , where  $\Gamma$  is a *context*, M is a raw-term, and B is a type, generated by the following *typing rules*:

$$\begin{split} \mathsf{A} &\equiv \mathsf{A}_{1} \Rightarrow \mathsf{A}_{2} \Rightarrow \dots \Rightarrow \mathsf{A}_{k} \Rightarrow o \ \ \mathsf{\Gamma} \equiv \Delta, \Theta \\ &\forall i \in \overline{k}. \ \mathsf{\Gamma} \vdash \mathsf{V}_{i} : \mathsf{A}_{i} \land \sharp(\mathsf{V}_{i}) = 0 \land \mathsf{x} \not\in \mathscr{FV}(\mathsf{V}_{i}) \\ \end{split} \\ (\mathsf{B}) \frac{\mathsf{b} \in \{\mathsf{tt}, \mathsf{ff}\}}{\mathsf{\Gamma} \vdash \mathsf{b} : o} \quad (\mathsf{C1}) \frac{\forall j \in \overline{2}. \ \mathsf{\Gamma} \vdash \mathsf{W}_{j} : o \land \sharp(\mathsf{W}_{j}) = 0 \land \mathsf{x} \notin \mathscr{FV}(\mathsf{W}_{j})}{\Delta, \mathsf{x} : \mathsf{A}, \Theta \vdash \mathsf{case}(\mathsf{x}\mathsf{V}_{1}\mathsf{V}_{2} \dots \mathsf{V}_{k})[\mathsf{W}_{1}; \mathsf{W}_{2}] : o} \end{split}$$

$$(C2)\frac{\Gamma \vdash \mathsf{M}: o \quad \forall j \in \overline{2}. \Gamma \vdash \mathsf{P}_{j}: o}{\Gamma \vdash \mathsf{case}(\mathsf{M})[\mathsf{P}_{1}; \mathsf{P}_{2}]: o} \quad (L)\frac{\Gamma, \mathsf{x}: \mathsf{A} \vdash \mathsf{M}: \mathsf{B}}{\Gamma \vdash \lambda \mathsf{x}^{\mathsf{A}}. \mathsf{M}: \mathsf{A} \Rightarrow \mathsf{B}}$$

$$(A) \frac{\Gamma \vdash \mathsf{M}_1 : A \Rightarrow B \ \Gamma \vdash \mathsf{M}_2 : A}{\Gamma \vdash \mathsf{M}_1 \mathsf{M}_2 : B}$$

where  $\sharp(\Gamma \vdash M : B) \in \mathbb{N}$ , often abbreviated as  $\sharp(M)$ , is the *execution number* of each term  $\Gamma \vdash M : B$  defined by the following induction on  $\Gamma \vdash M : B$ :

- $\quad \sharp(\mathsf{b}) \stackrel{\mathrm{df.}}{=} 0 \text{ if } \mathsf{b} \in \{\mathsf{tt},\mathsf{ff}\};$
- $\quad \sharp(\mathsf{case}(\mathsf{xV}_1\mathsf{V}_2\ldots\mathsf{V}_k)[\mathsf{W}_1;\mathsf{W}_2]) \stackrel{\mathrm{df.}}{=} 0;$
- $\ \ \sharp(\mathsf{case}(\mathsf{M})[\mathsf{P}_1;\mathsf{P}_2]) \stackrel{\mathrm{df.}}{=} 0;$

$$- \quad \sharp(\lambda \mathsf{x}^{\mathsf{A}}.\mathsf{M}) \stackrel{\mathrm{df.}}{=} \sharp(\mathsf{M});$$

 $- \ \sharp(\mathsf{M}_1\mathsf{M}_2) \stackrel{\mathrm{df.}}{=} \max(\sharp(\mathsf{M}_1),\sharp(\mathsf{M}_1)) + 1.$ 

We write  $\Gamma \vdash \{M\}_e : B$  for the term  $\Gamma \vdash M : B$  such that  $\sharp(M) = e$ . Also, we often omit the context and/or the type of a term if it does not bring confusion. A **program** (resp. a **value**) is a term generated by the rules B, C1, L and A (resp. B, C1 and L). A **subterm** of a term  $\Gamma \vdash M : B$  is a term that occurs in the deduction of  $\Gamma \vdash M : B$ , where note that a deduction (tree) of each term of FPCF is clearly *unique*.

**Remark.** The rules C2 above and  $\vartheta_4$  below are necessary for 'intermediate terms' during an evaluation of a program into a value.

- ( $\beta\vartheta$ -REDUCTION). The  $\beta\vartheta$ -reduction  $\rightarrow_{\beta\vartheta}$  on terms is the *contextual closure*, i.e., the closure with respect to the typing rules, of the union of the following five rules:

$$\begin{split} & (\lambda x.\ \mathsf{M})\mathsf{P} \rightarrow_{\beta} \mathsf{M}[\mathsf{P}/x] \\ & \mathsf{case}(\mathsf{tt})[\mathsf{M}_1;\mathsf{M}_2] \rightarrow_{\vartheta_1} \mathsf{M}_1 \\ & \mathsf{case}(\mathsf{ff})[\mathsf{M}_1;\mathsf{M}_2] \rightarrow_{\vartheta_2} \mathsf{M}_2 \\ & \mathsf{case}(\mathsf{case}(x\mathbf{V})[\mathsf{W}_1;\mathsf{W}_2])[\mathsf{M}_1;\mathsf{M}_2] \rightarrow_{\vartheta_3} \mathsf{case}(x\mathbf{V})[\mathsf{case}(\mathsf{W}_1)[\mathsf{M}_1;\mathsf{M}_2];\mathsf{case}(\mathsf{W}_2)[\mathsf{M}_1;\mathsf{M}_2]] \\ & \mathsf{case}(\mathsf{case}(\mathsf{M})[\mathsf{P}_1;\mathsf{P}_2])[\mathsf{Q}_1;\mathsf{Q}_2] \rightarrow_{\vartheta_4} \mathsf{case}(\mathsf{M})[\mathsf{case}(\mathsf{P}_1)[\mathsf{Q}_1;\mathsf{Q}_2];\mathsf{case}(\mathsf{P}_2)[\mathsf{Q}_1;\mathsf{Q}_2]] \end{split}$$

where M[P/x] denotes the *capture-free substitution* (Hankin, 1994) of P for x in M, and xV abbreviates  $xV_1V_2...V_k$  of the rule C1. We write nf(M) for the normal form of each term M with respect to  $\rightarrow_{\beta\vartheta}$ , i.e., nf(M) is a term such that  $M \rightarrow^*_{\beta\vartheta} nf(M)$ and  $nf(M) \not\rightarrow_{\beta\vartheta} M'$  for any term M', which uniquely exists by Theorems 2.2 and 2.3 given below. The *parallel*  $\beta \vartheta$ -reduction  $\rightrightarrows_{\beta \vartheta}$  on terms evaluates each term M in a single-step to its normal form nf(M).

-- (OPERATIONAL SEMANTICS). The *(small-step) operational semantics* (or the *reduction strategy*)  $\rightarrow$  on programs M is the 'simultaneous execution' of  $\rightrightarrows_{\beta\vartheta}$  on all subterms of M with the execution number 1, or more precisely  $\rightarrow$  is defined by:

$$\mathsf{M} \to \begin{cases} \mathsf{V} & \text{if } \mathsf{M} \equiv \mathsf{M}_1\mathsf{M}_2, \, \sharp(\mathsf{M}_1\mathsf{M}_2) = 1 \text{ and } \mathsf{M}_1\mathsf{M}_2 \rightrightarrows_{\beta\vartheta} \mathsf{V}; \\ \mathsf{M}'_1\mathsf{M}'_2 & \text{if } \mathsf{M} \equiv \mathsf{M}_1\mathsf{M}_2, \, \sharp(\mathsf{M}_1\mathsf{M}_2) \geqslant 2 \text{ and } \mathsf{M}_i \to \mathsf{M}'_i \text{ for } i = 1, 2; \\ \lambda \mathsf{x}^\mathsf{A}. \, \tilde{\mathsf{M}}' & \text{if } \mathsf{M} \equiv \lambda \mathsf{x}^\mathsf{A}. \, \tilde{\mathsf{M}} \text{ and } \tilde{\mathsf{M}} \to \tilde{\mathsf{M}}'. \end{cases}$$

**Remark.** The operational semantics  $\rightarrow$  of FPCF might appear a bit unusual, but as we shall see, it has a natural game-semantic counterpart, i.e., it makes sense from the game-semantic point of view.

**Eq(FPCF)** is the equational theory that consists of judgements  $\Gamma \vdash M = M' : B$ , where  $\Gamma \vdash M : B$  and  $\Gamma \vdash M' : B$  are terms of FPCF such that  $nf(M) \equiv nf(M')$ .

Note that values of FPCF are PCF Böhm trees except that the 'bottom term'  $\perp$  and the *natural number type*  $\iota$  are excluded; the  $\beta\vartheta$ -reduction  $\rightarrow_{\beta\vartheta}$  is essentially taken from Section 6 of the book (Amadio and Curien, 1998).

**Remark.** Let  $A \equiv A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_k \Rightarrow o$  be an arbitrary type of FPCF. Note that an expression of the form  $\Delta, x : A, \Theta \vdash x : A$  is *not* a term of FPCF, but instead there is another  $\Delta, x : A, \Theta \vdash \underline{x}^A : A$ , where  $\underline{x}^A \stackrel{\text{df.}}{\equiv} \lambda x_1^{A_1} x_2^{A_2} \dots x_k^{A_k} \cdot \mathsf{case}(x \underline{x_1}^{A_1} \underline{x_2}^{A_2} \dots \underline{x_k}^{A_k})[\mathsf{tt};\mathsf{ff}]$ , which *is* a term of FPCF. We often write  $\underline{x}$  for  $\underline{x}^A$  if it does not bring confusion.

Thus, FPCF computes as follows. Given a program  $\Gamma \vdash \{M\}_e : B$ , it produces a *finite* chain of *finitary* rewriting

$$\mathsf{M} \to \mathsf{M}_1 \to \mathsf{M}_2 \to \dots \to \mathsf{M}_{\mathsf{e}} \tag{1}$$

where  $M_e$  is a value. Note that the program M is constructed from values by a finite number of applications, and the computation (1) is executed in the *first-applications-first-evaluated* fashion, e.g., if  $M \equiv (V_1V_2)((V_3V_4)(V_5V_6))$  and e = 3, where  $V_1, V_2, \ldots, V_6$  are values, then the computation (1) would be of the form

$$(\mathsf{V}_1\mathsf{V}_2)((\mathsf{V}_3\mathsf{V}_4)(\mathsf{V}_5\mathsf{V}_6)) \to \mathsf{V}_7(\mathsf{V}_8\mathsf{V}_9) \to \mathsf{V}_7\mathsf{V}_{10} \to \mathsf{V}_{11}$$

where  $V_7 \equiv nf(V_1V_2)$ ,  $V_8 \equiv nf(V_3V_4)$ ,  $V_9 \equiv nf(V_5V_6)$ ,  $V_{10} \equiv nf(V_8V_9)$  and  $V_{11} \equiv nf(V_7V_{10})$ .

The rest of the present section is devoted to showing that the computation (1) of FPCF in fact correctly works (Corollary 2.1).

First, by the following Proposition 2.1 and Theorem 2.1, it makes sense that  $\rightarrow_{\beta\vartheta}$  is defined *on terms* (not on raw-terms):

**Proposition 2.1 (Unique typing).** If  $\Gamma \vdash \{M\}_e : B \text{ and } \Gamma \vdash \{M\}_{e'} : B'$ , then e = e' and  $B \equiv B'$ .

*Proof.* By induction on the construction of  $\Gamma \vdash M : B$ .

**Lemma 2.1 (Free variable lemma).** If  $\Gamma \vdash M : B$ , and  $x \in \mathcal{V}$  occurs free in M, then x : A occurs in  $\Gamma$  for some type A.

*Proof.* By induction on the construction of  $\Gamma \vdash M : B$ .

**Lemma 2.2 (EW-lemma).** If  $x_1 : A_1, x_2 : A_2, ..., x_k : A_k \vdash \{M\}_e : B$ , then:

- 1  $\mathsf{x}_{\sigma(1)} : \mathsf{A}_{\sigma(1)}, \mathsf{x}_{\sigma(2)} : \mathsf{A}_{\sigma(2)}, \dots, \mathsf{x}_{\sigma(k)} : \mathsf{A}_{\sigma(k)} \vdash \{\mathsf{M}\}_e : \mathsf{B} \text{ for any permutation } \sigma \text{ of } \overline{k};$
- $\begin{array}{ll} 2 & x_1: \mathsf{A}_1, x_2: \mathsf{A}_2, \dots, x_k: \mathsf{A}_k, x_{k+1}: \mathsf{A}_{k+1} \vdash \{\mathsf{M}\}_e: \mathsf{B} \text{ for any variable } x_{k+1} \in \mathscr{V} \text{ and type } \\ \mathsf{A}_{k+1} \text{ such that } x_{k+1} \not\equiv x_i \text{ for } i = 1, 2, \dots, k. \end{array}$

*Proof.* By induction on the construction of  $x_1 : A_1, x_2 : A_2, \ldots, x_k : A_k \vdash M : B$ .

**Lemma 2.3 (Substitution lemma).** If  $\Gamma, x : A \vdash \{P\}_e : B \text{ and } \Gamma \vdash Q : A$ , then  $\Gamma \vdash \{P[Q/x]\}_e : B$ .

*Proof.* By induction on  $|\mathsf{P}|$  with the help of Lemmata 2.1 and 2.2.

**Theorem 2.1 (Subject reduction).** If  $\Gamma \vdash M : B$  and  $M \rightarrow_{\beta\vartheta} R$ , then  $\Gamma \vdash R : B$ .

*Proof.* By induction on the structure  $\mathsf{M} \to_{\beta\vartheta} \mathsf{R}$  with the help of Lemma 2.3.

Next, we show that  $\rightrightarrows_{\beta\vartheta}$  is well-defined (Theorems 2.2 and 2.3).

**Lemma 2.4 (Hindley-Rosen).** Let  $R_1$  and  $R_2$  be binary relations on the set  $\mathcal{T}$  of all terms, and let us write  $\rightarrow_{R_i}$  for the contextual closure of  $R_i$  for i = 1, 2. If  $\rightarrow_{R_1}$  and  $\rightarrow_{R_2}$  are Church-Rosser, and satisfy  $\forall M, P, Q \in \mathcal{T}$ .  $M \rightarrow^*_{R_1} P \land M \rightarrow^*_{R_2} Q \Rightarrow \exists R \in \mathcal{T}$ .  $P \rightarrow^*_{R_2} R \land Q \rightarrow^*_{R_1} R$ , then  $\rightarrow_{R_1 \cup R_2}$  is Church-Rosser.

*Proof.* By simple 'diagram chase'; see (Hankin, 1994) for the details.

**Theorem 2.2 (Church-Rosser).** The  $\beta \vartheta$ -reduction  $\rightarrow_{\beta \vartheta}$  is Church-Rosser.

*Proof.* First, it is easy to see that the  $\vartheta$ -reduction  $\to_{\vartheta} \stackrel{\text{df.}}{=} \bigcup_{i=1}^{4} \to_{\vartheta_i}$  satisfies the diamond-property, and thus it is Church-Rosser.

Also, we may show that:

$$\mathsf{M} \to_{\beta} \mathsf{P} \land \mathsf{M} \to_{\vartheta} \mathsf{Q} \Rightarrow \exists \mathsf{R}. \mathsf{P} \to_{\vartheta}^{*} \mathsf{R} \land \mathsf{Q} \to_{\beta} \mathsf{R}$$
(2)

for all terms M, P and Q, where note the asymmetry of  $\rightarrow_{\vartheta}$  and  $\rightarrow_{\beta}$ , by a case analysis on the relation between  $\beta$ - and  $\vartheta$ -redexes in M:

— If the  $\beta$ -redex is inside the  $\vartheta$ -redex, then it is easy to see that (2) holds;

- If the  $\vartheta$ -redex is inside the body of the function subterm of the  $\beta$ -redex, then it suffices to show that  $\rightarrow_{\vartheta}$  commutes with substitution, but it is straightforward;
- If  $\vartheta$ -redex is inside the argument of the  $\beta$ -redex, then it may be duplicated by a finite number n, but whatever the number n is, (2) clearly holds;
- If the  $\beta$  and  $\vartheta$ -redexes are disjoint, then (2) trivially holds.

It then follows from (2) that:

$$\mathsf{M} \to_{\beta} \mathsf{P} \land \mathsf{M} \to_{\vartheta}^{*} \mathsf{Q} \Rightarrow \exists \mathsf{R}. \mathsf{P} \to_{\vartheta}^{*} \mathsf{R} \land \mathsf{Q} \to_{\beta} \mathsf{R}$$
(3)

Dynamic Game Semantics

which in turn implies that:

$$\mathsf{M} \to^*_{\beta} \mathsf{P} \land \mathsf{M} \to^*_{\vartheta} \mathsf{Q} \Rightarrow \exists \mathsf{R}. \mathsf{P} \to^*_{\vartheta} \mathsf{R} \land \mathsf{Q} \to^*_{\beta} \mathsf{R}$$
(4)

for all terms M, P and Q. Applying Lemma 2.4 to (4) (or equivalently by the well-known 'diagram chase' argument on  $\rightarrow^*_{\beta}$  and  $\rightarrow^*_{\vartheta}$ ), we may conclude that the  $\beta\vartheta$ -reduction  $\rightarrow_{\beta\vartheta} = \rightarrow_{\beta} \cup \rightarrow_{\vartheta}$  is Church-Rosser, completing the proof.

Now, we show strong normalization of  $\rightarrow_{\beta\vartheta}$ , i.e., there is no infinite chain of  $\rightarrow_{\beta\vartheta}$ :

**Theorem 2.3 (SN).** The  $\beta \vartheta$ -reduction  $\rightarrow_{\beta \vartheta}$  is strongly normalizing (SN).

*Proof.* By a slight, straightforward modification of the proof of strong normalization of the simply-typed  $\lambda$ -calculus in (Hankin, 1994).

Thus, it follows from Theorems 2.2 and 2.3 that the normal form  $nf(\mathsf{M})$  of each term  $\mathsf{M}$  of FPCF (with respect to  $\rightarrow_{\beta\vartheta}$ ) uniquely exists. Moreover, we have:

**Theorem 2.4 (Normal forms are values).** The normal form nf(M) of every program M (with respect to  $\rightarrow_{\beta\vartheta}$ ) is a value.

*Proof.* It has been shown in (Amadio and Curien, 1998) during the proof to show that PCF Böhm trees are closed under composition.  $\Box$ 

Therefore, we have shown that the operational semantics  $\rightarrow$  is well-defined:

Corollary 2.1 (Correctness of operational semantics). If  $\Gamma \vdash \{M\}_e$ : B is a program, and e > 1 (resp. e = 1), then there exists a unique program (resp. value)  $\Gamma \vdash \{M'\}_{e-1}$ : B that satisfies  $M \to M'$ .

*Proof.* By Theorems 2.1, 2.2, 2.3 and 2.4.

# 2.3. Dynamic Bicategorical Semantics of Finitary PCF

Next, we present a general, categorical recipe to give semantics of FPCF in a CCBoC in such a way that satisfies the DCP.

**Definition 2.4 (Structures for FPCF).** A *structure* for FPCF in a CCBoC  $C = (C, \mathcal{E})$  is a tuple  $S = (\mathscr{B}, 1, \times, \pi, \Rightarrow, ev, \underline{tt}, \underline{ff}, \vartheta)$  such that:

- $-\mathscr{B}\in\mathcal{C};$
- 1,  $(\times, \pi_1, \pi_2)$  and  $(\Rightarrow, ev)$  are respectively a  $\beta$ -terminal object, a  $\beta$ -product (with  $\beta$ -projections) and a  $\beta$ -exponential (with  $\beta$ -evaluations) in C;

 $--\underline{tt},\underline{ff}:1\to \mathscr{B} \text{ and } \vartheta:\mathscr{B}\times(\mathscr{B}\times\mathscr{B})\to \mathscr{B} \text{ are values in } \mathcal{C}.$ 

The *interpretation*  $[\![ - ]\!]_{\mathcal{C}}^{\mathcal{S}}$  of FPCF induced by  $\mathcal{S}$  in  $\mathcal{C}$  assigns an object  $[\![ \mathsf{A} ]\!]_{\mathcal{C}}^{\mathcal{S}} \in \mathcal{C}$  to each type  $\mathsf{A}$ , an object  $[\![ \mathsf{\Gamma} ]\!]_{\mathcal{C}}^{\mathcal{S}} \in \mathcal{C}$  to each context  $\mathsf{\Gamma}$ , and a  $\beta$ -morphism  $[\![ \mathsf{M} ]\!]_{\mathcal{C}}^{\mathcal{S}} : [\![ \mathsf{\Gamma} ]\!]_{\mathcal{C}}^{\mathcal{S}} \to [\![ \mathsf{B} ]\!]_{\mathcal{C}}^{\mathcal{S}}$  to each term  $\mathsf{\Gamma} \vdash \mathsf{M} : \mathsf{B}$  as follows:

 $\begin{array}{l} -- \ (\mathrm{Types}). \ \llbracket o \rrbracket_{\mathcal{C}}^{\mathcal{S}} \stackrel{\mathrm{df.}}{=} \mathscr{B} \ \mathrm{and} \ \llbracket \mathsf{A} \Rightarrow \mathsf{B} \rrbracket_{\mathcal{C}}^{\mathcal{S}} \stackrel{\mathrm{df.}}{=} \llbracket \mathsf{A} \rrbracket_{\mathcal{C}}^{\mathcal{S}} \Rightarrow \llbracket \mathsf{B} \rrbracket_{\mathcal{C}}^{\mathcal{S}}; \\ -- \ (\mathrm{CONTEXTS}). \ \llbracket \epsilon \rrbracket_{\mathcal{C}}^{\mathcal{S}} \stackrel{\mathrm{df.}}{=} 1 \ \mathrm{and} \ \llbracket \mathsf{\Gamma}, \mathsf{x} : \mathsf{A} \rrbracket_{\mathcal{C}}^{\mathcal{S}} \stackrel{\mathrm{df.}}{=} \llbracket \mathsf{\Gamma} \rrbracket_{\mathcal{C}}^{\mathcal{S}} \times \llbracket \mathsf{A} \rrbracket_{\mathcal{C}}^{\mathcal{S}}; \end{array}$ 

- (Terms).

$$\begin{split} \|\Gamma \vdash \mathsf{tt} : o\|_{\mathcal{C}}^{\mathcal{S}} \stackrel{\mathrm{df.}}{=} \mathcal{E}^{\omega}(!_{\|\Gamma\|_{\mathcal{C}}^{\mathcal{S}}}; \underline{tt}) \\ & \|\Gamma \vdash \mathsf{ff} : o\|_{\mathcal{C}}^{\mathcal{S}} \stackrel{\mathrm{df.}}{=} \mathcal{E}^{\omega}(!_{\|\Gamma\|_{\mathcal{C}}^{\mathcal{S}}}; \underline{ff}) \\ & \|\Gamma \vdash \lambda \mathsf{x}.\mathsf{M} : \mathsf{A} \Rightarrow \mathsf{B}\|_{\mathcal{C}}^{\mathcal{S}} \stackrel{\mathrm{df.}}{=} \Lambda_{\|\Gamma\|_{\mathcal{C}}^{\mathcal{S}}, \|\mathbb{A}\|_{\mathcal{C}}^{\mathcal{S}}, \|\mathbb{B}\|_{\mathcal{C}}^{\mathcal{S}}} (\|\Gamma, \mathsf{x} : \mathsf{A} \vdash \mathsf{M} : \mathsf{B}\|_{\mathcal{C}}^{\mathcal{S}}) \\ & \|\Gamma \vdash \mathsf{MN} : \mathsf{B}\|_{\mathcal{C}}^{\mathcal{S}} \stackrel{\mathrm{df.}}{=} \langle \|\Gamma \vdash \mathsf{M} : \mathsf{A} \Rightarrow \mathsf{B}\|_{\mathcal{C}}^{\mathcal{S}}, \|\Gamma \vdash \mathsf{N} : \mathsf{A}\|_{\mathcal{C}}^{\mathcal{S}} \rangle_{\|\mathbb{A} \Rightarrow \mathsf{B}\|_{\mathcal{C}}^{\mathcal{S}}, \|\mathbb{A}\|_{\mathcal{C}}^{\mathcal{S}}}; ev_{\|\mathbb{A}\|_{\mathcal{C}}^{\mathcal{S}}, \|\mathbb{B}\|_{\mathcal{C}}^{\mathcal{S}}} \\ & \|\Gamma \vdash \mathsf{case}(\mathsf{x}\mathbf{V})[\mathsf{W}_{1};\mathsf{W}_{2}] : o\|_{\mathcal{C}}^{\mathcal{S}} \stackrel{\mathrm{df.}}{=} \mathcal{E}^{\omega}(\langle \|\Gamma \vdash \mathsf{x}\mathbf{V} : o\|_{\mathcal{C}}^{\mathcal{S}}, \langle \|\Gamma \vdash \mathsf{W}_{1} : o\|_{\mathcal{C}}^{\mathcal{S}}, \|\Gamma \vdash \mathsf{W}_{2} : o\|_{\mathcal{C}}^{\mathcal{S}} \rangle ; \vartheta) \\ & \|\Gamma \vdash \mathsf{case}(\mathsf{M})[\mathsf{P}_{1};\mathsf{P}_{2}] : o\|_{\mathcal{C}}^{\mathcal{S}} \stackrel{\mathrm{df.}}{=} \mathcal{E}^{\omega}(\langle \|\Gamma \vdash \mathsf{M} : o\|_{\mathcal{C}}^{\mathcal{S}}, \langle \|\Gamma \vdash \mathsf{P}_{1} : o\|_{\mathcal{C}}^{\mathcal{S}}, \|\Gamma \vdash \mathsf{P}_{2} : o\|_{\mathcal{C}}^{\mathcal{S}} \rangle ; \vartheta) \end{split}$$

where  $\llbracket \Gamma \vdash x : A \rrbracket_{\mathcal{C}}^{\mathcal{S}} : \llbracket \Gamma \rrbracket_{\mathcal{C}}^{\mathcal{S}} \to \llbracket A \rrbracket_{\mathcal{C}}^{\mathcal{S}}$  (n.b.,  $\Gamma \vdash x : A$  is not a term of FPCF, but we need it for the application  $x \mathbf{V}$ ) is the obvious (possibly iterated)  $\beta$ -projection.

Moreover, the structure S is *standard* iff it satisfies the following five axioms:

- 1 The maps  $\Lambda_{A,B,C}$  and  $\langle \_, \_ \rangle_{A,B}^C$  in C are bijections for each triple  $A, B, C \in C$ ;
- 2 The object  $\mathscr{B}$ , a  $\beta$ -product and a  $\beta$ -exponential of  $\mathcal{C}$  are pairwise distinct;
- 3 Each  $\beta$ -composition that occurs as the interpretation of a term is not a value;
- 4 A  $\beta$ -currying and a  $\beta$ -composition of C that occur as the interpretations of terms never coincide;
- 5 The  $\beta$ -evaluation  $ev_{A,B}$  for any  $A, B \in \mathcal{C}$  is a mono with respect to the  $\beta$ -composition, i.e.,  $f; ev_{A,B} = f'; ev_{A,B} \Rightarrow f = f'$  for any  $C \in \mathcal{C}$  and  $f, f': C \to B^A \times A$  in  $\mathcal{C}$ .

Clearly, the interpretation  $\llbracket\_\rrbracket_{\mathcal{C}}^{\mathcal{S}}$  followed by  $\mathcal{E}^{\omega}$ , i.e.,  $\mathcal{E}^{\omega}(\llbracket\_\rrbracket_{\mathcal{C}}^{\mathcal{S}})$ , coincides with the standard categorical interpretation of the equational theory Eq(FPCF) in the CCC  $\mathcal{V}_{\mathcal{C}}$  (Lambek and Scott, 1988; Pitts, 2001; Crole, 1993; Jacobs, 1999). In this sense, we have refined the standard categorical semantics of type theories.

At this point, let us recall the DCP (see Section 1) specifically for the interpretation of FPCF induced by a structure in a CCBoC:

**Definition 2.5 (DCP for FPCF).** The interpretation  $[\![-]\!]_{\mathcal{C}}^{\mathcal{S}}$  of FPCF induced by a structure  $\mathcal{S}$  for FPCF in a CCBoC  $\mathcal{C} = (\mathcal{C}, \mathcal{E})$  satisfies the *dynamic correspondence property (DCP)* iff for any programs M<sub>1</sub> and M<sub>2</sub> of FPCF we have:

$$\mathsf{M}_1 \to \mathsf{M}_2 \Leftrightarrow \llbracket \mathsf{M}_1 \rrbracket^{\mathcal{S}}_{\mathcal{C}} \neq \llbracket \mathsf{M}_2 \rrbracket^{\mathcal{S}}_{\mathcal{C}} \wedge \mathcal{E}(\llbracket \mathsf{M}_1 \rrbracket^{\mathcal{S}}_{\mathcal{C}}) = \llbracket \mathsf{M}_2 \rrbracket^{\mathcal{S}}_{\mathcal{C}}.$$

Now, we reduce the DCP for FPCF to the following:

**Definition 2.6 (PDCP for FPCF).** The interpretation  $\llbracket\_\rrbracket_{\mathcal{C}}^{\mathcal{S}}$  of FPCF induced by a structure  $\mathcal{S}$  for FPCF in a CCBoC  $\mathcal{C} = (\mathcal{C}, \mathcal{E})$  satisfies the *pointwise dynamic correspondence property (PDCP)* iff for each term  $\Gamma \vdash \{M\}_e : B$  it satisfies:

 $\mathcal{E}(\llbracket M \rrbracket_{\mathcal{C}}^{\mathcal{S}}) = \begin{cases} \Lambda \circ \mathcal{E}(\llbracket P \rrbracket_{\mathcal{C}}^{\mathcal{S}}) & \text{if } \mathsf{M} \equiv \lambda \mathsf{x}.\mathsf{P}; \\ \llbracket W \rrbracket_{\mathcal{C}}^{\mathcal{S}} \text{ such that } \llbracket W \rrbracket_{\mathcal{C}}^{\mathcal{S}} \neq \llbracket \mathsf{M} \rrbracket_{\mathcal{C}}^{\mathcal{S}} & \text{if } \mathsf{M} \equiv \mathsf{UV}, \ e = 1 \text{ and } \mathsf{UV} \to \mathsf{W}; \\ \langle \mathcal{E}(\llbracket \mathsf{L} \rrbracket_{\mathcal{C}}^{\mathcal{S}}), \mathcal{E}(\llbracket \mathsf{R} \rrbracket_{\mathcal{C}}^{\mathcal{S}}) \rangle; ev & \text{if } \mathsf{M} \equiv \mathsf{LR} \text{ and } e > 1; \\ \llbracket \mathsf{M} \rrbracket_{\mathcal{C}}^{\mathcal{S}} & \text{otherwise.} \end{cases}$ 

**Lemma 2.5 (P-lemma).** If the interpretation  $[\![-]\!]_{\mathcal{C}}^{\mathcal{S}}$  induced by a standard structure  $\mathcal{S}$  for FPCF in a CCBoC  $\mathcal{C} = (\mathcal{C}, \mathcal{E})$  satisfies the PDCP, then  $\mathcal{E}([\![M]\!]_{\mathcal{C}}^{\mathcal{S}}) \neq [\![M]\!]_{\mathcal{C}}^{\mathcal{S}} \Leftrightarrow \sharp(\mathsf{M}) \ge 1$  for all terms  $\mathsf{M}$ .

*Proof.* By induction on the construction of M, where the first and the fifth axioms on standardness of S is essential.

**Theorem 2.5 (Standard bicategorical semantics of FPCF).** The interpretation  $\llbracket\_\rrbracket^S_{\mathcal{C}}$  of FPCF induced by a standard structure S for FPCF in a CCBoC  $\mathcal{C} = (\mathcal{C}, \mathcal{E})$  satisfies the DCP if it satisfies the PDCP.

*Proof.* In the following, we abbreviate  $\llbracket - \rrbracket_{\mathcal{C}}^{S}$  as  $\llbracket - \rrbracket$ . Assume that  $\llbracket - \rrbracket$  satisfies the PDCP. We show  $\mathsf{M} \to \mathsf{M}' \Leftrightarrow \llbracket \mathsf{M} \rrbracket \neq \llbracket \mathsf{M}' \rrbracket \land \mathcal{E}(\llbracket \mathsf{M} \rrbracket) = \llbracket \mathsf{M}' \rrbracket$  for any programs  $\Gamma \vdash \{\mathsf{M}\}_e : \mathsf{B}$  and  $\Gamma \vdash \{\mathsf{M}'\}_{e'} : \mathsf{B}$  of FPCF by induction on the construction of  $\mathsf{M}$ :

- If  $\mathsf{M} \equiv \mathsf{tt}$  or  $\mathsf{M} \equiv \mathsf{ff}$ , then there is no term  $\mathsf{M}'$  such that  $\mathsf{M} \to \mathsf{M}'$ , and there is no  $\beta$ -morphism f' in  $\mathcal{C}$  such that  $\llbracket \mathsf{M} \rrbracket \neq f' \land \mathcal{E}(\llbracket \mathsf{M} \rrbracket) = f'$  because  $\mathcal{E}(\llbracket \mathsf{M} \rrbracket) = \llbracket \mathsf{M} \rrbracket$ .
- If  $\Gamma \vdash M \equiv case(xV_1V_2...V_k)[W_1; W_2] : o$ , then it can be handled in the same manner as the above case.
- If  $\Gamma \vdash M \equiv \lambda x^A . P : A \Rightarrow C$ , then we have:

$$\begin{split} \mathsf{M} &\to \mathsf{M}' \Leftrightarrow \mathsf{M}' \equiv \lambda x.\mathsf{P}' \land \mathsf{P} \to \mathsf{P}' \text{ for some program } \mathsf{P}' \text{ and variable } x \\ &\Leftrightarrow \mathsf{M}' \equiv \lambda x.\mathsf{P}' \land \llbracket \mathsf{P} \rrbracket \neq \llbracket \mathsf{P}' \rrbracket \land \mathcal{E}(\llbracket \mathsf{P} \rrbracket) = \llbracket \mathsf{P}' \rrbracket \text{ for some } \mathsf{P}' \text{ and } x \\ &(\text{by the induction hypothesis}) \\ &\Leftrightarrow \llbracket \mathsf{P} \rrbracket \neq \Lambda^{-1}(\llbracket \mathsf{M}' \rrbracket) \land \mathcal{E}(\llbracket \mathsf{P} \rrbracket) = \Lambda^{-1}(\llbracket \mathsf{M}' \rrbracket) \\ &(\text{n.b., for } \Leftarrow, \Lambda^{-1}(\llbracket \mathsf{M}' \rrbracket) \downarrow \text{ implies that } \mathsf{M}' \text{ must be a currying as } \mathcal{S} \text{ is standard}) \\ &\Leftrightarrow \Lambda^{-1}(\llbracket \mathsf{M} \rrbracket) \neq \Lambda^{-1}(\llbracket \mathsf{M}' \rrbracket) \land \Lambda^{-1} \circ \mathcal{E}(\llbracket \mathsf{M} \rrbracket) = \Lambda^{-1}(\llbracket \mathsf{M}' \rrbracket) \text{ (as } \mathcal{E}(\llbracket \mathsf{M} \rrbracket) = \Lambda \circ \mathcal{E}(\llbracket \mathsf{P} \rrbracket)) \\ &\Leftrightarrow \llbracket \mathsf{M} \rrbracket \neq \llbracket \mathsf{M}' \rrbracket \land \mathcal{E}(\llbracket \mathsf{M} \rrbracket) = \llbracket \mathsf{M}' \rrbracket \text{ (by the bijectivity of } \Lambda). \end{split}$$

— If  $M \equiv LR$ ,  $\sharp(L) \ge 1$  and  $\sharp(R) \ge 1$ , then we have:

$$\begin{split} \mathsf{M} &\to \mathsf{M}' \Leftrightarrow \mathsf{M}' \equiv \mathsf{L}'\mathsf{R}' \land \mathsf{L} \to \mathsf{L}' \land \mathsf{R} \to \mathsf{R}' \text{ for some programs } \mathsf{L}' \text{ and } \mathsf{R}' \\ &\Leftrightarrow \mathsf{M}' \equiv \mathsf{L}'\mathsf{R}' \land \llbracket \mathsf{L} \rrbracket \neq \llbracket \mathsf{L}' \rrbracket \land \mathcal{E}(\llbracket \mathsf{L} \rrbracket) = \llbracket \mathsf{L}' \rrbracket \land \llbracket \mathsf{R} \rrbracket \neq \llbracket \mathsf{R}' \rrbracket \land \mathcal{E}(\llbracket \mathsf{R} \rrbracket) = \llbracket \mathsf{R}' \rrbracket \\ &\text{ for some } \mathsf{L}' \text{ and } \mathsf{R}' \text{ (by the induction hypothesis)} \\ &\Leftrightarrow \llbracket \mathsf{M}' \rrbracket = \langle \mathcal{E}(\llbracket \mathsf{L} \rrbracket), \mathcal{E}(\llbracket \mathsf{R} \rrbracket) \rangle; ev \land \llbracket \mathsf{L} \rrbracket \neq \mathcal{E}(\llbracket \mathsf{L} \rrbracket) \land \llbracket \mathsf{R} \rrbracket \neq \mathcal{E}(\llbracket \mathsf{R} \rrbracket) \\ &\text{ (n.b., } \Leftarrow \text{ holds by the third and the fourth axioms on standardness of } \mathcal{S}) \\ &\Leftrightarrow \llbracket \mathsf{M}' \rrbracket = \mathcal{E}(\llbracket \mathsf{LR} \rrbracket) \land \llbracket \mathsf{L} \rrbracket \neq \mathcal{E}(\llbracket \mathsf{L} \rrbracket) \land \llbracket \mathsf{R} \rrbracket \neq \mathcal{E}(\llbracket \mathsf{R} \rrbracket) \\ &\text{ (because the interpretation } \llbracket \ttL \rrbracket \neq \mathcal{E}(\llbracket \mathsf{L} \rrbracket) \land \llbracket \mathsf{R} \rrbracket \neq \mathcal{E}(\llbracket \mathsf{R} \rrbracket) \\ &\Leftrightarrow \llbracket \mathsf{M}' \rrbracket = \mathcal{E}(\llbracket \mathsf{M} \rrbracket) \land \llbracket \mathsf{L} \rrbracket \neq \mathcal{E}(\llbracket \mathsf{M} \rrbracket) \land \llbracket \mathsf{R} \rrbracket \neq \mathcal{E}(\llbracket \mathsf{R} \rrbracket) \\ &\Leftrightarrow \llbracket \mathsf{M}' \rrbracket = \mathcal{E}(\llbracket \mathsf{M} \rrbracket) \text{ (by Lemma 2.5)} \end{aligned}$$

 $-- \text{ If } \mathsf{M} \equiv \mathsf{LR}, \, \sharp(\mathsf{L}) = 0 \text{ and } \sharp(\mathsf{R}) \ge 1, \text{ then we have:} \\ \mathsf{M} \to \mathsf{M}' \Leftrightarrow \mathsf{M}' \equiv \mathsf{LR}' \land \mathsf{R} \to \mathsf{R}' \text{ for some program } \mathsf{R}' \\ \Leftrightarrow \mathsf{M}' \equiv \mathsf{LR}' \land \llbracket \mathsf{R} \rrbracket \neq \llbracket \mathsf{R}' \rrbracket \land \mathcal{E}(\llbracket \mathsf{R} \rrbracket) = \llbracket \mathsf{R}' \rrbracket \text{ for some } \mathsf{R}' \\ (\text{by the induction hypothesis}) \\ \Leftrightarrow \llbracket \mathsf{M}' \rrbracket = \langle \llbracket \mathsf{L} \rrbracket, \mathcal{E}(\llbracket \mathsf{R} \rrbracket) \rangle; \, ev \land \llbracket \mathsf{R} \rrbracket \neq \mathcal{E}(\llbracket \mathsf{R} \rrbracket) \\ (n.b., \leftarrow \text{ holds as in the above case}) \\ \Leftrightarrow \llbracket \mathsf{M}' \rrbracket = \langle \llbracket \mathsf{L} \rrbracket, \mathcal{E}(\llbracket \mathsf{R} \rrbracket) \rangle; \, ev (\text{ by Lemma 2.5}) \\ \Leftrightarrow \llbracket \mathsf{M}' \rrbracket = \mathcal{E}(\llbracket \mathsf{M} \rrbracket) (\text{by Lemma 2.5 and the PDCP of the interpretation } \llbracket - \rrbracket) \\ \Leftrightarrow \llbracket \mathsf{M}' \rrbracket = \mathcal{E}(\llbracket \mathsf{M} \rrbracket) \land \mathcal{E}(\llbracket \mathsf{M} \rrbracket) \neq \llbracket \mathsf{M} \rrbracket (\text{again by Lemma 2.5}). \\ -- \text{ If } \mathsf{M} \equiv \mathsf{LR}, \, \sharp(\mathsf{L}) \ge 1 \text{ and } \sharp(\mathsf{R}) = 0, \text{ then it is handled similarly to the above case.} \\ -- \text{ If } \mathsf{M} \equiv \mathsf{LR}, \, \sharp(\mathsf{L}) \ge 1 \text{ and } \sharp(\mathsf{R}) = 0, \text{ then we have:} \\ \mathsf{M} \to \mathsf{M}' \Leftrightarrow \mathcal{E}(\llbracket \mathsf{M} \rrbracket) = \llbracket \mathsf{M}' \rrbracket (\text{since the interpretation } \llbracket - \rrbracket) \text{ satisfies the PDCP} \\ \Leftrightarrow \mathcal{E}(\llbracket \mathsf{M} \rrbracket) = \llbracket \mathsf{M}' \rrbracket \land \mathcal{E}(\llbracket \mathsf{M} \rrbracket) \neq \llbracket \mathsf{M} \rrbracket (\text{by Lemma 2.5}). \end{cases}$ 

which completes the proof.

To summarize the present section, we have defined bicategorical 'universes' of dynamic, intensional computations, viz., (CC)BoCs, presented the simple functional programming language FPCF, and given an interpretation of the latter in the former as well as a sufficient condition, namely, the PDCP, for the interpretation to satisfy the DCP. Hence, our research problem (described in Section 1) has been reduced to giving a standard structure for FPCF in a game-semantic CCBoC that satisfies the PDCP.

### 3. Dynamic Games and Strategies

The present section introduces dynamic games and strategies and studies their algebraic structures. The main idea of dynamic games and strategies is to introduce the distinction between *internal* and *external* moves to conventional games and strategies; internal moves constitute 'internal communication' between dynamic strategies, representing *intensionality* of computation, and they are to be *a posteriori* 'hidden' by the *hiding operation*, capturing *dynamics* of computation. Conceptually, external moves are 'official' ones for the underlying game, while internal moves are supposed to be 'invisible' to Opponent for they represent how Player 'internally' computes the next external move.

Dynamic games and strategies are based on the variant given in (Abramsky and McCusker, 1999), which we call *static* games and strategies (more generally, to distinguish our 'dynamic concepts' from conventional ones, we add the word *static* in front of the corresponding notions in (Abramsky and McCusker, 1999), e.g., static arenas, static legal positions, etc.); this choice is because the variant combines good points of the two best-known variants: *AJM-games* (Abramsky et al., 2000) and *HO-games* (Hyland and Ong, 2000): It interprets the *linear decomposition* of implication (Girard, 1987), and it is *flexible* enough to model a wide range of programming features (Abramsky and McCusker, 1999). We have

chosen this variant with the hope that our framework is also applicable to various formal systems and programming languages.

# 3.1. Dynamic Arenas and Legal Positions

Just like static games (Abramsky and McCusker, 1999), dynamic games are based on (the 'dynamic generalizations' of) *arenas* and *legal positions*. An arena defines the basic components of a game, which in turn induces a set of legal positions that specifies the basic rules of the game. Let us first introduce these preliminary concepts.

## Definition 3.1 (Dynamic arenas). A dynamic arena is a triple

$$G = (M_G, \lambda_G, \vdash_G)$$

such that:

- $M_G$  is a set, whose elements are called **moves**;
- $\lambda_G \text{ is a function } M_G \to \{\mathsf{O},\mathsf{P}\} \times \{\mathsf{Q},\mathsf{A}\} \times \mathbb{N}, \text{ called the$ *labeling function* $, that satisfies <math>\mu(G) \stackrel{\text{df.}}{=} \mathsf{Sup}(\{\lambda_G^{\mathbb{N}}(m) \mid m \in M_G\}) \in \mathbb{N};$
- $\vdash_G$  is a subset of  $(\{\star\} \cup M_G) \times M_G$ , where  $\star$  is an arbitrary element such that  $\star \notin M_G$ , called the *enabling relation*, that satisfies:
  - (E1). If  $\star \vdash_G m$ , then  $\lambda_G(m) = \mathsf{OQ0}$  and  $n = \star$  whenever  $n \vdash_G m$ ;
  - (E2). If  $m \vdash_G n$  and  $\lambda_G^{\mathsf{QA}}(n) = \mathsf{A}$ , then  $\lambda_G^{\mathsf{QA}}(m) = \mathsf{Q}$  and  $\lambda_G^{\mathbb{N}}(m) = \lambda_G^{\mathbb{N}}(n)$ ;
  - (E3). If  $m \vdash_G n$  and  $m \neq \star$ , then  $\lambda_G^{\mathsf{OP}}(m) \neq \lambda_G^{\mathsf{OP}}(n)$ ;
  - (E4). If  $m \vdash_G n, m \neq \star$  and  $\lambda_G^{\mathbb{N}}(m) \neq \lambda_G^{\mathbb{N}}(n)$ , then  $\lambda_G^{\mathsf{OP}}(m) = \mathsf{O}$

in which  $\lambda_G^{\mathsf{OP}} \stackrel{\text{df.}}{=} \pi_1 \circ \lambda_G : M_G \to \{\mathsf{O},\mathsf{P}\}, \ \lambda_G^{\mathsf{QA}} \stackrel{\text{df.}}{=} \pi_2 \circ \lambda_G : M_G \to \{\mathsf{Q},\mathsf{A}\} \text{ and } \lambda_G^{\mathbb{N}} \stackrel{\text{df.}}{=} \pi_3 \circ \lambda_G : M_G \to \mathbb{N}. \text{ A move } m \in M_G \text{ is initial if } \star \vdash_G m, \text{ an } O\text{-move (resp. a } P\text{-move)} \text{ if } \lambda_G^{\mathsf{OP}}(m) = \mathsf{O} \text{ (resp. if } \lambda_G^{\mathsf{OP}}(m) = \mathsf{P}), \text{ a } question \text{ (resp. an } answer) \text{ if } \lambda_G^{\mathsf{QA}}(m) = \mathsf{Q} \text{ (resp. if } \lambda_G^{\mathsf{QA}}(m) = \mathsf{A}), \text{ and } internal \text{ or } \lambda_G^{\mathbb{N}}(m)\text{-internal (resp. } external) \text{ if } \lambda_G^{\mathbb{N}}(m) > 0 \text{ (resp. if } \lambda_G^{\mathbb{Q}}(m) = 0). \text{ Any } s \in M_G^* \text{ is } d\text{-complete if it ends with a move } m \text{ such that } \lambda_G^{\mathbb{N}}(m) = 0 \lor \lambda_G^{\mathbb{N}}(m) > d, \text{ where } d \in \mathbb{N} \cup \{\omega\}, \text{ and } \omega \text{ is the least transfinite ordinal.}$ 

Recall that a static arena G (Abramsky and McCusker, 1999) determines possible *moves* of a game, each of which is Opponent's/Player's question/answer, where the third parity  $\lambda_G^{\mathbb{N}}$  is not included, and specifies which move n can be performed for each move m by the relation  $m \vdash_G n$  (and  $\star \vdash_G m$  means that m can *initiate* a play). The axioms on a static arena are the following:

- (E1). An initial move must be Opponent's question, and an initial move cannot be enabled by any move;
- (The first point of E2). An answer must be performed for a question;
- (E3). An O-move must be performed for a P-move, and vice versa.

Thus, a dynamic arena is a static arena equipped with the **priority order**  $\lambda_G^{\mathbb{N}}$  on moves that satisfies additional axioms on the priority order; it is called so for it determines the 'priority order' of moves to be 'hidden' by the hiding operations on dynamic games (Definition 3.14) and on dynamic strategies (Definition 3.26). We need all natural

numbers for  $\lambda_G^{\mathbb{N}}$ , not only the internal/external (I/E) distinction, to define a *step-by-step* execution of the hiding operations. Conversely, dynamic arenas are generalized static arenas: A static arena is equivalent to a dynamic arena whose moves are all external.

The additional axioms for dynamic areas G are intuitively natural ones:

- We require a *finite* upper bound  $\mu(G)$  of the priority orders for it is conceptually natural and technically necessary for concatenation of dynamic games (Definition 3.21) to be well-defined and for the hiding operation on dynamic games to terminate;
- --- The axiom E1 adds the equation  $\lambda_G^{\mathbb{N}}(m_0) = 0$  for all  $m_0 \in M_G^{\text{lnit}} \stackrel{\text{df.}}{=} \{m \in M_G \mid \star \vdash m\}$  since Opponent cannot 'see' internal moves;
- The second requirement of the axiom E2 states that the priority orders between a 'QA-pair' must coincide, which is intuitively reasonable;
- The additional axiom E4 states that only Player can make a move for a previous one if they have different priority orders for internal moves are 'invisible' to Opponent (as we shall see, if  $\lambda_G^{\mathbb{N}}(m_1) = k_1 < k_2 = \lambda_G^{\mathbb{N}}(m_2)$ , then after the  $k_1$ -many iteration of the hiding operation,  $m_1$  and  $m_2$  become external and internal, respectively, i.e., the I/E-parity of moves is *relative*, which is why E4 is not only concerned with I/E-parity but more fine-grained priority orders).

Convention. Henceforth, an *arena* refers to a dynamic arena by default.

**Example 3.1.** The *terminal arena* T is given by  $T \stackrel{\text{df.}}{=} (\emptyset, \emptyset, \emptyset)$ .

**Example 3.2.** The *flat arena* flat(S) on a given set S is given by  $M_{flat(S)} \stackrel{\text{df.}}{=} \{q\} \cup S$ , where q is any element with  $q \notin S$ ;  $\lambda_{flat(S)} : q \mapsto \mathsf{OQ0}, (m \in S) \mapsto \mathsf{PA0}; \vdash_{flat(S)} \stackrel{\text{df.}}{=} \{(\star,q)\} \cup \{(q,m) \mid m \in S\}$ . For instance,  $N \stackrel{\text{df.}}{=} flat(\mathbb{N})$  is the arena of natural numbers, and  $\mathbf{2} \stackrel{\text{df.}}{=} flat(\mathbb{B})$ , where  $\mathbb{B} \stackrel{\text{df.}}{=} \{tt, ff\}$ , is the arena of booleans.

As already mentioned, interactions between Opponent and Player in a (dynamic or static) game are represented by certain finite sequences of moves of the underlying arena, equipped with *pointers* (Definition 3.3) that specify the occurrence of a move in the sequence for which each occurrence of a non-initial move in the sequence is performed. Technically, pointers are to distinguish similar but different computations; see (Abramsky and McCusker, 1999; Curien, 2006) for this point.

**Definition 3.2 (Occurrences of moves).** Given a finite sequence  $s \in M_G^*$  of moves of an arena G, an *occurrence (of a move)* in s is a pair (s(i), i) such that  $i \in \overline{|s|}$ . More specifically, we call the pair (s(i), i) an *initial occurrence* (resp. a *non-initial occurrence*) in s if  $\star \vdash_G s(i)$  (resp. otherwise).

Definition 3.3 (J-sequences (Hyland and Ong, 2000; Abramsky and McCusker, 1999)). A justified (j-) sequence of an areaa G is a pair  $s = (s, \mathcal{J}_s)$  of a finite sequence  $s \in M_G^*$ and a map  $\mathcal{J}_s : \overline{|s|} \to \{0\} \cup \overline{|s|-1}$  such that for all  $i \in \overline{|s|} \mathcal{J}_s(i) = 0$  if  $\star \vdash_G s(i)$ , and  $0 < \mathcal{J}_s(i) < i \land s(\mathcal{J}_s(i)) \vdash_G s(i)$  otherwise. The occurrence  $(s(\mathcal{J}_s(i)), \mathcal{J}_s(i))$  is called the justifier of a non-initial occurrence (s(i), i) in s. We also say that (s(i), i) is justified by  $(s(\mathcal{J}_s(i)), \mathcal{J}_s(i))$ , or there is a pointer from the former to the latter.

### Dynamic Game Semantics

The idea is that each non-initial occurrence in a j-sequence must be performed for a specific previous occurrence, viz., its justifier, in the j-sequence.

**Convention.** By abuse of notation, we usually keep the pointer structure  $\mathcal{J}_s$  of each j-sequence  $s = (s, \mathcal{J}_s)$  implicit and often abbreviate occurrences (s(i), i) in s as s(i). Also, we usually write  $\mathcal{J}_s(s(i)) = s(j)$  if  $\mathcal{J}_s(i) = j$ . This convention is mathematically imprecise, but it does not bring any serious confusion in practice.

**Notation.** We write  $\mathscr{J}_G$  for the set of all j-sequences of an areaa G. We write s = t for any  $s, t \in \mathscr{J}_G$  if s and t are the same j-sequence of G, i.e., s = t and  $\mathcal{J}_s = \mathcal{J}_t$ .

**Definition 3.4 (J-subsequences).** Given an arena G and a j-sequence  $s \in \mathcal{J}_G$ , a *j-subsequence* of s is a j-sequence  $t \in \mathcal{J}_G$  that satisfies:

- t is a subsequence of s, for which we write  $t = (s(i_1), s(i_2), \ldots, s(i_{|t|}));$
- $\mathcal{J}_{\boldsymbol{t}}(\boldsymbol{s}(i_r)) = \boldsymbol{s}(i_l) \text{ iff there are occurrences } \boldsymbol{s}(j_1), \boldsymbol{s}(j_2), \dots, \boldsymbol{s}(j_k) \text{ in } \boldsymbol{s} \text{ eliminated in } \boldsymbol{t}, \\ \text{where } l, r, k \in \mathbb{N} \text{ and } 1 \leqslant l < r \leqslant |\boldsymbol{t}|, \text{ such that } \mathcal{J}_{\boldsymbol{s}}(\boldsymbol{s}(i_r)) = \boldsymbol{s}(j_1) \land \mathcal{J}_{\boldsymbol{s}}(\boldsymbol{s}(j_1)) = \\ \boldsymbol{s}(j_2) \land \dots \land \mathcal{J}_{\boldsymbol{s}}(\boldsymbol{s}(j_{k-1})) = \boldsymbol{s}(j_k) \land \mathcal{J}_{\boldsymbol{s}}(\boldsymbol{s}(j_k)) = \boldsymbol{s}(i_l).$

We now consider justifiers, j-sequences and arenas from the 'external point of view':

**Definition 3.5 (External justifiers).** Let G be an arena, and assume  $s \in \mathscr{J}_G$  and  $d \in \mathbb{N} \cup \{\omega\}$ . Each non-initial occurrence n in s has a unique sequence of justifiers  $mm_1m_2\ldots m_kn \ (k \ge 0)$ , i.e.,  $\mathcal{J}_s(n) = m_k, \ \mathcal{J}_s(m_k) = m_{k-1}, \ldots, \ \mathcal{J}_s(m_2) = m_1$  and  $\mathcal{J}_s(m_1) = m$ , such that  $\lambda_G^{\mathbb{N}}(m) = 0 \lor \lambda_G^{\mathbb{N}}(m) > d$  and  $0 < \lambda_G^{\mathbb{N}}(m_i) \le d$  for  $i = 1, 2, \ldots, k$ . We call m the *d*-external justifier of n in s.

**Notation.** We write  $\mathcal{J}_{s}^{\odot d}(n)$  for the *d*-external justifier of *n* in a j-sequence *s*.

Note that *d*-external justifiers are a simple generalization of justifiers because 0-external justifiers coincide with justifiers (as there is no '0-internal' move). More generally, *d*-external justifiers are justifiers after the *d*-times iteration of the hiding operation, as we shall see shortly.

**Definition 3.6 (External j-subsequences).** Let G be an area,  $s \in \mathscr{J}_G$  and  $d \in \mathbb{N} \cup \{\omega\}$ . The *d-external j-subsequence*  $\mathcal{H}^d_G(s)$  of s is obtained from s by deleting occurrences of internal moves m such that  $0 < \lambda^{\mathbb{N}}_G(m) \leq d$  and equipping it with the pointers  $\mathcal{J}_{\mathcal{H}^d_G(s)} : n \mapsto \mathcal{J}^{\odot d}_s(n)$  (more precisely,  $\mathcal{J}_{\mathcal{H}^d_G(s)}$  is the obvious *restriction* of  $\mathcal{J}^{\odot d}_s$ ).

**Definition 3.7 (External arenas).** Let G be an arena, and  $d \in \mathbb{N} \cup \{\omega\}$ . The *d*-*external arena*  $\mathcal{H}^d(G)$  of G is given by:

$$\begin{split} &- M_{\mathcal{H}^{d}(G)} \stackrel{\mathrm{df.}}{=} \{ m \in M_{G} \mid \lambda_{G}^{\mathbb{N}}(m) = 0 \lor \lambda_{G}^{\mathbb{N}}(m) > d \, \}; \\ &- \lambda_{\mathcal{H}^{d}(G)} \stackrel{\mathrm{df.}}{=} \lambda_{G}^{\odot d} \upharpoonright M_{\mathcal{H}^{d}(G)}, \, \mathrm{where} \, \lambda_{G}^{\odot d} \stackrel{\mathrm{df.}}{=} \langle \lambda_{G}^{\mathsf{OP}}, \lambda_{G}^{\mathsf{QA}}, n \mapsto \lambda_{G}^{\mathbb{N}}(n) \odot d \rangle, \, \mathrm{and} \, n \odot d \stackrel{\mathrm{df.}}{=} \\ & \begin{cases} n - d & \mathrm{if} \, n \geqslant d; \\ 0 & \mathrm{otherwise} \end{cases} \text{ for all } n \in \mathbb{N}; \end{split}$$

 $- m \vdash_{\mathcal{H}^{d}(G)} n \stackrel{\text{df.}}{\Leftrightarrow} \exists k \in \mathbb{N}, m_{1}, m_{2}, \dots, m_{2k-1}, m_{2k} \in M_{G} \setminus M_{\mathcal{H}^{d}(G)}. m \vdash_{G} m_{1} \land \forall i \in \overline{k}. m_{2i-1} \vdash_{G} m_{2i} \land m_{2k} \vdash_{G} n \ (\Leftrightarrow m \vdash_{G} n \text{ if } k = 0).$ 

That is, the *d*-external arena  $\mathcal{H}^d(G)$  is obtained from the arena *G* by deleting internal moves *m* such that  $0 < \lambda_G^{\mathbb{N}}(m) \leq d$ , decreasing by *d* the priority orders of the remaining moves and 'concatenating' the enabling relation to form the '*d*-external' one.

**Convention.** Given  $d \in \mathbb{N} \cup \{\omega\}$ , we regard  $\mathcal{H}^d$  as an operation on dynamic arenas G, and  $\mathcal{H}^d_G$  as an operation on j-sequences  $s \in \mathscr{J}_G$ .

Now, let us establish:

Lemma 3.1 (External closure lemma). If G is an arena, then, for all  $d \in \mathbb{N} \cup \{\omega\}$ , so is  $\mathcal{H}^d(G)$ , and  $\mathcal{H}^d_G(s) \in \mathscr{J}_{\mathcal{H}^d(G)}$  for all  $s \in \mathscr{J}_G$ .

*Proof.* The case d = 0 is trivial; thus, assume d > 0. Clearly, the set  $M_{\mathcal{H}^d(G)}$  of moves and the labeling function  $\lambda_{\mathcal{H}^d(G)}$  are well-defined. Now, let us verify the axioms for the enabling relation  $\vdash_{\mathcal{H}^d(G)}$ :

- -- (E1). Note that  $\star \vdash_{\mathcal{H}^d(G)} m \Leftrightarrow \star \vdash_G m$  (because  $\leftarrow$  is immediate, and  $\Rightarrow$  holds by E4 on G as initial moves are all external). Thus, if  $\star \vdash_{\mathcal{H}^d(G)} m$ , then  $\lambda_{\mathcal{H}^d(G)}(m) = \lambda_G^{\odot d}(m) = \mathsf{OQ0}$ , and  $n \vdash_{\mathcal{H}^d(G)} m \Rightarrow n = \star$ .
- $\lambda_{G}^{\bigcirc d}(m) = \mathsf{O}\mathsf{Q}\mathsf{0}, \text{ and } n \vdash_{\mathcal{H}^{d}(G)} m \Rightarrow n = \star.$   $(E2). \text{ Assume } m \vdash_{\mathcal{H}^{d}(G)} n \text{ and } \lambda_{\mathcal{H}^{d}(G)}^{\bigcirc \mathsf{Q}\mathsf{A}}(n) = \mathsf{A}. \text{ If } m \vdash_{G} n, \text{ then } \lambda_{\mathcal{H}^{d}(G)}^{\bigcirc \mathsf{Q}\mathsf{A}}(m) = \lambda_{G}^{\bigcirc \mathsf{Q}\mathsf{A}}(m) = \mathsf{Q} \text{ and } \lambda_{\mathcal{H}^{d}(G)}^{\mathbb{N}}(m) = \lambda_{G}^{\mathbb{N}}(m) \odot d = \lambda_{G}^{\mathbb{N}}(n) \odot d = \lambda_{\mathcal{H}^{d}(G)}^{\mathbb{N}}(n). \text{ Otherwise, i.e.,}$ there are some  $k \in \mathbb{N}^{+} \stackrel{\text{df.}}{=} \{n \in \mathbb{N} \mid n > 0\}$  and  $m_{1}, m_{2}, \ldots, m_{2k} \in M_{G} \setminus M_{\mathcal{H}^{d}(G)}$  such that  $m \vdash_{G} m_{1} \land \forall i \in \overline{k}. m_{2i-1} \vdash_{G} m_{2i} \land m_{2k} \vdash_{G} n$ , then in particular  $m_{2k} \vdash_{G} n$  with  $\lambda_{G}^{\bigcirc \mathsf{Q}\mathsf{A}}(n) = \mathsf{A}, \text{ but } \lambda_{G}^{\mathbb{N}}(m_{2k}) \neq \lambda_{G}^{\mathbb{N}}(n), \text{ a contradiction.}$
- (E3). Assume  $m \vdash_{\mathcal{H}^d(G)} n$  and  $m \neq \star$ . If  $m \vdash_G n$ , then  $\lambda_{\mathcal{H}^d(G)}^{\mathsf{OP}}(m) = \lambda_G^{\mathsf{OP}}(m) \neq \lambda_G^{\mathsf{OP}}(n) = \lambda_{\mathcal{H}^d(G)}^{\mathsf{OP}}(n)$ . If  $\exists k \in \mathbb{N}^+, m_1, m_2, \dots, m_{2k} \in M_G \setminus M_{\mathcal{H}^d(G)}, m \vdash_G m_1 \land \forall i \in \overline{k}, m_{2i-1} \vdash_G m_{2i} \land m_{2k} \vdash_G n$ , then  $\lambda_{\mathcal{H}^d(G)}^{\mathsf{OP}}(m) = \lambda_G^{\mathsf{OP}}(m) = \lambda_G^{\mathsf{OP}}(m_2) = \lambda_G^{\mathsf{OP}}(m_4) = \dots = \lambda_G^{\mathsf{OP}}(m_{2k}) \neq \lambda_G^{\mathsf{OP}}(n) = \lambda_{\mathcal{H}^d(G)}^{\mathsf{OP}}(n).$
- (E4). Assume  $m \vdash_{\mathcal{H}^d(G)} n, m \neq \star$  and  $\lambda_{\mathcal{H}^d(G)}^{\mathbb{N}}(m) \neq \lambda_{\mathcal{H}^d(G)}^{\mathbb{N}}(n)$ . Then, we have  $\lambda_G^{\mathbb{N}}(m) \neq \lambda_G^{\mathbb{N}}(n)$ . If  $m \vdash_G n$ , then it is trivial; otherwise, i.e., there are some  $k \in \mathbb{N}^+$ ,  $m_1, m_2, \ldots, m_{2k} \in M_G \setminus M_{\mathcal{H}^d(G)}$  with the same property as in the case of E3 above,  $\lambda_{\mathcal{H}^d(G)}^{\mathsf{OP}}(m) = \lambda_G^{\mathsf{OP}}(m) = \mathsf{O}$  by E3 on G since  $\lambda_G^{\mathbb{N}}(m) \neq \lambda_G^{\mathbb{N}}(m_1)$ .

Hence, we have shown that the structure  $\mathcal{H}^d(G)$  forms a well-defined arena.

Next, let  $\mathbf{s} \in \mathscr{J}_G$ ; we have to show  $\mathcal{H}^d_G(\mathbf{s}) \in \mathscr{J}_{\mathcal{H}^d(G)}$ . Assume that m is a non-initial occurrence in  $\mathcal{H}^d_G(\mathbf{s})$ . By the definition, the d-external justifier  $m_0 \stackrel{\text{df.}}{=} \mathcal{J}_{\mathcal{H}^d_G(\mathbf{s})}(m)$  occurs in  $\mathcal{H}^d_G(\mathbf{s})$ . If m is a P-move, then the sequence of justifiers  $m_0 \vdash_G m_1 \vdash_G \cdots \vdash_G m_k \vdash m$  satisfies  $\mathsf{Even}(k)$  by the axioms E3 and E4 on G, so that  $m_0 \vdash_{\mathcal{H}^d(G)} m$  by the definition. If m is an O-move, then the justifier  $m'_0 \stackrel{\text{df.}}{=} \mathcal{J}_{\mathbf{s}}(m)$  satisfies  $\lambda^{\mathbb{N}}_G(m'_0) = \lambda^{\mathbb{N}}_G(m)$  by the axiom E4 on G, and so  $m'_0 \vdash_{\mathcal{H}^d(G)} m$  by the definition. Since m is arbitrary, we have shown that  $\mathcal{H}^d_G(\mathbf{s}) \in \mathscr{J}_{\mathcal{H}^d(G)}$ , completing the proof.

Next, let us introduce a useful lemma:

**Lemma 3.2 (Stepwise hiding on arenas).** Given an arena G, we have  $\widetilde{\mathcal{H}}^i(G) = \mathcal{H}^i(G)$  for all  $i \in \mathbb{N}$ , where  $\widetilde{\mathcal{H}}^i$  denotes the *i*-times iteration of  $\mathcal{H}^1$ .

*Proof.* By induction on i.

Thus, we may just focus on  $\mathcal{H}^1$ : Henceforth, we write  $\mathcal{H}$  for  $\mathcal{H}^1$  and call it the *hiding* operation (on arenas);  $\mathcal{H}^i$  for each  $i \in \mathbb{N}$  denotes the *i*-times iteration of  $\mathcal{H}$ .

We may establish a similar inductive property for j-sequences:

Lemma 3.3 (Stepwise hiding on j-sequences). Given a j-sequence  $s \in \mathscr{J}_G$  of an arena G, we have  $\mathcal{H}_G^{i+1}(s) = \mathcal{H}_{\mathcal{H}^i(G)}^1(\mathcal{H}_G^i(s))$  for all  $i \in \mathbb{N}$ .

*Proof.* By induction on i, where note that  $\mathcal{H}_{G}^{i+1}(s), \mathcal{H}_{\mathcal{H}^{i}(G)}^{1}(\mathcal{H}_{G}^{i}(s)) \in \mathscr{J}_{\mathcal{H}^{i+1}(G)}$  by Lemmata 3.1 and 3.2.

Lemma 3.3 implies that the equation

$$\mathcal{H}_{G}^{i}(\boldsymbol{s}) = \mathcal{H}_{\mathcal{H}^{i-1}(G)}^{1} \circ \mathcal{H}_{\mathcal{H}^{i-2}(G)}^{1} \circ \cdots \circ \mathcal{H}_{\mathcal{H}^{1}(G)}^{1} \circ \mathcal{H}_{G}^{1}(\boldsymbol{s})$$
(5)

holds for any arena  $G, s \in \mathscr{J}_G$  and  $i \in \mathbb{N}$  (n.b., the equation (5) means s = s if i = 0). Thus, we may focus on the operation  $\mathcal{H}^1_G$  on j-sequences, where G ranges over all arenas. Henceforth, we write  $\mathcal{H}_G$  for  $\mathcal{H}^1_G$  and call it the *hiding operation on j-sequences of*  $G; \mathcal{H}^i_G$  for each  $i \in \mathbb{N}$  denotes the operation on the right-hand side of (5).

Now, to deal with external j-subsequences in a mathematically rigorous manner, let us extend the hiding operation on j-sequences to that on j-subsequences (Definition 3.4):

**Definition 3.8 (Point-wise hiding on j-sequences).** Let  $s \in \mathscr{J}_G$  be a j-sequence of an arena G. Given an occurrence m in s, we define  $\widehat{\mathcal{H}}_G^m(s)$  to be the j-subsequence of s that consists of occurrences in s different from m if m is 1-internal, and s otherwise. Moreover, given a subsequence  $t = m_1 m_2 \dots m_k$  of (the underlying finite sequence of) sand a permutation  $\sigma$  on  $\overline{k}$ , we define  $\widehat{\mathcal{H}}_G^{t,\sigma}(s) \stackrel{\text{df.}}{=} \widehat{\mathcal{H}}_G^{m_{\sigma(k)}} \circ \dots \circ \widehat{\mathcal{H}}_G^{m_{\sigma(2)}} \circ \widehat{\mathcal{H}}_G^{m_{\sigma(1)}}(s)$ .

The point here is that the hiding operation on j-sequences can be executed in the 'move-wise' fashion in any order:

**Lemma 3.4 (Move-wise lemma).** Let G be an arena, and  $s \in \mathcal{J}_G$ .

1  $\widehat{\mathcal{H}}_{G}^{\boldsymbol{t},\sigma_{1}}(\boldsymbol{s}) = \widehat{\mathcal{H}}_{G}^{\boldsymbol{t},\sigma_{2}}(\boldsymbol{s})$  for any subsequence  $\boldsymbol{t}$  of  $\boldsymbol{s}$  and permutations  $\sigma_{1}$  and  $\sigma_{2}$  on  $|\boldsymbol{t}|$ ; 2  $\widehat{\mathcal{H}}_{G}^{\boldsymbol{s},\sigma}(\boldsymbol{s}) = \mathcal{H}_{G}(\boldsymbol{s})$  for any permutation  $\sigma$  on  $|\boldsymbol{s}|$ .

*Proof.* Immediate from the definition.

By Lemma 3.4, we have established the 'move-wise' procedure to execute the hiding operation  $\mathcal{H}_G$  on j-sequences of a given arena G, where the order of deleting moves is irrelevant. Then, e.g., it follows that  $\mathcal{H}_G(stuv) = \hat{\mathcal{H}}_G^{v,\nu} \circ \hat{\mathcal{H}}_G^{u,\mu} \circ \hat{\mathcal{H}}_G^{t,\tau} \circ \hat{\mathcal{H}}_G^{s,\sigma}(stuv)$  for any arena G and  $stuv \in \mathscr{J}_G$ , where  $\sigma, \tau, \mu$  and  $\nu$  are arbitrary permutations on |s|, |t|, |u| and |v|, respectively, which will be useful in the rest of the paper.

**Convention.** Thanks to Lemma 3.4, we henceforth dispense with the notation  $\widehat{\mathcal{H}}_{G}^{s,\sigma}$ , where G ranges over arenas, s over j-sequences of G, and  $\sigma$  over permutations on |s|, implicitly admitting any order of 'move-wise' execution of the operation  $\mathcal{H}_G$ . Also, we write, abusing notation,  $\mathcal{H}_G(s).\mathcal{H}_G(t).\mathcal{H}_G(u).\mathcal{H}_G(v)$  for  $\widehat{\mathcal{H}}_{G}^{v,\nu} \circ \widehat{\mathcal{H}}_{G}^{t,\mu} \circ \widehat{\mathcal{H}}_{G}^{t,\sigma} \circ \widehat{\mathcal{H}}_{G}^{s,\sigma}(stuv)$  given above, so that  $\mathcal{H}_G(stuv) = \mathcal{H}_G(s).\mathcal{H}_G(t).\mathcal{H}_G(u).\mathcal{H}_G(v)$ .

Next, let us recall the notion of 'relevant part' of previous moves, called views:

**Definition 3.9 (Views (Abramsky and McCusker, 1999)).** Given a j-sequence s of an arena G, the *Player (P-) view*  $\lceil s \rceil_G$  and the *Opponent (O-) view*  $\lfloor s \rfloor_G$  (we often omit the subscript G) are given by the following induction on |s|:

$$\begin{split} &- [\boldsymbol{\epsilon}]_{G} \stackrel{\text{df.}}{=} \boldsymbol{\epsilon}; \\ &- [\boldsymbol{s}m]_{G} \stackrel{\text{df.}}{=} [\boldsymbol{s}]_{G}.m \text{ if } m \text{ is a P-move}; \\ &- [\boldsymbol{s}m]_{G} \stackrel{\text{df.}}{=} m \text{ if } m \text{ is initial}; \\ &- [\boldsymbol{s}m\boldsymbol{t}n]_{G} \stackrel{\text{df.}}{=} [\boldsymbol{s}]_{G}.mn \text{ if } n \text{ is an O-move with } \mathcal{J}_{\boldsymbol{s}m\boldsymbol{t}n}(n) = m; \\ &- [\boldsymbol{\epsilon}]_{G} \stackrel{\text{df.}}{=} \boldsymbol{\epsilon}; \\ &- [\boldsymbol{s}m]_{G} \stackrel{\text{df.}}{=} [\boldsymbol{s}]_{G}.m \text{ if } m \text{ is an O-move}; \\ &- [\boldsymbol{s}m\boldsymbol{t}n]_{G} \stackrel{\text{df.}}{=} [\boldsymbol{s}]_{G}.mn \text{ if } n \text{ is a P-move with } \mathcal{J}_{\boldsymbol{s}m\boldsymbol{t}n}(n) = m \end{split}$$

where the justifiers of the remaining occurrences in  $\lceil s \rceil$  (resp.  $\lfloor s \rfloor$ ) are unchanged if they occur in  $\lceil s \rceil$  (resp.  $\lfloor s \rfloor$ ), and undefined otherwise. A *view* is a P- or O-view.

The idea behind Definition 3.9 is as follows. For a j-sequence tm of an arena G such that m is a P-move (resp. an O-move), the P-view  $\lceil t \rceil$  (resp. the O-view  $\lfloor t \rfloor$ ) is intended to be the currently 'relevant part' of t for Player (resp. Opponent). That is, Player (resp. Opponent) is concerned only with the last O-move (resp. P-move), its justifier and that justifier's P-view (resp. O-view), which then recursively proceeds.

We are now ready to introduce a 'dynamic generalization' of static legal positions:

**Definition 3.10 (Dynamic legal positions).** Given an arena G, a *dynamic legal position* of G is a j-sequence  $s \in \mathscr{J}_G$  that satisfies:

- (ALTERNATION). If  $s = s_1 mn s_2$ , then  $\lambda_G^{\mathsf{OP}}(m) \neq \lambda_G^{\mathsf{OP}}(n)$ ;
- (GENERALIZED VISIBILITY). If  $\boldsymbol{s} = \boldsymbol{t} m \boldsymbol{u}$  with m non-initial, and  $d \in \mathbb{N} \cup \{\omega\}$  satisfy  $\lambda_G^{\mathbb{N}}(m) = 0 \lor \lambda_G^{\mathbb{N}}(m) > d$ , then  $\mathcal{J}_{\boldsymbol{s}}^{\odot d}(m)$  occurs in  $[\mathcal{H}_G^d(\boldsymbol{t})]_{\mathcal{H}^d(G)}$  if m is a P-move, and it occurs in  $[\mathcal{H}_G^d(\boldsymbol{t})]_{\mathcal{H}^d(G)}$  if m is an O-move;
- (IE-SWITCH). If  $s = s_1 m n s_2$  with  $\lambda_G^{\mathbb{N}}(m) \neq \lambda_G^{\mathbb{N}}(n)$ , then m is an O-move.

Notation.  $\mathscr{L}_G$  denotes the set of all dynamic legal positions of a dynamic arena G.

Recall that a static legal position (Abramsky and McCusker, 1999) of a static arena is a j-sequence of the arena that satisfies alternation and *visibility* (i.e., generalized visibility only for d = 0). It specifies the basic rules of a static game in the sense that every 'development' or *(valid) position* of the game must be a legal position of the underlying arena (but the converse does not necessarily hold):

- In a position of the static game, Opponent always makes the first move by a question, and then Player and Opponent alternately play (by alternation), in which every noninitial move must be made for a specific previous move;
- The justifier of each non-initial move occurring in the position must belong to the 'relevant' part of previous moves occurring in the position (by visibility).

The additional axioms on dynamic legal positions are conceptually natural ones:

- Generalized visibility is a generalization of visibility, which requires that visibility must hold after any iteration of the hiding operation on j-sequences;
- IE-switch states that only Player can change a priority order during a play as internal moves are 'invisible' to Opponent, where the same remark as the one in the axiom E4 is applied for the finer distinction of priority orders than the I/E-parity.

Note that a dynamic legal position of a static arena, seen as a dynamic arena whose moves are all external, is clearly a static legal position, and vice versa. Hence, dynamic legal positions are in fact a generalization of static legal positions.

Convention. Henceforth, a *legal position* refers to a dynamic legal position by default.

### 3.2. Dynamic Games

We are now ready to define the central notion of *dynamic games*:

Definition 3.11 (Dynamic games). A dynamic game is a quintuple

$$G = (M_G, \lambda_G, \vdash_G, P_G, \simeq_G)$$

such that:

- The triple  $(M_G, \lambda_G, \vdash_G)$  forms an arena (Definition 3.1);
- $P_G$  is a subset of  $\mathscr{L}_G$ , whose elements are called *(valid) positions* of G, that satisfies:
  - (P1).  $P_G$  is non-empty and prefix-closed;
  - (DP2). If  $smn \in P_G^{\mathsf{Even}}$  and  $\lambda_G^{\mathbb{N}}(n) > 0$ , then  $\exists r \in M_G. smnr \in P_G;$
  - (DP3). Given  $tr, t'r' \in P_G^{\mathsf{Odd}}$  and  $i \in \mathbb{N}$  such that  $i < \lambda_G^{\mathbb{N}}(r) = \lambda_G^{\mathbb{N}}(r')$ , if  $\mathcal{H}_G^i(t) = \mathcal{H}_G^i(t')$ , then  $\mathcal{H}_G^i(tr) = \mathcal{H}_G^i(t'r')$ ;
- $\simeq_G$  is an equivalence relation on  $P_G$ , called the *identification of (valid) positions*, that satisfies:
  - (I1).  $\boldsymbol{s} \simeq_G \boldsymbol{t} \Rightarrow |\boldsymbol{s}| = |\boldsymbol{t}|;$
  - (I2).  $sm \simeq_G tn \Rightarrow s \simeq_G t \wedge \lambda_G(m) = \lambda_G(n) \wedge (m, n \in M_G^{\text{lnit}} \vee (\exists i \in \overline{|s|}. \mathcal{J}_{sm}(m) = s(i) \wedge \mathcal{J}_{tn}(n) = t(i)));$
  - $(DI3). \ \forall d \in \mathbb{N} \cup \{\omega\}. \ s \simeq^d_G t \land sm \in P_G \Rightarrow \exists tn \in P_G. \ sm \simeq^d_G tn, \text{ where} u \simeq^d_G v \stackrel{\text{df.}}{\Leftrightarrow} \exists u', v' \in P_G. u' \simeq_G v' \land \mathcal{H}^d_G(u') = \mathcal{H}^d_G(u) \land \mathcal{H}^d_G(v') = \mathcal{H}^d_G(v) \text{ for all} u, v \in P_G.$

A **play** of G is an finitely or infinitely increasing sequence of positions  $(\epsilon, m_1, m_1 m_2, ...)$  of G. A dynamic game whose moves are all external is said to be **normalized**.

Recall that a static game (Abramsky and McCusker, 1999) is a quintuple similar to a dynamic game except that the underlying arena is static, and it only satisfies the axioms P1, I1, I2 and I3 (i.e., DI3 only for d = 0). The axiom P1 corresponds to the natural phenomenon that a non-empty 'moment' or position of a game must have the previous 'moment'. Identifications of positions are originally introduced in (Abramsky et al., 2000) and also employed in Section 3.6 of (McCusker, 1998). They are to identify positions up

to inessential details of 'tags' for disjoint union, particularly for *exponential* ! (Definition 3.19); each position  $\mathbf{s} \in P_G$  of a game G is a representative of the equivalence class  $[\mathbf{s}] \stackrel{\text{df.}}{=} \{\mathbf{t} \in P_G \mid \mathbf{t} \simeq_G \mathbf{s}\} \in P_G / \simeq_G$  which we take as primary. For this underlying idea, the three axioms I1, I2 and I3 should make sense.

The additional axioms DP2 and DP3 are in order to enable Player to 'play alone', i.e., Opponent does not have to choose odd-length positions, for the internal part of a play since conceptually Opponent cannot 'see' internal moves; technically, the axiom DP2 is to preserve totality of dynamic strategies under the hiding operation (Corollary 3.4), and the axiom DP3 is for *external consistency* of dynamic strategies: A dynamic strategy behaves always in the same manner from the viewpoint of Opponent, i.e., the external part of a play by a dynamic strategy does not depend on the internal part (Theorem 3.7). Note that the axiom DP2 is slightly involved to be preserved under the hiding operation (Theorem 3.1); it is necessary to generalize the axiom I3 to DI3 for the same reason.

**Remark.** It is certainly simpler to dispense with the identification  $\simeq_G$  of positions for each game G by adopting a simpler formulation of exponential ! as in (McCusker, 1998); however, it would be mathematically *ad-hoc* because the cartesian closed structure of games and strategies would not arise via the standard Girard translation. Recall that the aim of the present work is to establish *mathematics* of dynamics and intensionality of logic and computation, where 'good' mathematics should be robust and general, not ad-hoc; also, it is interesting as future work to extend the present work to *linear* logic and computation. For these reasons, we have decided to retain  $\simeq_G$  as a structure of each game G. Moreover, we shall establish various reasonable properties on identification of positions, which adds credibility of the notions of dynamic games and strategies.

Convention. Henceforth, a *game* refers to a dynamic game by default.

**Example 3.3.** The *terminal game*  $T \stackrel{\text{df.}}{=} (\emptyset, \emptyset, \emptyset, \{\epsilon\}, \{(\epsilon, \epsilon)\})$  is the simplest game.

**Example 3.4.** The *flat game* flat(S) on a given set S is defined as follows. The triple  $flat(S) = (M_{flat(S)}, \lambda_{flat(S)}, \vdash_{flat(S)})$  is the flat arena in Example 3.2,  $P_{flat(S)} \stackrel{\text{df.}}{=} \{\epsilon, q\} \cup \{qm \mid m \in S\}$ , and  $\simeq_{flat(S)} \stackrel{\text{df.}}{=} \{(s, s) \mid s \in P_{flat(S)}\}$ . For instance,  $N \stackrel{\text{df.}}{=} flat(\mathbb{N})$  is the game of natural numbers sketched in the introduction, and  $\mathbf{2} \stackrel{\text{df.}}{=} flat(\mathbb{B})$  is the game of booleans. Also,  $\mathbf{0} \stackrel{\text{df.}}{=} flat(\emptyset)$  is the *empty game*.

Also, let us define a substructure relation between games:

**Definition 3.12 (Subgames).** Given games G and H, we say that H is a *(dynamic)* subgame of G, written  $H \leq G$ , iff  $M_H \subseteq M_G$ ,  $\lambda_H = \lambda_G \upharpoonright M_H$ ,  $\vdash_H \subseteq \vdash_G \cap ((\{\star\} \cup M_H) \times M_H)$ ,  $P_H \subseteq P_G$ ,  $\forall d \in \mathbb{N} \cup \{\omega\}$ .  $\simeq_H^d = \simeq_G^d \cap (P_H \times P_H)$  and  $\mu(H) = \mu(G)$ .

For  $H \leq G$ , the condition on the identifications of positions is required for all numbers  $d \in \mathbb{N} \cup \{\omega\}$  so that the dynamic subgame relation  $\leq$  is preserved under the hiding operation (Theorem 3.1); the last equation  $\mu(H) = \mu(G)$  is to preserve the relation  $\leq$  under concatenation of dynamic games (Definition 3.21).

We shall later focus on *well-founded* games:

**Definition 3.13 (Well-founded games (Clairambault and Harmer, 2010)).** A game G is *well-founded* if  $\vdash_G$  is well-founded *downwards*, i.e., there is no countably infinite sequence  $(m_i)_{i\in\mathbb{N}}$  of moves  $m_i \in M_G$  such that  $\star \vdash_G m_0 \land \forall i \in \mathbb{N}$ .  $m_i \vdash_G m_{i+1}$ .

Now, let us define the *hiding operation* on games:

**Definition 3.14 (Hiding operation on games).** Given  $d \in \mathbb{N} \cup \{\omega\}$ , the *d*-hiding operation (on games) maps each game G to its *d*-external game  $\mathcal{H}^d(G)$  given by:

- The triple  $(M_{\mathcal{H}^d(G)}, \lambda_{\mathcal{H}^d(G)}, \vdash_{\mathcal{H}^d(G)})$  is the *d*-external areaa  $\mathcal{H}^d(G)$  of the underlying areaa *G* (Definition 3.7);
- $P_{\mathcal{H}^d(G)} \stackrel{\mathrm{df.}}{=} \{ \mathcal{H}^d_G(\boldsymbol{s}) \mid \boldsymbol{s} \in P_G \};$
- $\mathcal{H}^d_G(\boldsymbol{s}) \simeq_{\mathcal{H}^d(G)} \mathcal{H}^d_G(\boldsymbol{t}) \stackrel{\mathrm{df.}}{\Leftrightarrow} \boldsymbol{s} \simeq^d_G \boldsymbol{t}.$

Now, we give the first main theorem of the present work:

**Theorem 3.1 (External closure of games).** Given  $d \in \mathbb{N} \cup \{\omega\}$ , (resp. well-founded) games are closed under the operation  $\mathcal{H}^d$ , and  $H \leq G$  implies  $\mathcal{H}^d(H) \leq \mathcal{H}^d(G)$ .

Proof. Let G be a game, and assume  $d \in \mathbb{N} \cup \{\omega\}$ ; we have to show that  $\mathcal{H}^d(G)$  is a game. By Lemma 3.1, it suffices to show that j-sequences in  $P_{\mathcal{H}^d(G)}$  are legal positions of the arena  $\mathcal{H}^d(G)$ , the set  $P_{\mathcal{H}^d(G)}$  satisfies the axioms P1, DP2 and DP3, and the relation  $\simeq_{\mathcal{H}^d(G)}$  is an equivalence relation on  $P_{\mathcal{H}^d(G)}$  that satisfies the axioms I1, I2 and DI3. Since  $\mu(G) \in \mathbb{N}$ , we assume  $d \in \mathbb{N}$ .

For alternation, assume  $s_1mns_2 \in P_{\mathcal{H}^d(G)}$ ; we have to show  $\lambda_{\mathcal{H}^d(G)}^{\mathsf{OP}}(m) \neq \lambda_{\mathcal{H}^d(G)}^{\mathsf{OP}}(n)$ . We have  $\mathcal{H}^d_G(\mathbf{t}_1mm_1m_2\dots m_kn\mathbf{t}_2) = s_1mns_2$  for some  $\mathbf{t}_1mm_1m_2\dots m_kn\mathbf{t}_2 \in P_G$ , where  $\mathcal{H}^d_G(\mathbf{t}_1) = \mathbf{s}_1, \mathcal{H}^d_G(\mathbf{t}_2) = \mathbf{s}_2$  and  $\mathcal{H}^d_G(m_1m_2\dots m_k) = \epsilon$ . Note that  $(\lambda_G^{\mathbb{N}}(m) = 0 \lor \lambda_G^{\mathbb{N}}(m) > d) \land (\lambda_G^{\mathbb{N}}(n) = 0 \lor \lambda_G^{\mathbb{N}}(n) > d)$  and  $0 < \lambda_G^{\mathbb{N}}(m_i) \leq d$  for  $i = 1, 2, \dots, k$ . By the axioms E3 and E4 on G, k must be an even number, and thus  $\lambda_{\mathcal{H}^d(G)}^{\mathsf{OP}}(m) = \lambda_G^{\mathsf{OP}}(m) = \lambda_G^{\mathsf{OP}}(m_2) = \lambda_G^{\mathsf{OP}}(m_4) = \dots = \lambda_G^{\mathsf{OP}}(m_k) \neq \lambda_G^{\mathsf{OP}}(n) = \lambda_G^{\mathsf{OP}}(n).$ 

For generalized visibility, let  $tmu \in P_{\mathcal{H}^d(G)}$  with m non-initial. We have to show, for each  $e \in \mathbb{N} \cup \{\omega\}$ , that if tm is e-complete, then:

— if m is a P-move, then the justifier  $(\mathcal{J}_{\boldsymbol{s}}^{\odot d})^{\odot e}(m)$  occurs in  $[\mathcal{H}_{\mathcal{H}^{d}(G)}^{e}(\boldsymbol{t})]_{\mathcal{H}^{e}(\mathcal{H}^{d}(G))};$ 

— if m is an O-move, then the justifier  $(\mathcal{J}_{\boldsymbol{s}}^{\odot d})^{\odot e}(m)$  occurs in  $[\mathcal{H}_{\mathcal{H}^{d}(G)}^{e}(\boldsymbol{t})]_{\mathcal{H}^{e}(\mathcal{H}^{d}(G))}$ .

Again, for  $\mu(G) \in \mathbb{N}$ , we may assume without loss of generality that  $e \in \mathbb{N}$ . Note that the condition is then equivalent to:

- if *m* is a P-move, then the justifier  $\mathcal{J}_{\boldsymbol{s}}^{\odot(d+e)}(m)$  occurs in  $[\mathcal{H}_{G}^{d+e}(\boldsymbol{t}')]_{\mathcal{H}^{d+e}(G)}$ ; - if *m* is an O-move, then the justifier  $\mathcal{J}_{\boldsymbol{s}}^{\odot(d+e)}(m)$  occurs in  $[\mathcal{H}_{G}^{d+e}(\boldsymbol{t}')]_{\mathcal{H}^{d+e}(G)}$ where  $\boldsymbol{t}'m \in P_{G}$  such that  $\mathcal{H}_{G}^{d}(\boldsymbol{t}'m) = \boldsymbol{t}m$ . It holds by generalized visibility on *G*.

For IE-switch, let  $s_1mns_2 \in P_{\mathcal{H}^d(G)}$  such that  $\lambda_{\mathcal{H}^d(G)}^{\mathbb{N}}(m) \neq \lambda_{\mathcal{H}^d(G)}^{\mathbb{N}}(n)$ . Then, there is some  $t_1munt_2 \in P_G$  such that  $\mathcal{H}_G^d(t_1munt_2) = s_1mns_2$ , where note that  $\lambda_G^{\mathbb{N}}(m) \neq \lambda_G^{\mathbb{N}}(n)$ . Therefore, if  $u = \epsilon$ , then we clearly have  $\lambda_{\mathcal{H}^d(G)}^{\mathsf{OP}}(m) = \mathsf{O}$  by IE-switch on G; otherwise, i.e., u = lu', then we have the same conclusion as  $\lambda_G^{\mathbb{N}}(m) \neq \lambda_G^{\mathbb{N}}(l)$ .

We have established  $P_{\mathcal{H}^d(G)} \subseteq \mathscr{L}_{\mathcal{H}^d(G)}$ . Next, we verify the axioms P1, DP2 and DP3:

- -- (P1). Because  $\boldsymbol{\epsilon} \in P_G$ , we have  $\boldsymbol{\epsilon} = \mathcal{H}^d_G(\boldsymbol{\epsilon}) \in P_{\mathcal{H}^d(G)}$ ; thus,  $P_{\mathcal{H}^d(G)}$  is non-empty. For prefix-closure, let  $\boldsymbol{s}m \in P_{\mathcal{H}^d(G)}$ ; we have to show  $\boldsymbol{s} \in P_{\mathcal{H}^d(G)}$ . There must be some  $\boldsymbol{t}m \in P_G$  such that  $\boldsymbol{s}m = \mathcal{H}^d_G(\boldsymbol{t}m) = \mathcal{H}^d_G(\boldsymbol{t})m$ . Thus,  $\boldsymbol{s} = \mathcal{H}^d_G(\boldsymbol{t}) \in P_{\mathcal{H}^d(G)}$ .
- -- (DP2). If  $smn \in P_{\mathcal{H}^d(G)}^{\mathsf{Even}}$  and  $\lambda_{\mathcal{H}^d(G)}^{\mathbb{N}}(n) > 0$ , then there is some  $tmun \in P_G^{\mathsf{Even}}$  such that  $\mathcal{H}^d_G(tmun) = smn$  and  $\lambda_G^{\mathbb{N}}(n) > d > 0$ . Hence, by the axiom DP2 on G, there is some  $tmunr \in P_G$  such that  $\lambda_G^{\mathbb{N}}(r) = \lambda_G^{\mathbb{N}}(n) > d$  by IE-switch on G. Therefore, we have found  $smnr = \mathcal{H}^d(tmunr) \in P_{\mathcal{H}^d(G)}$ , establishing DP2 on  $\mathcal{H}^d(G)$ .
- (DP3). Assume  $\mathbf{tr}, \mathbf{t'r'} \in P_{\mathcal{H}^d(G)}^{\mathsf{Odd}}$  and  $i \in \mathbb{N}$  such that  $i < \lambda_{\mathcal{H}^d(G)}^{\mathbb{N}}(r) = \lambda_{\mathcal{H}^d(G)}^{\mathbb{N}}(r')$ and  $\mathcal{H}_{\mathcal{H}^d(G)}^i(\mathbf{t}) = \mathcal{H}_{\mathcal{H}^d(G)}^i(\mathbf{t'})$ . We have some  $\mathbf{ur}, \mathbf{u'r'} \in P_G$  with  $\mathcal{H}_G^d(\mathbf{u}) = \mathbf{t}$  and  $\mathcal{H}_G^d(\mathbf{u'}) = \mathbf{t'}$ . Then,  $\mathcal{H}_G^{d+i}(\mathbf{u}) = \mathcal{H}_{\mathcal{H}^d(G)}^i(\mathcal{H}_G^d(\mathbf{u})) = \mathcal{H}_{\mathcal{H}^d(G)}^i(\mathbf{t}) = \mathcal{H}_{\mathcal{H}^d(G)}^i(\mathbf{t'}) =$  $\mathcal{H}_{\mathcal{H}^d(G)}^i(\mathcal{H}_G^d(\mathbf{u'})) = \mathcal{H}_G^{d+i}(\mathbf{u'})$ . Hence, by the axiom DP3 on G, we have r = r' and  $\mathcal{J}_{\mathbf{tr}}^{\odot i}(r) = \mathcal{J}_{\mathbf{ur}}^{\odot (d+i)}(r) = \mathcal{J}_{\mathbf{u'r'}}^{\odot (d+i)}(r') = \mathcal{J}_{\mathbf{t'r'}}^{\odot i}(r')$ , establishing DP3 on  $\mathcal{H}^d(G)$ .

Next,  $\simeq_{\mathcal{H}^d(G)}$  is a well-defined relation on  $P_{\mathcal{H}^d(G)}$  since  $\mathcal{H}^d_G(\mathbf{s}) \simeq_{\mathcal{H}^d(G)} \mathcal{H}^d_G(\mathbf{t})$  does not depend on the choice of representatives  $\mathbf{s}, \mathbf{t} \in P_G$ . Also, it is straightforward to see that  $\simeq_{\mathcal{H}^d(G)}$  is an equivalence relation. Now, we show that  $\simeq_{\mathcal{H}^d(G)}$  satisfies the axioms I1, I2 and DI3. Note that I1 and I2 on  $\simeq_{\mathcal{H}^d(G)}$  immediately follow from those on  $\simeq_G$ . For DI3 on  $\simeq_{\mathcal{H}^d(G)}$ , if  $\mathcal{H}^d_G(\mathbf{s}) \simeq^e_{\mathcal{H}^d(G)} \mathcal{H}^d_G(\mathbf{t})$ , and  $\mathcal{H}^d_G(\mathbf{s}).m \in P_{\mathcal{H}^d(G)}$ , where we may assume  $e \neq \omega$ , then  $\exists \mathbf{s'}m \in P_G$ .  $\mathcal{H}^d_G(\mathbf{s'}m) = \mathcal{H}^d_G(\mathbf{s}).m$ , and so  $\mathcal{H}^{d+e}_G(\mathbf{s'}) = \mathcal{H}^{d+e}_G(\mathbf{s}) \simeq_{\mathcal{H}^{d+e}(G)} \mathcal{H}^{d+e}_G(\mathbf{t})$ . By DI3 on  $\simeq_G$ , we may conclude that  $\exists \mathbf{t}n \in P_G$ .  $\mathbf{s'}m \simeq^{d+e}_G \mathbf{t}n$ , whence we obtain  $\mathcal{H}^d_G(\mathbf{t}).n \in P_{\mathcal{H}^d(G)}$  such that  $\mathcal{H}^d_G(\mathbf{s}).m = \mathcal{H}^d_G(\mathbf{s'}m) \simeq^e_{\mathcal{H}^d(G)} \mathcal{H}^d_G(\mathbf{t}n) = \mathcal{H}^d_G(\mathbf{t}).n$ .

Finally, the preservation of the dynamic subgame relation  $\triangleleft$  under the operation  $\mathcal{H}^d$  is clear from the definition, completing the proof.

Corollary 3.1 (Stepwise hiding on games). For any game G, we have  $\mathcal{H}^1(\mathcal{H}^i(G)) = \mathcal{H}^{i+1}(G)$  for all  $i \in \mathbb{N}$ .

*Proof.* By Lemmata 3.2 and 3.3, it suffices to show the equation  $\simeq_{\mathcal{H}^1(\mathcal{H}^i(G))} = \simeq_{\mathcal{H}^{i+1}(G)}$ . Then, given  $s, t \in P_G$ , we have:

$$\begin{array}{l} \Leftrightarrow \mathcal{H}^{1}_{\mathcal{H}^{i}(G)}(\mathcal{H}^{i}_{G}(\boldsymbol{s})) \simeq_{\mathcal{H}^{1}(\mathcal{H}^{i}(G))} \mathcal{H}^{1}_{\mathcal{H}^{i}(G)}(\mathcal{H}^{i}_{G}(\boldsymbol{t})) \\ \Leftrightarrow \exists \mathcal{H}^{i}(\boldsymbol{s}'), \mathcal{H}^{i}(\boldsymbol{t}') \in P_{\mathcal{H}^{i}(G)}, \mathcal{H}^{i}(\boldsymbol{s}') \simeq_{\mathcal{H}^{i}(G)} \mathcal{H}^{i}(\boldsymbol{t}') \land \mathcal{H}^{1}_{\mathcal{H}^{i}(G)}(\mathcal{H}^{i}(\boldsymbol{s}')) = \mathcal{H}^{1}_{\mathcal{H}^{i}(G)}(\mathcal{H}^{i}(\boldsymbol{s})) \\ \land \mathcal{H}^{1}_{\mathcal{H}^{i}(G)}(\mathcal{H}^{i}(\boldsymbol{t}')) = \mathcal{H}^{1}_{\mathcal{H}^{i}(G)}(\mathcal{H}^{i}(\boldsymbol{t})) \\ \Leftrightarrow \exists \boldsymbol{s''}, \boldsymbol{t''} \in P_{G}, \boldsymbol{s''} \simeq_{G} \boldsymbol{t''} \land \mathcal{H}^{i+1}_{G}(\boldsymbol{s''}) = \mathcal{H}^{i+1}_{G}(\boldsymbol{s}) \land \mathcal{H}^{i+1}_{G}(\boldsymbol{t}') = \mathcal{H}^{i+1}_{G}(\boldsymbol{t}) \\ \Leftrightarrow \mathcal{H}^{i+1}_{G}(\boldsymbol{s}) \simeq_{\mathcal{H}^{i+1}(G)} \mathcal{H}^{i+1}_{G}(\boldsymbol{t}) \end{array}$$

which establishes the required equation.

By the corollary, we may just focus on  $\mathcal{H}^1$ :

**Convention.** We write  $\mathcal{H}$  for  $\mathcal{H}^1$  and call it the *hiding operation (on games)*;  $\mathcal{H}^i$  denotes the *i*-times iteration of  $\mathcal{H}$  for all  $i \in \mathbb{N}$ .

Corollary 3.2 (Hiding operation on legal positions). Given an arena G and a number  $d \in \mathbb{N} \cup \{\omega\}$ , we have  $\{\mathcal{H}_G^d(s) \mid s \in \mathscr{L}_G\} = \mathscr{L}_{\mathcal{H}^d(G)}$ .

### Dynamic Game Semantics

*Proof.* Since there is an upper bound  $\mu(G) \in \mathbb{N}$ , it suffices to consider the case  $d \in \mathbb{N}$ . Then, by Lemmata 3.2 and 3.3, we may just focus on the case d = 1.

The inclusion  $\{\mathcal{H}_G(s) \mid s \in \mathscr{L}_G\} \subseteq \mathscr{L}_{\mathcal{H}(G)}$  is immediate by Theorem 3.1. For the other inclusion, let  $t \in \mathscr{L}_{\mathcal{H}(G)}$ ; we shall find some  $s \in \mathscr{L}_G$  such that

- 1  $\mathcal{H}_G(\boldsymbol{s}) = \boldsymbol{t};$
- 2 1-internal moves in s occur as even-length consecutive segments  $m_1m_2...m_{2k}$ , where  $m_i$  justifies  $m_{i+1}$  for i = 1, 2, ..., 2k 1;
- 3 s is 1-complete.

We proceed by induction on |t|. The base case  $t = \epsilon$  is trivial. For the inductive step, let  $tm \in \mathscr{L}_{\mathcal{H}(G)}$ . Then,  $t \in \mathscr{L}_{\mathcal{H}(G)}$ , and by the induction hypothesis there is some  $s \in \mathscr{L}_{G}$  that satisfies the three conditions (n.b., the first one is for t).

If m is initial, then  $sm \in \mathscr{L}_G$ , and sm satisfies the three conditions. Thus, assume that m is non-initial; we may write  $tm = t_1nt_2m$ , where m is justified by n.

We then need a case analysis:

— Assume  $n \vdash_G m$ . We take sm, where m points to n. Then,  $sm \in \mathscr{L}_G$  since:

- (JUSTIFICATION). It is immediate because  $n \vdash_G m$ .
- (ALTERNATION). By the condition 3 on s, the last moves of s and t just coincide. Thus, the alternation condition holds for sm.
- (GENERALIZED VISIBILITY). It suffices to establish the visibility on sm, as the other cases are included as the generalized visibility on tm. It is straightforward to see that, by the condition 2 on s, if the view of t contains n, then so does the view of s. And since  $tm \in \mathscr{L}_{\mathcal{H}(G)}$ , the view of t contains n. Hence, the view of s contains n as well.
- (IE-SWITCH). Again, the last moves of s and t coincide by the condition 3 on s; thus, IE-switch on tm can be directly applied.

Also, it is easy to see that sm satisfies the three conditions.

— Assume  $n \neq \star$  and  $\exists k \in \mathbb{N}^+, m_1, m_2, \ldots, m_{2k} \in M_G \setminus M_{\mathcal{H}(G)}$  such that

 $n \vdash_G m_1 \land \forall i \in \overline{k}. \ m_{2i-1} \vdash_G m_{2i} \land m_{2k} \vdash_G m.$ 

We then take  $sm_1m_2...m_{2k}m$ , in which  $m_1$  points to n,  $m_i$  points to  $m_{i-1}$  for i = 2, 3, ..., 2k, and m points to  $m_{2k}$ . Then,  $sm_1m_2...m_{2k}m \in \mathscr{L}_G$  because:

- (JUSTIFICATION). Obvious.
- (ALTERNATION). By the condition 3 on s, the last moves of s and t just coincide. Thus, the alternation condition holds for  $sm_1m_2...m_{2k}m$ .
- (GENERALIZED VISIBILITY). By the same argument as the above case.
- (IE-SWITCH). It clearly holds by the axiom E4.

Finally, it is easy to see that  $sm_1m_2...m_{2k}m$  satisfies the three conditions.

We have completed the case analysis.

# 3.3. Constructions on Dynamic Games

Next, we show that dynamic games accommodate all the standard constructions on static games (Abramsky and McCusker, 1999), i.e., they preserve the additional axioms for dynamic games, as well as some new constructions. This result implies that the notion of dynamic games (Definition 3.11) is in some sense 'correct'.

**Convention.** For brevity, we usually omit 'tags' for disjoint union of sets. For instance, we write  $x \in A + B$  iff  $x \in A$  or  $x \in B$  (not both); also, given relations  $R_A \subseteq A \times A$  and  $R_B \subseteq B \times B$ , we write  $R_A + R_B$  for the relation on the disjoint union A + B such that  $(x, y) \in R_A + R_B \stackrel{\text{df.}}{\Leftrightarrow} (x, y) \in R_A \vee (x, y) \in R_B$  (not both).

Let us begin with *tensor* (product)  $\otimes$ . Roughly, a position of the tensor  $A \otimes B$  of games A and B is an interleaving mixture of a position of A and a position of B, in which an AB-parity change is made always by Opponent. Formally:

**Definition 3.15 (Tensor of games (Abramsky and McCusker, 1999)).** Given games A and B, the *tensor (product)*  $A \otimes B$  of A and B is defined by:

$$\begin{array}{l} - M_{A \otimes B} \stackrel{\text{df.}}{=} M_A + M_B; \\ - \lambda_{A \otimes B} \stackrel{\text{df.}}{=} [\lambda_A, \lambda_B]; \\ - \vdash_{A \otimes B} \stackrel{\text{df.}}{=} \lfloor_A + \vdash_B; \\ - P_{A \otimes B} \stackrel{\text{df.}}{=} \{ \boldsymbol{s} \in \mathscr{L}_{A \otimes B} \mid \boldsymbol{s} \upharpoonright A \in P_A, \boldsymbol{s} \upharpoonright B \in P_B \}; \\ - \boldsymbol{s} \simeq_{A \otimes B} \boldsymbol{t} \stackrel{\text{df.}}{\Leftrightarrow} \boldsymbol{s} \upharpoonright A \simeq_A \boldsymbol{t} \upharpoonright A \land \boldsymbol{s} \upharpoonright B \simeq_B \boldsymbol{t} \upharpoonright B \land \forall i \in \mathbb{N}. \boldsymbol{s}(i) \in M_A \Leftrightarrow \boldsymbol{t}(i) \in M_A \\ - \boldsymbol{s} \simeq_{A \otimes B} \boldsymbol{t} \stackrel{\text{df.}}{\Leftrightarrow} \boldsymbol{s} \upharpoonright A \simeq_A \boldsymbol{t} \upharpoonright A \land \boldsymbol{s} \upharpoonright B \simeq_B \boldsymbol{t} \upharpoonright B \land \forall i \in \mathbb{N}. \boldsymbol{s}(i) \in M_A \Leftrightarrow \boldsymbol{t}(i) \in M_A \end{cases}$$

where  $s \upharpoonright A$  (resp.  $s \upharpoonright B$ ) denotes the j-subsequence of s that consists of occurrences of moves of A (resp. B).

In fact, as explained in (Abramsky et al., 1997), in a position of a tensor  $A \otimes B$ , only Opponent can switch between the component games A and B (by alternation).

**Example 3.5.** Consider the tensor  $N \otimes N$  of the natural number game N with itself, whose maximal position is either of the following forms:

$N_{[0]}$	$\otimes$	$N_{[1]}$	_	$N_{[0]}$	$\otimes$	$N_{[1]}$
$\left( egin{array}{c} q_{[0]} \ n_{[0]} \end{array}  ight)$						${q_{[1]} \choose m_{[1]}}$
	(	$m_{[1]} m_{[1]}$		$\langle { \scriptstyle q_{[0]} \atop \scriptstyle n_{[0]} }$		

where  $n, m \in \mathbb{N}$ , and  $(\_)_{[i]}$  (i = 0, 1) are again arbitrary, unspecified 'tags' such that  $[0] \neq [1]$  to distinguish the two copies of N, and the arrows represent pointers. Henceforth, however, we usually omit 'tags'  $(\_)_{[i]}$  unless it is strictly necessary.

**Theorem 3.2 (Well-defined tensor of games).** (Resp. well-founded) games are closed under tensor  $\otimes$ .

*Proof.* Since static games are closed under tensor  $\otimes$  (Abramsky and McCusker, 1999), it suffices to show that  $\otimes$  preserves the condition on labeling function and the axioms

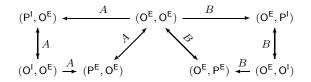


Table 1. The double parity diagram for the tensor  $A \otimes B$ .

E1, E2, E4, DP2, DP3 and DI3 (n.b.,  $\otimes$  clearly preserves well-foundedness of games). However, non-trivial ones are just DP3 and DI3; thus, we just focus on these two axioms. Let A and B be any games. To verify DP3 on  $A \otimes B$ , let  $slmn, s'l'm'n' \in P_{A\otimes B}^{Odd}$  and  $i \in \mathbb{N}$  such that  $\mathcal{H}_{A\otimes B}^{i}(slm) = \mathcal{H}_{A\otimes B}^{i}(s'l'm')$  and  $i < \lambda_{A\otimes B}^{\mathbb{N}}(n) = \lambda_{A\otimes B}^{\mathbb{N}}(n')$ . Note that  $\lambda_{A\otimes B}^{\mathbb{N}}(m) = \lambda_{A\otimes B}^{\mathbb{N}}(n) = \lambda_{A\otimes B}^{\mathbb{N}}(n') = \lambda_{A\otimes B}^{\mathbb{N}}(m')$  by IE-switch. At a first glance, it seems that  $A \otimes B$  does not satisfy DP3 as Opponent may choose to play in A or B at will. It is, however, not the case for internal moves for  $slmn \in P_{A\otimes B}^{Odd}$  with m internal implies  $m, n \in M_A$  or  $m, n \in M_B$ . This property immediately follows from Table 1 which shows all the possible transitions of OP- and IE-parities for a play of  $A \otimes B$ , where a state  $(X^Y, Z^W)$  indicates that the next move of A (resp. B) has the OP-parity X (resp. Z) and the IE-parity Y (resp. W). Note that m = m' and  $\mathcal{J}_{slm}^{\odot i}(m) = \mathcal{J}_{s'l'm'}^{\odot i}(m')$  as  $\mathcal{H}_{A\otimes B}^{i}(sl).m = \mathcal{H}_{A\otimes B}^{i}(slm) = \mathcal{H}_{A\otimes B}^{i}(s'l'm') = \mathcal{H}_{A\otimes B}^{i}(s'l').m'$ . Thus, m, n, m' and n'belong to the same component game. If  $m, n, m', n' \in M_A$ , then  $(sl \upharpoonright A).mn, (s'l' \upharpoonright A).m'n' \in P_A^{Odd}$ ,  $\mathcal{H}_A^{i}((sl \upharpoonright A).m) = \mathcal{H}_{A\otimes B}^{i}(slm) \upharpoonright \mathcal{H}^{i}(A) = \mathcal{H}_{A\otimes B}^{i}(s'l'm') \upharpoonright \mathcal{H}^{i}(A) =$   $\mathcal{H}_A^{i}((s'l' \upharpoonright A).m')$  and  $i < \lambda_A^{\mathbb{N}}(n) = \mathcal{J}_{(s'l' \upharpoonright A).m'n'}^{\odot i}(n')$ . The other case is completely analogous, showing that  $A \otimes B$  satisfies DP3.

Finally, to show that  $A \otimes B$  satisfies DI3, assume  $s \simeq_{A \otimes B}^{d} t$  and  $sm \in P_{A \otimes B}$  for some  $d \in \mathbb{N}$ ; we have to find some  $tn \in P_{A \otimes B}$  such that  $sm \simeq_{A \otimes B}^{d} tn$ . Assume  $m \in M_A$  for the other case is symmetric. Since  $s \simeq_{A \otimes B}^{d} t$ , we have some  $s' \simeq_{A \otimes B} t'$  such that  $\mathcal{H}_{A \otimes B}^{d}(s') = \mathcal{H}_{A \otimes B}^{d}(s)$  and  $\mathcal{H}_{A \otimes B}^{d}(t') = \mathcal{H}_{A \otimes B}^{d}(t)$ . Thus,  $s' \upharpoonright A \simeq_A t' \upharpoonright A$ ,  $\mathcal{H}_{A}^{d}(s \upharpoonright A) = \mathcal{H}_{A \otimes B}^{d}(s) \upharpoonright \mathcal{H}^{d}(A) = \mathcal{H}_{A \otimes B}^{d}(s') \upharpoonright \mathcal{H}^{d}(A) = \mathcal{H}_{A}^{d}(s' \upharpoonright A)$  and  $\mathcal{H}_{A}^{d}(t \upharpoonright A) = \mathcal{H}_{A \otimes B}^{d}(t) \upharpoonright \mathcal{H}^{d}(A) = \mathcal{H}_{A}^{d}(t') \upharpoonright \mathcal{H}^{d}(A) = \mathcal{H}_{A}^{d}(t' \upharpoonright A)$ , whence  $s \upharpoonright A \simeq_{A}^{d} t \upharpoonright A$ . Similarly,  $s \upharpoonright B \simeq_{B}^{d} t \upharpoonright B$ . Now, since  $(s \upharpoonright A).m = sm \upharpoonright A \in P_A$ , we have some  $(t \upharpoonright A).n \in P_A$  such that  $(s \upharpoonright A).m \simeq_{A}^{d}(t \upharpoonright A).n$ , i.e., some  $u \simeq_A v$  such that  $\mathcal{H}_{A}^{d}(u) = \mathcal{H}_{A}^{d}((s \upharpoonright A).m)$  and  $\mathcal{H}_{A}^{d}(v) = \mathcal{H}_{A}^{d}((t \upharpoonright A).n)$ . By Table 1, we may obtain a unique  $\tilde{s} \in P_{A \otimes B}$  from u and  $s' \upharpoonright B$  and  $\mathcal{H}_{A \otimes B}^{d}(\tilde{t}) = \mathcal{H}_{A \otimes B}^{d}(tn)$ .

Next, let us recall *linear implication*  $\multimap$ , which has been illustrated by examples in Section 1. The linear implication  $A \multimap B$  is intended to be the 'space' of *linear functions* from A to B in the sense of linear logic (Girard, 1987), i.e., they consume exactly one input in A to produce an output in B (strictly speaking, they consume at most one input since it is possible that no moves of A are performed at all during a play of  $A \multimap B$ ).

One additional point for *dynamic* games is that we need to apply the  $\omega$ -hiding operation  $\mathcal{H}^{\omega}$  to the domain A since otherwise the linear implication  $A \to B$  may not satisfy the

axiom DP2 or DP3. It conceptually makes sense too for the roles of Player and Opponent in A are exchanged, and thus Player should not be able to 'see' internal moves of A.

Definition 3.16 (Linear implication between games (Abramsky and McCusker, 1999)). The *linear implication*  $A \multimap B$  from a game A to another B is defined by:

$$\begin{split} &- M_{A \to B} \stackrel{\text{df.}}{=} M_{\mathcal{H}^{\omega}(A)} + M_B; \\ &- \lambda_{A \to B} \stackrel{\text{df.}}{=} [\overline{\lambda_{\mathcal{H}^{\omega}(A)}}, \lambda_B], \text{ where } \overline{\lambda_{\mathcal{H}^{\omega}(A)}} \stackrel{\text{df.}}{=} \langle \overline{\lambda_{\mathcal{H}^{\omega}(A)}}, \lambda_{\mathcal{H}^{\omega}(A)}^{\mathsf{OP}}, \lambda_{\mathcal{H}^{\omega}(A)}^{\mathsf{OP}} \rangle, \text{ and } \overline{\lambda_G^{\mathsf{OP}}}(m) \stackrel{\text{df.}}{=} \\ &\begin{cases} \mathsf{P} & \text{if } \lambda_G^{\mathsf{OP}}(m) = \mathsf{O}; \\ \mathsf{O} & \text{otherwise} \end{cases} \text{ for any game } G; \\ &- \star \vdash_{A \to B} m \stackrel{\text{df.}}{\Leftrightarrow} \star \vdash_B m; \\ &- m \vdash_{A \to B} n (m \neq \star) \stackrel{\text{df.}}{\Leftrightarrow} (m \vdash_{\mathcal{H}^{\omega}(A)} n) \lor (m \vdash_B n) \lor (\star \vdash_B m \land \star \vdash_{\mathcal{H}^{\omega}(A)} n); \\ &- P_{A \to OB} \stackrel{\text{df.}}{=} \{ s \in \mathscr{L}_{\mathcal{H}^{\omega}(A) \to B} \mid s \upharpoonright \mathcal{H}^{\omega}(A) \in P_{\mathcal{H}^{\omega}(A)}, s \upharpoonright B \in P_B \}; \\ &- s \simeq_{A \to OB} t \stackrel{\text{df.}}{\Leftrightarrow} s \upharpoonright \mathcal{H}^{\omega}(A) \simeq_{\mathcal{H}^{\omega}(A)} t \upharpoonright \mathcal{H}^{\omega}(A) \land s \upharpoonright B \simeq_B t \upharpoonright B \land \forall i \in \mathbb{N}. s(i) \in M_{\mathcal{H}^{\omega}(A)} \end{cases}$$

where pointers from an initial occurrence of  $\mathcal{H}^{\omega}(A)$  to that of B in s are deleted.

Dually to  $A \otimes B$ , it is easy to see that during a play of  $A \multimap B$  only Player may switch between  $\mathcal{H}^{\omega}(A)$  and B (again by alternation); see (Abramsky et al., 1997) for the details.

**Example 3.6.** See again the examples of linear implication in Section 1 to see how Definition 3.16 actually works.

**Theorem 3.3 (Well-defined linear implication between games).** (Resp. well-founded) games are closed under linear implication.

*Proof.* Again, it suffices to show the preservation property of the additional conditions on the labeling function and the axioms E1, E2, E4, DP2, DP3 and DI3. For brevity, assume that A is normalized and consider  $A \rightarrow B$ . Again, non-trivial conditions are just DP3 and DI3, but DI3 may be shown in a way similar to the case of tensor.

To verify DP3, let  $i \in \mathbb{N}$  and  $slmn, s'l'm'n' \in P_{A \multimap B}^{\mathsf{Odd}}$  such that  $\mathcal{H}_{A \multimap B}^i(slm) = \mathcal{H}_{A \multimap B}^i(s'l'm')$  and  $i < \lambda_{A \multimap B}^{\mathbb{N}}(n) = \lambda_{A \multimap B}^{\mathbb{N}}(n')$ . Again, m and m' are both internal, and so m, n, m' and n' all belong to B. Thus,  $(sl \upharpoonright B).mn, (s'l' \upharpoonright B).m'n' \in P_B^{\mathsf{Odd}}$  such that  $\mathcal{H}_B^i((sl \upharpoonright B).m) = \mathcal{H}_{A \multimap B}^i(slm) \upharpoonright \mathcal{H}^i(B) = \mathcal{H}_{A \multimap B}^i(s'l'm') \upharpoonright \mathcal{H}^i(B) = \mathcal{H}_B^i((s'l' \upharpoonright B).m')$  and  $i < \lambda_B^{\mathbb{N}}(n) = \lambda_B^{\mathbb{N}}(n')$ ; thus, by DP2 on B, we may conclude that n = n' and  $\mathcal{J}_{slmn}^{\odot i}(n) = \mathcal{J}_{(sl \upharpoonright B).mn}^{\odot i}(n) = \mathcal{J}_{(sl \upharpoonright B).mn}^{\odot i}(n') = \mathcal{J}_{slmn}^{\odot i}(n')$ .

Next, product & forms the categorical product in the categories of static games and strategies (Abramsky and McCusker, 1999). A position of the product A&B is simply a position of A or B:

**Definition 3.17 (Product of games (Abramsky and McCusker, 1999)).** Given games A and B, the *product* A&B of A and B is defined by:

$$- M_{A\&B} \stackrel{\text{df.}}{=} M_A + M_B;$$
$$- \lambda_{A\&B} \stackrel{\text{df.}}{=} [\lambda_A, \lambda_B];$$

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$$\begin{array}{l} -- \vdash_{A\&B} \stackrel{\mathrm{df.}}{=} \vdash_{A} + \vdash_{B}; \\ -- P_{A\&B} \stackrel{\mathrm{df.}}{=} \{ \boldsymbol{s} \in \mathscr{L}_{A\&B} \mid (\boldsymbol{s} \upharpoonright A \in P_{A} \land \boldsymbol{s} \upharpoonright B = \boldsymbol{\epsilon}) \lor (\boldsymbol{s} \upharpoonright A = \boldsymbol{\epsilon} \land \boldsymbol{s} \upharpoonright B \in P_{B}) \}; \\ -- \boldsymbol{s} \simeq_{A\&B} \boldsymbol{t} \stackrel{\mathrm{df.}}{\Leftrightarrow} \boldsymbol{s} \simeq_{A} \boldsymbol{t} \lor \boldsymbol{s} \simeq_{B} \boldsymbol{t}. \end{array}$$

**Example 3.7.** A maximal position of the product 2&N is either of the following forms:

where  $b \in \mathbb{B}$  and  $n \in \mathbb{N}$ .

Now, for our game-semantic CCBoC (given in Section 4), let us generalize product:

**Notation.** Given a function  $f: X \to Y$  and a subset  $Z \subseteq X$ , we write  $f \mid Z: X \setminus Z \to Y$ for the restriction of f to the subset  $X \setminus Z \subseteq X$ .

**Definition 3.18 (Pairing of games).** The *pairing*  $\langle L, R \rangle$  of games L and R such that  $\mathcal{H}^{\omega}(L) \leq C \multimap A$  and  $\mathcal{H}^{\omega}(R) \leq C \multimap B$  for some normalized games A, B and C is defined by:

- $M_{\langle L,R \rangle} \stackrel{\text{df.}}{=} M_C + (M_L \setminus M_C) + (M_R \setminus M_C)$ , where 'tags' for the disjoint union is chosen in such a way that  $\mathcal{H}^{\omega}(\langle L,R \rangle) \triangleleft C \multimap A\&B$  holds;
- $\lambda_{\langle L, R \rangle} \stackrel{\text{df.}}{=} [\overline{\lambda_C}, \lambda_L \mid M_C, \lambda_R \mid M_C];$  $- m \vdash_{\langle L,R \rangle} n \stackrel{\text{df.}}{\Leftrightarrow} (att_{\langle L,R \rangle}(m) = att_{\langle L,R \rangle}(n) \lor att_{\langle L,R \rangle}(m) = C \lor att_{\langle L,R \rangle}(n) = C) \land (peel_{\langle L,R \rangle}(m) \vdash_L peel_{\langle L,R \rangle}(n) \lor peel_{\langle L,R \rangle}(m) \vdash_R peel_{\langle L,R \rangle}(n));$  $--P_{(L,R)} \stackrel{\text{df.}}{=} \{ \boldsymbol{s} \in \mathscr{L}_{L\&R} \mid (\boldsymbol{s} \upharpoonright L \in P_L \land \boldsymbol{s} \upharpoonright R = \boldsymbol{\epsilon}) \lor (\boldsymbol{s} \upharpoonright L = \boldsymbol{\epsilon} \land \boldsymbol{s} \upharpoonright R \in P_R) \};$  $- s \simeq_{(L,R)} t \stackrel{\mathrm{df.}}{\Leftrightarrow} (s \upharpoonright L = \epsilon \Leftrightarrow t \upharpoonright L = \epsilon) \land s \upharpoonright L \simeq_L t \upharpoonright L \land s \upharpoonright R \simeq_R t \upharpoonright R$

where the map  $peel_{\langle L,R\rangle}$  :  $M_{\langle L,R\rangle} \rightarrow M_L \cup M_R$  is the obvious left inverse of the 'tagging' for  $M_{(L,R)}$ ,  $s \upharpoonright L$  (resp.  $s \upharpoonright R$ ) is the j-subsequence of s that consists of moves x such that  $peel_{(L,R)}(x) \in M_L$  (resp.  $peel_{(L,R)}(x) \in M_R$ ) yet changed into  $peel_{\langle L,R\rangle}(x)$ , and the map  $att_{\langle L,R\rangle}: M_{\langle L,R\rangle} \to \{L,R,C\}$  is given by  $att_{\langle L,R\rangle}(m) \stackrel{\text{df.}}{=}$ 

- $\begin{cases} L & \text{if } peel_{\langle L,R \rangle}(m) \in M_L \setminus M_C; \\ R & \text{if } peel_{\langle L,R \rangle}(m) \in M_R \setminus M_C; \\ C & \text{otherwise (i.e., if } peel_{\langle L,R \rangle}(m) \in M_C). \end{cases}$

Pairing of games is indeed a generalization of product for we have  $\langle T \multimap A, T \multimap B \rangle =$  $T \rightarrow A\&B$  for any games A and B, where note that each game G coincides with the linear implication  $T \multimap G$  up to 'tags'. Also, we shall see that the *(generalized) pairing*  $\langle \sigma, \tau \rangle$  of strategies  $\sigma : L$  and  $\tau : R$  forms a strategy on the pairing  $\langle L, R \rangle$  (Definition 3.35).

**Theorem 3.4 (Well-defined pairing of games).** If (resp. well-founded) games L and R satisfy  $\mathcal{H}^{\omega}(L) \leq C \multimap A$  and  $\mathcal{H}^{\omega}(R) \leq C \multimap B$  for normalized games A, B and C, then the pairing  $\langle L, R \rangle$  is a (resp. well-founded) game that satisfies  $\mathcal{H}^{\omega}(\langle L, R \rangle) \triangleleft C \multimap A\&B$ .

*Proof.* Similar to and simpler than the case of tensor.

Now, let us recall *exponential* !, which is essentially the countably-infinite iteration of tensor, i.e., !A and  $A \otimes A \otimes \ldots$  coincide up to 'tags'. Precisely, it is defined as follows:

**Definition 3.19 (Exponential of games (Abramsky et al., 2000; McCusker, 1998)).** Given a game A, the *exponential* !A of A is defined by:

$$\begin{array}{l} & - M_{!A} \stackrel{\text{df.}}{=} M_A \times \mathbb{N}; \\ & - \lambda_{!A} : (a,i) \mapsto \lambda_A(a); \\ & - \star \vdash_{!A} (a,i) \stackrel{\text{df.}}{\leftrightarrow} \star \vdash_A a; \\ & - (a,i) \vdash_{!A} (a',i') \stackrel{\text{df.}}{\Leftrightarrow} i = i' \wedge a \vdash_A a'; \\ & - P_{!A} \stackrel{\text{df.}}{=} \{ \boldsymbol{s} \in \mathscr{L}_{!A} \mid \forall i \in \mathbb{N}. \, \boldsymbol{s} \upharpoonright i \in P_A \, \}; \\ & - \boldsymbol{s} \simeq_{!A} \boldsymbol{t} \stackrel{\text{df.}}{\Leftrightarrow} \exists \varphi \in \mathcal{P}(\mathbb{N}). \, \forall i \in \mathbb{N}. \, \boldsymbol{s} \upharpoonright \varphi(i) \simeq_A \boldsymbol{t} \upharpoonright i \wedge \pi_2^*(\boldsymbol{s}) = (\varphi \circ \pi_2)^*(\boldsymbol{t}) \end{array}$$

where  $s \upharpoonright i$  is the j-subsequence of s that consists of occurrences of moves of the form (a, i) yet changed into a, and  $\mathcal{P}(\mathbb{N})$  is the set of all permutations of natural numbers.

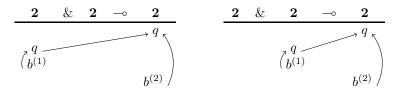
Example 3.8. A typical position of the exponential !2 is as follows:

$$\begin{array}{c} !2 \\ \hline (q,10) \\ (tt,10) \\ (q,100) \\ (ff,100) \end{array}$$

Now, it should be clear, from the definition of  $\simeq_{!A}$ , why we have equipped each game with an identification of positions: A particular choice of the 'tag' (\_, i) for an exponential !A should not matter; since this identification may occur *locally* in games in a *nested* form, e.g., !(! $A \otimes B$ ), ! $A \multimap B$ , etc., it gives a neat solution to define a *tailored* identification  $\simeq_G$  of positions as part of the structure of each game G. It was first introduced by (Abramsky et al., 2000) and also employed in (McCusker, 1998).

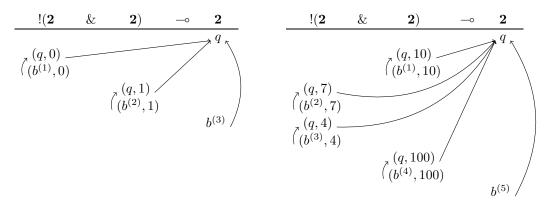
Exponential enables us, via *Girard's translation* (Girard, 1987)  $A \Rightarrow B \stackrel{\text{df.}}{=} !A \multimap B$ , to model the construction  $\Rightarrow$  of the usual *implication* (or the *function space*).

**Example 3.9.** In the linear implication  $2\&2 \rightarrow 2$ , Player may play at most only in one 2 out of the domain 2&2:



where  $b^{(1)}, b^{(2)} \in \mathbb{B}$ . On the other hand, however, positions of the implication  $2\&2 \Rightarrow 2 = !(2\&2) \multimap 2$  are of the expected form; for instance:

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where  $b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}, b^{(5)} \in \mathbb{B}$ . Hence, e.g., Player may play as conjunction  $\wedge : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$  or disjunction  $\vee : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$  on the implication  $2\& 2 \Rightarrow 2$  in the obvious manner, but not on the linear implication  $2\& 2 \multimap 2$ . This example illustrates why the standard notion of functions corresponds in game semantics to implication  $\Rightarrow$ , not linear one  $\multimap$ .

For the game-semantic CCBoC, let us generalize exponential of games as follows:

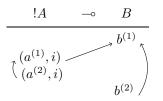
**Definition 3.20 (Promotion of games).** Given a game G such that  $\mathcal{H}^{\omega}(G) \leq A \to B$  for some normalized games A and B, the **promotion**  $G^{\dagger}$  of G is defined by:

$$\begin{split} &- M_{G^{\dagger}} \stackrel{\mathrm{df.}}{=} ((M_G \setminus M_{!A}) \times \mathbb{N}) + M_{!A}; \\ &- \lambda_{G^{\dagger}} : ((m,i) \in (M_G \setminus M_{!A}) \times \mathbb{N}) \mapsto \lambda_G(m), ((a,j) \in M_{!A}) \mapsto \lambda_G(a,j); \\ &- \star \vdash_{G^{\dagger}} (m,i) \stackrel{\mathrm{df.}}{\Leftrightarrow} \star \vdash_G m \text{ for all } i \in \mathbb{N}; \\ &- (m,i) \vdash_{G^{\dagger}} (n,j) \stackrel{\mathrm{df.}}{\Leftrightarrow} (i = j \wedge m, n \in M_G \setminus M_{!A} \wedge m \vdash_G n) \\ &\vee (i = j \wedge m \vdash_A n) \vee (m \in M_G \setminus M_{!A} \wedge (n,j) \in M_{!A} \wedge m \vdash_G (n,j)); \\ &- P_{G^{\dagger}} \stackrel{\mathrm{df.}}{=} \{ s \in \mathscr{L}_{G^{\dagger}} \mid \forall i \in \mathbb{N}. s \upharpoonright i \in P_G \}; \\ &- s \simeq_{G^{\dagger}} t \stackrel{\mathrm{df.}}{\Leftrightarrow} \exists \varphi \in \mathcal{P}(\mathbb{N}). \forall i \in \mathbb{N}. s \upharpoonright \varphi(i) \simeq_G t \upharpoonright i \wedge \pi_2^*(s) = (\varphi \circ \pi_2)^*(t) \end{split}$$

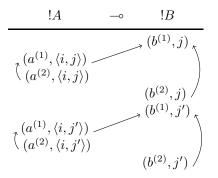
where  $s \upharpoonright i$  is the j-subsequence of s that consists of moves (m, i) with  $m \in M_G \setminus M_{!A}$ , or  $(a, \langle i, j \rangle)$  with  $a \in M_A \land j \in \mathbb{N}$ , yet changed into m or (a, j), respectively.

Note that we have  $(T \multimap A)^{\dagger} = !T \multimap !A$  for any game A, and therefore promotion of games is indeed a generalization of exponential. Also, we shall see later that the *(generalized) promotion*  $\phi^{\dagger}$  of a strategy  $\phi : G$  forms a strategy on the promotion  $G^{\dagger}$ .

**Example 3.10.** Let us consider the promotion  $(!A \multimap B)^{\dagger}$ , where A and B are arbitrary normalized games. If there is the following position of  $!A \multimap B$ :



then there is the following position of the promotion  $(!A \multimap B)^{\dagger}$ , where note that  $j, j' \in \mathbb{N}$  are arbitrarily chosen by Opponent:



**Theorem 3.5 (Well-defined promotion of games).** If a (resp. well-founded) game G satisfies  $\mathcal{H}^{\omega}(G) \leq A \to B$  for some normalized games A and B, then  $G^{\dagger}$  is a (resp. well-founded) game that satisfies  $\mathcal{H}^{\omega}(G)^{\dagger} \leq A \to B$ .

*Proof.* Similar to the case of tensor.

Now, let us introduce a new, central construction on games, which formalizes the construction for 'internal communication' between strategies sketched in Section 1:

**Definition 3.21 (Concatenation and composition of games).** Given games J and K that satisfies  $\mathcal{H}^{\omega}(J) \leq A \multimap B$  and  $\mathcal{H}^{\omega}(K) \leq B \multimap C$  for some normalized games A, B and C, the *concatenation*  $J \ddagger K$  of J and K is defined by:

- $M_{J\ddagger K} \stackrel{\text{df.}}{=} M_J + M_K$ , where 'tags' for the disjoint union is chosen in such a way that  $\mathcal{H}^{\omega}(J\ddagger K) \leq A \multimap C$  holds;
- $\lambda_{J\ddagger K} \stackrel{\text{df.}}{=} [\lambda_J \downarrow M_{B_{[1]}}, \lambda_J^{+\mu} \upharpoonright M_{B_{[1]}}, \lambda_K^{+\mu} \upharpoonright M_{B_{[2]}}, \lambda_K \downarrow M_{B_{[2]}}], \text{ where } B_{[1]} \text{ and } B_{[2]} \text{ are the copies of } B \text{ that belong to } J \text{ and } K, \text{ respectively, } \lambda_G^{+\mu} \stackrel{\text{df.}}{=} \langle \lambda_G^{\mathsf{OP}}, \lambda_G^{\mathsf{QA}}, n \mapsto \lambda_G^{\mathbb{N}}(n) + \mu \rangle$  (G is J or K), and  $\mu \stackrel{\text{df.}}{=} \max(\mu(J), \mu(K)) + 1;$

$$- \star \vdash_{J \ddagger K} m \stackrel{\text{df.}}{\Leftrightarrow} \star \vdash_K m;$$

$$-- m \vdash_{J\ddagger K} n \ (m \neq \star) \stackrel{\text{dif.}}{\Leftrightarrow} m \vdash_J n \lor m \vdash_K n \lor (\star \vdash_{B_{[2]}} m \land \star \vdash_{B_{[1]}} n);$$

$$- P_{J\ddagger K} \stackrel{\text{di.}}{=} \{ \boldsymbol{s} \in \mathscr{J}_{J\ddagger K} \mid \boldsymbol{s} \upharpoonright J \in P_J, \boldsymbol{s} \upharpoonright K \in P_K, \boldsymbol{s} \upharpoonright B_{[1]}, B_{[2]} \in pr_B \};$$

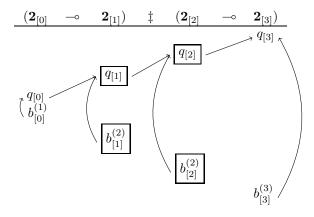
$$-- \mathbf{s} \simeq_{J \ddagger K} \mathbf{t} \stackrel{\text{div}}{\Leftrightarrow} (\forall i \in \mathbb{N}. \mathbf{s}(i) \in M_J \Leftrightarrow \mathbf{t}(i) \in M_J) \land \mathbf{s} \upharpoonright J \simeq_J \mathbf{t} \upharpoonright J \land \mathbf{s} \upharpoonright K \simeq_K \mathbf{t} \upharpoonright K$$

where  $pr_B \stackrel{\text{df.}}{=} \{ s \in P_{B_{[1]} \multimap B_{[2]}} \mid \forall t \leq s. \text{ Even}(t) \Rightarrow t \upharpoonright B_{[1]} = t \upharpoonright B_{[2]} \}$ . Moreover, the *composition* J; K (or  $K \circ J$ ) of J and K is given by:

$$J; K \stackrel{\mathrm{df.}}{=} \mathcal{H}^{\omega}(J \ddagger K).$$

**Example 3.11.** A typical maximal position of the concatenation  $(2 - 2) \ddagger (2 - 2)$  is:

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where  $b^{(1)}, b^{(2)}, b^{(3)} \in \mathbb{B}$ . We have marked internal moves by a square box just for clarity.

We shall see that the 'non-hiding composition' or *concatenation*  $\iota \ddagger \kappa$  of strategies  $\iota : J$ and  $\kappa : K$  forms a strategy on the concatenation  $J \ddagger K$ . It generalizes the particular case, where  $J = A \multimap B$  and  $K = B \multimap C$ , so that  $\iota; \kappa = \mathcal{H}^{\omega}(\iota \ddagger \kappa) : \mathcal{H}^{\omega}(J \ddagger K) = J; K =$  $A \multimap C$  (as we shall establish shortly), which reformulates conventional composition of static strategies as *concatenation plus hiding* of dynamic strategies.

Theorem 3.6 (Well-defined concatenation and composition of games). (Resp. well-founded) games are closed under concatenation and composition.

*Proof.* By Theorem 3.1, it suffices to focus on concatenation, where well-foundedness is clearly preserved under concatenation. We first show that the arena  $J \ddagger K$  is welldefined. The set  $M_{J\ddagger K}$  and the function  $\lambda_{J\ddagger K}$  are clearly well-defined, where the *finite* upper bounds  $\mu(J)$  and  $\mu(K)$  are crucial. For the relation  $\vdash_{J\ddagger K}$ , the axioms E1 and E3 clearly hold. For the axiom E2, if  $m \vdash_{J\ddagger K} n$  and  $\lambda_{J\ddagger K}^{\mathsf{QA}}(n) = \mathsf{A}$ , then  $m, n \in M_K \setminus M_{B_{[2]}}$ ,  $m, n \in M_{B_{[2]}}, m, n \in M_{B_{[1]}}$  or  $m, n \in M_J \setminus M_{B_{[1]}}$ . In either case,  $\lambda_{J\ddagger K}^{\mathsf{QA}}(m) = \mathsf{Q}$  and  $\lambda_{J\ddagger K}^{\mathbb{N}}(m) = \lambda_{J\ddagger K}^{\mathbb{N}}(n)$ .

For the axiom E4, let  $m \vdash_{J\ddagger K} n$ ,  $m \neq \star$  and  $\lambda_{J\ddagger K}^{\mathbb{N}}(m) \neq \lambda_{J\ddagger K}^{\mathbb{N}}(n)$ . We proceed by a case analysis. If  $(m \vdash_K n) \land (m, n \in M_K \setminus M_{B_{[2]}} \lor m, n \in M_{B_{[2]}})$ , then we may just apply E4 on K. It is similar if  $(m \vdash_J n) \land (m, n \in M_J \setminus M_{B_{[1]}} \lor m, n \in M_{B_{[1]}})$ . Note that the case  $\star \vdash_{B_{[2]}} m \land \star \vdash_{B_{[1]}} n$  cannot happen. Now, consider the case  $m \vdash_K n \land m \in$  $M_K \setminus M_{B_{[2]}} \land n \in M_{B_{[2]}}$ . If m is external, then  $m \in M_C$ , and so E4 on  $J \ddagger K$  is satisfied by the definition of  $B \multimap C$ ; if m is internal, then we may apply E4 on K. The case  $m \vdash_K n \land n \in M_K \setminus M_{B_{[2]}} \land m \in M_{B_{[2]}}$  is simpler as m must be internal. The remaining cases  $m \vdash_J n \land m \in M_J \setminus M_{B_{[1]}} \land n \in M_{B_{[1]}}$  and  $m \vdash_J n \land n \in M_J \setminus M_{B_{[1]}} \land m \in M_{B_{[1]}}$ are analogous. Hence, we have shown that the arena  $J \ddagger K$  is well-defined.

Next, we show that  $P_{J\ddagger K} \subseteq \mathscr{L}_{J\ddagger K}$ . For justification, let  $sm \in P_{J\ddagger K}$  with m non-initial. The non-trivial case is when m is initial in J. But in this case, m is initial in  $B_{[1]}$ , and so it has a justifier in  $B_{[2]}$ . For alternation and IE-switch, similarly to Table 1 for tensor  $\otimes$ , we have Table 2 for  $J\ddagger K$ , in which the first (resp. the second) component of each state is about the OP- and IE-parities of the next move of J (resp. K). For readability, some states are written twice, and the dotted arrow indicates two necessarily consecutive

$$(\mathsf{O}^{\mathsf{E}},\mathsf{O}^{\mathsf{E}}) \xrightarrow{C} (\mathsf{O}^{\mathsf{E}},\mathsf{P}^{\mathsf{I}})$$

$$C \downarrow \uparrow C \qquad K \downarrow \uparrow K$$

$$(\mathsf{P}^{\mathsf{I}},\mathsf{O}^{\mathsf{E}}) \xrightarrow{B_{[1]}B_{[2]}} (\mathsf{O}^{\mathsf{E}},\mathsf{P}^{\mathsf{E}}) \xleftarrow{K} (\mathsf{O}^{\mathsf{E}},\mathsf{O}^{\mathsf{I}})$$

$$J \downarrow \uparrow J \qquad B_{[2]} \stackrel{B_{[2]}}{\underset{B_{[1]}}{\overset{B_{[2]}}{\underset{B_{[1]}}{\overset{B_{[2]}}{\underset{B_{[1]}}{\overset{B_{[1]}}{\underset{B_{[1]}}{\overset{B_{[2]}}{\underset{B_{[1]}}{\underset{B_{[2]}}{\underset{B_{[1]}}{\underset{B_{[2]}}{\underset{B_{[1]}}{\underset{B_{[2]}}{\underset{B_{[1]}}{\underset{B_{[2]}}{\underset{B_{[1]}}{\underset{B_{[2]}}{\underset{B_{[1]}}{\underset{B_{[2]}}{\underset{B_{[1]}}{\underset{B_{[2$$

Table 2. The double parity diagram for the concatenation  $J \ddagger K$ .

moves of B. Then, alternation and IE-switch on  $J \ddagger K$  immediately follows from this diagram and the corresponding axioms on J and K.

For generalized visibility, let  $\mathbf{sm} \in P_{J\ddagger K}$  with m non-initial and  $d \in \mathbb{N} \cup \{\omega\}$  such that  $\mathbf{sm}$  is d-complete. Without loss of generality, we may assume  $d \in \mathbb{N}$  as  $\mathbf{s}$  is finite. It is not hard to see that  $\mathcal{H}_{J\ddagger K}^d(\mathbf{sm}) \in P_{\mathcal{H}^d(J)\ddagger \mathcal{H}^d(K)}$  if  $\mathcal{H}^d(J\ddagger K)$  is not normalized; thus, this case is reduced to the (usual) visibility on  $\mathcal{H}^d(J)\ddagger \mathcal{H}^d(K)$ . Otherwise, it is no harm to select the least  $d \in \mathbb{N}^+$  such that  $\mathcal{H}^d(J\ddagger K)$  is normalized; then  $\mathcal{H}_{J\ddagger K}^{d-1}(\mathbf{sm}) \in P_{(A \multimap B_{[1]})\ddagger (B_{[2]} \multimap C)}$ , and thus the visibility of  $\mathcal{H}_{J\ddagger K}^d(\mathbf{sm}) = \mathcal{H}_{\mathcal{H}^{d-1}(J\ddagger K)}(\mathcal{H}_{J\ddagger K}^{d-1}(\mathbf{sm}))$  can be shown completely in the same way as the proof that shows the composition of strategies is well-defined (in particular it satisfies visibility) (McCusker, 1998; Harmer, 2004). Consequently, it suffices to consider the case d = 0, i.e., to show the (usual) visibility.

For this, we need the following:

Lemma 3.5 (Visibility lemma). Assume that  $t \in P_{J\ddagger K}$  and  $t \neq \epsilon$ .

- 1 If the last move of t is of  $M_J \setminus M_{B_{[1]}}$ , then  $\lceil t \upharpoonright J \rceil_J \preceq \lceil t \rceil_{J\ddagger K} \upharpoonright J$  and  $\lfloor t \upharpoonright J \rfloor_J \preceq \lfloor t \rfloor_{J\ddagger K} \upharpoonright J$ ;
- 2 If the last move of  $\boldsymbol{t}$  is of  $M_K \setminus M_{B_{[2]}}$ , then  $[\boldsymbol{t} \upharpoonright K]_K \preceq [\boldsymbol{t}]_{J\ddagger K} \upharpoonright K$  and  $[\boldsymbol{t} \upharpoonright K]_K \preceq [\boldsymbol{t}]_{J\ddagger K} \upharpoonright K$ ;
- 3 If the last move of  $\boldsymbol{t}$  is an O-move of  $M_{B_{[1]}} \cup M_{B_{[2]}}$ , then  $[\boldsymbol{t} \upharpoonright B_{[1]}, B_{[2]}]_{B_{[1]} \multimap B_{[2]}} \preceq [\boldsymbol{t}]_{J\ddagger K} \upharpoonright B_{[1]}, B_{[2]}]_{B_{[1]} \multimap B_{[2]}} \preceq [\boldsymbol{t}]_{J\ddagger K} \upharpoonright B_{[1]}, B_{[2]}$ .

*Proof of the lemma* By induction on |t| with case analysis on the last move of t.

Note that we may write  $sm = s_1 n s_2 m$ , where n justifies m. If  $s_2 = \epsilon$ , then it is trivial; so assume  $s_2 = s'_2 r$ . We then proceed by a case analysis on m:

- Assume  $m \in M_J \setminus M_{B_{[1]}}$ . Then,  $n \in M_J$  and  $r \in M_J$  by Table 2. By Lemma 3.5,  $\lceil \boldsymbol{s} \upharpoonright J \rceil \preceq \lceil \boldsymbol{s} \rceil \upharpoonright J$  and  $\lfloor \boldsymbol{s} \upharpoonright J \rfloor \preceq \lfloor \boldsymbol{s} \rfloor \upharpoonright J$ . Also, for  $(\boldsymbol{s} \upharpoonright J).m \in P_J$  and visibility on J,

*n* occurs in  $\lceil \boldsymbol{s} \upharpoonright J \rceil$  if *m* is a P-move;

n occurs in  $|\boldsymbol{s} \mid J|$  if m is an O-move.

Hence we may conclude that n occurs in  $\lceil s \rceil$  (resp. |s|) if m is a P- (resp. O-) move.

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- Assume  $m \in M_K \setminus M_{B_{[2]}}$ . This case can be handled in a completely analogous way to the above case.
- Assume  $m \in M_{B_{[1]}}$ . If m is a P-move, then  $n, r \in M_J$  and so it can be handled in the same way as the case  $m \in M_J \setminus M_{B_{[1]}}$ ; thus, assume that m is an O-move. Then,  $r \in M_{B_{[2]}}$  and it is a 'copy' of m. Since r is an O-move of  $B_{[1]} \multimap B_{[2]}$ , by Lemma 3.5,  $\lceil \mathbf{s} \upharpoonright B_{[1]}, B_{[2]} \rceil \preceq \lfloor \mathbf{s} \rfloor \upharpoonright B_{[1]}, B_{[2]}$ . Note that n is a move of  $B_{[1]}$  or an initial move of  $B_{[2]}$ . In either case, we have  $(\mathbf{s} \upharpoonright B_{[1]}, B_{[2]}).m \in P_{B_{[1]} \multimap B_{[2]}}$ ; thus, n occurs in  $\lceil \mathbf{s} \upharpoonright B_{[1]}, B_{[2]} \rceil$ . Hence we may conclude that n occurs in  $\lfloor \mathbf{s} \rfloor$ .
- --- Assume  $m \in M_{B_{[2]}}$ . If m is a P-move, then  $n, r \in M_K$ ; so it can be dealt with in the same way as the case  $m \in M_K \setminus M_{B_{[2]}}$ . Thus, assume m is an O-move. By Table 2, we have  $r \in M_{B_{[1]}}$ , and it is an O-move of  $B_{[1]} \multimap B_{[2]}$ . Thus by Lemma 3.5,  $\lceil \boldsymbol{s} \upharpoonright B_{[1]}, B_{[2]} \rceil \preceq \lfloor \boldsymbol{s} \rfloor \upharpoonright B_{[1]}, B_{[2]}$ . Then again,  $(\boldsymbol{s} \upharpoonright B_{[1]}, B_{[2]}) \cdot m \in P_{B_{[1]} \multimap B_{[2]}}$ ; thus, n occurs in  $\lceil \boldsymbol{s} \upharpoonright B_{[1]}, B_{[2]} \rceil$ , and so n occurs in  $\lfloor \boldsymbol{s} \rfloor$ .

Next, we verify the axioms P1, DP2 and DP3. For P1,  $\epsilon \in P_{J\ddagger K}$  is clear; for prefixclosure, let  $sm \in P_{J\ddagger K}$ . If  $m \in M_J \setminus M_{B_{[1]}}$ , then  $(s \upharpoonright J).m = sm \upharpoonright J \in P_J$ ; thus,  $s \upharpoonright J \in P_J$ ,  $s \upharpoonright K = sm \upharpoonright K \in P_K$  and  $s \upharpoonright B_{[1]}, B_{[2]} = sm \upharpoonright B_{[1]}, B_{[2]} \in pr_B$ , whence  $s \in P_{J\ddagger K}$ . The other cases may be handled similarly. For DP2, assume  $smn \in P_{J\ddagger K}^{\mathsf{Even}}$  and  $\lambda_{J\ddagger K}^{\mathbb{N}}(n) > 0$ . If  $n \notin M_{B_{[1]}} \cup M_{B_{[2]}}$ , then we may just apply DP2 on J or K; and the remaining case is trivial by the definition of  $J \ddagger K$ .

For DP3, let  $i \in \mathbb{N}$  and  $sm, s'm' \in P_{J^{\ddagger}K}^{\text{Odd}}$  such that  $i < \lambda_{J^{\ddagger}K}^{\mathbb{N}}(m) = \lambda_{J^{\ddagger}K}^{\mathbb{N}}(m')$ and  $\mathcal{H}_{J^{\ddagger}K}^{i}(s) = \mathcal{H}_{J^{\ddagger}K}^{i}(s')$ . Without loss of generality, we may assume i = 0 and  $\lambda_{J^{\ddagger}K}^{\mathbb{N}}(m) = 1 = \lambda_{J^{\ddagger}K}^{\mathbb{N}}(m')$  because if  $\lambda_{J^{\ddagger}K}^{\mathbb{N}}(m) = \lambda_{J^{\ddagger}K}^{\mathbb{N}}(m') = j > 1$ , then we may consider  $\mathcal{H}_{J^{\ddagger}K}^{j-1}(sm), \mathcal{H}_{J^{\ddagger}K}^{j-1}(s'm') \in P_{\mathcal{H}^{j-1}(J)^{\ddagger}\mathcal{H}^{j-1}(K)}$  (n.b., the justifiers of m and m' have the same priority order). Thus, s = s' and  $m, m' \in M_J \lor m, m' \in M_K$ . If  $m, m' \in M_J$  (resp.  $m, m' \in M_K$ ), then  $(s \upharpoonright J).m, (s' \upharpoonright J).m' \in P_J^{\text{Odd}}$  (resp.  $(s \upharpoonright K).m, (s' \upharpoonright K).m' \in P_K^{\text{Odd}})$ , and so we may just apply DP3 on J (resp. K).

Finally, the axioms I1, I2 and DI3 on  $\simeq_{J\ddagger K}$  can be verified similarly to the case of tensor, completing the proof.

For completeness, let us explicitly define the rather trivial *currying* of games:

**Definition 3.22 (Currying of games).** Given a game G such that  $\mathcal{H}^{\omega}(G) \leq A \otimes B \multimap C$  for some normalized games A, B and C, the *currying*  $\Lambda(G)$  of G is G up to 'tags' that satisfies  $\mathcal{H}^{\omega}(\Lambda(G)) \leq A \multimap (B \multimap C)$ .

Trivially, (resp. well-founded) games are closed under currying.

Next, we show that these constructions as well as the hiding operation preserve the subgame relation  $\leq$  (Definition 3.12):

**Notation.** We write  $\mathbf{A}_{i \in I}$ , where *I* is {1} or {1,2}, for any of the constructions on games introduced so far, i.e.,  $\mathbf{A}_{i \in I}$  is either  $\otimes$ ,  $\neg \otimes$ ,  $\langle \neg \rangle$ ,  $\langle \neg \rangle$ ,  $\langle \neg \rangle^{\dagger}$ ,  $\ddagger$  or  $\Lambda$ .

**Lemma 3.6 (Preservation of subgames).** Let  $\mathbf{a}_{i \in I}$  be a construction on games, and assume  $H_i \leq G_i$  for all  $i \in I$ . Then,  $\mathbf{a}_{i \in I} H_i \leq \mathbf{a}_{i \in I} G_i$ .

Proof. Let us first consider tensor. It is trivial to check the conditions on the sets of

moves and the labeling functions, and so we omit them. For the enabling relations:

$$\begin{split} \vdash_{H_1 \otimes H_2} &= \vdash_{H_1} + \vdash_{H_2} \\ &\subseteq (\vdash_{G_1} \cap ((\{\star\} \cup M_{H_1}) \times M_{H_1})) + (\vdash_{G_2} \cap ((\{\star\} \cup M_{H_2}) \times M_{H_2})) \\ &= (\vdash_{G_1} \cap ((\{\star\} \cup M_{H_1 \otimes H_2}) \times M_{H_1 \otimes H_2})) + (\vdash_{G_2} \cap ((\{\star\} \cup M_{H_1 \otimes H_2}) \times M_{H_1 \otimes H_2})) \\ &= (\vdash_{G_1} + \vdash_{G_2}) \cap ((\{\star\} \cup M_{H_1 \otimes H_2}) \times M_{H_1 \otimes H_2}) \\ &= \vdash_{G_1 \otimes G_2} \cap ((\{\star\} \cup M_{H_1 \otimes H_2}) \times M_{H_1 \otimes H_2}). \end{split}$$

For the positions, we have:

$$P_{H_1 \otimes H_2} = \{ \boldsymbol{s} \in \mathscr{L}_{H_1 \otimes H_2} \mid \forall i \in \{1, 2\}. \, \boldsymbol{s} \upharpoonright H_i \in P_{H_i} \}$$
$$\subseteq \{ \boldsymbol{s} \in \mathscr{L}_{G_1 \otimes G_2} \mid \forall i \in \{1, 2\}. \, \boldsymbol{s} \upharpoonright G_i \in P_{G_i} \}$$
$$= P_{G_1 \otimes G_2}.$$

For the identifications of positions, given  $d \in \mathbb{N} \cup \{\omega\}$ , we have:

$$s \simeq_{H_1 \otimes H_2}^d t \Leftrightarrow \exists s', t' \in P_{H_1 \otimes H_2}. s' \simeq_{H_1 \otimes H_2} t' \wedge \mathcal{H}_{H_1 \otimes H_2}^d (s') = \mathcal{H}_{H_1 \otimes H_2}^d (s)$$
  
 
$$\wedge \mathcal{H}_{H_1 \otimes H_2}^d (t') = \mathcal{H}_{H_1 \otimes H_2}^d (t)$$
  
 
$$\Leftrightarrow \forall j \in \{1, 2\}. \exists s'_j, t'_j \in P_{H_j}. s'_j \simeq_{H_j} t'_j \wedge \mathcal{H}_{H_j}^d (s'_j) = \mathcal{H}_{H_j}^d (s \upharpoonright H_j)$$
  
 
$$\wedge \mathcal{H}_{H_j}^d (t'_j) = \mathcal{H}_{H_j}^d (t \upharpoonright H_j) \wedge \forall k \in \mathbb{N}. s_k \in M_{H_1} \Leftrightarrow t_k \in M_{H_1}$$
  
 
$$\Leftrightarrow \forall j \in \{1, 2\}. s \upharpoonright H_j \simeq_{H_j}^d t \upharpoonright H_j \wedge \forall k \in \mathbb{N}. s_k \in M_{H_1} \Leftrightarrow t_k \in M_{H_1}$$
  
 
$$\Leftrightarrow \forall j \in \{1, 2\}. s \upharpoonright G_j, t \upharpoonright G_j \in P_{H_j} \wedge s \upharpoonright G_j \simeq_{G_j}^d t \upharpoonright G_j$$
  
 
$$\wedge \forall k \in \mathbb{N}. s_k \in M_{G_1} \Leftrightarrow t_k \in M_{G_1}$$
  
 
$$\Leftrightarrow s, t \in P_{H_1 \otimes H_2} \wedge s \simeq_{G_1 \otimes G_2}^d t.$$

Finally, we have  $\mu(H_1 \otimes H_2) = \max(\mu(H_1), \mu(H_2)) = \max(\mu(G_1), \mu(G_2)) = \mu(G_1 \otimes G_2)$ , showing that  $H_1 \otimes H_2 \triangleleft G_1 \otimes G_2$ .

Linear implication and promotion are similar, and pairing and currying are even simpler; thus, we omit them. Next, let us consider concatenation. Assume that  $\mathcal{H}^{\omega}(H_1) \leq A \multimap B$ ,  $\mathcal{H}^{\omega}(H_2) \leq B \multimap C$ ,  $\mathcal{H}^{\omega}(G_1) \leq D \multimap E$ ,  $\mathcal{H}^{\omega}(G_2) \leq E \multimap F$  for some normalized games A, B, C, D, E and F; without loss of generality, we assume that these normalized games are the least ones with respect to  $\leq$ . By Theorem 3.1,  $\mathcal{H}^{\omega}(H_1) \leq \mathcal{H}^{\omega}(G_1) \leq D \multimap E$  and  $\mathcal{H}^{\omega}(H_2) \leq \mathcal{H}^{\omega}(G_2) \leq E \multimap F$ , which in turn implies  $A \leq D, B \leq E$  and  $C \leq F$ . First, we clearly have  $M_{H_1 \ddagger H_2} \subseteq M_{G_1 \ddagger G_2}$  and  $\lambda_{G_1 \ddagger G_2} \upharpoonright M_{H_1 \ddagger H_2} = \lambda_{H_1 \ddagger H_2}$ , where  $\mu(H_i) = \mu(G_i)$  for i = 1, 2 ensures that the priority orders of moves of B coincide.

Next, for the enabling relations, we have:

$$\star \vdash_{H_1 \ddagger H_2} m \Leftrightarrow \star \vdash_{H_2} m \Leftrightarrow \star \vdash_C m \Rightarrow \star \vdash_F m \Leftrightarrow \star \vdash_{G_1 \ddagger G_2} m$$

as well as:

$$\begin{split} m \vdash_{H_1 \ddagger H_2} n \Leftrightarrow m \vdash_{H_1} n \lor m \vdash_{H_2} n \lor (\star \vdash_{B_{[2]}} m \land \star \vdash_{B_{[1]}} n) \\ \Rightarrow m \vdash_{G_1} n \lor m \vdash_{G_2} n \lor (\star \vdash_{E_{[2]}} m \land \star \vdash_{E_{[1]}} n) \\ \Leftrightarrow m \vdash_{G_1 \ddagger G_2} n \end{split}$$

for any  $m, n \in M_{H_1 \ddagger H_2}$ . For the positions, we have:

$$\begin{aligned} P_{H_1 \ddagger H_2} &= \{ \boldsymbol{s} \in \mathscr{J}_{H_1 \ddagger H_2} \mid \boldsymbol{s} \upharpoonright H_1 \in P_{H_1}, \boldsymbol{s} \upharpoonright H_2 \in P_{H_2}, \boldsymbol{s} \upharpoonright B_{[1]}, B_{[2]} \in pr_B \} \\ &\subseteq \{ \boldsymbol{s} \in \mathscr{J}_{G_1 \ddagger G_2} \mid \boldsymbol{s} \upharpoonright G_1 \in P_{G_1}, \boldsymbol{s} \upharpoonright G_2 \in P_{G_2}, \boldsymbol{s} \upharpoonright E_{[1]}, E_{[2]} \in pr_E \} \\ &= P_{G_1 \ddagger G_2}. \end{aligned}$$

Finally, we may show, in the same manner as in the case of tensor, the required condition on the identifications of positions, completing the proof.  $\Box$ 

At the end of the present section, we establish the following useful lemma:

**Lemma 3.7 (Hiding lemma on games).** Let  $\clubsuit_{i \in I}$  be a construction on games and  $G_i$  a game for all  $i \in I$ . For each  $d \in \mathbb{N} \cup \{\omega\}$ , we have:

- 1  $\mathcal{H}^d(\clubsuit_{i\in I}G_i) = \clubsuit_{i\in I}\mathcal{H}^d(G_i)$  if  $\clubsuit_{i\in I} \neq \ddagger;$
- 2  $\mathcal{H}^d((G_1) \ddagger (G_2)) \leq A \multimap C$  if  $\mathcal{H}^d(G_1 \ddagger G_2)$  is normalized, where A, B and C are normalized games such that  $\mathcal{H}^{\omega}(G_1) \leq A \multimap B$  and  $\mathcal{H}^{\omega}(G_2) \leq B \multimap C$ , and in particular  $(A \multimap B); (B \multimap C) = A \multimap C;$
- 3  $\mathcal{H}^d(G_1 \ddagger G_2) = \mathcal{H}^d(G_1) \ddagger \mathcal{H}^d(G_2)$  otherwise.

*Proof.* Since there is an upper bound of the priority orders of each game, it suffices to consider the case  $d \in \mathbb{N}$ . But then, as  $\mathcal{H}^{i+1} = \mathcal{H} \circ \mathcal{H}^i$  for all  $i \in \mathbb{N}$ , we may focus on d = 1. We focus on tensor as the other constructions may be handled similarly.

We have to show  $\mathcal{H}(G_1 \otimes G_2) \leq \mathcal{H}(G_1) \otimes \mathcal{H}(G_2)$ . Their sets of moves and labeling functions clearly coincide. For the enabling relations, we have:

as well as:

$$\begin{split} m \vdash_{\mathcal{H}(G_1 \otimes G_2)} n \ (m \neq \star) \\ \Leftrightarrow \ (m \vdash_{G_1 \otimes G_2} n) \lor \exists k \in \mathbb{N}^+, m_1, m_2, \dots, m_{2k} \in M_{G_1 \otimes G_2} \setminus M_{\mathcal{H}(G_1 \otimes G_2)}. \\ m \vdash_{G_1 \otimes G_2} m_1 \land \forall i \in \overline{k}. m_{2i-1} \vdash_{G_1 \otimes G_2} m_{2i} \land \land m_{2k} \vdash_{G_1 \otimes G_2} n \\ \Leftrightarrow \ (m \vdash_{G_1} n \lor m \vdash_{G_2} n) \lor \exists i \in \{1, 2\}, k \in \mathbb{N}^+, m_1, m_2, \dots, m_{2k} \in M_{G_i} \setminus M_{\mathcal{H}(G_i)}. \\ m \vdash_{G_i} m_1 \land \forall i \in \overline{k}. m_{2i-1} \vdash_{G_1 \otimes G_2} m_{2i} \land m_{2k} \vdash_{G_i} n \\ \Leftrightarrow \ \exists i \in \{1, 2\}. m \vdash_{G_i} n \lor \exists k \in \mathbb{N}^+, m_1, m_2, \dots, m_{2k} \in M_{G_i} \setminus M_{\mathcal{H}(G_i)}. \\ m \vdash_{G_i} m_1 \land \forall i \in \overline{k}. m_{2i-1} \vdash_{G_1 \otimes G_2} m_{2i} \land m_{2k} \vdash_{G_i} n \\ \Leftrightarrow \ m \vdash_{\mathcal{H}(G_1) \otimes \mathcal{H}(G_2)} n. \end{split}$$

Thus, the arenas  $\mathcal{H}(G_1 \otimes G_2)$  and  $\mathcal{H}(G_1) \otimes \mathcal{H}(G_2)$  coincide.

. .

For the positions, we have:

 $\boldsymbol{s} \in P_{\mathcal{H}(G_1 \otimes G_2)}$  $\Leftrightarrow \exists t \in \mathscr{L}_{G_1 \otimes G_2}. \mathcal{H}_{G_1 \otimes G_2}(t) = s \land \forall i \in \{1, 2\}. t \upharpoonright G_i \in P_{G_i}$  $\Leftrightarrow \exists t \in \mathscr{L}_{G_1 \otimes G_2}, \mathcal{H}_{G_1 \otimes G_2}(t) = s \land \forall i \in \{1, 2\}, \mathcal{H}_{G_i}(t \upharpoonright G_i) \in P_{\mathcal{H}(G_i)}$  $\Leftrightarrow \exists t \in \mathscr{L}_{G_1 \otimes G_2}. \mathcal{H}_{G_1 \otimes G_2}(t) = s \land \forall i \in \{1, 2\}. \mathcal{H}_{G_1 \otimes G_2}(t) \upharpoonright \mathcal{H}(G_i) \in P_{\mathcal{H}(G_i)}$ (n.b.,  $\Leftarrow$  is by induction on |t|)  $\Leftrightarrow \boldsymbol{s} \in \mathscr{L}_{\mathcal{H}(G_1 \otimes G_2)} = \mathscr{L}_{\mathcal{H}(G_1) \otimes \mathcal{H}(G_2)} \land \forall i \in \{1, 2\}. \, \boldsymbol{s} \upharpoonright \mathcal{H}(G_i) \in P_{\mathcal{H}(G_i)}$  $\Leftrightarrow \boldsymbol{s} \in P_{\mathcal{H}(G_1) \otimes \mathcal{H}(G_2)}.$ 

Finally, for the identifications of positions, given  $d \in \mathbb{N} \cup \{\omega\}$ , we have:

$$\begin{aligned} \mathcal{H}_{G_1 \otimes G_2}(\boldsymbol{s}) \simeq^{d}_{\mathcal{H}(G_1 \otimes G_2)} \mathcal{H}_{G_1 \otimes G_2}(\boldsymbol{t}) \\ \Leftrightarrow \exists \boldsymbol{s}', \boldsymbol{t}' \in P_{G_1 \otimes G_2}. \mathcal{H}_{G_1 \otimes G_2}(\boldsymbol{s}') \simeq_{\mathcal{H}(G_1 \otimes G_2)} \mathcal{H}_{G_1 \otimes G_2}(\boldsymbol{t}') \land \mathcal{H}^{d+1}_{G_1 \otimes G_2}(\boldsymbol{s}') = \mathcal{H}^{d+1}_{G_1 \otimes G_2}(\boldsymbol{s}) \\ \land \mathcal{H}^{d+1}_{G_1 \otimes G_2}(\boldsymbol{t}') = \mathcal{H}^{d+1}_{G_1 \otimes G_2}(\boldsymbol{t}) \\ \Leftrightarrow \forall j \in \{1, 2\}. \exists \boldsymbol{s}'_{\boldsymbol{j}}, \boldsymbol{t}'_{\boldsymbol{j}} \in P_{G_j}. \mathcal{H}_{G_j}(\boldsymbol{s}'_{\boldsymbol{j}}) \simeq_{\mathcal{H}(G_j)} \mathcal{H}_{G_j}(\boldsymbol{t}'_{\boldsymbol{j}}) \land \mathcal{H}^{d+1}_{G_j}(\boldsymbol{s}'_{\boldsymbol{j}}) = \mathcal{H}^{d+1}_{G_j}(\boldsymbol{s} \upharpoonright G_j) \\ \land \mathcal{H}^{d+1}_{G_j}(\boldsymbol{t}'_{\boldsymbol{j}}) = \mathcal{H}^{d+1}_{G_j}(\boldsymbol{t} \upharpoonright G_j) \land \forall k \in \mathbb{N}. \mathcal{H}^{d+1}_{G_1 \otimes G_2}(\boldsymbol{s}(k)) \in M_{\mathcal{H}^{d+1}(G_1)} \Leftrightarrow \mathcal{H}^{d+1}_{G_1 \otimes G_2}(\boldsymbol{t}(k)) \in M_{\mathcal{H}^{d+1}(G_1)} \\ \Leftrightarrow \forall \boldsymbol{j} \in \{1, 2\}. \boldsymbol{s} \upharpoonright G_{\boldsymbol{j}} \simeq^{d+1}_{G_j} \boldsymbol{t} \upharpoonright G_{\boldsymbol{j}} \land \forall k \in \mathbb{N}. \mathcal{H}^{d+1}_{G_1 \otimes G_2}(\boldsymbol{s}(k)) \in M_{\mathcal{H}^{d+1}(G_1)} \Leftrightarrow \mathcal{H}^{d+1}_{G_1 \otimes G_2}(\boldsymbol{t}(k)) \in M_{\mathcal{H}^{d+1}(G_1)} \\ \Leftrightarrow \mathcal{H}_{G_1 \otimes G_2}(\boldsymbol{s}) \simeq^{d}_{\mathcal{H}(G_1) \otimes \mathcal{H}(G_2)} \mathcal{H}_{G_1 \otimes G_2}(\boldsymbol{t}) \land \mathcal{H}_{G_1 \otimes G_2}(\boldsymbol{s}), \mathcal{H}_{G_1 \otimes G_2}(\boldsymbol{t}) \in P_{\mathcal{H}(G_1 \otimes G_2)} \end{aligned}$$

which completes the proof.

# 3.4. Dynamic Strategies

Dynamic strategies, another central notion of the present work, is just static strategies (Abramsky and McCusker, 1999) on dynamic games:

Definition 3.23 (Dynamic strategies). A dynamic strategy on a (dynamic) game G is a subset  $\sigma \subseteq P_G^{\mathsf{Even}}$ , written  $\sigma : G$ , that satisfies:

- (S1). It is non-empty and *even-prefix-closed* (i.e.,  $smn \in \sigma \Rightarrow s \in \sigma$ );
- (S2). It is deterministic on even-length positions (i.e.,  $smn, s'm'n' \in \sigma \land sm =$  $s'm' \Rightarrow smn = s'm'n').$

A dynamic strategy  $\sigma : G$  is said to be **normalized** if  $\forall s \in \sigma, \forall i \in \overline{|s|}, \lambda_G^{\mathbb{N}}(s(i)) = 0$ .

Clearly, a normalized dynamic strategy on a normalized dynamic game is equivalent to a static strategy.

**Convention.** Henceforth, a *strategy* refers to a dynamic strategy by default.

As positions of a game G are identified up to  $\simeq_G$ , we must identify strategies on G if they behave in the same manner up to  $\simeq_G$ , leading to:

# Definition 3.24 (Identification of strategies (Abramsky et al., 2000; McCusker, 1998)).

The *identification of strategies* on a game G, written  $\simeq_G$ , is the relation between strategies  $\sigma, \tau : G$  given by:

$$\sigma \simeq_G \tau \stackrel{\text{df.}}{\Leftrightarrow} \forall s \in \sigma, t \in \tau, sm, tl \in P_G. sm \simeq_G tl \Rightarrow \forall smn \in \sigma. \exists tlr \in \tau. smn \simeq_G tlr$$
$$\land \forall tlr \in \tau. \exists smn \in \sigma. tlr \simeq_G smn.$$

We are particularly concerned with strategies identified with themselves:

**Definition 3.25 (Validity of strategies).** A strategy  $\sigma$  : G is *valid* if  $\sigma \simeq_G \sigma$ .

Since internal moves are conceptually 'invisible' to Opponent, a strategy  $\sigma : G$  must be externally consistent: If  $\mathbf{s}mn, \mathbf{s}'m'n' \in \sigma, \lambda_G^{\mathbb{N}}(n) = \lambda_G^{\mathbb{N}}(n') = 0$  and  $\mathcal{H}_G^{\omega}(\mathbf{s}m) = \mathcal{H}_G^{\omega}(\mathbf{s}'m')$ , then n = n' and  $\mathcal{J}_{\mathbf{s}mn}^{\odot\omega}(n) = \mathcal{J}_{\mathbf{s}'m'n'}^{\odot\omega}(n')$ . Moreover, external consistency of strategies should hold with respect to identification of positions as well. In fact, we now proceed to establish a stronger property (Theorem 3.7).

**Lemma 3.8 (O-determinacy).** Let  $\sigma, \tau : G$  such that  $\sigma \simeq_G \tau$ , and  $d \in \mathbb{N} \cup \{\omega\}$ .

- 1 If  $sm, s'm' \in P_G$  are *d*-complete,  $s, s' \in \sigma$ , and  $\mathcal{H}^d_G(sm) = \mathcal{H}^d_G(s'm')$ , then sm = s'm';
- 2 If  $sm, tl \in P_G$  are *d*-complete,  $s \in \sigma$ ,  $t \in \tau$ , and  $\mathcal{H}^d_G(sm) \simeq_{\mathcal{H}^d(G)} \mathcal{H}^d_G(tl)$ , then  $sm \simeq_G tl$ .

*Proof.* Let us focus on the first statement for the second one can be proved similarly. We proceed by induction on  $|\mathbf{s}|$ . The base case  $\mathbf{s} = \boldsymbol{\epsilon}$  is trivial: For any  $d \in \mathbb{N} \cup \{\omega\}$ , if  $\mathcal{H}_G^d(\mathbf{s}m) = \mathcal{H}_G^d(\mathbf{s}'m')$ , then  $\mathcal{H}_G^d(\mathbf{s}'m') = \mathcal{H}_G^d(\mathbf{s}m) = m$ , and so  $\mathbf{s}'m' = m = \mathbf{s}m$ .

For the induction step, let  $d \in \mathbb{N} \cup \{\omega\}$  be fixed, and assume  $\mathcal{H}_G^d(sm) = \mathcal{H}_G^d(s'm')$ . We may suppose that sm = tlrm, where l is the rightmost O-move occurring on the left of m in s such that  $\lambda_G^{\mathbb{N}}(l) = 0 \lor \lambda_G^{\mathbb{N}}(l) > d$ . Then,  $\mathcal{H}_G^d(s'm') = \mathcal{H}_G^d(sm) = \mathcal{H}_G^d(t).l.\mathcal{H}_G^d(rm)$ , and so we may write  $s'm' = t'_1.l.t'_2.m'$ . Now,  $t, t'_1 \in \sigma$ ,  $tl, t'_1l \in P_G$ ,  $\mathcal{H}_G^d(tl) = \mathcal{H}_G^d(t'_1l)$ , and tl and t'l' are both d-complete; thus, by the induction hypothesis,  $tl = t'_1l$ . Thus,  $\mathcal{H}_G^d(t).l.\mathcal{H}_G^d(t'_2m') = \mathcal{H}_G^d(s'm') = \mathcal{H}_G^d(sm) = \mathcal{H}_G^d(t).l.\mathcal{H}_G^d(rm)$ , whence  $t'_2$  is of the form  $rt''_2$  by the determinacy of  $\sigma$ . Hence, sm = tlrm and  $s'm' = tlrt''_2m'$ . Finally, if r is external, then so is m by IE-switch, and so s'm' = sm; if r is j-internal (j > d), then so is m, and we apply the axiom DP2 for i = j - 1 to s and s', whence sm = s'm'.

**Theorem 3.7 (External consistency).** Let  $\sigma, \tau : G$  such that  $\sigma \simeq_G \tau$ , and  $d \in \mathbb{N} \cup \{\omega\}$ .

- 1 If  $smn, s'm'n' \in \sigma$  are *d*-complete, and  $\mathcal{H}^d_G(sm) = \mathcal{H}^d_G(s'm')$ , then smn = s'm'n';
- 2 If  $smn \in \sigma$ ,  $tlr \in \tau$  are *d*-complete, and  $\mathcal{H}^{\vec{d}}_{G}(sm) \simeq_{\mathcal{H}^{d}(G)} \mathcal{H}^{d}_{G}(tl)$ , then  $smn \simeq_{G} tlr$ .

Proof. Let us first prove the first statement. Let  $\sigma : G$  be a strategy,  $smn, s'm'n' \in \sigma$ and  $d \in \mathbb{N} \cup \{\omega\}$ , and assume that smn, s'm'n' are both *d*-complete and  $\mathcal{H}^d_G(sm) = \mathcal{H}^d_G(s'm')$ . By the first statement of Lemma 3.8, we have sm = s'm'; thus, by the axiom S2 on  $\sigma$ , we have n = n' and  $\mathcal{J}_{smn}(n) = \mathcal{J}_{s'm'n'}(n')$ , whence  $\mathcal{J}^{\odot d}_{smn}(n) = \mathcal{J}^{\odot d}_{s'm'n'}(n')$ .

Similarly, the second statement is proved by the second statement of Lemma 3.8, completing the proof.  $\hfill \Box$ 

Corollary 3.3 (Stepwise identification of strategies). Any strategies  $\sigma, \tau : G$  such

that  $\sigma \simeq_G \tau$  satisfy  $\sigma \simeq_G^d \tau$  for all  $d \in \mathbb{N} \cup \{\omega\}$ , where:

$$\sigma \simeq^d_G \tau \stackrel{\text{df.}}{\Leftrightarrow} \forall s \in \sigma, t \in \tau, sm, tl \in P_G. sm \simeq^d_G tl \Rightarrow \forall smn \in \sigma. \exists tlr \in \tau. smn \simeq^d_G tlr \land \forall tlr \in \tau. \exists smn \in \sigma. tlr \simeq^d_G smn.$$

Proof. Immediate from Theorem 3.7.

Hence, for any strategies  $\sigma, \tau : G$ , we have:

$$\sigma \simeq_G \tau \Leftrightarrow \forall d \in \mathbb{N} \cup \{\omega\}. \, \sigma \simeq^d_G \tau$$

which will be useful later in the paper.

Let us proceed to show that the relation  $\simeq_G$  on strategies on any game G is a PER.

**Lemma 3.9 (PER lemma).** Given  $\sigma, \tau : G$  such that  $\sigma \simeq_G \tau$ , we have:

 $(\forall s \in \sigma. \exists t \in \tau. s \simeq_G t) \land (\forall t \in \tau. \exists s \in \sigma. t \simeq_G s).$ 

*Proof.* By symmetry, it suffices to show  $\forall s \in \sigma. \exists t \in \tau. s \simeq_G t$ . We prove it by induction on |s|. The base case is trivial; for the inductive step, let  $smn \in \sigma$ . By the induction hypothesis, there exists some  $t \in \tau$  such that  $s \simeq_G t$ . Then, by the axiom DI3 on  $\simeq_G$ , there exists some  $tl \in \tau$  such that  $sm \simeq_G tl$ . Finally, since  $\sigma \simeq_G \tau$ , there exists some  $tlr \in \tau$  such that  $smn \simeq_G tlr$ , completing the proof.

**Proposition 3.1 (PERs on strategies).** Given a game G, the identification  $\simeq_G$  of strategies on G is a PER, i.e., a symmetric, transitive relation.

*Proof.* We just show the transitivity as the symmetry is obvious. Let  $\sigma, \tau, \mu : G$  such that  $\sigma \simeq_G \tau$  and  $\tau \simeq_G \mu$ . Assume that  $smn \in \sigma$ ,  $u \in \mu$  and  $sm \simeq_G up$ . By Lemma 3.9, there exists some  $t \in \tau$  such that  $s \simeq_G t$ . By the axiom DI3 on  $\simeq_G$ , there exists some  $tl \in P_G$  such that  $sm \simeq_G tl$ , whence  $tl \simeq_G up$ . Also, since  $\sigma \simeq_G \tau$ , there exists some  $tlr \in \tau$  such that  $smn \simeq_G tlr$ . Finally, since  $\tau \simeq_G \mu$ , there exists some  $upq \in \mu$  such that  $tlr \simeq_G upq$ , whence  $smn \simeq_G upq$ , completing the proof.

Therefore, given a game G, we may take the equivalence classes  $[\sigma] \stackrel{\text{df.}}{=} \{\tau : G \mid \sigma \simeq_G \tau\}$  of valid strategies  $\sigma : G$ ; these equivalence classes, rather than strategies themselves, have interpreted proofs and programs (Abramsky et al., 2000; McCusker, 1998).

At this point, let us note that even-length positions are not necessarily preserved under the hiding operation on j-sequences (Definition 3.6). For instance, let smnt be an evenlength position of a game G such that sm (resp. nt) consists of external (resp. internal) moves only. By IE-switch on G, m is an O-move, and so  $\mathcal{H}_{G}^{\omega}(smnt) = sm$  is of odd-length.

Taking into account this fact, we define:

**Definition 3.26 (Hiding operation on strategies).** Let G be a game, and  $d \in \mathbb{N} \cup \{\omega\}$ . Given  $s \in P_G$ , we define:

$$oldsymbol{s} rac{1}{s} rac{1}{s} \mathcal{H}^d_G(oldsymbol{s}) \quad ext{if } oldsymbol{s} ext{ is } d ext{-complete (Definition 3.1);} \ oldsymbol{t} oldsymbol{t} \ oldsymbol{i} \ oldsymbol{t} \ oldsymbol{s} \ oldsymbol{t} \ oldsymb$$

The *d*-hiding operation  $\mathcal{H}^d$  (on strategies) is then given by:

$$\mathcal{H}^d: (\sigma:G) \mapsto \{ s \natural \mathcal{H}^d_G \mid s \in \sigma \}.$$

Let us proceed to establish a beautiful fact:  $\sigma : G \Rightarrow \mathcal{H}^d(\sigma) : \mathcal{H}^d(G)$  for all  $d \in \mathbb{N} \cup \{\omega\}$ . For this task, we need the following lemma:

Lemma 3.10 (Asymmetry lemma). Let  $\sigma$ : G be a strategy, and  $d \in \mathbb{N} \cup \{\omega\}$ . Assume that  $smn \in \mathcal{H}^d(\sigma)$ , where  $smn = tmunv \natural \mathcal{H}^d_G$  with  $tmunv \in \sigma$  not d-complete. Then,  $smn = \mathcal{H}^d(tmun) = \mathcal{H}^d(t).mn$ .

Proof. Since  $tmunv \in \sigma$  is not d-complete, we may write  $v = v_1 lv_2 r$  with  $\lambda_G^{\mathbb{N}}(l) = 0 \lor \lambda_G^{\mathbb{N}}(l) > d, \ 0 < \lambda_G^{\mathbb{N}}(r) \leqslant d \text{ and } 0 < \lambda_G^{\mathbb{N}}(x) \leqslant d \text{ for all moves } x \text{ in } v_1 \text{ or } v_2$ . Then, we have  $smn = tmunv_1 lv_2 r \natural \mathcal{H}_G^d = \mathcal{H}_G^d(t)m\mathcal{H}_G^d(u)n = \mathcal{H}_G^d(t)mn$ .

We are now ready to establish:

# **Theorem 3.8 (Hiding theorem).** If $\sigma : G$ , then $\mathcal{H}^d(\sigma) : \mathcal{H}^d(G)$ for all $d \in \mathbb{N} \cup \{\omega\}$ .

*Proof.* We first show  $\mathcal{H}^d(\sigma) \subseteq P_{\mathcal{H}^d(G)}^{\mathsf{Even}}$ . Let  $s \in \mathcal{H}^d(\sigma)$ , i.e.,  $s = t \natural \mathcal{H}_G^d$  for some  $t \in \sigma$ . Let us write t = t'm as the case  $t = \epsilon$  is trivial.

- If t is d-complete, then  $s = t \not\models \mathcal{H}_G^d = \mathcal{H}_G^d(t) \in P_{\mathcal{H}^d(G)}$ . Also, since  $s = \mathcal{H}_G^d(t')m$  and m is a P-move, s must be of even-length by alternation on  $\mathcal{H}^d(G)$ .
- If  $\boldsymbol{t}$  is not d-complete, then we may write  $\boldsymbol{t} = \boldsymbol{t''}m_0m_1\dots m_k$ , where  $m_k = m, \boldsymbol{t''}m_0$  is d-complete, and  $0 < \lambda_G^{\mathbb{N}}(m_i) \leq d$  for  $i = 1, 2, \dots, k$ . By IE-switch,  $m_0$  is an O-move, and thus  $\boldsymbol{s} = \mathcal{H}_G^d(\boldsymbol{t''}) \in P_{\mathcal{H}^d(G)}$  is of even-length.

It remains to verify the axioms S1 and S2. For S1,  $\mathcal{H}^d(\sigma)$  is non-empty as  $\boldsymbol{\epsilon} \in \mathcal{H}^d(\sigma)$ . For the even-prefix-closure, let  $\boldsymbol{smn} \in \mathcal{H}^d(\sigma)$ ; we have to show  $\boldsymbol{s} \in \mathcal{H}^d(\sigma)$ . We have some  $\boldsymbol{tmunv} \in \sigma$  such that  $\boldsymbol{tmunv} \natural \mathcal{H}_G^d = \boldsymbol{smn}$ . By Lemma 3.10,  $\boldsymbol{smn} = \mathcal{H}_G^d(\boldsymbol{t})\boldsymbol{mn}$ , whence  $\boldsymbol{s} = \mathcal{H}_G^d(\boldsymbol{t})$ . For  $\boldsymbol{tm}$  is d-complete, so is  $\boldsymbol{t}$  by IE-switch. Thus,  $\boldsymbol{s} = \mathcal{H}_G^d(\boldsymbol{t}) = \boldsymbol{t} \natural \mathcal{H}_G^d \in \mathcal{H}^d(\sigma)$ .

Finally for S2, let  $smn, smn' \in \mathcal{H}^d(\sigma)$ ; we have to show n = n' and  $\mathcal{J}_{sm}^{\odot d}(n) = \mathcal{J}_{sm}^{\odot d}(n')$ . Clearly,  $smn = tmunv \natural \mathcal{H}_G^d$ ,  $smn' = t'mu'n'v' \natural \mathcal{H}_G^d$  for some  $tmunv, t'mu'n'v' \in \sigma$ . Then, by Lemma 3.10,  $smn = \mathcal{H}_G^d(tmu)n$  and  $smn' = \mathcal{H}_G^d(t'mu')n'$ . Therefore, by Theorem 3.7, n = n' and  $\mathcal{J}_{smn}^{\odot d}(n) = \mathcal{J}_{smn'}^{\odot d}(n')$ , completing the proof.

Next, let us review standard constraints on strategies. First, recall that a programming language is *total* if its computation always terminates in a finite period of time. This programming concept is interpreted in game semantics by *totality* of strategies in the sense similar to totality of partial functions:

**Definition 3.27 (Totality of strategies (Abramsky et al., 1997)).** A strategy  $\sigma$ : *G* is *total* if it satisfies  $\forall s \in \sigma, sm \in P_G$ .  $\exists smn \in \sigma$ .

Nevertheless, it is well-known that totality of strategies is *not* preserved under composition due to the problem of 'infinite chattering' (Abramsky et al., 1997; Clairambault and Harmer, 2010). For this point, one usually imposes a condition on strategies stronger than totality, e.g., *winning* (Abramsky et al., 1997), that is preserved under composition. We may certainly just apply the winning condition of (Abramsky et al., 1997), but it requires an additional

structure on games, which may be criticized as extrinsic and/or ad-hoc; thus, we prefer another, simpler solution. A natural idea is then to require that strategies should not contain any strictly increasing (with respect to  $\leq$ ) infinite sequence of positions. However, we have to relax this constraint: The dereliction  $der_A$  (Definition 3.34), the  $\beta$ -identity on a game A in the game-semantic CCBoC given in Section 4, satisfies it iff so does A, but we cannot impose it on games as the operation  $\Rightarrow = !(\_) \multimap (\_)$  on games, which is the  $\beta$ -exponential construction in the CCBoC, does not preserve it.

Thus, instead, we apply the same idea to *P*-views, arriving at:

**Definition 3.28 (Noetherianity of strategies (Clairambault and Harmer, 2010)).** A strategy  $\sigma$  : G is **noetherian** if it does not contain any strictly increasing (with respect to  $\preceq$ ) infinite sequence of P-views of G.

It has been shown in (Clairambault and Harmer, 2010) that total, noetherian static strategies are closed under composition.

Next, recall that one of the highlights of *HO-games* (Hyland and Ong, 2000) is to give a one-to-one correspondence between PCF Böhm trees and *innocent*, *well-bracketed* static strategies (on static games modeling types of PCF). That is, the two conditions narrow down the hom-sets of the codomain of the interpretation functor, i.e., the category of HO-games, so that the interpretation becomes *full*. Roughly, a strategy is innocent if its computation depends only on P-views, and well-bracketed if every 'question-answering' by the strategy is achieved in the 'last-question-first-answered' fashion. Formally:

**Definition 3.29 (Innocence of strategies (Hyland and Ong, 2000)).** A strategy  $\sigma : G$  is *innocent* if  $\forall smn, t \in \sigma, tm \in P_G$ .  $\lceil tm \rceil = \lceil sm \rceil \Rightarrow tmn \in \sigma \land \lceil tmn \rceil = \lceil smn \rceil$ .

Definition 3.30 (Well-bracketing of strategies (Hyland and Ong, 2000)). A strategy  $\sigma$ : G is well-bracketed (wb) if, given  $sqta \in \sigma$ , where  $\lambda_G^{QA}(q) = Q$ ,  $\lambda_G^{QA}(a) = A$  and  $\mathcal{J}_{sqta}(a) = q$ , each occurrence of a question in t', defined by  $\lceil sqt \rceil_G = \lceil sq \rceil_G \cdot t'$ , justifies an occurrence of an answer in t'.

Now, let us show that the standard constraints on strategies except totality are all preserved under the hiding operation, which implies that dynamic strategies are a reasonable generalization of static strategies in a certain sense.

Corollary 3.4 (Preservation of constraints on strategies under hiding). If a strategy  $\sigma : G$  is valid, innocent, we or noetherian, then so is  $\mathcal{H}^d(\sigma) : \mathcal{H}^d(G)$ , and if another  $\tau : G$  satisfies  $\sigma \simeq_G \tau$ , then  $\mathcal{H}^d(\sigma) \simeq_{\mathcal{H}^d(G)} \mathcal{H}^d(\tau)$ , for all  $d \in \mathbb{N} \cup \{\omega\}$ .

*Proof.* Let  $d \in \mathbb{N} \cup \{\omega\}$  be arbitrarily fixed. We have  $\mathcal{H}^d(\sigma) : \mathcal{H}^d(G)$  by Theorem 3.8.

- Preservation of validity is by Lemma 3.8, Corollary 3.3 and the axiom DI3 on  $\simeq_G$ ;
- Preservation of innocence and noetherianity holds because  $[\mathcal{H}^d_G(sm)]_{\mathcal{H}^d(G)}$  is a jsubsequence of  $\mathcal{H}^d_G([sm]_G)$  for any  $sm \in P^{\mathsf{Odd}}_G$ ;
- Well-bracketing is preserved under the *d*-hiding operation  $\mathcal{H}^d$  because *both* of the question and the answer of each 'QA-pair' are either deleted or retained.

Finally, preservation of identification of strategies is proved similarly to that of validity, completing the proof.

**Remark.** Totality of strategies is *not* preserved under the *d*-hiding operation  $\mathcal{H}^d$  on strategies for all  $d \in \mathbb{N} \cup \{\omega\}$ . For instance, consider any total strategy that always performs a 1-internal P-move, which is no longer total when  $\mathcal{H}$  is applied. As we shall see shortly, it is why totality is preserved under concatenation of strategies but not under composition (i.e., composition coincides with *concatenation plus hiding*).

At the end of the present section, we establish an inductive property of the *d*-hiding operation on strategies for each  $d \in \mathbb{N} \cup \{\omega\}$ :

**Notation.** Given  $\sigma : G$  and  $d \in \mathbb{N} \cup \{\omega\}$ , we define  $\sigma_{\downarrow}^{d \stackrel{\text{df.}}{=}} \{s \in \sigma \mid s \text{ is } d\text{-complete}\}$  and  $\sigma_{\uparrow}^{d \stackrel{\text{df.}}{=}} \sigma \setminus \sigma_{\downarrow}^{d}$ .

Lemma 3.11 (Hiding and complete positions). Let  $\sigma : G$ . Given  $i, d \in \mathbb{N}$  such that  $i \ge d$ , we have  $\mathcal{H}^i(\sigma) = \mathcal{H}^i(\sigma^d_{\perp}) \stackrel{\text{df.}}{=} \{ s \not\models \mathcal{H}^i_G \mid s \in \sigma^d_{\perp} \}.$ 

Proof.  $\mathcal{H}^{i}(\sigma_{\downarrow}^{d}) \subseteq \mathcal{H}^{i}(\sigma)$  is obvious. For the opposite inclusion, let  $s \in \mathcal{H}^{i}(\sigma)$ , i.e.,  $s = t \not\models \mathcal{H}_{G}^{i}$  for some  $t \in \sigma$ ; we have to show  $s \in \mathcal{H}^{i}(\sigma_{\downarrow}^{d})$ . If  $t \in \sigma_{\downarrow}^{d}$ , then we are done; thus, assume otherwise. If there is no external or *j*-internal move with j > i other than the first move  $m_{0}$  in t, then  $s = \epsilon \in \mathcal{H}^{i}(\sigma_{\downarrow}^{d})$ ; so assume otherwise. As a result, we may write  $t = m_{0}t_{1}mnt_{2}r$ , where  $t_{2}r$  consists only of *j*-internal moves with  $0 < j \leq i$ , and m and n are P- and O-moves, respectively, such that  $\lambda_{G}^{\mathbb{N}}(m) = \lambda_{G}^{\mathbb{N}}(n) = 0 \lor \lambda_{G}^{\mathbb{N}}(m) = \lambda_{G}^{\mathbb{N}}(n) > i$ . Take  $m_{0}t_{1}m \in \sigma_{\downarrow}^{d}$  such that  $m_{0}t_{1}m \not\models \mathcal{H}_{G}^{i} = m_{0}\mathcal{H}_{G}^{i}(t_{1})m = t \not\models \mathcal{H}_{G}^{i} = s$ , whence  $s \in \mathcal{H}^{i}(\sigma_{\downarrow}^{d})$ .

We are now ready to show:

Lemma 3.12 (Stepwise hiding on strategies). Given  $\sigma : G$ , we have  $\mathcal{H}^{i+1}(\sigma) = \mathcal{H}^1(\mathcal{H}^i(\sigma))$  for all  $i \in \mathbb{N}$ .

*Proof.* We first show the inclusion  $\mathcal{H}^{i+1}(\sigma) \subseteq \mathcal{H}^1(\mathcal{H}^i(\sigma))$ . By Lemma 3.11, we may write any element of the set  $\mathcal{H}^{i+1}(\sigma)$  as  $s \not\models \mathcal{H}^{i+1}_G$  for some  $s \in \sigma_{\perp}^{i+1}$ . Then observe that:

$$\mathbf{s} \natural \mathcal{H}_G^{i+1} = \mathcal{H}_G^{i+1}(\mathbf{s}) = \mathcal{H}_{\mathcal{H}^i(G)}(\mathcal{H}_G^i(\mathbf{s})) = (\mathbf{s} \natural \mathcal{H}_G^i) \natural \mathcal{H}_{\mathcal{H}^i(G)}^1 \in \mathcal{H}^1(\mathcal{H}^i(\sigma)).$$

For the opposite inclusion  $\mathcal{H}^1(\mathcal{H}^i(\sigma)) \subseteq \mathcal{H}^{i+1}(\sigma)$ , again by Lemma 3.11, we may write any element of  $\mathcal{H}^1(\mathcal{H}^i(\sigma))$  as  $(s \not\in \mathcal{H}^i_G) \not\in \mathcal{H}^{i+1}_{\mathcal{H}^i(G)}$  for some  $s \in \sigma^i_{\downarrow}$ . We have to show that  $(s \not\in \mathcal{H}^i_G) \not\in \mathcal{H}^{i+1}(\sigma)$ . If  $s \in \sigma^{i+1}_{\downarrow}$ , then it is completely analogous to the above argument; so assume otherwise. Also, if an external or *j*-internal move with j > i + 1 in s is only the first move  $m_0$ , then  $(s \not\in \mathcal{H}^i_G) \not\in \mathcal{H}^{i+1}(\sigma)$ ; thus assume othewise. Now, we may write:

$$s = s'mnm_1m_2\ldots m_{2k}r$$

where  $\lambda_G^{\mathbb{N}}(r) = i+1, m_1, m_2, \dots, m_{2k}$  are *j*-internal with  $0 < j \leq i+1$ , and *m* and *n* are

external or *j*-internal P- and O-moves with j > i + 1, respectively. Then,

$$\begin{aligned} (\mathbf{s} \natural \mathcal{H}_{G}^{i}) \natural \mathcal{H}_{\mathcal{H}^{i}(G)}^{1} &= \mathcal{H}_{G}^{i}(\mathbf{s}) \natural \mathcal{H}_{\mathcal{H}^{i}(G)}^{1} \\ &= \mathcal{H}_{\mathcal{H}^{i}(G)}(\mathcal{H}_{G}^{i}(\mathbf{s'})).m \\ &= \mathcal{H}_{G}^{i+1}(\mathbf{s'}).m \text{ (by Lemma 3.3)} \\ &= \mathbf{s} \natural \mathcal{H}_{G}^{i+1} \in \mathcal{H}^{i+1}(\sigma) \end{aligned}$$

which completes the proof.

Thus, as in the case of games, we may focus on the operation  $\mathcal{H}^1$ :

**Convention.** Henceforth, we write  $\mathcal{H}$  for  $\mathcal{H}^1$  and call it the *hiding operation (on strategies)*;  $\mathcal{H}^i$  denotes the *i*-times iteration of  $\mathcal{H}$  for all  $i \in \mathbb{N}$ .

#### 3.5. Constructions on Dynamic Strategies

Next, let us recall standard constructions on strategies (Abramsky and McCusker, 1999). Note that since (dynamic) strategies are simply 'static strategies on (dynamic) games', they are clearly closed under all the constructions on static strategies.

Nevertheless, the CCBoC of games and strategies given in Section 4 has normalized games as 0-cells and strategies  $\phi : G$  such that  $\mathcal{H}^{\omega}(G) \triangleleft A \Rightarrow B$  as 1-cells  $A \to B$ , and therefore we need to generalize *pairing* and *promotion* of static strategies; in fact, we have generalized product and exponential of static games respectively to pairing and promotion of dynamic games for this purpose. Also, we shall decompose and generalize composition of static strategies as *concatenation plus hiding* of dynamic strategies, for which we have introduced concatenation of dynamic games.

Let us begin with recalling *tensor*  $\otimes$  of strategies:

1.0

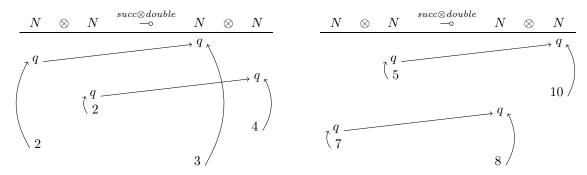
**Definition 3.31 (Tensor of strategies (Abramsky and McCusker, 1999)).** Given games A, B, C and D, and strategies  $\phi : A \multimap C$  and  $\psi : B \multimap D$ , the *tensor (product)*  $\phi \otimes \psi$  of  $\phi$  and  $\psi$  is given by:

$$\phi \otimes \psi \stackrel{\text{df.}}{=} \{ \boldsymbol{s} \in \mathscr{L}_{A \otimes B \multimap C \otimes D} \mid \boldsymbol{s} \upharpoonright A, C \in \phi, \boldsymbol{s} \upharpoonright B, D \in \psi \}.$$

Intuitively the tensor  $\phi \otimes \psi : A \otimes B \multimap C \otimes D$  of  $\phi : A \multimap C$  and  $\psi : B \multimap D$  plays by  $\phi$  if the last O-move is of A or C, and by  $\psi$  otherwise.

**Example 3.12.** The tensor  $succ \otimes double : N \otimes N \multimap N \otimes N$ , where succ,  $double : N \multimap N$  are given in Section 1, plays, e.g., as follows:

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**Lemma 3.13 (Well-defined tensor of strategies).** Given games A, B, C and D, and strategies  $\phi : A \multimap C$  and  $\psi : B \multimap D$ ,  $\phi \otimes \psi$  is a strategy on  $A \otimes B \multimap C \otimes D$ . If  $\phi$  and  $\psi$  are innocent (resp. wb, total, noetherian), then so is  $\phi \otimes \psi$ . Given  $\phi' : A \multimap C$  and  $\psi' : B \multimap D$  with  $\phi \simeq_{A \multimap C} \phi'$  and  $\psi \simeq_{B \multimap D} \psi', \phi \otimes \psi \simeq_{A \otimes B \multimap C \otimes D} \phi' \otimes \psi'$ .

Proof. Straightforward; see (McCusker, 1998; Abramsky et al., 2000).

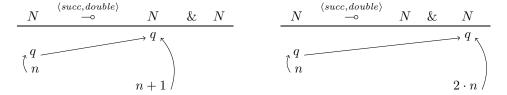
We proceed to recall *pairing* of strategies:

**Definition 3.32 (Pairing of strategies (Abramsky and McCusker, 1999)).** Given games A, B and C, and strategies  $\phi : C \multimap A$  and  $\psi : C \multimap B$ , the *pairing*  $\langle \phi, \psi \rangle$  of  $\phi$  and  $\psi$  is defined by:

$$\langle \phi, \psi \rangle \stackrel{\text{df.}}{=} \{ \boldsymbol{s} \in \mathscr{L}_{C \multimap A \& B} \mid (\boldsymbol{s} \upharpoonright C, A \in \phi \land \boldsymbol{s} \upharpoonright B = \boldsymbol{\epsilon}) \lor (\boldsymbol{s} \upharpoonright C, B \in \psi \land \boldsymbol{s} \upharpoonright A = \boldsymbol{\epsilon}) \}.$$

That is, the pairing  $\langle \phi, \psi \rangle : C \multimap A \& B$  of  $\phi : C \multimap A$  and  $\psi : C \multimap B$  plays by  $\phi$  if the play is of  $C \multimap A$ , and by  $\psi$  otherwise.

**Example 3.13.** The pairing  $(succ, double) : N \multimap N \& N$  plays as either of the following:



where  $n \in \mathbb{N}$ , depending on the first O-move.

**Lemma 3.14 (Well-defined pairing of strategies).** Given games A, B and C, and strategies  $\phi : C \multimap A$  and  $\psi : C \multimap B$ ,  $\langle \phi, \psi \rangle$  is a strategy on  $C \multimap A\&B$ . If  $\phi$  and  $\psi$  are innocent (resp. wb, total, noetherian), then so is  $\langle \phi, \psi \rangle$ . Given  $\phi' : C \multimap A$  and  $\psi' : C \multimap B$  with  $\phi \simeq_{C \multimap A} \phi'$  and  $\psi \simeq_{C \multimap B} \psi', \langle \phi, \psi \rangle \simeq_{C \multimap A\&B} \langle \phi', \psi' \rangle$ .

Proof. Straightforward; see (McCusker, 1998; Abramsky et al., 2000).

Next, let us recall *promotion* of strategies:

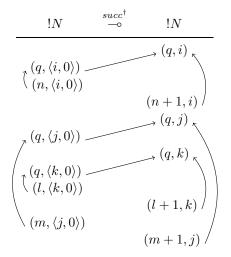
Definition 3.33 (Promotion of strategies (McCusker, 1998)). Given games A

and B, and a strategy  $\varphi : !A \multimap B$ , the **promotion**  $\varphi^{\dagger}$  of  $\varphi$  is defined by:

$$\varphi^{\dagger} \stackrel{\mathrm{df.}}{=} \{ \boldsymbol{s} \in \mathscr{L}_{!A \multimap !B} \mid \forall i \in \mathbb{N}. \, \boldsymbol{s} \upharpoonright i \in \varphi \, \}$$

That is, the promotion  $\varphi^{\dagger} : !A \multimap !B$  of  $\varphi : A \Rightarrow B$  plays, during a play s of  $!A \multimap !B$ , as  $\varphi$  for each j-subsequence  $s \upharpoonright i$  or *thread*. We could have defined noetherianity of strategies in terms of positions, but then it would not be preserved under promotion by the obvious reason; it is why we have defined it in terms of P-views (Definition 3.28).

**Example 3.14.** Let  $succ : N \Rightarrow N$  be the successor strategy (n.b., it is on the implication  $\Rightarrow$ , not the linear implication  $\neg$ ), which specifically selects, say, the 'tag' (\_,0) in the domain !N. Then, the promotion  $succ^{\dagger} : !N \multimap !N$  plays, e.g., as follows:



where  $i, j, k, n, m, l \in \mathbb{N}$  such that  $i \neq j, i \neq k$  and  $j \neq k$ , and they are all selected by Opponent. Note that  $succ^{\dagger}$  consistently plays as succ for each thread.

**Lemma 3.15 (Well-defined promotion of strategies).** Given games A and B, and a strategy  $\varphi : !A \multimap B$ , the promotion  $\varphi^{\dagger}$  is a strategy on  $!A \multimap !B$ . If  $\varphi$  is innocent (resp. wb, total, noetherian), then so is  $\varphi^{\dagger}$ . Given  $\tilde{\varphi} : !A \multimap B$  with  $\varphi \simeq_{!A \multimap B} \tilde{\varphi}, \varphi^{\dagger} \simeq_{!A \multimap !B} \tilde{\varphi}^{\dagger}$ .

Proof. Straightforward; see (McCusker, 1998; Abramsky et al., 2000).

We proceed to recall a simple kind strategies, which are  $\beta$ -identities of our gamesemantic CCBoC given in Section 4:

**Definition 3.34 (Derelictions (Abramsky et al., 2000; McCusker, 1998)).** The *dereliction*  $der_A : !A \multimap A$  on a normalized game A is defined by:

$$der_A \stackrel{\text{di.}}{=} \{ s \in P_{!A \multimap A}^{\mathsf{Even}} \mid \forall t \preceq s. \, \mathsf{Even}(t) \Rightarrow (t \upharpoonright !A) \upharpoonright 0 = t \upharpoonright A \}$$

Note that any 'tag' (-, i) such that  $i \in \mathbb{N}$  would work; our choice (-, 0) does not matter.

**Lemma 3.16 (Well-defined derelictions).** Given a normalized game A,  $der_A$  is a valid, innocent, wb, total strategy on  $!A \multimap A$ . It is noetherian if A is well-founded.

*Proof.* We just show that  $der_A$  is noetherian if A is well-founded for the other points are trivial, e.g., validity of  $der_A$  is immediate from the definition of  $\simeq_{!A\multimap A}$ . Given  $smm \in der_A$ , it is easy to see by induction on |s| that the P-view  $\lceil sm \rceil$  is of the form  $m_1m_1m_2m_2\ldots m_km_km$ , and thus there is a sequence  $\star \vdash_A m_1 \vdash_A m_2 \cdots \vdash_A m_k \vdash_A m$ of enabling pairs. Therefore, if A is well-founded, then  $der_A$  must be noetherian.

Let us proceed to introduce some generalizations of existing constructions. Note that tensor, pairing and promotion of static strategies have been already generalized slightly because they allow non-normalized dynamic games and strategies. However, for the gamesemantic CCBoC in Section 4, we need further generalizations:

**Definition 3.35 (Generalized pairing of strategies).** Given strategies  $\phi : L$  and  $\psi : R$  such that  $\mathcal{H}^{\omega}(L) \leq C \multimap A$  and  $\mathcal{H}^{\omega}(R) \leq C \multimap B$  for some normalized games A, B and C, the *(generalized) pairing*  $\langle \phi, \psi \rangle$  of  $\phi$  and  $\psi$  is defined by:

$$\langle \phi, \psi \rangle \stackrel{\text{df.}}{=} \{ \boldsymbol{s} \in \mathscr{L}_{\langle L, R \rangle} \mid (\boldsymbol{s} \upharpoonright L \in \phi \land \boldsymbol{s} \upharpoonright R = \boldsymbol{\epsilon}) \lor (\boldsymbol{s} \upharpoonright R \in \psi \land \boldsymbol{s} \upharpoonright L = \boldsymbol{\epsilon}) \}.$$

**Theorem 3.9 (Well-defined generalized pairing of strategies).** Given strategies  $\phi : L$  and  $\psi : R$  such that  $\mathcal{H}^{\omega}(L) \leq C \multimap A$  and  $\mathcal{H}^{\omega}(R) \leq C \multimap B$  for some normalized games A, B and  $C, \langle \phi, \psi \rangle$  is a strategy on  $\langle L, R \rangle$ . If  $\phi$  and  $\psi$  are innocent (resp. wb, total, noetherian), then so is  $\langle \phi, \psi \rangle$ . Given  $\phi' : L$  and  $\psi' : R$  such that  $\phi \simeq_L \phi'$  and  $\psi \simeq_R \psi'$ , we have  $\langle \phi, \psi \rangle \simeq_{\langle L, R \rangle} \langle \phi', \psi' \rangle$ .

Proof. Straightforward.

Convention. Henceforth, *pairing of strategies* refers to the generalized one.

**Definition 3.36 (Generalized promotion of strategies).** Given a strategy  $\varphi : G$  such that  $\mathcal{H}^{\omega}(G) \leq A \to B$  for some normalized games A and B, the *(generalized)* promotion  $\varphi^{\dagger}$  of  $\varphi$  is defined by:

$$\varphi^{\dagger} \stackrel{\text{dl.}}{=} \{ s \in \mathscr{L}_{G^{\dagger}} \mid \forall i \in \mathbb{N}. \ s \upharpoonright i \in \varphi \}.$$

**Theorem 3.10 (Well-defined generalized promotion on strategies).** Given a strategy  $\varphi : G$  such that  $\mathcal{H}^{\omega}(G) \leq A \multimap B$  for some normalized games A and B,  $\varphi^{\dagger}$  is a strategy on  $G^{\dagger}$ . If  $\varphi$  is innocent (resp. wb, total, noetherian), then so is  $\varphi^{\dagger}$ . Given  $\varphi' : G$  such that  $\varphi \simeq_G \varphi'$ , we have  $\varphi^{\dagger} \simeq_{G^{\dagger}} \varphi'^{\dagger}$ .

Proof. Straightforward.

Convention. Henceforth, promotion of strategies refers to the generalized one.

Next, let us introduce a new construction on strategies, which plays a fundamental role in the present work:

**Definition 3.37 (Concatenation of strategies).** Let  $\iota : J$  and  $\kappa : K$  be strategies such that  $\mathcal{H}^{\omega}(J) \leq A \multimap B$  and  $\mathcal{H}^{\omega}(K) \leq B \multimap C$  for some normalized games A, B and C. The *concatenation*  $\iota \ddagger \kappa$  of  $\iota$  and  $\kappa$  is defined by:

$$\iota \ddagger \kappa \stackrel{\mathrm{df.}}{=} \{ \boldsymbol{s} \in \mathscr{J}_{J\ddagger K} \mid \boldsymbol{s} \upharpoonright J \in \iota, \boldsymbol{s} \upharpoonright K \in \kappa, \boldsymbol{s} \upharpoonright B_{[1]}, B_{[2]} \in pr_B \}.$$

**Theorem 3.11 (Well-defined concatenation of strategies).** Let  $\iota : J$  and  $\kappa : K$  be strategies such that  $\mathcal{H}^{\omega}(J) \leq A \multimap B$  and  $\mathcal{H}^{\omega}(K) \leq B \multimap C$ , where A, B and C are normalized games. Then,  $\iota \ddagger \kappa : J \ddagger K$  and  $\mathcal{H}^{\omega}(\iota); \mathcal{H}^{\omega}(\kappa) = \mathcal{H}^{\omega}(\iota \ddagger \kappa) : A \multimap C$ , where  $\mathcal{H}^{\omega}(\iota); \mathcal{H}^{\omega}(\kappa)$  is the *composition* of  $\mathcal{H}^{\omega}(\iota) : A \multimap B$  and  $\mathcal{H}^{\omega}(\kappa) : B \multimap C$  (Abramsky and McCusker, 1999). If  $\iota$  and  $\kappa$  are innocent (resp. wb, noetherian, winning), then so is  $\iota \ddagger \kappa$ . Given  $\iota' : J$  and  $\kappa' : K$  with  $\iota \simeq_J \iota'$  and  $\kappa \simeq_K \kappa'$ , we have  $\iota \ddagger \kappa \simeq_{J \ddagger K} \iota' \ddagger \kappa'$ .

*Proof.* We just show the first statement as the other ones are straightforward. It then suffices to prove  $\iota \ddagger \kappa : J \ddagger K$  and  $\mathcal{H}^{\omega}(\iota \ddagger \kappa) = \iota; \kappa$  since it implies  $\iota; \kappa = \mathcal{H}^{\omega}(\iota \ddagger \kappa) : \mathcal{H}^{\omega}(J \ddagger K) \leq A \multimap C$  by Lemmata 3.7 and 3.8. However,  $\mathcal{H}^{\omega}(\iota \ddagger \kappa) = \iota; \kappa$  is immediate from the definition of concatenation; thus, we focus on  $\iota \ddagger \kappa : J \ddagger K$ .

First, we have  $\iota \ddagger \kappa \subseteq P_{J\ddagger K}$  as any  $\boldsymbol{s} \in \iota \ddagger \kappa$  satisfies  $\boldsymbol{s} \in \mathscr{J}_{J\ddagger K}$ ,  $\boldsymbol{s} \upharpoonright J \in \iota \subseteq P_J$ ,  $\boldsymbol{s} \upharpoonright K \in \kappa \subseteq P_K$  and  $\boldsymbol{s} \upharpoonright B_{[1]}, B_{[2]} \in pr_B$ . It is also immediate that such  $\boldsymbol{s}$  is of evenlength. It remains to verify the axioms S1 and S2. For this, we need:

 $(\diamond)$  Each  $s \in \iota \ddagger \kappa$  consists of adjacent pairs mn such that  $m, n \in M_J$  or  $m, n \in M_K$ .

Proof of the claim  $\diamondsuit$  By induction on |s|. The base case is trivial. For the inductive step, let  $smn \in \iota \ddagger \kappa$ . If  $m \in M_J$ , then  $(s \upharpoonright J).m.(n \upharpoonright J) \in \sigma$ , where  $s \upharpoonright J$  is of even-length by the induction hypothesis. Thus, we must have  $n \in M_J$ . If  $m \in M_K$ , then  $n \in M_K$  by the same argument.

- --- (S1). Since  $\boldsymbol{\epsilon} \in \iota \ddagger \kappa$ , we have  $\iota \ddagger \kappa \neq \emptyset$ . For even-prefix-closure, assume  $smn \in \iota \ddagger \kappa$ . By the claim  $\diamondsuit$ , either  $m, n \in M_J$  or  $m, n \in M_K$ . In either case, it is straightforward to see that  $\boldsymbol{s} \in P_{J\ddagger K}, \boldsymbol{s} \upharpoonright J \in \iota, \boldsymbol{s} \upharpoonright K \in \kappa$  and  $\boldsymbol{s} \upharpoonright B_{[1]}, B_{[2]} \in pr_B$ , i.e.,  $\boldsymbol{s} \in \iota \ddagger \kappa$ .

Therefore, we have shown that  $\iota \ddagger \kappa : J \ddagger K$ .

Note that totality of (dynamic) strategies is *not* preserved under composition, but it is preserved under concatenation. This phenomenon is essentially because totality is not preserved under the hiding operation as already remarked above.

For completeness, let us explicitly define the rather trivial *currying* of strategies:

**Definition 3.38 (Currying of strategies).** Given  $\sigma : G$  with  $\mathcal{H}^{\omega}(G) \leq A \otimes B \multimap C$  for some normalized games A, B and C, the *currying*  $\Lambda(\sigma) : \Lambda(G)$  of  $\sigma$  is  $\sigma$  up to 'tags'.

**Proposition 3.2 (Well-defined currying of strategies).** Strategies are closed under currying, and currying preserves totality, innocence, well-bracketing, noetherianity and identification of strategies.

Proof. Obvious.

Now, as in the case of games, we establish the *hiding lemma* on strategies (Lemma 3.18). We first need the following:

Lemma 3.17 (Hiding on legal positions in the second form). For any arena G and number  $d \in \mathbb{N} \cup \{\omega\}$ , we have  $\mathscr{L}_{\mathcal{H}^d(G)} = \{s \not\models \mathcal{H}_G^d \mid s \in \mathscr{L}_G\}$ .

*Proof.* Observe that:

$$\{ \boldsymbol{s} \models \mathcal{H}_{G}^{d} \mid \boldsymbol{s} \in \mathscr{L}_{G} \} = \{ \boldsymbol{s} \models \mathcal{H}_{G}^{d} \mid \boldsymbol{s} \in \mathscr{L}_{G}, \boldsymbol{s} \text{ is } d\text{-complete} \}$$
$$= \{ \mathcal{H}_{G}^{d}(\boldsymbol{s}) \mid \boldsymbol{s} \in \mathscr{L}_{G}, \boldsymbol{s} \text{ is } d\text{-complete} \}$$
$$= \{ \mathcal{H}_{G}^{d}(\boldsymbol{s}) \mid \boldsymbol{s} \in \mathscr{L}_{G} \} \text{ (by the same argument as above)}$$
$$= \mathscr{L}_{\mathcal{H}^{d}(G)} \text{ (by Corollary 3.2)}$$

completing the proof.

**Notation.** We write  $\blacklozenge_{i \in I}$ , where *I* is  $\{1\}$  or  $\{1, 2\}$ , for any of the constructions on strategies introduced so far, i.e.,  $\blacklozenge_{i \in I}$  is either  $\otimes$ ,  $(\_)^{\dagger}$ ,  $\langle\_,\_\rangle$ ,  $\ddagger$ , ; or  $\Lambda$ .

Lemma 3.18 (Hiding lemma on strategies). Let  $\blacklozenge_{i \in I}$  be a construction on strategies, and  $\sigma_i : G_i$  for each  $i \in I$ . Then, for all  $d \in \mathbb{N} \cup \{\omega\}$ , we have:

1  $\mathcal{H}^{d}(\blacklozenge_{i\in I}\sigma_{i}) = \diamondsuit_{i\in I}\mathcal{H}^{d}(\sigma_{i}) \text{ if } \diamondsuit_{i\in I} \text{ is } \otimes, (\_)^{\dagger}, \langle\_, \_\rangle \text{ or } \Lambda;$ 

2  $\mathcal{H}^d(\sigma_1 \ddagger \sigma_2) = \mathcal{H}^d(\sigma_1) \ddagger \mathcal{H}^d(\sigma_2)$  if  $\mathcal{H}^d(\sigma_1 \ddagger \sigma_2)$  is not normalized;

3  $\mathcal{H}^d(\sigma_1 \ddagger \sigma_2) = \mathcal{H}^d(\sigma_1); \mathcal{H}^d(\sigma_2)$  otherwise.

*Proof.* As in the case of games, it suffices to assume d = 1. Here, we just focus on pairing since the other constructions may be handled analogously.

Let  $\sigma_i : G_i, i = 1, 2$ , be strategies such that  $\mathcal{H}^{\omega}(G_1) \leq C \multimap A, \mathcal{H}^{\omega}(G_2) \leq C \multimap B$  for some normalized games A, B and C. For  $\mathcal{H}(\langle \sigma_1, \sigma_2 \rangle) \subseteq \langle \mathcal{H}(\sigma_1), \mathcal{H}(\sigma_2) \rangle$ , observe that:

$$\begin{split} \boldsymbol{s} \in \mathcal{H}(\langle \sigma_1, \sigma_2 \rangle) \Rightarrow \exists \boldsymbol{t} \in \langle \sigma_1, \sigma_2 \rangle. \, \boldsymbol{t} \natural \mathcal{H}^1_{\langle G_1, G_2 \rangle} = \boldsymbol{s} \\ \Rightarrow \exists \boldsymbol{t} \in \mathscr{L}_{\langle G_1, G_2 \rangle}. \, \boldsymbol{t} \natural \mathcal{H}^1_{\langle G_1, G_2 \rangle} = \boldsymbol{s} \wedge \left( (\boldsymbol{t} \upharpoonright G_1 \in \sigma_1 \wedge \boldsymbol{t} \upharpoonright G_2 = \boldsymbol{\epsilon}) \lor (\boldsymbol{t} \upharpoonright G_2 \in \sigma_2 \wedge \boldsymbol{t} \upharpoonright G_1 = \boldsymbol{\epsilon}) \right) \\ \Rightarrow \boldsymbol{s} \in \mathscr{L}_{\mathcal{H}(\langle G_1, G_2 \rangle)} \wedge \left( \boldsymbol{s} \upharpoonright \mathcal{H}(G_1) \in \mathcal{H}(\sigma_1) \land \boldsymbol{s} \upharpoonright \mathcal{H}(G_2) = \boldsymbol{\epsilon} \right) \\ \lor \left( \boldsymbol{s} \upharpoonright \mathcal{H}(G_2) \in \mathcal{H}(\sigma_2) \land \boldsymbol{s} \upharpoonright \mathcal{H}(G_1) = \boldsymbol{\epsilon} \right) \right) \text{ (by Lemma 3.17)} \\ \Rightarrow \boldsymbol{s} \in \langle \mathcal{H}(\sigma_1), \mathcal{H}(\sigma_2) \rangle. \end{split}$$

Next, we show the converse:

$$\begin{split} \boldsymbol{s} \in \langle \mathcal{H}(\sigma_1), \mathcal{H}(\sigma_2) \rangle \Rightarrow \boldsymbol{s} \in \mathscr{L}_{\mathcal{H}(\langle G_1, G_2 \rangle)} \wedge (\boldsymbol{s} \upharpoonright \mathcal{H}(G_1) \in \mathcal{H}(\sigma_1) \wedge \boldsymbol{s} \upharpoonright \mathcal{H}(G_2) = \boldsymbol{\epsilon}) \\ & \lor (\boldsymbol{s} \upharpoonright \mathcal{H}(G_2) \in \mathcal{H}(\sigma_2) \wedge \boldsymbol{s} \upharpoonright \mathcal{H}(G_1) = \boldsymbol{\epsilon})) \\ \Rightarrow (\exists \boldsymbol{u} \in \sigma_1. \, \boldsymbol{u} \natural \mathcal{H}_{G_1}^1 = \boldsymbol{s} \upharpoonright \mathcal{H}(G_1) \wedge \boldsymbol{u} \upharpoonright G_2 = \boldsymbol{\epsilon}) \\ & \lor (\exists \boldsymbol{v} \in \sigma_2. \, \boldsymbol{v} \natural \mathcal{H}_{G_2}^1 = \boldsymbol{s} \upharpoonright \mathcal{H}(G_2) \wedge \boldsymbol{v} \upharpoonright \mathcal{H}(G_1) = \boldsymbol{\epsilon}) \\ \Rightarrow \exists \boldsymbol{w} \in \langle \sigma_1, \sigma_2 \rangle. \, \boldsymbol{w} \natural \mathcal{H}_{\langle G_1, G_2 \rangle}^1 = \boldsymbol{s} \\ \Rightarrow \boldsymbol{s} \in \mathcal{H}(\langle \sigma_1, \sigma_2 \rangle) \end{split}$$

which completes the proof.

Finally, as a technical preparation for the next section, let us define:

**Definition 3.39 (Dereliction games).** The *dereliction game* on a game G is the

subgame  $\Xi_G \leq G \Rightarrow G$  given by  $M_{\Xi_G} \stackrel{\text{df.}}{=} M_{G\Rightarrow G}, \lambda_{\Xi_G} \stackrel{\text{df.}}{=} \lambda_{G\Rightarrow G}, \vdash_{\Xi_G} \stackrel{\text{df.}}{=} \vdash_{G\Rightarrow G}, P_{\Xi_G} \stackrel{\text{df.}}{=} \{s \in P_{G_{[0]}\Rightarrow G_{[1]}} \mid \forall t \preceq s. \text{Even}(t) \Rightarrow t \upharpoonright G_{[0]} = t \upharpoonright G_{[1]}\}, \text{ and } \simeq_{\Xi_G} \stackrel{\text{df.}}{=} \simeq_{G\Rightarrow G} \upharpoonright P_{\Xi_G} \times P_{\Xi_G}.$ Given normalized games A and B, we define:

 $\begin{array}{l} & - & \varPi_1^{A,B} \leqslant A \& B \Rightarrow A \text{ to be } \varXi_A \text{ up to 'tags', where we often abbreviate it as } \varPi_1; \\ & - & \varPi_2^{A,B} \leqslant A \& B \Rightarrow B \text{ to be } \varXi_B \text{ up to 'tags', where we often abbreviate it as } \varPi_2; \\ & - & \varUpsilon_{A,B} \leqslant B^A \& A \Rightarrow B \text{ to be } \varXi_{A \Rightarrow B} \text{ up to 'tags', where we often abbreviate it as } \varUpsilon_2. \end{array}$ 

That is, the dereliction game  $\Xi_G$  on a game G is the subgame of  $G \Rightarrow G$ , in which only plays by the dereliction  $der_G$  are possible.

**Lemma 3.19 (D-lemma).** Given normalized games  $A, B, C, L \leq C \Rightarrow A, R \leq C \Rightarrow B$ ,  $P \leq C \Rightarrow A\&B, U \leq A\&B \Rightarrow C$  and  $V \leq A \Rightarrow C^B$ , we have:

$$\langle L, R \rangle^{\dagger}; \Pi_1^{A,B} = L \langle L, R \rangle^{\dagger}; \Pi_2^{A,B} = R \langle P^{\dagger}; \Pi_1^{A,B}, P^{\dagger}; \Pi_2^{A,B} \rangle = P \langle (\Pi_1^{A,B})^{\dagger}; \Lambda(U), \Pi_2^{A,B} \rangle^{\dagger}; \Upsilon_{B,C} = U \Lambda(\langle (\Pi_1^{A,B})^{\dagger}; V, \Pi_2^{A,B} \rangle^{\dagger}; \Upsilon_{B,C}) = V.$$

Proof. Straightforward.

# 4. Dynamic Game Semantics of Finitary PCF

This section is the climax of the present work. We first define a game-semantic CCBoC  $\mathcal{LDG}$  (Definition 4.1) and a standard structure  $\mathcal{S}_{\mathcal{G}}$  for FPCF in  $\mathcal{LDG}$  (Definition 4.2) in Section 4.1. Then, as the main result, we show that the induced interpretation  $[-]_{\mathcal{LDG}}^{\mathcal{S}_{\mathcal{G}}}$  satisfies the PDCP (Theorem 4.2), and thus the DCP by Theorem 2.5, in Section 4.2, giving the first instance of dynamic game semantics.

#### 4.1. Dynamic Game Semantics of Finitary PCF

Let us give the CCBoC  $\mathcal{LDG}$  of dynamic games and strategies:

**Definition 4.1 (The CCBoC \mathcal{LDG}).** The CCBoC  $\mathcal{LDG} = (\mathcal{LDG}, \mathcal{H})$  is defined by:

- Objects are normalized, well-founded games;
- A  $\beta$ -morphisms  $A \to B$  is a pair  $(J, [\phi]_W)$  of a game J such that  $\mathcal{H}^{\omega}(J) \leq A \Rightarrow B$ and the equivalence class  $[\phi]_W \stackrel{\text{df.}}{=} \{\psi : J \mid \psi \text{ is winning}, \psi \simeq_J \phi\}$  of a valid, winning strategy  $\phi : J$ ;
- The  $\beta$ -composition  $A \xrightarrow{(J,[\phi]_W)} B \xrightarrow{(K,[\psi]_W)} C$  is the pair  $(J^{\dagger} \ddagger K, [\phi^{\dagger} \ddagger \psi]_W);$
- The  $\beta$ -identity  $id_A : A \to A$  on each object A is the pair  $(\Xi_A, [der_A]_W);$
- The evaluation  $\mathcal{H}$  maps morphisms  $(J, [\phi]_{\mathsf{W}}) : A \to B$  to  $\mathcal{H}(J, [\phi]_{\mathsf{W}}) \stackrel{\text{df.}}{=} (\mathcal{H}(J), [\mathcal{H}(\phi)]_{\mathsf{W}});$
- The  $\beta$ -terminal object is the terminal game T (Example 3.3);
- ---  $\beta$ -product and  $\beta$ -exponential are respectively given by  $A \times B \stackrel{\text{df.}}{=} A \& B$  and  $B^A \stackrel{\text{df.}}{=} A \Rightarrow B = !A \multimap B$  for any objects  $A, B \in \mathcal{LDG}$ ;

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- $\beta$ -pairing is given by  $\langle (L, [\alpha]_{\mathsf{W}}), (R, [\beta]_{\mathsf{W}}) \rangle \stackrel{\text{df.}}{=} (\langle L, R \rangle, [\langle \alpha, \beta \rangle]_{\mathsf{W}}) : C \to A\&B \text{ for any objects } A, B, C \in \mathcal{LDG}, \text{ and morphisms } (L, [\alpha]_{\mathsf{W}}) : C \to A \text{ and } (R, [\beta]_{\mathsf{W}}) : C \to B;$
- The  $\beta$ -projections  $\pi_1 : A\&B \to A$  and  $\pi_2 : A\&B \to B$  are respectively the pairs  $(\Pi_1^{A,B}, [\varpi_1^{A,B}]_{\mathsf{W}})$  and  $(\Pi_2^{A,B}, [\varpi_2^{A,B}]_{\mathsf{W}})$  for any objects  $A, B \in \mathcal{LDG}$ , where  $\varpi_1^{A,B} : \Pi_1^{A,B}$  and  $\varpi_2^{A,B} : \Pi_2^{A,B}$  are respectively the derelictions  $der_A$  and  $der_B$  up to 'tags';
- $\beta \text{-currying is given by } \Lambda(G, [\varphi]_{\mathsf{W}}) \stackrel{\text{df.}}{=} (\Lambda(G), [\Lambda(\varphi)]_{\mathsf{W}}) : A \to (B \Rightarrow C) \text{ for any objects} \\ A, B, C \in \mathcal{LDG}, \text{ and morphism } (G, [\varphi]_{\mathsf{W}}) : A\& B \to C;$
- The  $\beta$ -evaluation  $ev_{B,C} : C^B \& B \to C$  for any objects  $B, C \in \mathcal{LDG}$  is the pair  $(\Upsilon_{B,C}, [v_{B,C}]_W)$ , where  $v_{B,C} : \Upsilon_{B,C}$  is the dereliction  $der_{B\Rightarrow C}$  up to 'tags'.

Note that we have made the *underlying game* of each  $\beta$ -morphism in  $\mathcal{LDG}$  explicit in order to take the equivalence class of strategies. Also, we have focused on *well-founded* games and *winning* strategies for the full completeness result (Corollary 4.2), where note that games must be well-founded for derelictions to be noetherian (Lemma 3.16).

# **Theorem 4.1 (Well-defined \mathcal{LDG}).** The structure $\mathcal{LDG}$ forms a CCBoC.

Proof. First, for  $\beta$ -composition, let  $A, B, C \in \mathcal{LDG}$ ,  $(J, [\phi]_W) : A \to B$  and  $(K, [\psi]_W) : B \to C$  in  $\mathcal{LDG}$ . Then,  $\phi^{\dagger} : J^{\dagger}$  by Theorem 3.10, and  $\mathcal{H}^{\omega}(J^{\dagger}) \leq !A \multimap !B$  by Theorem 3.5; thus, we may form  $\phi^{\dagger} \ddagger \psi : J^{\dagger} \ddagger K$  such that  $\mathcal{H}^{\omega}(J^{\dagger} \ddagger K) \leq A \Rightarrow C$  by Theorem 3.11. Also, promotion and concatenation both preserve validity and winning of strategies (by Theorems 3.10 and 3.11). Hence, the pair  $(J^{\dagger} \ddagger K, [\phi^{\dagger} \ddagger \psi]_W)$  is a  $\beta$ -morphism  $A \to C$  in  $\mathcal{LDG}$ . Note that the composition does not depend on the representatives  $\phi$  and  $\psi$ .

Moreover,  $\beta$ -composition preserves  $\simeq$ : For any  $A, B, C \in \mathcal{LDG}$ ,  $(J, [\iota]_{\mathsf{W}}), (\tilde{J}, [\tilde{\iota}]_{\mathsf{W}})$ :  $A \to B$  and  $(K, [\kappa]_{\mathsf{W}}), (\tilde{K}, [\tilde{\kappa}]_{\mathsf{W}})$ :  $B \to C$  in  $\mathcal{LDG}$ , if  $\mathcal{H}^{\omega}(J, [\iota]_{\mathsf{W}}) = \mathcal{H}^{\omega}(\tilde{J}, [\tilde{\iota}]_{\mathsf{W}})$  and  $\mathcal{H}^{\omega}(K, [\kappa]_{\mathsf{W}}) = \mathcal{H}^{\omega}(\tilde{K}, [\tilde{\kappa}]_{\mathsf{W}})$ , then  $\mathcal{H}^{\omega}(J^{\dagger} \ddagger K) = \mathcal{H}^{\omega}(J)^{\dagger}; \mathcal{H}^{\omega}(K) = \mathcal{H}^{\omega}(\tilde{J})^{\dagger}; \mathcal{H}^{\omega}(\tilde{K}) =$   $\mathcal{H}^{\omega}(\tilde{J}^{\dagger} \ddagger \tilde{K})$  by Lemma 3.7, and  $\mathcal{H}^{\omega}(\iota^{\dagger} \ddagger \kappa) \simeq_{\mathcal{H}^{\omega}(J^{\dagger} \ddagger K)} \mathcal{H}^{\omega}(\tilde{\iota}^{\dagger} \ddagger \tilde{\kappa})$  by Corollary 3.4, whence  $\mathcal{H}^{\omega}(J^{\dagger} \ddagger K, [\iota^{\dagger} \ddagger \kappa]_{\mathsf{W}}) = \mathcal{H}^{\omega}(\tilde{J}^{\dagger} \ddagger \tilde{K}, [\tilde{\iota}^{\dagger} \ddagger \tilde{\kappa}]_{\mathsf{W}}).$ 

Then clearly, associativity of  $\beta$ -composition up to  $\simeq$  holds: Given  $D \in \mathcal{LDG}$ , and  $(G, [\varphi]) : C \to D$  in  $\mathcal{LDG}$ , by Lemma 3.7 we have:

$$\begin{aligned} \mathcal{H}^{\omega}((J^{\dagger} \ddagger K)^{\dagger} \ddagger G) &= (\mathcal{H}^{\omega}(J^{\dagger}); \mathcal{H}^{\omega}(K))^{\dagger}; \mathcal{H}^{\omega}(G) \\ &= (\mathcal{H}^{\omega}(J)^{\dagger}; \mathcal{H}^{\omega}(K)^{\dagger}); \mathcal{H}^{\omega}(G) \\ &= \mathcal{H}^{\omega}(J)^{\dagger}; (\mathcal{H}^{\omega}(K)^{\dagger}; \mathcal{H}^{\omega}(G)) \\ &= \mathcal{H}^{\omega}(J^{\dagger}); (\mathcal{H}^{\omega}(K^{\dagger}); \mathcal{H}^{\omega}(G)) \\ &= \mathcal{H}^{\omega}(J^{\dagger} \ddagger (K^{\dagger} \ddagger G)) \end{aligned}$$

as well as by Lemma 3.18:

$$\begin{aligned} \mathcal{H}^{\omega}((\phi^{\dagger} \ddagger \psi)^{\dagger} \ddagger \varphi) &= (\mathcal{H}^{\omega}(\phi^{\dagger}); \mathcal{H}^{\omega}(\psi))^{\dagger}; \mathcal{H}^{\omega}(\varphi) \\ &= (\mathcal{H}^{\omega}(\phi^{\dagger}); \mathcal{H}^{\omega}(\psi)^{\dagger}); \mathcal{H}^{\omega}(\varphi) \\ &= (\mathcal{H}^{\omega}(\phi^{\dagger}); (\mathcal{H}^{\omega}(\psi^{\dagger}); \mathcal{H}^{\omega}(\varphi)) \\ &= \mathcal{H}^{\omega}(\phi^{\dagger} \ddagger (\psi^{\dagger} \ddagger \varphi)) \end{aligned}$$

whence  $((J, [\phi]_{\mathsf{W}}); (K, [\psi]_{\mathsf{W}})); (G, [\varphi]_{\mathsf{W}}) \simeq (J, [\phi]_{\mathsf{W}}); ((K, [\psi]_{\mathsf{W}}); (G, [\varphi]_{\mathsf{W}})).$ 

Similarly, unit law up to  $\simeq$  holds; we leave the details to the reader.

Also,  $\mathcal{H}$  clearly satisfies the four axioms of BoC (Definition 2.2), having shown that  $\mathcal{LDG}$  is a BoC. It remains to verify its cartesian closed structure up to  $\simeq$ .

The universal property of the  $\beta$ -terminal game T up to  $\simeq$  is obvious, where we define  $!_A \stackrel{\text{df.}}{=} (A \Rightarrow T, [\{\epsilon\}]_W) : A \to T$  for each  $A \in \mathcal{LDG}$ . The  $\beta$ -projections are clearly values in  $\mathcal{LDG}$ . Given  $\beta$ -morphisms  $(L, [\alpha]_W) : C \to A$  and  $(R, [\beta]_W) : C \to B$  in  $\mathcal{LDG}$ , i.e.,  $\alpha : L, \beta : R, \mathcal{H}^{\omega}(L) \leq C \Rightarrow A$  and  $\mathcal{H}^{\omega}(R) \leq C \Rightarrow B$ , we may obtain the valid, winning pairing  $\langle \alpha, \beta \rangle : \langle L, R \rangle$  such that  $\mathcal{H}^{\omega}(\langle L, R \rangle) \leq C \Rightarrow A\&B$  by Theorem 3.4. Hence, the pair  $(\langle L, R \rangle, [\langle \alpha, \beta \rangle]_W)$  is a  $\beta$ -morphism  $C \to A\&B$  in  $\mathcal{LDG}$ , which does not depend on the representatives  $\alpha$  and  $\beta$ . Note also that the  $\beta$ -pairing clearly preserves values in  $\mathcal{LDG}$ .

Also, we have by Lemmata 3.7 and 3.19:

$$\mathcal{H}^{\omega}(\langle L, R \rangle^{\dagger} \ddagger \Pi_{1}^{A,B}) = \langle \mathcal{H}^{\omega}(L), \mathcal{H}^{\omega}(R) \rangle^{\dagger}; \Pi_{1}^{A,B} = \mathcal{H}^{\omega}(L)$$

as well as by Lemma 3.18:

$$\mathcal{H}^{\omega}(\langle \alpha, \beta \rangle^{\dagger} \ddagger \varpi_{1}^{A,B}) = \langle \mathcal{H}^{\omega}(\alpha)^{\dagger}, \mathcal{H}^{\omega}(\beta)^{\dagger} \rangle; \varpi_{1}^{A,B}$$
$$= \mathcal{H}^{\omega}(\alpha).$$

Similarly,  $\mathcal{H}^{\omega}(\langle L, R \rangle^{\dagger} \ddagger \Pi_{2}^{A,B}) = \mathcal{H}^{\omega}(R)$  and  $\mathcal{H}^{\omega}(\langle \alpha, \beta \rangle^{\dagger} \ddagger \varpi_{2}^{A,B}) = \mathcal{H}^{\omega}(\beta)$ . Hence,  $\langle (L, [\alpha]_{\mathsf{W}}), (R, [\beta]_{\mathsf{W}}) \rangle; \pi_{1} \simeq (L, [\alpha]_{\mathsf{W}})$  and  $\langle (L, [\alpha]_{\mathsf{W}}), (R, [\beta]_{\mathsf{W}}) \rangle; \pi_{2} = (R, [\beta]_{\mathsf{W}})$  hold. Next, given any  $\beta$ -morphism  $(P, [\rho]_{\mathsf{W}}) : C \to A\&B$  in  $\mathcal{LDG}$ , we have:

$$\begin{aligned} \mathcal{H}^{\omega}(\langle P^{\dagger} \ddagger \Pi_{1}^{A,B}, P^{\dagger} \ddagger \Pi_{2}^{A,B} \rangle) &= \langle \mathcal{H}^{\omega}(P)^{\dagger}; \Pi_{1}^{A,B}, \mathcal{H}^{\omega}(P)^{\dagger}; \Pi_{2}^{A,B} \rangle \\ &= \mathcal{H}^{\omega}(P) \end{aligned}$$

again by Lemmata 3.7 and 3.19, as well as by Lemma 3.18:

$$\begin{aligned} \mathcal{H}^{\omega}(\langle \rho^{\dagger} \ddagger \varpi_{1}^{A,B}, \rho^{\dagger} \ddagger \varpi_{2}^{A,B} \rangle) &= \langle \mathcal{H}^{\omega}(\rho)^{\dagger}; \varpi_{1}^{A,B}, \mathcal{H}^{\omega}(\rho)^{\dagger}; \varpi_{2}^{A,B} \rangle \\ &= \mathcal{H}^{\omega}(\rho). \end{aligned}$$

Hence,  $\langle (P, [\rho]); \pi_1, (P, [\rho]); \pi_2 \rangle \simeq (P, [\rho])$  holds.

It is also straightforward to check that  $\beta$ -pairing in  $\mathcal{LDG}$  preserves  $\simeq$ : Given any  $\beta$ morphisms  $(L, [\alpha]_W), (\tilde{L}, [\tilde{\alpha}]_W) : C \to A$  and  $(R, [\beta]_W), (\tilde{R}, [\tilde{\beta}]_W) : C \to B$  in  $\mathcal{LDG}$  such that  $\mathcal{H}^{\omega}(L, [\alpha]_W) = \mathcal{H}^{\omega}(\tilde{L}, [\tilde{\alpha}]_W)$  and  $\mathcal{H}^{\omega}(R, [\beta]_W) = \mathcal{H}^{\omega}(\tilde{R}, [\tilde{\beta}]_W)$ , we have:

$$\begin{aligned} \mathcal{H}^{\omega}(\langle (L, [\alpha]_{\mathsf{W}}), (R, [\beta]_{\mathsf{W}}) \rangle) &= (\mathcal{H}^{\omega}(\langle L, R \rangle), [\mathcal{H}^{\omega}(\langle \alpha, \beta \rangle)]_{\mathsf{W}}) \\ &= (\langle \mathcal{H}^{\omega}(L), \mathcal{H}^{\omega}(R) \rangle, [\langle \mathcal{H}^{\omega}(\alpha), \mathcal{H}^{\omega}(\beta) \rangle]_{\mathsf{W}}) \\ &= (\langle \mathcal{H}^{\omega}(\tilde{L}), \mathcal{H}^{\omega}(\tilde{R}) \rangle, [\langle \mathcal{H}^{\omega}(\tilde{\alpha}), \mathcal{H}^{\omega}(\tilde{\beta}) \rangle]_{\mathsf{W}}) \\ &= (\mathcal{H}^{\omega}(\langle \tilde{L}, \tilde{R} \rangle), [\mathcal{H}^{\omega}(\langle \tilde{\alpha}, \tilde{\beta} \rangle)]_{\mathsf{W}}) \\ &= \mathcal{H}^{\omega}(\langle (\tilde{L}, [\tilde{\alpha}]_{\mathsf{W}}), (\tilde{R}, [\tilde{\beta}]_{\mathsf{W}}) \rangle). \end{aligned}$$

Finally, the requirements for  $\beta$ -exponentials,  $\beta$ -currying and  $\beta$ -evaluations are proved more or less similarly to the case of  $\beta$ -products,  $\beta$ -pairing and  $\beta$ -projections, and thus we leave the details to the leader.

We proceed to give a standard structure (Definition 2.4) for FPCF in  $\mathcal{LDG}$ :

## Definition 4.2 (Standard structure in $\mathcal{LDG}$ ). The standard structure

$$\mathcal{S}_{\mathcal{G}} = (\mathbf{2}, T, \&, \pi, \Rightarrow, ev, \underline{tt}, \underline{ff}, \vartheta)$$

of games and strategies for FPCF in  $\mathcal{LDG}$  is given by:

- -2 is the game of booleans (Example 3.4), and T is the terminal game (Example 3.3);
- & is product of games, and  $\pi_1^{A,B} \stackrel{\text{df.}}{=} (\Pi_1^{A,B}, [\varpi_1^{A,B}]_{\mathsf{W}}) \ (i = 1, 2)$  for any  $A, B \in \mathcal{LDG}$ ;
- $\begin{array}{l} \longrightarrow \text{ is function space of games, and } ev_{A,B} \stackrel{\text{df.}}{=} (\Upsilon_{A,B}, [v_{A,B}]_{\mathsf{W}}) \text{ for any } A, B \in \mathcal{LDG}; \\ \longrightarrow \underline{tt} \stackrel{\text{df.}}{=} (T \Rightarrow \mathbf{2}, [\mathsf{Pref}(\{q.tt\})^{\mathsf{Even}}]_{\mathsf{W}}), \underline{ff} \stackrel{\text{df.}}{=} (T \Rightarrow \mathbf{2}, [\mathsf{Pref}(\{q.ff\})^{\mathsf{Even}}]_{\mathsf{W}}) : T \to \mathbf{2}; \end{array}$
- $-\vartheta \stackrel{\text{df.}}{=} (2\&(2\&2) \Rightarrow 2, [case]_W) : 2\&(2\&2) \rightarrow 2, \text{ where } case : 2\&(2\&2) \Rightarrow 2,$ is the standard game semantics of the case-construction (Hyland and Ong, 2000; Abramsky and McCusker, 1999) modified to a normalized strategy in the obvious manner.

Lemma 4.1 (Standardness of  $\mathcal{S}_{\mathcal{G}}$ ). The structure  $\mathcal{S}_{\mathcal{G}}$  for FPCF in  $\mathcal{LDG}$  is standard in the sense defined in Definition 2.4.

*Proof.* Straightforward.

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# 4.2. Game-Semantic Dynamic Correspondence Property for FPCF

At last, we are now ready to prove that our game semantics satisfies a DCP:

**Theorem 4.2 (PDCP-theorem).** The interpretation  $[-]_{\mathcal{LDG}}^{\mathcal{S}_{\mathcal{G}}}$  of FPCF (Definitions 2.4) and 4.1) satisfies the PDCP (Definition 2.6).

*Proof.* To establish the PDCP, the only non-trivial case is to show for any reduction of FPCF of the form  $(\lambda x^{A}, V)W \rightarrow U$ , where V, W and U are values,  $\mathcal{H}(\llbracket(\lambda x^{A}, V)W\rrbracket_{\mathcal{LDG}}^{\mathcal{S}_{\mathcal{G}}}) = \llbracket U\rrbracket_{\mathcal{LDG}}^{\mathcal{S}_{\mathcal{G}}}$  (n.b.,  $\llbracket(\lambda x^{A}, V)W\rrbracket_{\mathcal{LDG}}^{\mathcal{S}_{\mathcal{G}}} \neq \llbracket U\rrbracket_{\mathcal{LDG}}^{\mathcal{S}_{\mathcal{G}}}$  is immediate from the first component of each  $\beta$ -morphism in  $\mathcal{LDG}$  and the third axiom on standardness of  $\mathcal{S}_{\mathcal{G}}$ ); the other conditions for the PDCP follow from Lemmata 3.7 and 3.18. Let us focus on the non-trivial case, for which we define the **height**  $Ht(\mathsf{B}) \in \mathbb{N}$  of each type  $\mathsf{B}$  by  $Ht(o) \stackrel{\text{df.}}{=} 0$  and  $Ht(\mathsf{B}_1 \Rightarrow \mathsf{B}_2) \stackrel{\text{df.}}{=}$  $\max(Ht(B_1)+1, Ht(B_2))$ . We proceed by induction on the height of the type A of W.

Below, given  $\beta$ -morphisms  $(H, [\tau]_W) : C \to (A \Rightarrow B)$  and  $(G, [\sigma]_W) : C \to A$  in  $\mathcal{LDG}$ . we define the  $\beta$ -morphism  $(H, [\tau]_{\mathsf{W}}) \lfloor (G, [\sigma]_{\mathsf{W}}) \rfloor \stackrel{\text{df.}}{=} (\langle G, H \rangle^{\dagger} \ddagger \Upsilon_{A,B}, [\langle \tau, \sigma \rangle^{\dagger} \ddagger v_{A,B}]_{\mathsf{W}}) : C \to B \text{ in } \mathcal{LDG}.$  If  $(H, [\tau]_{\mathsf{W}}) : C \to (A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_k \Rightarrow B)$  and  $(G_i, [\sigma_i]_{\mathsf{W}}) : C \to C \to C$  $A_i$  for i = 1, 2, ..., k, then we write  $(H, [\tau]_W) \lfloor (G_1, [\sigma_1]_W), (G_2, [\sigma_2]_W), ..., (G_k, [\sigma_k]_W) \rfloor$ for  $(H, [\tau]_W) \lfloor (G_1, [\sigma_1]_W) \rfloor \lfloor (G_2, [\sigma_2]_W) \rfloor \dots \lfloor (G_k, [\sigma_k]_W) \rfloor : C \to B$ . We abbreviate in this proof the interpretation  $[-]_{\mathcal{LDG}}^{S_{\mathcal{G}}}$  as [-]. Let  $\Gamma$  be the context of  $(\lambda x^{\mathsf{A}}, \mathsf{V})\mathsf{W}$  (as well as  $\mathsf{U}$ ). In the following, we abbreviate each  $\beta$ -morphism  $(G, [\sigma]_W)$  in  $\mathcal{LDG}$  as  $[\sigma]$  for brevity, and focus on the second components (i.e., the equivalence classes of strategies); the corresponding equations on the first components (i.e., games) may be obtained, thanks to Lemmata 3.7 and 3.19, similarly to the ways for the first components shown below.

For the base case, assume Ht(A) = 0, i.e.,  $A \equiv o$ . By induction on |V|, we have:

- If  $\mathsf{V} \equiv \mathsf{tt}$ , then  $(\lambda \mathsf{x}^{\mathsf{A}}, \mathsf{tt})\mathsf{W} \to \mathsf{tt}$ , and clearly  $\mathcal{H}(\llbracket (\lambda \mathsf{x}^{\mathsf{A}}, \mathsf{tt})\mathsf{W} \rrbracket) = \langle \Lambda(\llbracket \mathsf{tt} \rrbracket), \llbracket \mathsf{W} \rrbracket \rangle^{\dagger}; [v] =$ **[tt]**. The case of  $V \equiv ff$  is analogous.

$$\begin{split} \mathcal{H}(\llbracket \mathsf{VW} \rrbracket) &= \mathcal{H}(\langle \Lambda_{\llbracket \mathsf{A} \rrbracket}(\Lambda_{\llbracket \mathsf{C} \rrbracket}(\llbracket \mathsf{V}' \rrbracket)), \llbracket \mathsf{W} \rrbracket)^{\dagger} \ddagger [\upsilon]) \\ &= \langle \Lambda_{\llbracket \mathsf{A} \rrbracket}(\Lambda_{\llbracket \mathsf{C} \rrbracket}(\llbracket \mathsf{V}' \rrbracket)), \llbracket \mathsf{W} \rrbracket)^{\dagger} ; [\upsilon] \text{ (by Lemma 3.18)} \\ &= \Lambda_{\llbracket \mathsf{C} \rrbracket}(\langle \Lambda_{\llbracket \mathsf{A} \rrbracket}(\llbracket \mathsf{V}' \rrbracket), \llbracket \mathsf{W} \rrbracket)^{\dagger} ; [\upsilon]) \\ &= \Lambda_{\llbracket \mathsf{C} \rrbracket}(\mathcal{H}(\langle \Lambda_{\llbracket \mathsf{A} \rrbracket}(\llbracket \mathsf{V}' \rrbracket), \llbracket \mathsf{W} \rrbracket)^{\dagger} \ddagger [\upsilon])) \\ &= \Lambda_{\llbracket \mathsf{C} \rrbracket}(\mathcal{H}(\llbracket (\lambda \mathsf{x}^{\mathsf{A}}.\mathsf{V}')\mathsf{W} \rrbracket)) \\ &= \Lambda_{\llbracket \mathsf{C} \rrbracket}(\mathfrak{U}(\llbracket (\lambda \mathsf{x}^{\mathsf{A}}.\mathsf{V}')\mathsf{W} \rrbracket)) \\ &= \llbracket \lambda_{\varPsi \mathsf{C} \rrbracket}(\llbracket \mathsf{U}' \rrbracket). \end{split}$$

— If  $V \equiv case(yV_1 \dots V_k)[\tilde{V}_1; \tilde{V}_2]$  with  $x \neq y$ , then  $(\lambda x^A.V)W \rightarrow U$ , where

$$U \equiv \mathsf{case}(\mathsf{yn}f(\mathsf{V}_1[\mathsf{W}/\mathsf{x}]) \dots nf(\mathsf{V}_k[\mathsf{W}/\mathsf{x}]))[nf(\tilde{\mathsf{V}}_1[\mathsf{W}/\mathsf{x}]); nf(\tilde{\mathsf{V}}_1[\mathsf{W}/\mathsf{x}])].$$

By the induction hypothesis and the interpretation of the variable y, we have:

$$\begin{split} & [[(\lambda \mathbf{x}^{\mathbf{A}}.\mathbf{V})\mathbf{W}]] \\ &= \mathcal{H}^{\omega}(\Lambda_{\llbracket A \rrbracket}(\langle \llbracket \mathbf{y} \rrbracket \lfloor \llbracket \mathbf{V}_{1} \rrbracket, \dots, \llbracket \mathbf{V}_{k} \rrbracket \rfloor, \langle \llbracket \tilde{\mathbf{V}}_{1} \rrbracket, \llbracket \tilde{\mathbf{V}}_{2} \rrbracket \rangle \rangle^{\dagger} \ddagger [case]) \lfloor \llbracket \mathbf{W} \rrbracket \rfloor) \\ &= \mathcal{H}^{\omega}(\langle \Lambda_{\llbracket A \rrbracket}(\llbracket \mathbf{y} \rrbracket) \lfloor \llbracket \mathbf{W} \rrbracket \rfloor \lfloor \Lambda_{\llbracket A \rrbracket}(\llbracket \mathbf{V}_{1} \rrbracket) \lfloor \llbracket \mathbf{W} \rrbracket \rfloor, \dots, \Lambda_{\llbracket A \rrbracket}(\llbracket \mathbf{V}_{k} \rrbracket) \lfloor \llbracket \mathbf{W} \rrbracket \rfloor \rfloor, \langle \Lambda_{\llbracket A \rrbracket}(\llbracket \tilde{\mathbf{V}}_{1} \rrbracket) \lfloor \llbracket \mathbf{W} \rrbracket \rfloor, \dots, \Lambda_{\llbracket A \rrbracket}(\llbracket \mathbf{V}_{k} \rrbracket) \lfloor \llbracket \mathbf{W} \rrbracket \rfloor \rfloor, \langle \Lambda_{\llbracket A \rrbracket}(\llbracket \tilde{\mathbf{V}}_{1} \rrbracket) \lfloor \llbracket \mathbf{W} \rrbracket \rfloor, \dots, \Lambda_{\llbracket A \rrbracket}(\llbracket \mathbf{V}_{k} \rrbracket) \lfloor \llbracket \mathbf{W} \rrbracket \rfloor \rfloor, \langle \Lambda_{\llbracket A \rrbracket}(\llbracket \tilde{\mathbf{V}}_{1} \rrbracket) \lfloor \llbracket \mathbf{W} \rrbracket \rfloor \rfloor, \dots, \mathbb{I}(\lambda \mathbf{x}, \mathbf{V}_{k}) \mathbf{W} \rrbracket \rfloor, \langle \Lambda_{\llbracket A \rrbracket}(\llbracket \tilde{\mathbf{V}}_{1} \rrbracket) \lfloor \llbracket \mathbf{W} \rrbracket \rfloor, \dots, \mathbb{I}(\lambda \mathbf{x}, \mathbf{V}_{k}) \mathbf{W} \rrbracket \rfloor, \langle \Lambda_{\llbracket A \rrbracket}(\llbracket \tilde{\mathbf{V}}_{1} \rrbracket) \rangle \rangle^{\dagger} \ddagger [case]) \\ &= \mathcal{H}^{\omega}(\langle \llbracket (\lambda \mathbf{x}, \mathbf{y}) \mathbf{W} \rrbracket \lfloor \llbracket (\lambda \mathbf{x}, \mathbf{V}_{1}) \mathbf{W} \rrbracket, \dots, \llbracket nf(\mathbf{V}_{k} [\mathbf{W}/\mathbf{x}]) \rrbracket \rfloor, \langle \llbracket nf(\tilde{\mathbf{V}}_{1} [\mathbf{W}/\mathbf{x}]) \rrbracket \rangle \rangle^{\dagger} \ddagger [case]) \\ &= \mathcal{H}^{\omega}(\langle \llbracket \mathbf{y} \rrbracket \lfloor \llbracket nf(\mathbf{V}_{1} [\mathbf{W}/\mathbf{x}]) \rrbracket, \dots, \llbracket nf(\mathbf{V}_{k} [\mathbf{W}/\mathbf{x}]) \rrbracket \rfloor, \langle \llbracket nf(\tilde{\mathbf{V}}_{1} [\mathbf{W}/\mathbf{x}]) \rrbracket \rangle \rangle^{\dagger} \ddagger [case]) \\ &= \llbracket \mathbf{U} \rrbracket. \end{split}$$

— If  $V \equiv \mathsf{case}(x)[\tilde{V}_1;\tilde{V}_2]$ , then  $(\lambda x^A.V)W \to U$ , where

$$U \equiv \mathsf{case}(\mathsf{W})[\mathit{nf}(\tilde{\mathsf{V}}_1[\mathsf{W}/\mathsf{x}]); \mathit{nf}(\tilde{\mathsf{V}}_2[\mathsf{W}/\mathsf{x}])].$$

By the same reasoning as the above case, we get  $\mathcal{H}(\llbracket(\lambda x^{A}.V)W\rrbracket) = \llbracket U\rrbracket$ .

Next, for the inductive step, assume Ht(A) = h + 1. We may proceed in the same way as the base case, i.e., by induction on |V|, except that the last case is generalized to  $V \equiv \mathsf{case}(\mathsf{xV}_1 \dots \mathsf{V}_k)[\tilde{\mathsf{V}}_1; \tilde{\mathsf{V}}_2]$ , where  $A \equiv A_1 \Rightarrow A_2 \Rightarrow \dots \Rightarrow A_k \Rightarrow o \ (k \ge 0)$ . We have to consider the additional case of  $k \ge 1$ ; then we have  $(\lambda \mathsf{x}^A, \mathsf{V})\mathsf{W} \to \mathsf{U}$ , where

$$\mathsf{U} \equiv \mathsf{case}(nf(\mathsf{W}(\mathsf{V}_1[\mathsf{W}/\mathsf{x}])\dots(\mathsf{V}_k[\mathsf{W}/\mathsf{x}])))[nf(\mathsf{V}_1[\mathsf{W}/\mathsf{x}]); nf(\mathsf{V}_2[\mathsf{W}/\mathsf{x}])].$$

We then have the following chain of equations:

$$\begin{aligned} \mathcal{H}\llbracket(\lambda x. V) W\rrbracket \\ &= \mathcal{H}(\Lambda(\mathcal{H}^{\omega}(\langle\llbracket x \rrbracket \lfloor \llbracket V_1 \rrbracket, \dots, \llbracket V_k \rrbracket \rfloor, \langle\llbracket \tilde{V}_1 \rrbracket, \llbracket \tilde{V}_2 \rrbracket \rangle\rangle^{\dagger} \ddagger [case])) \lfloor\llbracket W\rrbracket \rfloor) \\ &= \mathcal{H}^{\omega}(\langle \Lambda(\llbracket x \rrbracket) \lfloor\llbracket W \rrbracket \rfloor \lfloor \Lambda(\llbracket V_1 \rrbracket) \lfloor\llbracket W \rrbracket \rfloor, \dots, \Lambda(\llbracket V_k \rrbracket) \lfloor\llbracket W \rrbracket \rfloor \rfloor, \langle \Lambda(\llbracket \tilde{V}_1 \rrbracket) \lfloor\llbracket W \rrbracket \rfloor, \Lambda(\llbracket \tilde{V}_2 \rrbracket) \lfloor\llbracket W \rrbracket \rfloor \rangle\rangle^{\dagger} \ddagger [case]) \\ &= \mathcal{H}^{\omega}(\langle\llbracket (\lambda x. x) W \rrbracket \lfloor \llbracket (\lambda x. V_1) W \rrbracket, \dots, \llbracket (\lambda x. V_k) W \rrbracket \rfloor, \langle \llbracket (\lambda x. \tilde{V}_1) W \rrbracket, \llbracket (\lambda x. \tilde{V}_2) W \rrbracket \rangle\rangle^{\dagger} \ddagger [case]) \\ &= \mathcal{H}^{\omega}(\langle\llbracket W \rrbracket \lfloor \llbracket nf(V_1 \llbracket V / x]) \rrbracket, \dots, \llbracket nf(V_k \llbracket V / x]) \rrbracket \rfloor, \langle \llbracket nf(\tilde{V}_1 \llbracket V / x]) \rrbracket, \llbracket nf(\tilde{V}_2 \llbracket V / x]) \rrbracket \rangle\rangle^{\dagger} \ddagger [case]) \\ &= \mathcal{H}^{\omega}(\langle \llbracket nf(W(V_1 \llbracket W / x]) \ldots, (V_k \llbracket W / x])) \rrbracket, \langle \llbracket nf(\tilde{V}_1 \llbracket W / x]) \rrbracket, \llbracket nf(\tilde{V}_2 \llbracket W / x]) \rrbracket \rangle\rangle^{\dagger} \ddagger [case]) \\ &= \mathcal{H}^{\omega}(\langle \llbracket nf(W(V_1 \llbracket W / x]) \ldots (V_k \llbracket W / x])) \rrbracket, \langle \llbracket nf(\tilde{V}_1 \llbracket W / x]) \rrbracket, \llbracket nf(\tilde{V}_2 \llbracket W / x]) \rrbracket \rangle\rangle^{\dagger} \ddagger [case]) \\ &= \llbracket U \rrbracket \end{aligned}$$

which completes the proof.

Corollary 4.1 (Dynamic game semantics of FPCF). The interpretation  $[-]_{\mathcal{LDG}}^{\mathcal{S}_{\mathcal{G}}}$  of FPCF and the hiding operation  $\mathcal{H}$  satisfy the DCP in the sense of Definition 2.5.

Proof. By Lemma 4.1, and Theorems 2.5, 4.1 and 4.2.

The relation between the syntax and the semantics of FPCF is actually tighter than Corollary 4.1: Exploiting the strong definability result (Amadio and Curien, 1998; Hyland and Ong, 2000), FPCF can be seen as a *formal calculus* for computations in the CCBoC  $\mathcal{LDG}$ . In addition, FPCF represents every computation in  $\mathcal{LDG}$  by the following full completeness result (Curien, 2007): Any strategy on a game that interprets a type of FPCF is the denotation of some term of FPCF:

Corollary 4.2 (Dynamic full completeness). Let G be a game such that for some strategy  $\sigma$  : G the pair  $(G, [\sigma]_W)$  is the interpretation  $\llbracket \Gamma \vdash M : B \rrbracket_{\mathcal{LDG}}^{\mathcal{S}_G}$  of a program  $\Gamma \vdash M : B$  of FPCF. Then, for any strategy  $\tilde{\sigma} : G$ , there is a program  $\Gamma \vdash \tilde{M} : B$  of FPCF such that  $\llbracket \Gamma \vdash \tilde{M} : B \rrbracket_{\mathcal{LDG}}^{\mathcal{S}_G} = (G, [\tilde{\sigma}]_W)$ .

*Proof.* Note that the game G is constructed along with the construction of type B of FPCF. We proceed by induction on the construction of G (or B).

First, since values of FPCF are PCF Böhm trees except that the natural number type  $\iota$  is replaced with the boolean type o, and the bottom term  $\perp$  is deleted, the conventional full completeness and the strong definability hold for values of FPCF in the same way as that of the conventional game semantics of PCF, where the winning condition on strategies excludes the denotation of the bottom term  $\perp$ ; see (Abramsky and McCusker, 1999; Curien, 2006) for the details.

It remains to consider the rule A for applications, i.e., the case where G is of the form  $\langle U, V \rangle^{\dagger} \ddagger \Upsilon$ . But then, note that only plays by the dereliction (up to 'tags') are possible in  $\Upsilon$  (Definition 3.39), and therefore we may just apply the induction hypothesis. 

# 5. Conclusion and Future Work

We have presented a *mathematical* (and *syntax-independent*) formulation of dynamics and intensionality of computation in terms of bicategories as well as games and strategies. From the opposite angle, we have developed bicategorical and game-semantic frameworks for dynamic, intensional computation with a convenient formal calculus.

Let us emphasize that the dynamic, intensional nature of our semantics stands in sharp contrast to the static, extensional nature of conventional (categorical or game) semantics. In particular, our semantics satisfies the highly non-trivial DCP with respect to FPCF.

Note also that the present work refines and generalizes standard categorical and game semantics of type theories. For instance, composition of static strategies is decomposed and generalized as *concatenation plus hiding* of dynamic strategies. Also, standard constructions and constraints on static games and strategies are naturally accommodated in the framework of dynamic games and strategies. Moreover, from the category-theoretic point, the present work refines the standard CCC-interpretation of type theories by the CCBoC-interpretation. In this sense, our approach is natural and general, achieving *mathematics* of dynamics and intensionality of computation as promised in Section 1.

Let us remark that our result does not contradict the standard result (Danos et al., 1996), i.e., the correspondence between the execution of linear head reduction (LHR) and the step-by-step 'internal communication' between conventional strategies. In fact, LHR is a finer reduction strategy than the operational semantics of FPCF (Definition 2.3), and the work by Danos et al. implies that LHR corresponds in conventional game semantics what should be called a 'move-wise' execution of the hiding operation. On the other hand, our operational semantics is executed in a much coarser, 'type-wise' fashion, and thus it may be seen as executing at a time a certain 'chunk' of LHR in a specific order. Our dynamic game semantics captures such a coarser dynamics of computation, and therefore it does not contradict the work (Danos et al., 1996). Of course, it is highly interesting to refine the present work to capture LHR or another, finer reduction strategy such as *explicit substitution* (Rose, 1996) and the *differential*  $\lambda$ -calculus (Ehrhard and Regnier, 2003), which we leave as future work.

More generally, the most immediate future work is to apply the framework of dynamic game semantics to various logics and computations as in the case of conventional game semantics. Also, it would be interesting to see how accurately our game-semantic approach can measure the computational complexity of (higher-order) programming.

Finally, the notion of (CC)BoCs can be a concept of interest in its own right. For instance, it might be fruitful to develop it further to accommodate various models of computations in the same spirit of (Longley and Normann, 2015) but on *computation*, not computability. Also, it might be interesting to consider their relation with *computations as monads* in the sense introduced by Eugenio Moggi (Moggi, 1991).

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