A CATEGORICAL VIEW OF VARIETIES OF ORDERED ALGEBRAS

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ABSTRACT. It is well known that classical varieties of Σ -algebras correspond bijectively to finitary monads on Set. We present an analogous result for varieties of ordered Σ algebras, i.e., categories presented by inequations between Σ -terms. We prove that they correspond bijectively to strongly finitary monads on Pos. That is, those finitary monads which preserve reflexive coinserters. We deduce that strongly finitary monads have a coinserter presentation, analogous to the coequaliser presentation of finitary monads due to Kelly and Power. We also show that these monads are liftings of finitary monads on Set.

> Dedicated to John Power, from whom we have learned so much, on the occasion of his 60th birthday

1. INTRODUCTION

Varieties of ordered algebras, i.e., classes of ordered Σ -algebras (for a finitary signature Σ) presented by inequations between Σ -terms, play an important role in universal algebra and computer science. Example: ordered monoids with bottom as the unit, e, are presented by the inequation $e \leq x$ and the usual equations for classical monoids. For every variety \mathcal{V} free algebras exist on all posets, that is, the forgetful functor $\mathcal{V} \to \mathsf{Pos}$ has a left adjoint. The corresponding monad \mathbf{T} on Pos will be proved to be *strongly finitary*, which means that its underlying endofunctor T preserves

(1) filtered colimits, and

(2) coinserters of reflexive pairs.

In the above example of ordered monoids **T** is a lifting of the word monad (of monoids) on Set. For every poset X we have the poset $TX = X^*$ with the following order: a word $x_0 \ldots x_{n-1}$ is smaller or equal to a word w iff w decomposes as $w = w_0 \ldots w_{n-1}$ and each w_i contains a letter $y_i \in X$ with $x_i \leq y_i$ in X.

Conversely, given a strongly finitary monad \mathbf{T} on Pos, its Eilenberg-Moore category $\mathsf{Pos}^{\mathbf{T}}$ will be proved to be isomorphic to a variety of ordered algebras. This leads to the following main result of our paper:

Theorem. The category of varieties of ordered algebras (with concrete functors as morphisms) is dually equivalent to the category of strongly finitary monads on Pos.

We thus obtain a bijective correspondence between varieties of ordered algebras and strongly finitary monads on Pos. This is analogous to the well-known correspondence between (classical) varieties and finitary monads on Set, up to natural isomorphism.

Moreover, every variety of ordered algebras is a lifting of a classical variety. This follows from the above bijective correspondence and the fact we prove that every strongly finitary monad \mathbf{T} on Pos is a lifting of a finitary monad $\widetilde{\mathbf{T}}$ on Set: for every poset X the underlying set of TX is $\widetilde{T}|X|$, and the underlying maps of η_X and μ_X are $\widetilde{\eta}_{|X|}$ and $\widetilde{\mu}_{|X|}$, respectively.

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Naturally, one classical variety can have many liftings, consider e.g. ordered monods (a 'minimal' lifting of the variety of monoids), compared with our example above.

Related results. The bijective correspondence between varieties of ordered algebras and strongly finitary monads has been established already by Kurz and Velebil [15]. However, the proof there was derived from technically involved results concerning the exactness (in **Pos**-enriched sense) of these varieties. Our present proof is much simpler.

Strongly finitary monads on enriched categories were studied by Kelly and Lack [12]. When specialised to Pos (enriched over itself as a cartesian closed category), their results yield a bijection between strongly finitary monads and equationally (!) presented classes of Σ -algebras. However, here Σ means a much more complex concept of signature, following the paper of Kelly and Power [13]: let Pos_f be a set of finite posets representing all of them up to isomorphism. The signatures in Pos introduced in [13] are collections $\Sigma = (\Sigma_n)_{n \in \mathsf{Pos}_f}$ of posets Σ_n . In the recent paper [1] finitary (ordinary as well as enriched) monads on Pos are studied. They are related to inequationally specified classes of Σ -algebras for signatures Σ that present a compromise between the classical signatures (used in the present paper) and those of Kelly and Power: they are collections of sets Σ_n indexed by $n \in \mathsf{Pos}_f$.

2. FINITARY AND STRONGLY FINITARY FUNCTORS

In the present section we recall finitary and strongly finitary endofunctors of Pos. We observe that a finitary endofunctor is strongly finitary iff it preserves reflexive coinserters.

Remark 2.1.

- (1) Throughout the paper we view Pos as the cartesian closed category with the homsets Pos(X, Y) ordered pointwise. All categories are understood to be enriched over Pos. That is, hom-sets carry partial orders such that composition is monotone. All functors, limits, colimits and adjunctions are understood as enriched over Pos. Thus when we say 'endofunctor H of Pos' we automatically mean that it is locally monotone. Its underlying ordinary functor is denoted by H_0 .
- (2) Colimits are understood to be weighted. Let us recall that for a given scheme, i.e., a small category \mathscr{D} , a weight is a functor $\varphi : \mathscr{D}^{op} \to \mathsf{Pos}$. Example: given a poset X and a diagram $D : \mathscr{D} \to \mathsf{Pos}$, the functor $\mathsf{Pos}(D-, X) : \mathscr{D}^{op} \to \mathsf{Pos}$ is a weight. The category of all weights is simply the functor category $[\mathscr{D}^{op}, \mathsf{Pos}]$.

A weighted colimit of a diagram $D : \mathscr{D} \to \mathsf{Pos}$ of weight φ is a poset $\varphi * D$ together with an isomorphism

$$\mathsf{Pos}(\varphi * D, X) \cong [\mathscr{D}^{op}, \mathsf{Pos}](\varphi, \mathsf{Pos}(D-, X))$$

natural in $X \in \mathsf{Pos}$.

(3) Every set is considered as a poset with the discrete order. In particular, every natural number n is the discrete poset on the set $\{0, 1, \ldots, n-1\}$.

Example 2.2. Coinserters are colimits of the scheme \mathscr{D} given by a parallel pair



The weight $\varphi : \mathscr{D}^{op} \to \mathsf{Pos}$ is as follows:



Thus, a diagram in **Pos** is a parallel pair

$$f_0, f_1: A \to B$$

of monotone maps (considered as an ordered pair (f_0, f_1) , of course). And the coinserter is a morphism $c: B \to C$ universal w.r.t. $c \cdot f_0 \leq c \cdot f_1$.



That is:

- (1) for every morphism $u: B \to D$ with $u \cdot f_0 \leq u \cdot f_1$ there exists a unique morphism $v: C \to D$ with $u = v \cdot c$, and
- (2) the map $u \mapsto v$ is monotone: given $u' = v' \cdot c$, then $u \leq u'$ implies $v \leq v'$.

Remark 2.3. Every finite poset *P* is a *canonical coinserter* of a parallel pair

$$k \underbrace{\overset{p_1}{\overbrace{\qquad p_0 \qquad }} n$$

of morphisms in \mathcal{N} . Let *n* be the number of elements of *P* and *k* the number of comparable pairs in *P*. Thus we can assume that *P* has elements $0, \ldots, n-1$, and we can index all comparable pairs as follows

$$p_0(t) \leq p_1(t)$$
 for $t = 0, \dots, k - 1$.

This defines functions $p_0, p_1 : k \to n$. The coinserter of this pair is carried by the identity map:

$$k \underbrace{\overset{p_1}{\underset{p_0}{\longrightarrow}}}_{p_0} n \xrightarrow{id} P$$

Notation 2.4. Denote by

$$J: \mathsf{Pos}_{\mathsf{f}} \to \mathsf{Pos}$$

the full embedding of a subcategory $\mathsf{Pos}_{\mathsf{f}}$ representing all finite posets up to isomorphism.

Remark 2.5.

(1) Pos is a free completion of $\mathsf{Pos}_{\mathsf{f}}$ under filtered conical colimits. In the realm of ordinary categories this follows from [3] (Theorem 1.46) since Pos is a locally finitely presentable category with finite posets precisely the finitely presentable objects. Thus, given an ordinary category \mathscr{K} with filtered colimits, for every ordinary functor $H : \mathsf{Pos}_{\mathsf{f}} \to \mathscr{K}$ there exists an extension $H' : \mathsf{Pos} \to \mathscr{K}$ preserving filtered colimits, unique up to natural isomorphism. Filtered conical colimits in Pos have the property that given a colimit cocone $c_i : C_i \to C$ $(i \in I)$, then for two morphisms $u, v : C \to X$ we have $u \leq v$ iff $u \cdot c_i \leq v \cdot c_i$ for all $i \in I$. It follows that H' is locally monotone whenever H is. Thus, the statement above holds also in the enriched sense.

(2) Following Kelly [11] we call an endofunctor of Pos *finitary* iff its underlying ordinary endofunctor is finitary (i.e., preserves ordinary filtered colimits).

We now turn to strongly finitary functors.

Notation 2.6. The full subcategory of Pos on natural numbers (Remark 2.1 (3)) is denoted by \mathcal{N} , and the full embedding by

$$I: \mathcal{N} \to \mathsf{Pos}.$$

Definition 2.7 ([12]). An endofunctor H of Pos is called *strongly finitary* if it is the left Kan extension of its restriction to \mathcal{N} . More precisely:

$$H = \operatorname{Lan}_{I} H \cdot I.$$

Remark 2.8. In ordinary categories *sifted colimits* are colimits of diagrams whose schemes \mathscr{D} are (small) sifted categories. This means categories such that colimits of diagrams $D: \mathscr{D} \to \mathsf{Set}$ commute with finite products.

In our enriched setting, sifted colimits are introduced analogously. A weight $\varphi : \mathscr{D}^{op} \to \mathsf{Pos}$ is called *sifted* if the functor $\varphi * - : [\mathscr{D}, \mathsf{Pos}] \to \mathsf{Pos}$ preserves finite (conical) products. Sifted colimits then are colimits weighted by sifted weights.

Example 2.9.

- (1) Filtered colimits are clearly sifted (the corresponding weighted colimits in Pos commute with finite limits).
- (2) A pair $f_0, f_1 : A \to B$ is called reflexive if there exists $i : B \to A$ with $f_0 \cdot i = id_B = f_1 \cdot i$. Coinserters of reflexive pairs are sifted colimits. The proof is completely analogous to the fact that in ordinary categories coequalisers of reflexive pairs are sifted colimits ([4], Example 1.2). We speak about *reflexive coinserters*. Example: the canonical coinserters (Remark 2.3) are clearly reflexive.

Theorem 2.10 ([8], Corollary 8.45). The following conditions are equivalent for endofunctors H of Pos:

- (1) H is strongly finitary,
- (2) H preserves sifted colimits,
- (3) H is finitary and preserves reflexive coinserters, and
- (4) $H = \operatorname{Lan}_I H \cdot I$.

Proof. Every poset is a filtered colimit of its finite subposets, each of which is a coinserter as in Remark 2.3.

Consequently, starting with the subcategory \mathscr{N} we obtain all of Pos by reflexive coinserters and filtered colimits. In the terminology of [10] (Theorem 5.29), this states that the embedding $I : \mathscr{N} \to \mathsf{Pos}$ has a codensity presentation formed by filtered colimits and reflexive coinserters. By that theorem properties (1)-(4) are equivalent.

Remark 2.11. In (3) we can substitute reflexive coinserters by canonical coinserters, as is clear from the above proof.

Remark 2.12. The above theorem is completely analogous to the fact proved in [4] for ordinary endofunctors of categories with finite coproducts: preservation of sifted colimits is equivalent to the preservation of filtered colimits and reflexive coequalisers.

Example 2.13.

- (1) The endofunctor $X \mapsto X^m$ (for $m \in \mathbb{N}$) of Pos is strongly finitary: it clearly preserves filtered colimits, and we verify that it also preserves the canonical coinserters of Remark 2.3. Suppose m = 2. Then a comparable pair in $P \times P$ is a pair (a, b) where the left-hand components of a and b are comparable in P, and thus have the form $x_{p_0(i)} \leq x_{p_1(i)}$ for some $i \leq k - 1$. And the right-hand components have the form $x_{p_0(j)} \leq x_{p_1(j)}$ for some $j \leq k - 1$. Thus the only comparable pairs of $P \times P$ are $(x_{p_0(i)}, x_{p_0(j)}), (x_{p_1(i)}, x_{p_1(j)})$. We conclude that the canonical coinserter of the poset $P \times P$ is given by $p_0 \times p_0, p_1 \times p_1 : k \times k \to n \times n$. Analogously for m > 2.
- (2) Coproducts of strongly finitary endofunctors are strongly finitary. Example: given a signature Σ , the corresponding polynomial functor $X \mapsto \coprod_{m \in \mathbb{N}} \Sigma_m \times X^m$ is strongly finitary.
- (3) (Weighted) colimits of strongly finitary endofunctors are strongly finitary.
- (4) A composite of strongly finitary endofunctors is strongly finitary.

Remark 2.14. Every strongly finitary endofunctor H of Pos generates a free monad whose underlying functor \hat{H} is also strongly finitary. Indeed, following [17], \hat{H} is a colimit in [Pos, Pos] of the following ω -chain

$$Id \xrightarrow{w_0} H + Id \xrightarrow{w_1} H(H + Id) + Id \xrightarrow{w_2} \dots$$

That is, the chain $W: \omega \to [\mathsf{Pos}, \mathsf{Pos}]$ has objects

 $W_0 = Id$ and $W_{n+1} = HW_n + Id$

and morphisms

 $w_0: Id \to H + Id$ the coproduct injection

and

$$w_{n+1} = Hw_n + id.$$

Thus if H is strongly finitary, so is each W_n (by the preceding example). Consequently, $\hat{H} = \operatorname{colim} W_n$ is strongly finitary.

Notation 2.15. A monad whose endofunctor is strongly finitary is called a *strongly finitary monad*. We denote by

 $Mnd_{sf}(Pos)$

the category of strongly finitary monads and monad morphisms.

Example 2.16. The endofunctor H_{Σ} generates the following free monad \mathbf{T}_{Σ} on Pos: to every poset X (of variables) it assigns the poset $T_{\Sigma}X$ of Σ -terms with variables from X. That is, the underlying set is the smallest set containing X and such that for every $\sigma \in \Sigma_n$ and every *n*-tuple t_i in $T_{\Sigma}X$ we have $\sigma(t_i)$ in $T_{\Sigma}X$. This yields a structure of a Σ -algebra on $T_{\Sigma}X$. The ordering of $T_{\Sigma}X$ is the smallest one such that $T_{\Sigma}X$ contains X as a subposet, and all operations are monotone. It follows from 2.14 that \mathbf{T}_{Σ} is strongly finitary.

Remark 2.17. It follows from Example 2.13 and Remark 2.14 that $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Pos})$ has (weighted) colimits. Indeed, given a diagram D and a weight, the underlying diagram D_0 in $\mathsf{End}_{\mathsf{sf}}(\mathsf{Pos})$ has a colimit H which is strongly finitary. The free monad \hat{H} is then a colimit of D in $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Pos})$, and it is strongly finitary.

3. FROM VARIETIES OF ORDERED ALGEBRAS TO STRONGLY FINITARY MONADS

Notation 3.1. Let Σ be a signature, i.e., a collection of sets Σ_n (of *n*-ary operation symbols) indexed by $n \in \mathbb{N}$. An ordered Σ -algebra is a poset A together with a monotone map $\sigma_A : A^n \to A$ for every $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$. The category of ordered Σ -algebras and homomorphisms (i.e., monotone functions preserving the given operations) is denoted by $\mathsf{Alg}(\Sigma)$.

Remark 3.2.

(1) With Σ we associate the polynomial functor $H_{\Sigma} : \mathsf{Pos} \to \mathsf{Pos}$ given on objects by

$$H_{\Sigma}X = \coprod_{n \in \mathbb{N}} \Sigma_n \times X^n$$

and analogously on morphisms. By Example 2.13 (2), H_{Σ} is strongly finitary.

- (2) $\operatorname{Alg}(\Sigma)$ is clearly equivalent to the category of algebras for H_{Σ} , i.e., pairs (A, α) where A is a poset and $\alpha : H_{\Sigma}A \to A$ is a monotone function. (Morphisms are monotone maps making the obvious square commutative.)
- (3) It follows from [5] that the category of algebras for an ordinary endofunctor H is equivalent to the category of Eilenberg-Moore algebras for the free monad \hat{H} (see Remark 2.14). The same result holds for enriched endofunctors. In particular, we conclude

$$\mathsf{Alg}(\Sigma) \simeq \mathsf{Pos}^{\mathbf{T}_{\Sigma}}.$$

(4) The algebra $T_{\Sigma}X$ of terms (Example 2.16) is a free Σ -algebra on $\eta_X : X \to T_{\Sigma}X$, the inclusion of variables: for every Σ -algebra A and every monotone function $f: X \to A$ the unique extension to a homomorphism $f^{\sharp}: T_{\Sigma}X \to A$ is given by

$$f^{\sharp}(\sigma(t_i)) = \sigma_A(f^{\sharp}(t_i)).$$

Definition 3.3. Let V be a countably infinite set (of variables), $V = \{x_n \mid n \in \mathbb{N}\}$. An ordered pair of terms in $T_{\Sigma}V$ is called an *inequation* and is written as $u \leq v$. A Σ -algebra A satisfies $u \leq v$ iff every map $f : V \to |A|$ (interpretation of variables) fulfills $f^{\sharp}(u) \leq f^{\sharp}(v)$.

By a variety of ordered Σ -algebras we understand a full subcategory of $\mathsf{Alg}(\Sigma)$ specified by a set of inequations.

Example 3.4.

(1) Ordered monoids are specified by the usual signature $\Sigma = \{\cdot, e\}$ and the usual equations for monoids. The corresponding algebras are monoids with a partial order making the multiplication monotone (in both variables).

This leads to the monad \mathbf{T} on Pos lifting the word monad on Set as follows:

$$TX = X^*$$

the poset of words on |X| ordered pointwise:

 $x_0 x_1 \dots x_{n-1} \leq y_0 y_1 \dots y_{m-1}$ iff n = m and $x_i \leq y_i$ (i < n).

(2) If we add to the equations above the inequation

 $x \leqslant x \cdot y$

we obtain the variety of ordered monoids with e the smallest element. That is, the above inequation is equivalent to

 $e \leq y$.

Indeed the first inequation yields the latter one by putting x = e. Conversely, from $e \leq y$ we get $x = x \cdot e \leq x \cdot y$.

The corresponding monad is the lifting of the word monad

 $TX = X^*$

ordered as follows:

 $x_0 x_1 \dots x_{n-1} \leq w$ iff $w = w_0 w_1 \dots w_{n-1}$ and w_i contains y_i with $x_i \leq y_i$ (i < n).

(3) Bounded posets (with a least element 0 and a largest element 1) form a variety with Σ given by nullary operations 0,1 and the variety is presented by the inequations

 $0 \leq x$ and $x \leq 1$.

This is a lifting of the variety of non-ordered algebras with two nullary operations.

Remark 3.5. Every variety of Σ -algebras is a reflective subcategory of $Alg(\Sigma)$ with surjective reflections.

Indeed, since H_{Σ} is a finitary endofunctor on a locally finitely presentable category, Alg(Σ) \cong H_{Σ} -Alg is also locally finitely presentable, see [3], Remark 2.78. In particular, it is complete and cowellpowered. The factorisation system (epi, embedding) on **Pos** lifts, since H_{Σ} preserves epimorphisms, to Alg(Σ). Since a variety \mathscr{V} is easily seen to be closed under products and subalgebras carried by embeddings, the surjective reflections follow, see [2], Theorem 16.8.

Construction 3.6 (see [7]). For every variety \mathcal{V} of ordered algebras the free algebra $T_{\mathcal{V}}X$ of \mathcal{V} on a poset X can be constructed as follows.

Let \mathcal{E}_X be the collection of all inequations $s \leq t$ satisfied by all algebras of \mathcal{V} , where $s, t \in T_{\Sigma}X$ are terms in variables from X. Then \mathcal{E}_X is a preorder, i.e., a reflexive and transitive relation on $T_{\Sigma}X$. Moreover, it is *admissible* in the sense of Bloom [7]: given an *n*-ary symbol $\sigma \in \Sigma$ and *n* pairs $s_i \leq t_i$ (i < n) in \mathcal{E}_X , it follows that the pair $\sigma(s_i) \leq \sigma(t_i)$ also lies in \mathcal{E}_X . Indeed, given an algebra $A \in \mathcal{V}$ and an interpretation $f : X \to |A|$, we know that the homomorphism $f^{\sharp}: T_{\Sigma}X \to A$ fulfils $f^{\sharp}(s_i) \leq f^{\sharp}(t_i)$ for all *i*, thus

$$f^{\sharp}(\sigma(s_i)) = \sigma_{T_{\mathcal{V}}X}(f^{\sharp}(s_i)) \leqslant \sigma_{T_{\mathcal{V}}X}(f^{\sharp}(t_i)) = f^{\sharp}(\sigma(t_i)).$$

Consequently, for the induced equivalence relation

$$\mathcal{E}_X^o = \mathcal{E}_X \cap \mathcal{E}_X^{-1}$$

we obtain a Σ -algebra $T_{\mathcal{V}}X$ on the quotient set

$$|T_{\mathcal{V}}X| = |T_{\Sigma}X|/\mathcal{E}_X^o$$

(of all equivalence classes [t] of terms $t \in T_{\Sigma}X$). The operations are as expected:

$$\sigma_{T_{\mathcal{V}}X}(\lfloor t_0 \rfloor, \ldots, \lfloor t_{n-1} \rfloor) = \lfloor \sigma(t_0, \ldots, t_{n-1}) \rfloor$$

for every *n*-ary σ and all *n*-tuples $t_0, \ldots, t_{n-1} \in T_{\Sigma}X$. Finally, we consider $T_{\mathcal{V}}X$ as a poset via

$$[s] \leq [t]$$
 iff $(s,t) \in \mathcal{E}_X$.

The following theorem was stated by Bloom ([7], Theorem 2.2). We present a full proof since we need it later, and the original proof was only a sketch.

Theorem 3.7. The above ordered algebra $T_{\mathcal{V}}X$ is a free algebra of the variety \mathcal{V} on the poset X w.r.t. $\eta_X : X \to T_{\mathcal{V}}X$ given by $x \mapsto [x]$.

Proof.

- (1) $T_{\mathcal{V}}X$ is a well-defined ordered Σ -algebra. This follows easily from the fact that \mathcal{E}_X is an admissible preorder.
- (2) \mathcal{V} has a free algebra on X which is given by an admissible preorder \sqsubseteq on $T_{\Sigma}X$ (that is, for the induced equivalence relation ~ the underlying poset is $|T_{\mathcal{V}}X| = |T_{\Sigma}X|/\sim$ and the operations are induced by those of $T_{\Sigma}X$). This statement follows from Remark 3.5, which implies that a free algebra $T_{\mathcal{V}}X$ exists, and the unique homomorphism

$$e_X: T_\Sigma X \to T_\mathcal{V} X$$

extending the universal arrow is epic. Indeed, the desired preorder is simply

$$s \sqsubseteq t \text{ iff } e_X(s) \leqslant e_X(t)$$

(3) The preorder \mathcal{E}_X of the above construction coincides with \sqsubseteq of (2). Indeed, if $(s,t) \in \mathcal{E}_X$, then the algebra $T_{\mathcal{V}}X$ satisfies $s \leq t$ (since it lies in \mathcal{V}) and taking the universal map $(\eta_{\mathcal{V}})_X : X \to T_{\mathcal{V}}X$ as the interpretation, we have

$$e_X = (\eta_{\mathcal{V}})_X^{\sharp}$$

(because e_X is a Σ -homomorphism). Since $e_X(s) \leq e_X(t)$, we conclude that $s \equiv t$. Conversely, if $s \equiv t$, which means $e_X(s) \leq e_X(t)$, we verify that every algebra $A \in \mathcal{V}$ satisfies $s \leq t$. Let $f: X \to A$ be an interpretation, then the corresponding homomorphism $f^{\sharp}: T_{\Sigma}X \to A$ factorises through the reflection of $T_{\Sigma}X$ in \mathcal{V} in $\mathsf{Alg}(\Sigma)$:



Since h is monotone, the inequality $e_X(s) \leq e_X(t)$ implies $f^{\sharp}(s) \leq f^{\sharp}(t)$, as required.

Notation 3.8. For every variety \mathcal{V} of Σ -algebras we denote by

$$c_{\mathcal{V}}:\mathbf{T}_{\Sigma}\to\mathbf{T}_{\mathcal{V}}$$

the monad morphism whose components are the canonical quotient maps

$$|T_{\Sigma}X| \to |T_{\Sigma}X|/\mathcal{E}_X^o.$$

Lemma 3.9. For every variety \mathcal{V} the forgetful functor to Pos is strictly monadic: the comparison functor $K : \mathcal{V} \to \mathsf{Pos}^{\mathbf{T}_{\mathcal{V}}}$ is an isomorphism.

Proof. For classical varieties see [16], Theorem VI.8.1. The proof for varieties of ordered algebras is completely analogous, one just replaces the equation $\lambda_B = \mu_B$ with the inequation $\lambda_B \leq \mu B$.

Remark 3.10 (See [10]).

(1) Recall the *continuation monad* $\langle A, A \rangle$ on Pos associated with every poset A: to a poset X it assigns the power of A to the set $\mathsf{Pos}_0(X, A)$ of all monotone maps $f: X \to A$:

$$\langle A,A\rangle X=\prod_{\mathsf{Pos}_0(X,A)}A.$$

Denote by $\pi_f : \langle A, A \rangle X \to A$ the projection corresponding to $f : X \to A$. To every morphism $h : X \to Y$ the monad assigns the morphism $\langle A, A \rangle h$ determined by the following commutative triangles:



The unit is $\langle f \rangle_{f \in \mathsf{Pos}_0(X,A)} : X \to \langle A, A \rangle X$, and the multiplication μ_X is determined by the following commutative triangles:



(2) It follows from [9] that for every monad **T** and every poset A there is a bijection between monad morphisms $\mathbf{T} \to \langle A, A \rangle$ and algebras of $\mathsf{Pos}^{\mathbf{T}}$ on A. This bijection assigns to an algebra $\alpha : TA \to A$ the monad morphism

$$\hat{\alpha}: \mathbf{T} \to \langle A, A \rangle$$

with components determined by the following commutative squares:

$$\begin{array}{ccc} TX & \xrightarrow{\hat{\alpha}_X} & \langle A, A \rangle X \\ T_f & & & & \\ T_f & & & & \\ TA & \xrightarrow{\alpha} & A \end{array} \qquad f \in \mathsf{Pos}_0(X, A).$$

Thus if $\mathbf{T} = \mathbf{T}_{\Sigma}$, then $\hat{\alpha}_X$ assigns to a term $t \in T_{\Sigma}X$ the tuple $(f^{\sharp}(t))_{f:X \to A}$.

(3) Let $b : \mathbf{S} \to \mathbf{T}$ be a monad morphism. Every algebra (A, α) in $\mathsf{Pos}^{\mathbf{T}}$ then yields an algebra $(A, \alpha \cdot b_A)$ in $\mathsf{Pos}^{\mathbf{S}}$. The following triangle



commutes. Indeed, for every poset X and every $f \in \mathsf{Pos}_0(X, A)$ we have $\pi_f(\widehat{\alpha}_X \cdot b_X) = \alpha \cdot Tf \cdot b_X = \alpha \cdot b_A \cdot Sf.$

The same result is obtained by

$$\pi_f(\alpha \cdot b_{AX}) = \alpha \cdot b_A \cdot Sf.$$

(4) In particular, let $\mathbf{T} = \mathbf{T}_{\Sigma}$ for a signature Σ . Given a term u in $T_{\Sigma}n$, it corresponds to a monad morphism

$$\widetilde{u}: \mathbf{T}_{\Omega_n} \to \mathbf{T}_{\Sigma}$$

where Ω_n is a signature of a single operation ω of arity n. Its component $\widetilde{u}_X : T_{\Omega_n}X \to T_{\Sigma}X$ assigns to a term t over X (containing the unique operation symbol ω) the Σ -term obtained by replacing each ω by the term u. Thus if a Σ -algebra (A, α) satisfies an inequation $u_0 \leq u_1$, the inequation $(\widehat{\alpha} \cdot \widetilde{u}_0)_X \leq (\widehat{\alpha} \cdot \widetilde{u}_1)_X$ holds for all posets X. Shortly: $\widehat{\alpha} \cdot \widetilde{u}_0 \leq \widehat{\alpha} \cdot \widetilde{u}_1$.

Example 3.11. We describe the free-algebra monad of the variety given by a single inequation $u_0 \leq u_1$ in signature Σ . Let u_0, u_1 be terms with variables x_0, \ldots, x_{n-1} . For the signature Ω_n of a single operation of arity n they can be viewed (via Yoneda lemma) as natural transformations

$$u_0, u_1: H_{\Omega_n} \to T_{\Sigma}.$$

The corresponding monad morphisms

$$\widetilde{u}_0, \widetilde{u}_1 : \mathbf{T}_{\Omega_n} \to \mathbf{T}_{\Sigma}.$$

have, in the category of strongly finitary monads, a coinserter we denote as follows:

$$\mathbf{T}_{\Omega_n} \underbrace{\overset{\widetilde{u}_1}{\overbrace{}}}_{\widetilde{u}_0} \mathbf{T}_{\Sigma} \overset{c}{\longrightarrow} \mathbf{T}$$

We verify that this is precisely $c_{\mathcal{V}}$ above for the variety presented by $u_0 \leq u_1$.

Proposition 3.12. The above monad **T** is the free-algebra monad of the variety presented by the inequation $u_0 \leq u_1$.

Proof. The variety \mathcal{V} presented by $u_0 \leq u_1$ yields a free-algebra monad $\mathbf{T}_{\mathcal{V}}$. The proposition will be proved by verifying that $c_{\mathcal{V}}$ (Notation 3.8) is a coinserter of \tilde{u}_0, \tilde{u}_1 in $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Pos})$. From the definition of $c_{\mathcal{V}}$ we conclude

$$c_{\mathcal{V}} \cdot \widetilde{u}_0 \leqslant c_{\mathcal{V}} \cdot \widetilde{u}_1$$

(a) Given a strongly finitary monad $\mathbf{S} = (S, \mu^S, \eta^S)$ and a monad morphism $b : \mathbf{T}_{\Sigma} \to \mathbf{S}$ with

$$b \cdot \widetilde{u}_0 \leqslant b \cdot \widetilde{u}_1$$

we prove that b factorises through $c_{\mathcal{V}}$ via a monad morphism.

For every poset X, the free algebra (SX, μ_X^S) for **S** yields, since b is a monad morphism, the following algebra for \mathbf{T}_{Σ} on SX:

$$\beta_X := T_\Sigma SX \xrightarrow{b_{SX}} SSX \xrightarrow{\mu_X^S} SX$$

From $\alpha_X \cdot (\widetilde{u}_0)_X \leq \alpha_X \cdot (\widetilde{u}_1)_X$ we deduce, using Remark 3.10 (4), that the Σ -algebra (SX, β_X) satisfies the inequality $u_0 \leq u_1$. Since the free algebra (TX, μ_X^T) of \mathcal{V} on X corresponds to the Σ -algebra

$$T_{\Sigma}TX \xrightarrow{(c_{\mathcal{V}})_{TX}} TTX \xrightarrow{\mu_{X}^{T}} TX,$$

we obtain a unique Σ -homomorphism \overline{b}_X with $\overline{b}_X \cdot \eta_X^T = \eta_X^S$:



We verify that these morphisms \overline{b}_X form a monad morphism

$$\overline{b}: \mathbf{T} \to \mathbf{S}$$
 with $b = \overline{b} \cdot c_{\mathcal{V}}$.

(1) The equality $b_X = \overline{b}_X \cdot (c_{\mathcal{V}})_X : T_{\Sigma}X \to SX$ holds because both sides are homomorphisms of Σ -algebras and we have

$$b_X \cdot \eta_X^{\Sigma} = \eta_X^S = \overline{b}_X \cdot \eta_X^T = \overline{b}_X \cdot (c_{\mathcal{V}})_X \cdot \eta_X^{\Sigma}$$

(2) \overline{b}_X is natural in X. In fact, every morphism $f: X \to Y$ yields a Σ -homomorphism

$$Tf: (TX, \mu_X^T \cdot (c_{\mathcal{V}})_{TX}) \to (TY, \mu_Y^T \cdot (c_{\mathcal{V}})_{TY})$$

Thus, $\overline{b}_Y \cdot Tf$ is also a Σ -homomorphism, and so is $Sf \cdot \overline{b}_X : (TX, \mu_X^T \cdot (c_{\mathcal{V}})_{TX}) \to (SY, \alpha_Y)$. Since the domain of both composites is a free algebra of \mathcal{V} on X, for proving that they are equal we just need to verify

$$\overline{b}_Y \cdot Tf \cdot \eta_X^T = Sf \cdot \overline{b}_X \cdot \eta_X^T.$$

See the following diagram:



(3) The equality

$$\overline{b}\cdot\eta^T=\eta^S$$

follows from the right-hand triangle in the diagram defining \overline{b}_X above. (4) We finally prove

$$\overline{b} \cdot \mu^T = \mu^S \cdot S\overline{b} \cdot \overline{b}T.$$

Consider the following diagram



The outward rectangle is the definition of \overline{b}_X . The left-hand parts commute by (1) and (2). Consequently, the desired right-hand square commutes since it does when precomposed by the epimorphism $(c_{\mathcal{V}})_{TX}$.

(b) Finally for every monad morphism $b' : \mathbf{T}_{\Sigma} \to \mathbf{S}$ factorised as $b' = \overline{b'} \cdot c_{\mathcal{V}}$ we are to verify that

$$b \leq b'$$
 implies $\overline{b} \leq \overline{b'}$.

This is trivial since the components of $c_{\mathcal{V}}$ are surjective.

Construction 3.13. The above proposition immediately generalises to sets of inequations. For every variety \mathcal{V} of Σ -algebras the free-algebra monad $\mathbf{T}_{\mathcal{V}}$ is a canonical quotient $c_{\mathcal{V}}: \mathbf{T}_{\Sigma} \to \mathbf{T}_{\mathcal{V}}$ of the free- Σ -algebra monad, see Notation 3.8. We construct monad morphisms $\tilde{u}_0, \tilde{u}_1 : \mathbf{T}_\Omega \to \mathbf{T}_\Sigma$ for some signature Ω forming a coinserter in $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Pos})$ as follows:

$$\mathbf{T}_{\Omega} \underbrace{\overset{\widetilde{u}_1}{\overbrace{\underset{\widetilde{u}_0}{\widetilde{u}_0}}}}_{\widetilde{u}_0} \mathbf{T}_{\Sigma} \xrightarrow{c_{\mathcal{V}}} \mathbf{T}_{\mathcal{V}}$$

Given a collection

$$u_0^i \leqslant u_1^i, \qquad i \in I$$

of inequations specifying the variety \mathcal{V} , let n_i be the number of variables on both sides. We define a signature $\Omega = \{\gamma_i\}_{i \in I}$, where γ_i has arity n_i . By Yoneda lemma we obtain natural transformations $u_0, u_1 : H_\Omega \to T_\Sigma$, since we have $H_\Omega \cong \coprod_{i \text{ in } I} \mathsf{Pos}(n_i, -)$. Let $\widetilde{u}_0, \widetilde{u}_1 : \mathbf{T}_\Omega \to \mathbf{T}_\Sigma$ be the corresponding monad morphisms. In the category $\mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Pos})$ we form a coinserter



Proposition 3.14. For every variety \mathcal{V} of ordered algebras the above monad \mathbf{T} is the corresponding free-algebra monad $\mathbf{T}_{\mathcal{V}}$.

The proof is completely analogous to Proposition 3.12.

Corollary 3.15. The free-algebra monad $\mathbf{T}_{\mathcal{V}}$ of a variety of ordered algebras is strongly finitary. It follows from the above proposition that we have a coinserter



in [Pos, Pos]. Hence, $\mathbf{T}_{\mathcal{V}}$ is strongly finitary by Examples 2.13 (3) and 2.16.

Example 3.16. A finitary monad on Pos need not be strongly finitary. (In contrast, every finitary monad on Set is strongly finitary in the sense of preserving reflexive coequalisers, see [14].)

Denote by \mathcal{V} the category of partial algebras (A, α) where A is a poset and α a monotone function assigning to every pair $a_0 \leq a_1$ in A an element of A. Morphisms to (B, β) are monotone functions $h: A \to B$ such that

$$h\alpha(a_0, a_1) = \beta(h(a_0), h(a_1))$$

holds for all $a_0 \leq a_1$. This is a 'variety in context' as introduced in [1], from which it follows that the forgetful functor $U : \mathcal{V} \to \mathsf{Pos}$ is finitary monadic, see Theorem 3.24 in op. cit. The corresponding monad **T** assigns to a poset X the poset TX defined by induction as follows:

- (1) elements of X are terms; they are ordered as in X, and
- (2) given terms $u_0 \leq u_1$, then $\alpha(u_0, u_1)$ is a term and the ordering is pointwise: for terms $v_0 \leq v_1$ we have $\alpha(u_0, u_1) \leq \alpha(v_0, v_1)$ iff $u_i \leq v_i$ for i = 0, 1.

This monad is not strongly finitary because for the 2-chain P given by $x_0 \leq x_1$ it does not preserve its canonical reflexive coinserter (recall Remark 2.3):



Indeed, every coinserter is surjective, whereas Tc is not: the element $\alpha(x_0, x_1)$ of TP does not lie in the image of Tc.

4. FROM STRONGLY FINITARY MONADS TO VARIETIES

We now prove that the results of Section 3 can be reversed: for every strongly finitary monad \mathbf{T} a variety is presented with \mathbf{T} as the free-algebra monad.

Recall that given a monad **T** every morphism $f : X \to TY$ yields a homomorphism $f^* : (TX, \mu_X) \to (TY, \mu_Y)$ by $f^* = \mu_Y \cdot Tf$. Below we associate with every *n*-ary operation symbol σ the term $\sigma(x_i)_{i < n}$ over V (see Definition 3.3).

Definition 4.1. For every monad **T** on Pos the associated variety $\mathcal{V}_{\mathbf{T}}$ has the signature Σ whose *n*-ary symbols are the elements of $Tn \ (n \in \mathbb{N})$. The variety is presented by inequations as follows (with *n* and *m* ranging over \mathbb{N}):

- (1) $\sigma(x_i) \leq \tau(x_i)$ for all $\sigma \leq \tau$ in Tn;
- (2) $k^*(\sigma)(x_i) = \sigma(k_0(x_i), \dots, k_{m-1}(x_i))$ for all *m*-tuples $k : m \to Tn, k = (k_0, \dots, k_{m-1})$ and all $\sigma \in Tm$.

Example 4.2. Every algebra $\alpha : TA \to A$ in $\mathsf{Pos}^{\mathsf{T}}$ yields a Σ -algebra in \mathcal{V}_{T} : given an *n*-ary symbol $\sigma \in Tn$ and an *n*-tuple $f : n \to A$, let $f^+ = \alpha \cdot Tf : (Tn, \mu_n) \to (A, \alpha)$ be the corresponding homomorphism for T . We put

$$\sigma_A(f) = f^+(\sigma).$$

To verify that this Σ -algebra satisfies (1) in Definition 4.1, observe that for every *n*-tuple $f: n \to A$ the corresponding Σ -homomorphism $f^{\sharp}: T_{\Sigma}V \to A$ fulfills

(3)
$$f^{\sharp}(\sigma(x_i)) = f^+(\sigma) \text{ for all } \sigma \in Tn.$$

This equality holds since $\sigma(x_i)$ is the result of the operation σ in the algebra $T_{\Sigma}n$ (Example 2.16) on (x_i) , thus, $f^{\sharp}(\sigma(x_i)) = \sigma_A(f(x_i))$. Given $\sigma \leq \tau$ in Tn, then $f^+(\sigma) \leq f^+(\tau)$ since $f^+ = \alpha \cdot Tf$ is monotone, thus $f^{\sharp}(\sigma(x_i)) \leq f^{\sharp}(\tau(x_i))$ holds.

To verify (2), we need to prove

$$f^{\sharp}(k^*(\sigma)(x_i)) = f^{\sharp}(\sigma(k_0(x_i),\ldots,k_{m-1}(x_i)))$$

for every *n*-tuple $f: n \to A$. Due to (3) above, the left-hand side is

$$f^+(k^*(\sigma)).$$

Since f^{\sharp} is a homomorphism, the right-hand side is

$$\sigma_A(f^{\sharp}(k_0(x_i)),\ldots,f^{\sharp}(k_{m-1}(x_i)))$$

which due to (3) is equal to

$$\sigma_A(f^+ \cdot k) = (f^+ \cdot k)^+(\sigma)$$

Thus we only need to observe that

(4)
$$f^+ \cdot k^* = (f^+ \cdot k)^+ : (Tn, \mu_n) \to (A, \alpha).$$

Indeed, both sides are homomorphisms in $\mathsf{Pos}^{\mathbf{T}}$, and they are equal when precomposed with the universal map:

$$f^+ \cdot k^* \cdot \eta_n = f^+ \cdot k = (f^+ \cdot k)^+ \cdot \eta_n.$$

Remark 4.3. We can thus consider $\mathsf{Pos}^{\mathsf{T}}$ as a full subcategory of \mathcal{V}_{T} . Indeed, given two algebras (A, α) and (B, β) in $\mathsf{Pos}^{\mathsf{T}}$, then a monotone map $h : A \to B$ is a homomorphism in $\mathsf{Pos}^{\mathsf{T}}$ iff it is a Σ -homomorphism:

(1) Let $h \cdot \alpha = \beta \cdot Th$. Then

$$h \cdot f^+ = (h \cdot f)^+ : (Tn, \mu_n) \to (A, \alpha)$$

because both sides are homomorphisms of $\mathsf{Pos}^{\mathbf{T}}$ extending $h \cdot f$. For every $\sigma \in \Sigma_n$ and every *n*-tuple $f : n \to A$ we have

$$h(\sigma_A(f)) = h \cdot f^+(\sigma), \text{ by definition of } \sigma_A$$
$$= (h \cdot f)^+, \text{ as } h \cdot f^+ = (h \cdot f)^+$$
$$= \sigma_B(h \cdot f), \text{ by definition of } \sigma_B.$$

Thus h is a Σ -homomorphism.

(2) Let h be a Σ -homomorphism. To prove that h is a homomorphism of T-algebras, consider the diagram below for an arbitrary $n \in \mathbb{N}$ and $f: n \to A$. (Recall that n

is the discrete poset on $\{0, \ldots, n-1\}$.) Since **T** is finitary, it is sufficient to show that the desired square commutes when precomposed by Tf.

$$Tn \xrightarrow{Tf} TA \xrightarrow{\alpha} A$$

$$Th \qquad \qquad \downarrow \qquad \qquad \downarrow h$$

$$TB \xrightarrow{\beta} B$$

Indeed, given $\sigma \in Tn$ we have

$$\beta \cdot Th \cdot Tf(\sigma) = (h \cdot f)^+(\sigma), \text{ by definition of } (h \cdot f)^+$$
$$= \sigma_B(h \cdot f), \text{ by definition of } \sigma_B$$
$$= h(\sigma_A(f)), \text{ since } h \text{ is a } \Sigma\text{-homomorphism}$$
$$= h(f^+(\sigma)), \text{ by definition of } \sigma_A$$
$$= h(\alpha \cdot Tf(\sigma)), \text{ by definition of } f^+.$$

Theorem 4.4. Every strongly finitary monad on Pos is the free-algebra monad of the associated variety $\mathcal{V}_{\mathbf{T}}$.

Proof.

(1) For every poset X we prove that the free algebra (TX, μ_X) on X in $\mathsf{Pos}^{\mathbf{T}}$, considered as a Σ -algebra, is free on X in $\mathcal{V}_{\mathbf{T}}$ w.r.t. η_X as the universal map.

To verify this, we can restrict ourselves to finite posets X. Then it holds for all posets since T preserves filtered colimits: express $X = \operatorname{colim} X_i$ as a filtered colimit of finite posets, then $TX = \operatorname{colim} TX_i$, and from Remark 4.3 we conclude that the Σ -algebra TX is a filtered colimit of TX_i ($i \in I$) in $\operatorname{Alg}(\Sigma)$. Thus from TX_i being a free Σ -algebra on X_i in $\mathcal{V}_{\mathbf{T}}$ we conclude that TX is a free Σ -algebra on X.

Let P be a finite poset, say, on the set $\{x_0, \ldots, x_{n-1}\}$. Then its canonical coinserter (Remark 2.3) yields, since T is strongly finitary, the following coinserter

$$Tk \underbrace{\xrightarrow{Tp_1}}_{Tp_0} Tn \xrightarrow{id} TP$$

The free algebras Tk and Tn of $\mathsf{Pos}^{\mathbf{T}}$ are also free Σ -algebras in $\mathcal{V}_{\mathbf{T}}$: see Remark 4.3. Given an algebra A of $\mathcal{V}_{\mathbf{T}}$ and a monotone function $f: P \to A$, we thus have a unique Σ -homomorphism $f': Tn \to A$ with $f = f' \cdot \eta_n$. To prove that f' is also a Σ -homomorphism $f': TP \to A$, it is sufficient to verify

$$f' \cdot Tp_0 \leqslant f' \cdot Tp_1 : Tk \to A.$$



Thus we need to prove that for each $x \in Tk$ we have $f'(Tp_0(x)) \leq f'(Tp_1(x))$. Indeed, this holds for all variables $y_i \in k$:

> $f' \cdot Tp_0(\eta_k(y_i)) = f(p_0(y_i)), \text{ by the diagram above}$ $\leq f(p_1(y_i)), \text{ by } f \text{ being monotone}$ $= f' \cdot Tp_1(\eta_k(y_i)), \text{ by the diagram above.}$

And thus we only need to observe that the set of all $x \in Tk$ with the desired property is closed under the Σ -operations. For every $\sigma \in \Sigma_n$ and every n-tuple $(x_i)_{i < n}$ with $f' \cdot Tp_0(\sigma_{Tk}(x_i)) \leq f' \cdot Tp_1(x_i)$ we have (since Tp_i are homomorphisms of $\mathsf{Pos}^{\mathbf{T}}$)

- $f' \cdot Tp_0(\sigma_{Tk}(x_i)) = f'(\sigma_{Tn}(Tp_0(x_i))), \text{ by Remark 4.3}$ $= \sigma_A(f'(Tp_0(x_i))), \text{ since } f' \text{ is a } \Sigma\text{-homomorphism}$ $\leqslant \sigma_A(f'(Tp_1(x_i))), \text{ since } \sigma_A \text{ is monotone}$ $= f' \cdot Tp_1(\sigma_{Tk}(x_i)) \text{ as above.}$
- (2) The full embedding $E : \mathsf{Pos}^{\mathbf{T}} \to \mathcal{V}_{\mathbf{T}}$ of Remark 4.3 is concrete. That is, if $U : \mathsf{Pos}^{\mathbf{T}} \to \mathsf{Pos}$ and $V : \mathcal{V}_{\mathbf{T}} \to \mathsf{Pos}$ denote the forgetful functors, the triangle



commutes. Both U and V are monadic functors by Lemma 3.9. It follows from (1) that the corresponding monads are isomorphic.

Notation 4.5. Let Var(Pos) denote the category of varieties of ordered algebras and concrete functors. These are functors $F : \mathcal{V}_1 \to \mathcal{V}_2$ which commute (strictly) with the forgetful functors $U_i : \mathcal{V}_i \to Pos$:



Theorem 4.6. The category of varieties is dually equivalent to the category of strongly finitary monads:

$$\mathsf{Var}(\mathsf{Pos}) \cong \mathsf{Mnd}_{\mathsf{sf}}(\mathsf{Pos})^{op}$$

Proof.

(1) Let $F : \mathcal{V}_1 \to \mathcal{V}_2$ be a concrete functor. The comparison functors $K_i : \mathcal{V}_i \to \mathsf{Pos}^{\mathbf{T}_{\mathcal{V}_i}}$ are isomorphisms of categories by Lemma 3.9. These isomorphisms are concrete: if $U'_i : \mathsf{Pos}^{\mathbf{T}_{\mathcal{V}_i}} \to \mathsf{Pos}$ denotes the underlying functor, then $U_i = U'_i \cdot K_i$. From Fwe thus obtain a concrete functor



The passage $F \mapsto \overline{F}$ is bijective (with the inverse passage $K_2^{-1} \cdot (-) \cdot K_1$) and preserves composition and identity morphisms.

(2) Given monads \mathbf{T}_1 , \mathbf{T}_2 , monad morphisms $\rho : \mathbf{T}_2 \to \mathbf{T}_1$ bijectively correspond to concrete functors from $\mathsf{Pos}^{\mathbf{T}_1}$ to $\mathsf{Pos}^{\mathbf{T}_2}$: the bijection takes ρ to $H_{\rho} : \mathsf{Pos}^{\mathbf{T}_1} \to \mathsf{Pos}^{\mathbf{T}_2}$ assigning to an algebra $\alpha : T_1 A \to A$ in $\mathsf{Pos}^{\mathbf{T}_1}$ the algebra

$$T_2A \xrightarrow{\rho_A} T_1A \xrightarrow{\alpha} A$$

in $\mathsf{Pos}^{\mathbf{T}_2}$. This passage $\rho \mapsto H_{\rho}$ moreover preserves composition and indentity morphisms. See [6], Theorem 3.6.3.

(3) Define a functor

$$R: Var(Pos) \rightarrow Mnd_{sf}(Pos)^{op}$$

on objects by

$$R(\mathcal{V}) = \mathbf{T}_{\mathcal{V}}$$

and on morphisms $F: \mathcal{V}_1 \to \mathcal{V}_2$ by the following rule

$$R(F) = \rho \text{ iff } H_{\rho} = \overline{F}.$$

It follows from (1) and (2) that R is a well-defined full and faithful functor. Theorem 4.4 tells us that every strongly finitary monad is isomorphic to $R(\mathcal{V})$ for some variety \mathcal{V} . Therefore, R is an equivalence of categories.

5. LIFTING FINITARY MONADS FROM Set TO Pos

The examples of varieties of ordered algebras presented so far are all liftings of varieties of classical algebras (over Set). In the present section we prove that this is no coincidence: there are no other examples. Since varieties of ordered algebras are in a bijective correspondence with strongly finitary monads on Pos (and varieties of classical algebras are in

a bijective correspondence with finitary monads on Set), an equivalent statement is the following theorem.

Theorem 5.1. Every strongly finitary monad (T, μ, η) on Pos is a lifting of a finitary monad $(\tilde{T}, \tilde{\mu}, \tilde{\eta})$ on Set: for every poset X the underlying set of TX is $\tilde{T}|X|$, and the underlying maps of μ_X and η_X are $\tilde{\mu}_X$ and $\tilde{\eta}_X$, resp.

Before proving this theorem, we explain why we have decided for the above strict variant of lifting.

Remark 5.2.

(1) There is a less strict concept of a lifting of an ordinary monad $\widetilde{\mathbf{T}}$ on Set: denote by $U : \mathsf{Pos} \to \mathsf{Set}$ the forgetful functor. A monad \mathbf{T} on Pos is a *non-strict lifting* of $\widetilde{\mathbf{T}}$ iff there is a natural isomorphism φ



such that the following diagrams commute:



(2) Given φ as above, **T** is isomorphic to a monad \mathbf{T}_0 on Pos which is a strict lifting of $\widetilde{\mathbf{T}}$ (i.e., for which the conditions in the above theorem hold). Indeed, define $\mathbf{T}_0 = (T_0, \mu_0, \eta_0)$ by letting $T_0 X$ be the unique poset on the set $\widetilde{T}|X|$ for which φ_X carries an isomorphism $TX \cong T_0 X$ in Pos. Analogously define T_0 on morphisms $f: X \to Y$: the underlying map of $T_0 f$ is such that the square



commutes. The unit of \mathbf{T}_0 has components $\varphi_X \cdot \eta_X : X \to T_0 X$ and the multiplication $(\mu_0)_X : T_0 T_0 X \to T_0 X$ is the unique monotone map for which the square



commutes. It is easy to see that $\mathbf{T}_0 = (T_0, \eta_0, \mu_0)$ is a well-defined monad on Pos isomorphic to \mathbf{T} via φ , and that it is a strict lifting of $\widetilde{\mathbf{T}}$.

Proof of Theorem 5.1. In view of Theorem 4.6 it is sufficient to present, for every variety \mathcal{V} of ordered Σ -algebras, a variety $\widetilde{\mathcal{V}}$ of non-ordered algebras such that $\mathbf{T}_{\mathcal{V}}$ is a lifting of $\mathbf{T}_{\widetilde{\mathcal{V}}}$. Here $\mathbf{T}_{\mathcal{V}}$ is the ordinary \mathcal{V} -free-algebra monad on Pos, and $\mathbf{T}_{\widetilde{\mathcal{V}}}$ the $\widetilde{\mathcal{V}}$ -free-algebra (ordinary) monad on Set. Recall that we consider an arbitrary set as the poset with the trivial order.

- (1) For our standard set $V = \{x_0, x_1, x_2, ...\}$ of variables in Definition 3.3 we have defined a set $\mathcal{E}_V^o = \mathcal{E}_V \cap \mathcal{E}_V^{-1}$ of equations in Construction 3.6: they are those equations s = t which every ordered algebra in \mathcal{V} satisfies. (Since this is equivalent to satisfying both $s \leq t$ and $t \leq s$.) We denote by $\widetilde{\mathcal{V}}$ the variety of non-ordered algebras presented by \mathcal{E}_V^o . This clearly implies that every algebra in $\widetilde{\mathcal{V}}$ satisfies, for every set X, all equations s = t for pairs in \mathcal{E}_X^o . Moreover, \mathcal{E}_X^o is clearly a congruence on the non-ordered Σ -algebra $\widetilde{T}_{\Sigma}X$ of all Σ -terms on X.
- (2) Denote by $T_{\widetilde{\mathcal{V}}}X$ the free algebra of $\widetilde{\mathcal{V}}$ on the set X. It can be constructed as the quotient of the non-ordered algebra $\widetilde{T}_{\Sigma}X$ modulo the congruence \mathcal{E}_X^o :

$$T_{\widetilde{\mathcal{V}}}X = \widetilde{T}_{\Sigma}X/\mathcal{E}_X^o.$$

The proof is completely analogous to that of Theorem 3.7. We thus conclude that for an arbitrary poset X our choice of $T_{\mathcal{V}}X$ and $T_{\tilde{\mathcal{V}}}|X|$ can be such that the underlying set of $T_{\mathcal{V}}X$ is $T_{\tilde{\mathcal{V}}}|X|$ and all operations are equal. The universal arrows $(\eta_{\mathcal{V}})_X : X \to |T_{\mathcal{V}}X|$ and $(\eta_{\tilde{\mathcal{V}}})_{|X|} : |X| \to T_{\tilde{\mathcal{V}}}|X|$ are both given by forming the equivalence classes of $x \in X$ modulo \mathcal{E}_X^o , thus $\eta_{\tilde{\mathcal{V}}}$ is the underlying map of $\eta_{\mathcal{V}}$. The multiplication $(\mu_{\mathcal{V}})_X : T_{\mathcal{V}}T_{\mathcal{V}}X \to T_{\mathcal{V}}X$ is an interpretation of every term $t \in T_{\mathcal{V}}X$ over the poset $T_{\mathcal{V}}X$ of Σ -terms as a term $\mu_{\mathcal{V}}(t)$ over X modulo \mathcal{E}_X^o . This interpretation is independent of the ordering of X, shortly, the underlying function of $(\mu_{\mathcal{V}})_X$ is the corresponding interpretation $(\tilde{\mu}_{\mathcal{V}})_{|X|}$ of terms modulo \mathcal{E}_X^o w.r.t. $\tilde{\mathbf{T}}$.

Definition 5.3. A variety \mathcal{V} of ordered algebras is called a *lifting* of a variety $\widetilde{\mathcal{V}}$ of classical (non-ordered) algebras if a functor from \mathcal{V} to $\widetilde{\mathcal{V}}$ is given which is concrete over **Set** and takes the free algebra on any poset X to the free algebra on |X|.

Corollary 5.4. Every variety of ordered algebras is a lifting of some classical variety.

Indeed, given a variety \mathcal{V} , let $\widetilde{\mathbf{T}}$ be an ordinary monad of Set such that $\mathbf{T}_{\mathcal{V}}$ is a lifting of it. The comparison functor is an isomorphism $K : \mathcal{V} \to \mathsf{Pos}^{\mathbf{T}_{\mathcal{V}}}$ concrete over Pos (Lemma 3.9). And we have a classical variety $\widetilde{\mathcal{V}}$ with an analogous concrete isomorphism $\widetilde{K} : \widetilde{\mathcal{V}} \to \mathsf{Set}^{\widetilde{\mathbf{T}}}$ over Set. Define a concrete functor $H : \mathsf{Pos}^{\mathbf{T}} \to \mathsf{Set}^{\widetilde{\mathbf{T}}}$ over Set by the obvious rule: it sends an algebra $\alpha : TA \to A$ to $\alpha : \widetilde{T}|A| \to |A|$. The desired functor is $\widetilde{K}^{-1} \cdot H \cdot K : \mathcal{V} \to \widetilde{\mathcal{V}}$. **Example 5.5.** We present a finitary lifting of a monad on **Set** to **Pos** which is not strongly finitary.

Consider all ordered algebras on two binary operations + and *. The full subcategory on all algebras for which the implication

$$x \leqslant y \implies x + y \leqslant x * y$$

holds yields the following monad \mathbf{T} on Pos. Given a poset X, the poset TX contains all terms with variables in X using + and *, where the order on TX is the smallest one such that

- (1) $x \leq y$ in X implies $x \leq y$ in TX,
- (2) + and * are monotone, and
- (3) $t + s \leq t * s$ for all terms $t \leq s$ in TX.

Thus \mathbf{T} is a lifting of the monad on Set corresponding to two binary operations (and no equations).

The monad **T** is not strongly finitary. For example, it does not preserve the canonical coinserter (recall Remark 2.3) of the chain **2** given by 0 < 1:

$$3 \underbrace{\overset{p_1}{\overbrace{\qquad p_0 \qquad \qquad }} 2 \xrightarrow{id} \mathbf{2}$$

Indeed, in T2 we have 0 + 1 < 0 * 1. In contrast, this does not hold in the coinserter of Tp_0 and Tp_1 . We can describe the order of that coinserter as the smallest one that, besides conditions (1)-(3) above, also fulfills $t \leq s$ for terms such that s is obtained by changing some 0 in t to 1. The down-set of the term 0 * 1 in that coinserter consists of the following terms

$$0 + 0 < 0 * 0 < 0 * 1.$$

Thus, **T** does not preserve the coinserter of p_0 and p_1 .

6. CONCLUSIONS

Kelly and Power proved that every finitary monad \mathbf{T} on Pos has a presentation as a coequaliser of a parallel pair of monad morphisms between free monads on generalised signatures, see [13]. In the present paper we derive an analogous result for strongly finitary monads: each such monad has a presentation as a coinserter of a parallel pair of monad morphisms between free monads \mathbf{T}_{Σ} on (classical) signatures Σ , see Construction 3.13. The move from coequalisers to coinserters is needed since the signatures used in [13] were substantially more general than those we use here: they were collections $\Sigma = (\Sigma_{\Gamma})_{\Gamma \in \mathsf{Pos}_{\mathsf{F}}}$ of posets Σ_{Γ} indexed by finite posets. However, the proof method we use is closely related to that in [13].

We have proved that for (classical) varieties of ordered Σ -algebras the corresponding free-algebra monad on Pos is strongly finitary, i.e. finitary and preserving reflexive coinserters. Using this we proved that the category of varieties of ordered algebras is dually equivalent to the category of strongly finitary monads on Pos.

In the future we plan extending our results to strongly finitary monads on more general \mathscr{V} -categories for closed monoidal categories \mathscr{V} , e.g. the category of small categories. For general \mathscr{V} it is interesting to know under which conditions strongly finitary functors are precisely the finitary ones preserving reflexive coinserters. But the main question is whether strongly finitary monads correspond again to 'naturally' defined varieties of algebras in \mathscr{V} .

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