# Long Range Percolation Mixing Time 

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#### Abstract

We provide an estimate, sharp up to poly-logarithmic factors, of the asymptotic almost sure mixing time of the graph created by long-range percolation on the cycle of length $N(\mathbb{Z} / N \mathbb{Z})$. While it is known that the asymptotic almost sure diameter drops from linear to poly-logarithmic as the exponent $s$ decreases below 2 4, 10, the asymptotic almost sure mixing time drops from $N^{2}$ only to $N^{s-1}$ (up to poly-logarithmic factors).


## 1 Introduction

In this note we study mixing time of simple random walks on the random graph obtained by adding edges to a graph of a cycle, where edges are added between any two vertices with probability decaying with their distance, and independently for any two vertices (in fact, this probability is $1-\exp \left(-\beta\|x-y\|^{-s}\right)$, where $\beta$ and the exponent $s$ are parameters). See below for definitions of the model and of mixing times, as well as further remarks.

The bounds on the mixing time are presented in the coming sections.
Let us mention two interesting findings. First, the mixing time undergoes a phase transition as the exponent $s$ decreases below 2. Second, in this natural model the almost sure diameter is a.a.s. (asymptotically almost surely) poly-logarithmic, for certain range of the parameters, yet the a.a.s. mixing time is polynomial. Such a gap between the diameter and the mixing time cannot exist for vertex transitive graphs or in graphs where the isoperimetric dimension is determined by the volume growth function.

[^0]The long range percolation graphs gained some recent interest, see [15]. From an algorithmic viewpoint it is useful to consider the mixing times of these graphs.

### 1.1 The Model

The model we discuss is the finite long-range percolation model with polynomial decay. Let $N$ be a positive integer, let $\beta>0,1<s<2$, and consider the following random graph: Start with the cycle on $N$ vertices $(\mathbb{Z} / N \mathbb{Z})$. Define $\|x-y\|=\min \{|x-y|, N-|x-y|\}$, (which is the regular graph-theoretical distance). The following edges are randomly added: If $x \neq y$, then $x$ and $y$ will be attached with probability $1-\exp \left(-\beta\|x-y\|^{-s}\right)$. The different edges are all independent of each other. The probability of an edge between two (distant enough) vertices is very close to $\beta\|x-y\|^{-s}$. We call the graph created this way $G_{s, \beta}(N)$.

For updated background on long-range percolation see [10, 11 .

### 1.2 Mixing Time

Throughout this paper we consider random walks on a random graph. In order to avoid issues of convergence to the stationary distribution, we always consider the lazy random walk, i.e. the Markov chain on the vertices of the graph whose transition matrix is $P(x, y)=\frac{1}{2 \operatorname{deg}(x)}$ if $(x, y)$ is an edge in the graph, and $P(x, x)=\frac{1}{2}$. This insures that the random walk is ergodic and converges to the stationary distribution.

Also, throughout this paper we consider oriented edges. Thus, if $x$ and $y$ are adjacent vertices of some graph, we consider both $(x, y)$ and $(y, x)$ as edges in the edge set.

The variational distance of the random walk on a graph $G$ is defined

$$
\Delta_{x}(t)=\frac{1}{2} \sum_{y \in V(G)}\left|P^{t}(x, y)-\pi(x)\right|
$$

where $P^{t}$ is the $t^{\text {th }}$ power of the transition matrix of the walk, and $\pi$ is the stationary distribution, i.e. $\pi(x)=\frac{\operatorname{deg}(x)}{|E(G)|}(\operatorname{deg}(x)$ is the degree of the vertex $x$, and recall that $E(G)$ is the set of oriented edges). $\Delta_{x}(t)$ measures how close the distribution of the walk starting at $x$, at time $t$, is to the stationary distribution $\pi$.

Define

$$
\tau_{x}(\varepsilon)=\min \left\{t \mid \forall t^{\prime} \geq t \Delta_{x}\left(t^{\prime}\right) \leq \varepsilon\right\}
$$

Since we are interested in the time it takes the walk to converge to the stationary distribution, the mixing time is defined

$$
\tau(G)=\max _{x \in V(G)} \tau_{x}(1 / 4)
$$

It is well know that using the second eigenvalue of the transition matrix, one can bound the mixing time of the graph. Formally (though not in full generality), Diaconis and Strook in [12] prove that:

$$
\begin{equation*}
\tau(G) \leq \frac{\log (4|E(G)|)}{1-\lambda_{G}} \tag{1.1}
\end{equation*}
$$

where $\lambda_{G}$ is the second eigenvalue of $G$ (i.e. the eigenvalues of the transition matrix of the random walk on $G$ are $1>\lambda_{G} \geq \lambda_{3} \geq \cdots \geq \lambda_{|V(G)|} .1-\lambda_{G}$ is also called the spectral gap of G.)

For more on mixing see [1, 18].

### 1.3 Remarks

- In the following sections we will provide upper and lower bounds on the mixing time of $G_{s, \beta}(N)$, that match up to poly-logarithmic factors. It is interesting to note that using two different methods, we obtain matching bounds.
- As will be seen in Section 4 a phase transition occurs in the mixing time when $s$ passes from below 2 to above 2. Two open questions regarding the mixing time are:

1. What is the mixing time at $s=2$. We note that even the diameter is not known in this case.
2. When $s$ drops below 1, it is shown in [6] that the diameter is bounded. We conjecture that the mixing time is constant (independent of $N$ ) in this case.

- Long-range percolation gives natural examples of graphs with small diameter (polylogarithmic, see [10]) yet large polynomial mixing time. Long range percolation is a natural model of some social networks, in which the probability you know a person decays with distance. This suggests that while the diameter of such networks might be small, sampling from such network via random walk might take long time.
- For almost sure expansion of other models of random graphs see [2, 17. Regarding mixing for random walks on other models of random graphs see [7, 14] and [9].
- Another question related to mixing on random graphs is the following: Let $\left\{G_{n}\right\}$ be a family of vertex transitive graphs such that $\lim _{n \rightarrow \infty}\left|G_{n}\right|=\infty$. Assume that the average degree of the giant component $G_{n}^{\prime}$ of Bernoulli percolation on $G_{n}$ is uniformly bounded in $n$. Prove

$$
\tau\left(G_{n}^{\prime}\right) \leq O\left(\max \left\{\tau\left(G_{n}\right), \log ^{2}\left|G_{n}\right|\right\}\right)
$$

- Consider uniform spanning tree on the long range percolation graph over $\mathbb{Z}$. Our mixing time estimates show that the mixing time of long range percolation at $s=3 / 2$ is like that of a 4 dimensional torus. Since the transition from tree to forest in the uniform spanning tree on $\mathbb{Z}^{d}$ occurs at $d=4$, this suggests that perhaps the uniform spanning tree on the long range percolation graph over $\mathbb{Z}$, is supported on a tree a.s. iff $s \geq 3 / 2$. See [8] for background.


## 2 Upper Bound

### 2.1 Multicommodity Flow

Let $P$ be the transition matrix of a reversible Markov chain, with stationary distribution $\pi$. Let $V$ be the set of states of the chain, and let $E$ be the set of oriented edges; i.e.

$$
E=\{(x, y) \in V \times V: P(x, y)>0\}
$$

For $x, y \in V$ let $\mathcal{P}(x, y)$ be the set of all simple paths from $x$ to $y$. Let $\mathcal{P}=\cup_{x \neq y \in V} \mathcal{P}(x, y)$.
A flow is a function $f: \mathcal{P} \rightarrow[0,1]$ such that for all $x, y \in V$

$$
\sum_{p \in \mathcal{P}(x, y)} f(p)=\pi(x) \pi(y)
$$

The edge load of an edge $e \in E$ is defined as

$$
f(e)=\sum_{\substack{p \in \mathcal{P} \\ p \ni e}} f(p)|p|
$$

The congestion of a flow $f$ is defined as

$$
\rho(f)=\max _{(a, b) \in E} \frac{1}{\pi(a) P(a, b)} f((a, b))
$$

Theorem 5' of [18] states that if the eigenvalues of $P$ are $1>\lambda \geq \lambda_{3} \geq \cdots \geq \lambda_{n}$ (where $n=|V|)$, then for any flow $f,(1-\lambda)^{-1} \leq \rho(f)$. Furthermore, Theorem 8 in [18] shows that if $P$ induces an ergodic Markov chain (i.e. if $\lambda_{n}>-1$ ), then there exists a flow $f^{*}$ such that $\rho\left(f^{*}\right) \leq 16 \tau$, where $\tau$ is the mixing time of the chain. We call $f^{*}$ the optimal flow for $P$.

### 2.2 Upper Bound

We are now ready to prove an upper bound on the mixing time of $G_{s, \beta}(N)$.
Proposition 2.1. Let $G_{s, \beta}(N)$ be the graph obtained by long-range percolation on the cycle of length $N$. Then there exists $c=c(s, \beta)>0$ such that

$$
\lim _{N \rightarrow \infty} \mathbb{P}\left[\tau\left(G_{s, \beta}(N)\right) \leq \log ^{c}(N) \cdot N^{s-1}\right]=1
$$

Proof. Set $G=G_{s, \beta}(N)$. With hindsight, set $L=\left\lceil N^{s-1} \xi(N)\right\rceil$ for $\xi(N)=\alpha \log (N) / 2^{s} \beta$, and $\alpha>0$ some constant to be determined below. Let $\ell=N(\bmod L)$, and set $k=\frac{N-\ell}{L}$. Divide the cycle into $k$ intervals, $S_{1}, \ldots, S_{k}$, so that $S_{1}, \ldots, S_{k-1}$ are of length $L$, and $S_{k}$ is of length $L+\ell \leq 2 L$. Since $s<2$, we can take $N$ large enough so that $2^{s} \beta L^{2} \leq N^{s}$.

Let $1 \leq i \neq j \leq k$. Let $\mathcal{E}(i, j)$ be the event that there exist $x \in S_{i}$ and $y \in S_{j}$ such that $(x, y) \in E(G)$.

Let $\Gamma$ be the graph obtained from $G$ by contracting each of the intervals $S_{1}, \ldots, S_{k}$ to a vertex. That is $V(\Gamma)=\{1,2, \ldots, k\}$ and $(i, j) \in E(\Gamma)$ if $\mathcal{E}(i, j)$ occurs.
We bound from below the probability that $(i, j) \in E(\Gamma)$.

$$
\begin{aligned}
\mathbb{P}[(i, j) \notin E(\Gamma)] & =\mathbb{P}[\text { not } \mathcal{E}(i, j)]=\prod_{\substack{x \in S_{i} \\
y \in S_{j}}} \mathbb{P}[(x, y) \notin E(G)] \\
& \leq \prod_{\substack{x \in S_{i} \\
y \in S_{j}}} \exp \left(-2^{s} \beta N^{-s}\right) \leq \exp \left(-2^{s} \beta L^{2} N^{-s}\right)
\end{aligned}
$$

Using the inequality $1-e^{-\zeta} \geq \zeta / e($ valid for $\zeta \in[0,1])$, we get that for all $1 \leq i \neq j \leq k$,

$$
\begin{aligned}
\mathbb{P}[(i, j) \in E(\Gamma)] & \geq 1-\exp \left(-2^{s} \beta L^{2} N^{-s}\right) \\
& \geq \frac{2^{s} \beta}{e} \cdot \frac{L^{2}}{N}
\end{aligned}
$$

Since $k \geq \frac{N}{L}-1$, we have that for large enough $N$ (depending on $s, \beta$ ),

$$
\begin{equation*}
\mathbb{P}[(i, j) \in E(\Gamma)] \geq \frac{\alpha}{2 e} \cdot \frac{\log k}{k} \tag{2.1}
\end{equation*}
$$

for all $1 \leq i \neq j \leq k$.
Let $\Gamma^{\prime}$ be the Erdos-Renyi random graph on $k$ vertices, with edge probability $p=\frac{\alpha}{2 e} \cdot \frac{\log k}{k}$. That is, $V\left(\Gamma^{\prime}\right)=\{1,2, \ldots, k\}$ and $(i, j) \in E\left(\Gamma^{\prime}\right)$ and $(j, i) \in E\left(\Gamma^{\prime}\right)$ with probability $p$, all edges $\{i, j\}$ independently.

By (2.1), we can couple $G$ and $\Gamma^{\prime}$ so that $\Gamma^{\prime}$ will be a subgraph of $\Gamma$.
$\operatorname{deg}_{\Gamma^{\prime}}(j)$ has the binomiaml distribution with parameters $k, p$. Thus, a quick calculation shows that there exists a constant $c_{1}=c_{1}(\alpha)>0$ such that with probability tending to 1 ,

$$
\begin{equation*}
\max _{1 \leq j \leq k} \operatorname{deg}_{\Gamma^{\prime}}(j) \leq c_{1} \log N \tag{2.2}
\end{equation*}
$$

Furthermore, in [5] it is shown that for large enough $\alpha$, there exists a constant $c_{2}>0$ such that with probability tending to 1 ,

$$
\begin{equation*}
\tau\left(\Gamma^{\prime}\right) \leq c_{2} \log k \leq c_{2} \log N \tag{2.3}
\end{equation*}
$$

We now derive an upper bound on the mixing time of $G$, by constructing a flow on $G$, using the optimal flow for $\Gamma^{\prime}$.

Let $\pi_{G}, \pi_{\Gamma^{\prime}}$ denote the stationary distribution of $G, \Gamma^{\prime}$ respectively. Let $\mathcal{P}(G), \mathcal{P}\left(\Gamma^{\prime}\right)$ be the set of simple paths in $G, \Gamma^{\prime}$ respectively. Let $\mathcal{P}(x, y ; G), \mathcal{P}\left(i, j ; \Gamma^{\prime}\right)$ be the set of simple paths in $G, \Gamma^{\prime}$ respectively, from $x$ to $y, i$ to $j$, respectively. For a path $p$ let $p^{+}$be the starting vertex of $p$, and let $p^{-}$be the ending vertex of $p$ (specifically for edges $e=\left(e^{+}, e^{-}\right)$).

For $(i, j) \in E(\Gamma)$ let $e(i, j)$ be a specific edge such that $e(i, j)=(x, y) \in E(G)$ and $x \in S_{i}$ and $y \in S_{j}$ (by definition there always exists at least one such edge). For $1 \leq j \leq k$ let $G_{j}$ be the induced subgraph on $S_{j}$. For $x, y \in S_{j}$ let $p(x, y)$ be a path in $G_{j}$ that realizes the distance between $x$ and $y$ in $G_{j}$ (a geodesic). If $x=y$ let $p(x, x)$ be the empty path.

For $q \in \mathcal{P}\left(i, j ; \Gamma^{\prime}\right)$, and $x \in V_{i}, y \in V_{j}$, define $p(q, x, y) \in \mathcal{P}(x, y ; G)$ by interpolating $q$ using the specified edges $e(i, j)$ and geodesics $p(x, y)$; that is if $q=e_{1} e_{2} \cdots e_{|q|}$, then

$$
p(q, x, y)=p\left(x, e_{1}^{+}\right) e\left(e_{1}^{+}, e_{1}^{-}\right) p\left(e_{1}^{-}, e_{2}^{+}\right) e\left(e_{2}^{+}, e_{2}^{-}\right) \cdots e\left(e_{|q|}^{+}, e_{|q|}^{-}\right) p\left(e_{|q|}^{-}, y\right)
$$

Setting $\Delta=\max _{j} \operatorname{diam}\left(G_{j}\right)$ we get that $|p(q, x, y)| \leq(\Delta+1)|q|$.
Let $f^{*}$ be the optimal flow on $\Gamma^{\prime}$. As mentioned above, by Theorem 8 of [18], using also (2.3), there exists a constant $c_{3}>0$ such that with probability tending to 1 ,

$$
\begin{equation*}
\forall(i, j) \in E\left(\Gamma^{\prime}\right) \quad\left|E\left(\Gamma^{\prime}\right)\right| \sum_{\substack{q \in \mathcal{P}\left(\Gamma^{\prime}\right) \\ q \ni(i, j)}} f^{*}(q)|q| \leq 16 \tau\left(\Gamma^{\prime}\right) \leq c_{3} \log N \tag{2.4}
\end{equation*}
$$

We now define a flow $f$ on $G$ using $f^{*}$. Let $x, y \in V(G)$, and let $i, j$ be such that $x \in S_{i}$ and $y \in S_{j}$.

If $i=j$ set $f(p)=\pi_{G}(x) \pi_{G}(y)$ if $p=p(x, y)$ and 0 otherwise.
If $i \neq j$, then for any $q \in \mathcal{P}\left(i, j ; \Gamma^{\prime}\right)$ set

$$
f(p(q, x, y))=\frac{f^{*}(q)}{\pi_{\Gamma^{\prime}}(i) \pi_{\Gamma^{\prime}}(j)} \cdot \pi_{G}(x) \pi_{G}(y)
$$

and 0 otherwise.
We calculate the conestion of the flow $f$. Let $(x, y) \in E(G)$, and let $i, j$ be such that $x \in S_{i}$ and $y \in S_{j}$.

Case 1: $i \neq j$. In this case, any path $p$ that contains the edge $(x, y)$, such that $f(p)>0$, must be of the form $p=p(q, z, w)$ for some $q \in \mathcal{P}\left(\Gamma^{\prime}\right)$ that contains $(i, j)$. Thus, using (2.2) and (2.4),

$$
\begin{align*}
\sum_{\substack{p \in \mathcal{P}(G) \\
p \ni(x, y)}} f(p)|p| & \leq \sum_{\substack{q \in \mathcal{P}\left(\Gamma^{\prime}\right) \\
q \ni(i, j)}} \sum_{\substack{ \\
w \in S_{q^{+}} \\
w \in S_{q^{-}}}} \frac{\pi_{G}(z) \pi_{G}(w)}{\pi_{\Gamma^{\prime}}\left(q^{+}\right) \pi_{\Gamma^{\prime}}\left(q^{-}\right)} \cdot f^{*}(q)|q|(\Delta+1) \\
& \leq(\Delta+1) \cdot\left(\max _{j} \pi_{G}\left(S_{j}\right)\right)^{2} \cdot\left|E\left(\Gamma^{\prime}\right)\right|^{2} \cdot \sum_{\substack{q \in \mathcal{P}\left(\Gamma^{\prime}\right) \\
q \ni(i, j)}} f^{*}(q)|q| \\
& \leq \frac{1}{|E(G)|}(\Delta+1) \cdot\left(\max _{x} \operatorname{deg}_{G}(x)\right)^{2} \cdot \frac{L^{2} k}{N} \cdot c_{4} \log ^{c_{5}} N \\
& \leq \frac{1}{|E(G)|} \cdot c_{4} \log ^{c_{5}} N \cdot(\Delta+1) \cdot\left(\max _{x} \operatorname{deg}_{G}(x)\right)^{2} \cdot L \tag{2.5}
\end{align*}
$$

where $c_{4}, c_{5}>0$ are constants (independent of $N$ ).
Case 2: $i=j$. I this case, any path $p$ that contains the edge $(x, y)$, such that $f(p)>0$, is one of the follwing: Either it is of the form $p=p(q, z, w)$ for some $q \in \mathcal{P}\left(\Gamma^{\prime}\right)$ that contains the vertex $i$, or it is of the form $p=p(z, w)$ for some $z, w \in S_{i}$. Any path $q \in \mathcal{P}\left(\Gamma^{\prime}\right)$ that contains the vertex $i$ must contain some edge $(i, j) \in E\left(\Gamma^{\prime}\right)$. Thus, using (2.2) and (2.5),

$$
\begin{align*}
\sum_{\substack{p \in \mathcal{P}(G) \\
p \ni(x, y)}} f(p)|p| & \leq \sum_{z, w \in S_{i}} f(p(z, w))|p(z, w)|+\sum_{\substack{j:(i, j) \in E\left(\Gamma^{\prime}\right)}} \sum_{\substack{\begin{subarray}{c}{\text { P(P)} \\
q \ni\left(\Gamma^{\prime}\right)} }}\end{subarray}} \sum_{z \in S_{q^{+}}} f(p(q, z, w))|p(q, z, w)| \\
& \leq \sum_{z, w \in S_{q^{-}}} \pi_{G}(z) \pi_{G}(w) \Delta+\operatorname{deg}_{\Gamma^{\prime}}(i) \cdot \frac{1}{|E(G)|} \cdot c_{4} \log ^{c_{5}} N \cdot(\Delta+1) \cdot\left(\max _{x} \operatorname{deg}_{G}(x)\right)^{2} \cdot L \\
& \leq \frac{1}{|E(G)|} \cdot c_{6} \log ^{c_{7}} N \cdot(\Delta+1) \cdot\left(\max _{x} \operatorname{deg}_{G}(x)\right)^{2} \cdot L \tag{2.6}
\end{align*}
$$

where $c_{6}, c_{7}>0$ are constants (independent of $N$ ).
By our choice of $L$, and by (1.1), it suffices to show that there exists constants $c_{8}, c_{9}, c_{10}, c_{11}>$ 0 (that may depend on $s, \beta$ ) such that $\Delta \leq c_{8} \log ^{c 9} N$ and $\max _{x} \operatorname{deg}_{G}(x) \leq c_{10} \log ^{c_{11}} N$, with probability tending to 1 .

In [4], the following is shown: There exist $\delta=\delta(s, \beta)>0$ and $n_{0}=n_{0}(s, \beta)>0$ such that for all $n>n_{0}$,
$\mathbb{P}\left[\operatorname{diam}(\right.$ long-range percolation on the interval of length $\left.n)>\log ^{\delta}(n)\right]<\frac{1}{n^{2}}$.
Thus, since there are $k \leq N / L$ such intervals, each of length at least $L$, a union bound gives that for large enough $N$,

$$
\begin{equation*}
\Delta=\max _{j} \operatorname{diam}\left(G_{j}\right) \leq \log ^{\delta}(N) \tag{2.7}
\end{equation*}
$$

with probability at least $1-O\left(N^{4-3 s}\right)$, which tends to 1 .
Next, we show that with probability tending to 1 , the maximal degree in $G$ is bounded by $2 \log (N)$. This follows from the following considerations:
Fix a vertex $x$ in $G$. We can write $\operatorname{deg}_{G}(x)=2+\sum_{y \neq x} Z_{x y}$, where $Z_{x y}$ is the indicator function of the event that $x$ and $y$ are connected by an edge not in the cycle. The random variables $Z_{x y}$ are all independent. Note that $\mu_{x y}=\mathbb{E}\left[Z_{x y}\right]=1-e^{-\beta\|x-y\|^{-s}}$, and for any $t>0, \mathbb{E}\left[\exp \left(t Z_{x y}\right)\right]=1+\mu_{x y}\left(e^{t}-1\right) \leq \exp \left(\left(e^{t}-1\right) \mu_{x y}\right)$. So we can calculate using Markov's inequality, for all $\lambda>0$,

$$
\begin{aligned}
\mathbb{P}\left[\operatorname{deg}_{G}(x)>2+\lambda\right] & \leq \exp (-\lambda) \prod_{y \neq x} \exp \left((e-1) \mu_{x y}\right) \\
& \leq \exp \left(-\lambda+c(s, \beta) N^{1-s}\right)
\end{aligned}
$$

Thus, taking $\lambda=2 \log (N)-2$ and using a union bound, the probability that there exists a vertex $x$ with $\operatorname{deg}_{G}(x)>2 \log (N)$ is bounded by $N \frac{1+o(1)}{N^{2}} \leq \frac{1+o(1)}{N}$. Thus, with probability tending to 1 we have that

$$
\begin{equation*}
\max _{x \in V(G)} \operatorname{deg}_{G}(x) \leq 2 \log (N) \tag{2.8}
\end{equation*}
$$

Finally, combining Theorem 5 ' of [18] with the flow $f$, (1.1), (2.5), (2.6), (2.7) and (2.8), and by our choice of $L$, we conclude that there exist constants $c, c^{\prime}>0$, (independent of $N$, but perhaps depending on $s, \beta$ ), such that with probability tending to 1 ,

$$
\tau(G) \leq c \log c^{\prime}(N) \cdot N^{s-1}
$$

## 3 Lower Bound

In Corollary 5.2 of [4], it is shown that $\tau(G) \geq \Omega\left(N^{s-1}\right)$ with probability tending to 1 . This miraculously matches our upper bound up to poly-logarithmic factors. In [4] there is a typo in the parameters, so for completeness we provide the proof here.

Proposition 3.1. Let $G_{s, \beta}(N)$ be the graph obtained by long-range percolation on the cycle of length $N$. Then

$$
\lim _{N \rightarrow \infty} \mathbb{P}\left[\tau\left(G_{s, \beta}(N)\right) \leq c N^{s-1}\right]=0
$$

where $c=c(s, \beta)$ is a constant independent of $N$.

Proof. Let $G=G_{s, \beta}(N)$. It is well known (see e.g. [1, 12, 18]) that it is enough to bound from above the Cheeger constant of the graph $G$, which is defined as

$$
\mathcal{C}(G)=\min _{\substack{\emptyset \neq A \subset V(G) \\|A| \leq N / 2}} \frac{|\partial A|}{|A|} \quad \partial A=\{\{x, y\} \in E(G) \mid x \in A, y \notin A\}
$$

The natural subset to choose is $A=\{1,2, \ldots, N / 2\}$ (any arc of length $N / 2$ will suffice). For $x \in A$ and $y \notin A$ let $Z_{x y}$ be the indicator function of the event that $x$ and $y$ are connected by an edge not in the cycle. Then, $Z_{x y}$ are all independent. Set $\mu_{x y}=\mathbb{E}\left[Z_{x y}\right]=$ $1-\exp \left(-\beta\|x-y\|^{-s}\right)$. For any $t>0$,

$$
\mathbb{E}\left[\exp \left(t Z_{x y}\right)\right]=1+\mu_{x y}\left(e^{t}-1\right) \leq \exp \left(\mu_{x y}\left(e^{t}-1\right)\right)
$$

We have that $|\partial A|=2+\sum_{x \in A} \sum_{y \notin A} Z_{x y}$, and that

$$
\begin{aligned}
\sum_{x \in A} \sum_{y \notin A} \mu_{x y} & \leq \sum_{x \in A} \sum_{y \neq x} \mu_{x y} \\
& \leq \sum_{x \in A} c(s, \beta) N^{1-s} \leq c(s, \beta) N^{2-s}
\end{aligned}
$$

Thus, there exists $c=c(s, \beta)$ such that for any $t>0$ and any $\lambda>0$,

$$
\begin{aligned}
\mathbb{P}\left[|\partial A|>2+\lambda N^{2-s}\right] & \leq \frac{\mathbb{E}[\exp (t(|\partial A|-2))]}{\exp \left(t \lambda N^{2-s}\right)} \\
& =\exp \left(-t \lambda N^{2-s}\right) \prod_{x \in A} \prod_{y \notin A} \mathbb{E}\left[\exp \left(t Z_{x y}\right)\right] \\
& =\exp \left(\left(e^{t}-1\right) \sum_{x \in A} \sum_{y \notin A} \mu_{x y}\right) \exp \left(-t \lambda N^{2-s}\right) \\
& \leq \exp \left(N^{2-s} \cdot\left(c(s, \beta)\left(e^{t}-1\right)-t \lambda\right)\right)
\end{aligned}
$$

Choosing $t$ small enough, we get that for some fixed $c=c(s, \beta)$ we have that

$$
\lim _{N \rightarrow \infty} \mathbb{P}\left[|\partial A|>c N^{2-s}\right]=0
$$

which implies that with probability tending to 1 as $N$ tends to infinity, $\mathcal{C}(G) \leq c N^{1-s}$ for some (possibly different) $c=c(s, \beta)$ independent of $N$.

This gives a bound on the mixing time (see [18]):

$$
\tau(G) \geq \frac{1-\log 2}{2}\left(\frac{1}{2 \mathcal{C}(G)}-1\right) \geq c N^{s-1}
$$

with probability tending to 1 , and $c=c(s, \beta)$ independent of $N$.

## 4 A Phase Transition

In the previous sections we have shown that the mixing time of $G_{s, \beta}$ is $N^{s-1}$ (disregarding poly-logarithmic factors) for $1<s<2$. When $s$ tends to 2 , this quantity tends to $N$. We will show that a phase transition occurs at $s=2$, meaning that for $s>2$ the mixing time will "jump" to $N^{2}$.

Proposition 4.1. Let $G=G_{s, \beta}(N)$ for $s>2$. Then the mixing time of $G$ satisfies

$$
\lim _{N \rightarrow \infty} \mathbb{P}\left[\tau(G) \geq c N^{2}\right]=1
$$

for some constant $c=c(s, \beta)$, independent of $N$.

Proof. For simplicity we assume that $N$ is divisible by 8 . For other $N$ the proof is similar.
Set

$$
A=\left\{1,2, \ldots, \frac{N}{2}\right\} \quad B=\left\{\frac{N}{2}+1, \ldots, \frac{3 N}{4}\right\} \quad C=\left\{\frac{3 N}{4}+1, \ldots, N\right\}
$$

Also, for $i=1,2, \ldots, 8$ set $K_{i}=\left\{(i-1) \frac{N}{8}+1, \ldots, i \frac{N}{8}\right\}$.
By the proof of Theorem 3.1 (A) in [4], there exists $c_{1}=c_{1}(s, \beta)$ such that with probability $1-o(1)$, we have that all sets $K_{1}, \ldots, K_{8}$ each contain at least $c_{1} N$ vertices of degree 2 .

Further, by rotating the cycle, without loss of generality we can assume that $\pi(A) \geq \pi(B \cup C)$, and that $\pi(B) \geq \pi(C)$. This implies that $\pi(A \cup B) \geq \frac{3}{4}$.
Fix a vertex $x \notin A \cup B$, and let $\left(S_{t} ; t \geq 0\right)$ be a simple random walk on $G$ starting at $S_{0}=x$.
Let $T$ be the hitting time of the set $A \cup B$. Note that at any time $t \geq \tau(G)$, we have that

$$
\frac{3}{4}-\mathbb{P}\left[S_{t} \in A \cup B\right] \leq \sum_{y \in A \cup B}\left|\pi(y)-P^{t}(x, y)\right| \leq 2 \Delta_{x}(t) \leq \frac{2}{e}
$$

So we conclude that for any $x \notin A \cup B$ and $t \geq \tau(G)$ we have that

$$
\underset{x}{\mathbb{P}}[T \leq t] \geq \underset{x}{\mathbb{P}}\left[S_{t} \in A \cup B\right]>0.01
$$

This implies that for any $x \notin A \cup B$ and for any real $s, \mathbb{P}_{x}[T>s] \leq 0.99^{(s / \tau-1)}$. Thus, there exists $c_{2}>0$ independent of $N$, such that

$$
\underset{x}{\mathbb{E}}[T]=\sum_{t=0}^{\infty} \underset{x}{\mathbb{P}}[T>t] \leq c_{2} \tau
$$

Set $u=\frac{3}{8} N$. Recall that there are at least $c_{1} N$ vertices of degree 2 separating $u$ from $A \cup B$ (on each side of $u$ ). We will show that this implies that $\mathbb{E}_{u}[T] \geq c_{3} N^{2}$ for some $c_{3}=c_{3}(s, \beta)$ independent of $N$.
We use the language of electrical networks, see [13, 16] for background. We remark that for the reader not familiar with these notions, one can use the Varopoulos-Carne bounds (see e.g. [16]) to show that a linear diameter implies that the mixing time is at least $\frac{c N^{2}}{\log N}$.

We can write $T=\sum_{x \notin A \cup B} V_{x}$ where $V_{x}$ is the number of visits to the vertex $x$, up to time $T$. Ground the set $A \cup B$ (so that its voltage is 0 ), and set a potential to $u$ so that there is a unit current flowing into $u$. We get that for any $x$, we have the identity $\mathbb{E}_{u}\left[V_{x}\right]=v(x) d(x)$, where $v(x)$ is the voltage at $x$, and $d(x)$ is the degree of $x$ (this follows from noting that $\mathbb{E}_{u}\left[V_{x}\right] / d(x)$ is harmonic).
Let $x_{1}, x_{2}, \ldots, x_{c_{1} N}$ be the vertices of degree 2 on the side of $u$ with at least $1 / 2$ the current (without loss of generality say in the interval $\left\{\frac{3}{8} N+1, \ldots, N\right\}$ ). Fix $1 \leq i \leq c_{1} N$. Since $x_{i}$ is a cut point, the current flowing into and out of $x_{i}$ is at least $1 / 2$. Thus, the voltage at $x_{i}$, $v\left(x_{i}\right)$, is at least $1 / 2$ the resistance between $x_{i}$ and $A \cup B$. This resistance is bounded from below by the number of cut edges (which are resistors of resistance 1 connected serially), which in turn is bounded by the number of vertices of degree 2 between $x_{i}$ and the set $A \cup B$. That is, $v\left(x_{i}\right) \geq i / 2$. Thus,

$$
\underset{u}{\mathbb{E}}[T] \geq \sum_{\substack{x \notin A \cup B \\ d(x)=2}} \underset{\substack{\underset{u}{x}}}{\mathbb{E}}\left[V_{x}\right]=\sum_{\substack{x \notin A \cup B \\ d(x)=2}} d(x) v(x) \geq \sum_{i=1}^{c_{1} N} i=c_{3} N^{2}
$$

for some $c_{3}=c_{3}(s, \beta)$ independent of $N$.
We have shown that with probability $1-o(1)$ there are a linear number of vertices of degree 2 separating a vertex $u$ from a set of high weight under the stationary distribution. Thus, $\tau(G) \geq c_{4} \mathbb{E}_{u}[T] \geq c_{5} N^{2}$, where $c_{4}, c_{5}$ depend only on $s$ and $\beta$.

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