

# Three Sampling Formulas

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**Abstract.** Sampling formulas describe probability laws of exchangeable combinatorial structures like partitions and compositions. We give a brief account of two known parametric families of sampling formulas and add a new family to the list.

**1 Introduction.** By an integer composition of weight  $n$  and length  $\ell$  we shall mean an ordered collection of positive integer parts  $\lambda = (\lambda_1, \dots, \lambda_\ell)$ ; we write  $\lambda \vdash n$  for  $\sum \lambda_j = n$ . It will be convenient to also use variables  $\Lambda_k = \lambda_k + \dots + \lambda_\ell$ ,  $k \leq \ell$ , so that  $\lambda_j = \Lambda_j - \Lambda_{j-1}$ .

A *composition structure* is a nonnegative function  $q$  on compositions such that for each  $n$  the values  $\{q(\lambda) : \lambda \vdash n\}$  comprise a probability distribution, say  $q_n$ , and the  $q_n$ 's satisfy the following sampling consistency condition. Imagine an ordered series of randomly many nonempty boxes filled in randomly with balls, so that the distribution of occupancy numbers from the left to the right is  $q_n$ . The condition requires that if some  $k < n$  balls are sampled out uniformly at random then the distribution of the reduced occupancy numbers in nonempty boxes (in same order) must be exactly  $q_{n-k}$  (without loss of generality we can take  $k = 1$ ).

Ignoring the order of boxes yields Kingman's partition structure [7] (see [1] and [10] for systematic development of the theory of partition structures, their relation to exchangeability and many references). But the relation cannot be uniquely inverted, because for a given partition structure there are many ways to introduce the order in a consistent fashion.

Gnedin [4] showed that all composition structures can be uniquely represented by a version of the Kingman's paintbox construction [7]. Let  $U$  be a *paintbox* – a random open subset of  $[0, 1]$ . With a paintbox we associate an ordered partition of  $[0, 1]$  comprised of the intervals of  $U$  and of individual elements of the complement  $U^c$ , with the order of blocks induced by the order on reals. Suppose  $n$  independent uniform random points are sampled from  $[0, 1]$  independently of  $U$ . The sample points group somehow within the partition blocks and we obtain a random composition by writing the nonzero occupation numbers from the left to the right. With probability one there is no tie among the sample points and the consistency for various sample sizes follows from exchangeability in the sample.

From a topological viewpoint, the representation establishes a homeomorphism between the space of extreme composition structures and the compact space of open subsets of  $[0, 1]$  (endowed with a weak topology), and also identifies the generic composition structure with a unique mixture of extremes. Thus already the set of extremes is intrinsically infinite-dimensional, not to say about the mixtures. It is therefore a question of interest to find smaller parametric families which admit a reasonably simple description.

In this note we discuss briefly three such families: one is an ordered modification, due to Donnelly and Joyce [2], of the ubiquitous Ewens sampling formula (corresponding to  $(0, \theta)$ -

partition structure from the Ewens-Pitman two-parametric family [9]); another one, due to Pitman [8], is an ordered (symmetric) version of the  $(\alpha, 0)$ -partition structure, and the third composition structure is new. Despite the fact that the new composition structure is, in a sense, constructed from the beta distributions, like the first two, the corresponding partition structure does not fit in the Ewens-Pitman family. All three belong to a large infinite-dimensional family of regenerative compositions introduced and characterised in [6] and all three are mixed, i.e. generated by genuinely random paintboxes.

**2 Ordered ESF.** In their encyclopaedical exposition of the multivariate Ewens distribution Ewens and Tavaré presented the ordered version of ESF (see [3], Eqn. 41.6),

$$(1) \quad e(\lambda) = \frac{\theta^\ell n!}{[\theta]_n} \prod_{j=1}^{\ell} \frac{1}{\Lambda_j} \quad \theta > 0,$$

in connection with a size-biased permutation of the Ewens partition structure (here and forth,  $[\ ]$  is the Pochhammer factorial). The special case  $\theta = 1$  is well known to combinatorialists as the distribution of cycle lengths in a uniform random permutation, provided the cycles are ordered by increasing of their least elements.

Donnelly and Joyce [2] observed that the formula also defines a composition structure, i.e. that (1) determines a consistent sequence of random partitions taken together with an *intrinsic* ordering of classes, based neither on the sizes of classes nor on labeling of ‘balls in boxes’. They argued that the ordered structure is of some significance for biological applications, and proved the following paintbox representation of  $e$ .

Let  $Z_j$  be independent random variables with beta density

$$d\omega = \theta (1 - z)^{\theta-1} dz.$$

Let  $U_e$  be the open set complementary to the stick-breaking sequence

$$1 - \prod_{j=1}^k (1 - Z_j) \quad k = 1, 2, \dots$$

taken together with the endpoints of  $[0, 1]$ . Then rephrasing Theorem 10 from [2] we have

**Theorem 1** *The composition structure  $e$  can be derived from the paintbox  $U_e$ .*

The proof of this result given in [2] relied on the twin fact about weak convergence of the size-biased permutation of ESF. Next is a direct argument which offers some more insight and exemplifies the approach taken in this paper.

*Proof.* Introduce the binomial moments

$$(2) \quad \begin{aligned} w(n : m) &= \binom{n}{m} \int_0^1 z^m (1 - z)^{n-m} d\omega(z) \\ &= \theta \binom{n}{m} B(m + 1, n - m + \theta) \end{aligned}$$

For  $I$  the leftmost interval of  $U$  (adjacent to 0) the size of  $I$  equals  $Z_1$ . Denoting  $\hat{e}$  the composition structure derived from  $U_e$  we aim to show that  $\hat{e} = e$ .

The argument is based on two facts. Firstly, suppose  $n$  uniform points have been sampled from  $[0, 1]$  and  $I$  occurred to contain  $m$  sample points, then conditionally given  $I$  the configuration of other  $n - m$  points is as if it were a uniform sample from  $I^c$ . The second fact is that given  $I$  the set  $U_e \setminus I$  is a scaled distributional copy of  $U_e$ , as it is clear from the definition of the paintbox via stick-breaking.

Composition  $(\lambda_1, \dots, \lambda_\ell)$  can only appear if the interval  $J$ , defined to be the leftmost of the intervals of  $U_e$  discovered by the sample, contains exactly  $\lambda_1$  sample points. The chance that  $J$  coincides with  $I$  is  $w(n : m)$  and in this case the composition derived from the piece of  $U_e$  to the right from  $J$  must be  $(\lambda_2, \dots, \lambda_\ell)$ . Otherwise  $\lambda$  can appear only if  $I$  contains no sample points and all  $n$  group within  $U_e \setminus J$  in accord with  $\lambda$ .

Combining these facts we get equation

$$e(\lambda_1, \dots, \lambda_\ell) = w(n : \lambda_1) e(\lambda_2, \dots, \lambda_\ell) + w(n : 0) e(\lambda_1, \dots, \lambda_\ell)$$

leading to the recursion

$$e(\lambda_1, \dots, \lambda_\ell) = \frac{w(n : \lambda_1)}{1 - w(n : 0)} e(\lambda_2, \dots, \lambda_\ell).$$

which is solved as

$$(3) \quad \hat{e}(\lambda) = \prod_{j=1}^{\ell} q(\Lambda_j : \lambda_j)$$

where

$$(4) \quad \begin{aligned} q(n : m) &:= \frac{w(n : m)}{1 - w(n : 0)} \\ &= \frac{\theta}{n} \frac{n!}{(n - m)!} \frac{[\theta]_{n-m}}{[\theta]_n}. \end{aligned}$$

Cancelling common factors we arrive at (1), thus  $\hat{e} = e$ .  $\square$

There is a canonical correspondence between composition structures and probability distributions of exchangeable compositions of an infinite set  $\{\underline{1}, \underline{2}, \dots\}$  (see [4]). In terms of the paintbox representation the composition derived from  $U$  is obtained by sampling infinitely many uniform points and then assigning objects  $\underline{i}$  and  $\underline{j}$  to distinct classes if the closed interval spanned on the  $i$ th and the  $j$ th sample points has a nonempty intersection with  $U^c$ .

The infinite composition associated with  $e$ , call it  $\mathcal{E}$ , has a simply ordered collection of blocks, and the law of large numbers says that the asymptotic frequencies of the blocks (in a growing sample) coincide with the sizes of stick-breaking residuals, from the left to the right. When we view  $\mathcal{E}$  from the perspective of restrictions  $\mathcal{E}_n$  on finite sets  $\{\underline{1}, \dots, \underline{n}\}$ 's, the collection of blocks *stabilises* (with probability one) in the sense that for any  $k$  no *new* block appearing in  $\mathcal{E}_{n'}$ , for  $n' > n$ , will interlace with the collection of the first  $k$  blocks represented in  $\mathcal{E}_n$ , provided  $n$  is sufficiently large (a zero-one law). Compositions with this property were called

‘representable’ in [2] and the class of such compositions generated by a general stick-breaking paintbox was characterised in [6].

**3 PSF.** Pitman’s composition structure is given by Eqn. (30) in [8]:

$$(5) \quad p(\lambda) = \frac{n! \alpha^\ell}{[\alpha]_n} \prod_{j=1}^{\ell} \frac{[1 - \alpha]_{\lambda_j}}{\lambda_j!} \quad 0 < \alpha < 1.$$

This sampling formula was derived from the following paintbox representation.

**Theorem 2** *The paintbox  $U_p$  for  $p$  is the union of excursion intervals of the Bessel bridge of dimension  $2 - 2\alpha$ .*

Equivalently, the complement  $U_p^c$  is the set of zeroes of the Bessel bridge on  $[0, 1]$ . The case  $\alpha = 1/2$  corresponds to the Brownian bridge.

In fact,  $p$  is a conditional version of another Pitman’s composition structure  $p'$  derived from the set of zeroes of a Bessel process (which has final meander interval adjacent to the rightpoint of  $[0, 1]$ ). Pitman obtained a formula for  $p'$  akin to (5) (see [8], Eqn. (28)) using selfsimilarity of the Bessel process and distribution of the length of meander interval. The relation between the structures is that

$$p(\lambda) = \text{const}(n) p'(\lambda, 1) \quad \lambda \vdash n.$$

Gnedin and Pitman [6] give a characterisation of  $p$  related to the observation that this composition structure is also of the product form (similar to (3)) with

$$q(n : m) = - \frac{\binom{\alpha}{m} \binom{-\alpha}{n-m}}{\binom{-\alpha}{n}}$$

For  $\ell$  fixed,  $p$  is a symmetric function of the parts. This reflects in that  $U_p$  is *symmetric*, that is has component intervals ‘in random order’ (in [1] the open sets with this kind of invariance are called ‘exchangeable interval partitions’). Summing  $p(\lambda)$  over distinct permutations of parts yields a function on integer partitions which is the  $(\alpha, 0)$ -partition structure from the Ewens-Pitman family. It follows that  $p$  could be obtained from the partition structure by permuting the parts in uniform random order (this is the general device allowing to derive symmetric composition structures and symmetric open sets from their unordered relatives [5]).

Blocks of the Pitman’s composition  $\mathcal{P}$  on  $\{\underline{1}, \underline{2}, \dots, \}$  are ordered like the set of rational numbers and a such cannot be labeled by integers consistently with their intrinsic order. This happens each time a composition has infinitely many blocks (almost surely) and is symmetric. A consequence is that the infinite composition  $\mathcal{P}$  has no definite first, second, etc or the last block, in particular the first (hence  $k$ th) block in  $\mathcal{P}_n$  does not stabilise as  $n$  grows.

**4 A new sampling formula.** Here is a new composition structure

$$(6) \quad g(\lambda) = \frac{n!}{[\theta]_n} \prod_{j=1}^{\ell} \frac{1}{\lambda_j h_{\theta}(\Lambda_j)} \quad \theta > 0$$

where

$$h_\theta(n) = \sum_{k=1}^n \frac{1}{\theta + k - 1}$$

are the generalised harmonic numbers which coincide with the partial sums of the harmonic series when  $\theta = 1$ .

To explain the genesis of the formula consider stick-breaking with the general beta density

$$(7) \quad d\omega(z) = \text{const} \cdot z^{\alpha-1} (1-z)^{\theta-1} \quad \alpha, \theta > 0.$$

The resulting paintbox generates a composition structure given by the RHS of (3) with

$$(8) \quad q(n : m) = \binom{n}{m} \frac{[\alpha]_m [\theta]_{n-m}}{[\alpha + \theta]_n - [\theta]_n}$$

where (8) is obtained like (4) from the binomial moments of the beta density (7) (to see this just follow the lines in the proof of Theorem 1).

For general  $\alpha$  and  $\beta$  the induced composition structure cannot be expressed by a simple product formula, because the denominator has no good factorisation. One notable exception is the ESF appearing when  $\alpha = 1$ . Another exception is the case  $\alpha = 0$  giving rise to  $g$ ; but this should be interpreted properly because measure  $\omega$  becomes infinite.

**Theorem 3** *When  $\alpha \downarrow 0$  the stick-breaking composition structure directed by the beta density (7) converges to  $g$ .*

*Proof.* Expansions in powers of  $\alpha$  start with

$$[\alpha]_m = \alpha (m-1)! + \dots, \quad [\alpha + \theta]_n - [\theta]_n = \alpha h_\theta(n) + \dots$$

therefore when  $\alpha$  approaches 0 we get

$$(9) \quad q(n : m) = \frac{n!}{(n-m)!} \frac{[\theta]_{n-m}}{[\theta]_n} \frac{1}{m h_\theta(n)}.$$

which yields  $g$  as in (3).  $\square$

Distribution (8) underlying  $g$  is especially simple for  $\theta = 1$  when it gives a weight proportional to  $m^{-1}$  to each  $m = 1, \dots, n$ .

To determine the paintbox representation for  $g$  we will extend the classical stick-breaking procedure by embedding the process into continuous time and allowing infinitely many breaks within any time interval. Note that defining a composition structure via the RHS of (3), through the binomial moments of some measure  $\omega$  and

$$q(n : m) = \frac{w(n : m)}{w(n : 1) + \dots + w(n : n)}$$

we need not require that the measure  $\omega$  be finite and do need to only impose the condition

$$\int_0^1 z d\omega(z) < \infty$$

to have all binomial moments finite for  $1 \leq m \leq n < \infty$ .

In particular, our  $g$  appears when we take improper density

$$(10) \quad d\omega(z) = z^{-1}(1-z)^{\theta-1} dz$$

(see [6] for more examples). For this  $\omega$  consider a planar Poisson process (PPP) in the infinite strip  $[0, \infty] \times [0, 1]$  with Lebesgue  $\times \omega$  as intensity measure. The PPP has countably many atoms  $(\tau_j, \xi_j)$  (we adopt the conventional fake labeling of atoms which is not intended to say that  $\tau_j$  or  $\xi_j$  is a definite random variable for particular  $j$ ), and each location on the abscissa is a concentration point for the set of atoms. Define a pure-jump process with increasing cadlag paths

$$S_t = 1 - \prod_{(\tau_j, \xi_j): \tau_j \leq t} (1 - \xi_j)$$

where the product is over all PPP atoms to the left from  $t$ . For any  $t$  the product converges because  $z\omega(dz)$  is a finite measure. The process  $(S_t)$  is a geometric subordinator: for  $t' > t$  the ratio  $(1 - S_{t'})/(1 - S_t)$  is independent of the partial path on  $[0, t]$  and has same distribution as  $1 - S_{t'-t}$ .

(The reader feeling more comfort with breaking sticks from the right to the left should translate paintbox formulas using involution  $z \leftrightarrow 1 - z$  and also mirror the sampling formulas.)

**Theorem 4** *The paintbox  $U_g$  representing  $g$  is the complement to the closure of the random set  $\{S_t : t > 0\}$ , which is the range of the geometric subordinator.*

*Proof.* Fix  $\lambda \vdash n$  and consider a uniform sample of size  $n$ . The composition  $\lambda$  appears when for some  $\tau_j$  the interval  $[0, S_{\tau_j}]$  contains  $m$  sample points grouped in one component interval of  $U_g \cap [0, S_{\tau_j}]$  and the composition on the remaining  $(n - m)$  sample points is  $(\lambda_2, \dots, \lambda_\ell)$ . From the properties of uniform distribution and because PPP is ruled by a product measure follows that the composition structure induced by  $U_g$  is of the product form as in (3) and we only need to justify the formula (9) for  $q$  which is the distribution of the first part of composition of  $n$ .

To that end, let  $\pi(t)$  be the probability that some  $m$  sample points group in one interval of  $U_g \cap [0, S_t]$  and denote  $\epsilon_1, \dots, \epsilon_n$  the increasing order statistics of uniform sample. Considering a small time interval  $[0, dt]$  it is not hard to see that  $\pi$  satisfies the differential equation

$$\pi' = -a\pi + b, \quad \pi(0) = 0$$

with constant coefficients

$$a = E\omega[\epsilon_1, 1] = w(n : 1) + \dots + w(n : n) \quad \text{and} \quad b = E\omega[\epsilon_m, \epsilon_{m+1}] = w(n : m)$$

(with 1 in place of  $\epsilon_{m+1}$  in case  $m = n$ ) where  $w(n : m)$ 's are the binomial moments of (10). Solving the equation we obtain  $\phi(t) = (b/a)(1 - e^{-at}) \rightarrow b/a = q(n : m)$ , as  $t \rightarrow \infty$  whence  $q(n : m) = b/a$  and this is (9).  $\square$

The infinite composition  $\mathcal{G}$  associated with  $g$  has infinitely many blocks, and the set of blocks is order isomorphic to the set of rational numbers. Unlike Pitman's  $\mathcal{P}$  it is not symmetric, i.e.  $g$  is sensible to permutation of parts  $\lambda_j$  when  $\ell > 1$ , and the representing paintbox  $U_g$  is not an

‘exchangeable interval partition’. A combinatorialist might find natural to view  $g$  as a function on Young diagrams  $(\Lambda_1, \dots, \Lambda_\ell)$  with strictly decreasing parts.

Ignoring the order in  $\mathcal{G}$  yields a novel partition structure. For no  $\theta$  belongs this partition structure to the Ewens-Pitman two-parameter family, which had covered practically all explicit sampling formulas known to date. The distinction can be seen immediately by comparing the probability of one-class partition, our  $g(n) = q(n : n)$  given by (8) versus the analogous quantity computed via Eqn. (16) in [9] (the formulas do not match for  $n > 4$  whatever the values of parameters).

Taking other integer values of  $\alpha$  in (7) leads to formulas involving products of stereotypic polynomial factors, e.g. for  $\alpha = 2$  we have

$$g_2(\lambda) = \frac{n! \theta^\ell (1 + \theta)^\ell}{[\theta]_n} \prod_{j=1}^{\ell} \frac{\lambda_j + 1}{\Lambda_j + 2\theta + 1} .$$

The resulting infinite compositions have simply ordered blocks and thus are more in line with  $\mathcal{E}$ .

## References

- [1] Aldous, D.J. (1985) Exchangeability and related topics, In *École d’été de probabilités de Saint Flour XII*, Lecture Notes in Mathematics **1117**, Springer, NY.
- [2] Donnelly, P. and Joyce, P. (1991) Consistent ordered sampling distributions: characterization and convergence, *Adv. Appl. Probab.* **23**, 229-258.
- [3] Ewens, W.J. and Tavaré, S. (1995) The Ewens sampling formula. In *Multivariate Discrete Distributions* (Johnson, N.S. et al eds), Wiley, NY.
- [4] Gneden, A.V. (1997) The representation of composition structures, *Ann. Probab.* **25**, 1437-1450.
- [5] Gneden, A.V. (1998) On the Poisson-Dirichlet limit, *J. Multivariate Analysis* **67**, 90-98.
- [6] Gneden, A.V. and Pitman, J. (2002) Regenerative compositions, (paper in progress).
- [7] Kingman, J.F.C. (1978) The representation of partition structures, *J. London Math. Soc.* **18**, 374-380.
- [8] Pitman, J. (1997) Partition structures derived from Brownian motion and stable subordinators, *Bernoulli* **3**, 79-96.
- [9] Pitman, J. (1995) Exchangeable and partially exchangeable random partitions, *Probab. Th. Rel. Fields* **102**, 145-158.
- [10] Pitman, J. (2002) Combinatorial stochastic processes, (Lecture Notes for St. Flour course, July 2002).

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