# Decision trees and influences of variables over product probability spaces. 

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#### Abstract

A celebrated theorem of Friedgut says that every function $f:\{0,1\}^{n} \rightarrow\{0,1\}$ can be approximated by a function $g:\{0,1\}^{n} \rightarrow\{0,1\}$ with $\|f-g\|_{2}^{2} \leq \epsilon$ which depends only on $e^{O\left(I_{f} / \epsilon\right)}$ variables where $I_{f}$ is the sum of the influences of the variables of $f$. Dinur and Friedgut later showed that this statement also holds if we replace the discrete domain $\{0,1\}^{n}$ with the continuous domain $[0,1]^{n}$, under the extra assumption that $f$ is increasing. They conjectured that the condition of monotonicity is unnecessary and can be removed.

We show that certain constant-depth decision trees provide counter-examples to DinurFriedgut conjecture. This suggests a reformulation of the conjecture in which the function $g:[0,1]^{n} \rightarrow\{0,1\}$ instead of depending on a small number of variables has a decision tree of small depth. In fact we prove this reformulation by showing that the depth of the decision tree of $g$ can be bounded by $e^{O\left(I_{f} / \epsilon^{2}\right)}$.

Furthermore we consider a second notion of the influence of a variable, and study the functions that have bounded total influence in this sense. We use a theorem of Bourgain to show that these functions have certain properties. We also study the relation between the two different notions of influence.


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## 1 Introduction

The notion of the influence of a variable on a Boolean function over a product probability space plays an important role in computer science, combinatorics, statistical physics, economics and game theory (see for example KKL88, $\mathrm{BKK}^{+} 92$, Fri04, Bou02, Fri98, LMN93]). It is usually the case that the functions whose variables satisfy certain bounds on their influences have simple structures. Although results of this type are studied extensively (see for example Bou02, Fri98, Hat06, FKN02), still many basic questions are open (A handful of them can be found in (Fri04).

For a set $X$ and given a function $f: X^{n} \rightarrow \mathbb{R}$, roughly speaking the influence of the $i$-th variable measures the sensitivity of $f$ with respect to the changes in the $i$-th coordinate. There are numerous ways to measure this sensitivity and each gives rise to a different notion of the influence. It is usually the case that certain bounds on the influences imply some structure on $f$. For example there is a theorem of Friedgut (Theorem A below) in this flavor that concerns the case $X=\{0,1\}$. This theorem has many applications and provides a tool for proving hardness of approximation results using PCP reductions, and also serves as a tool in machine learning [DS05, KR03, DRS05, OS06]. Dinur and Friedgut later conjectured that this result holds in the continuous setting of $X=[0,1]$ as well. In this article we give a counter-example to their conjecture. Fortunately this does not mean that there is no continuous version of Friedgut's theorem. Indeed we characterize all functions that satisfy the conditions of DinurFriedgut conjecture, and prove that such functions enjoy very nice and simple structures.

A graph on $n$ vertices can be thought of as an element in $\{0,1\}\binom{n}{2}$, where each coordinate corresponds to one of the $\binom{n}{2}$ possible edges. A graph property on $n$ vertices is a subset of graphs on $n$ vertices which is invariant under the permutation of vertices. Hence such a graph property can be formulated as a function $f:\{0,1\}\left(\begin{array}{l}\binom{n}{2}\end{array} \rightarrow\{0,1\}\right.$ which is invariant under certain permutations of coordinates: the ones that come from the permutations of the vertices of a graph. Let $\mu_{p}(f)$ denote the probability that $f(G(n, p))=1$, where $G(n, p)$ is the random graph on $n$ vertices where each edge is present with probability $p$. It follows from a theorem of Margulis Mar74] and Russo [Rus81] that $\frac{d \mu_{p}}{d p}$ is closely related to the influences of variables for $f$. The derivative $\frac{d \mu_{p}}{d p}$ plays an important role in theory of random graphs and statistical physics. A bounded derivative shows that the property changes smoothly with respect to the changes in $p$. Friedgut [Fri99] used the connection to the influences to prove a beautiful theorem that characterizes all graph properties that satisfy $\frac{d \mu_{p}}{d p}<C$ for a constant $C$. He conjectured that being a graph property is irrelevant here and his characterization holds for every function $f:\{0,1\}^{n} \rightarrow\{0,1\}$. Bourgain [Fri99] proved a weaker version of Friedgut's conjecture. We will observe that Bourgain's proof works in the more general setting of functions $f: X^{n} \rightarrow \mathbb{R}$. Inspired by this we give a conjecture that is similar to Friedgut's conjecture but is for the general setting of $X^{n}$. We show that a positive answer to this conjecture would settle one of the conjectures in [Fri04].

## 2 Definitions and known facts

### 2.1 Notation

Throughout this paper, following Hardy notation, $C$ always denotes a universal constant, but not necessarily the same in different statements. Let $[n]=\{1, \ldots, n\}$ for every natural number
$n$. For two functions $f, g: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$, we write $f=O(g)$ if there exists a constant $C$ such that $f(x) \leq C g(x)$, for every $x \geq C$. We also write $f=o(g)$ if $\lim _{x \rightarrow \infty} f(x) / g(x)=0$. Sometimes we write $f=O_{a}(g)$ or $f=o_{a}(g)$ to specify that the involved constants may depend on $a$. When there is no subscript, the constants are universal. A function is called Boolean if its range is $\{0,1\}$.

Let $(X, \mathcal{F}, \mu)$ be a probability space, and let $X^{n}$ be the product space endowed with the product probability measure $\mu^{n}$. Our goal is to study measurable functions $f: X^{n} \rightarrow \mathbb{R}$. Sometimes we consider $f$ as a random variable and write $\mathbf{E}[f]$ (the expected value of $f$ ) for $\int f(x) d \mu^{n}(x)$. The variance of $f$ is defined as $\operatorname{Var}[f]=\mathbf{E}\left[f^{2}\right]-(\mathbf{E}[f])^{2}$. For $J \subseteq[n]$, and $a \in X$, we denote by $(J=a)$ the event $\left\{x: x_{i}=a \forall i \in J\right\}$. Thus we may write $\mathbf{E}[f \mid J=a]$ to denote the conditional expectation of $f$ on the event that all variables with indices in $J$ are set to be equal to $a$.

Suppose that $X$ is an ordered set. Then a function $f: X^{n} \rightarrow \mathbb{R}$ is called increasing if $x_{i} \leq y_{i}$ for all $i \in[n]$, implies $f(x) \leq f(y)$.

### 2.2 Fourier-Walsh expansion

Let $(X, \mathcal{F}, \mu)$ be a probability space, and let $X^{n}$ be the product space endowed with the product probability measure $\mu^{n}$. Consider a function $f: X^{n} \rightarrow \mathbb{R}$. Then it is possible to find functions $F_{S}: X^{n} \rightarrow \mathbb{R}$ for $S \subseteq[n]$ such that
(i) We have $f=\sum_{S \subseteq[n]} F_{S}$.
(ii) For $x=\left(x_{1}, \ldots, x_{n}\right) \in X^{n}$, the value of $F_{S}(x)$ depends only on the variables $\left\{x_{i}: i \in S\right\}$.
(iii) We have $\int F_{S} d \mu\left(x_{i}\right)=0$ for every $i \in S$.
(iv) We have $\int F_{S} F_{T} d \mu\left(x_{i}\right)=0$ for every $i \in T \backslash S$.

Then $f=\sum_{S \subseteq[n]} F_{S}$ is called the Fourier-Walsh expansion of $f$. Note that (iv) is implied by (iii) and from (i-iv) it is easy to see that

$$
\begin{equation*}
\|f\|_{2}^{2}=\int f^{2} d \mu^{n}=\sum_{S \subseteq[n]} \sum_{T \subseteq[n]} \int F_{S} F_{T} d \mu^{n}=\sum_{S \subseteq[n]} \int F_{S}^{2} d \mu^{n}=\sum_{S \subseteq[n]}\left\|F_{S}\right\|_{2}^{2} \tag{1}
\end{equation*}
$$

For uniform measure on $X=\{-1,1\}$, the Fourier-Walsh expansion becomes very simple. For $S \subseteq[n]$, let $w_{S}: X^{n} \rightarrow \mathbb{R}$ be defined as $w_{S}:\left(x_{1}, \ldots, x_{n}\right) \mapsto \prod_{i \in S} x_{i}$. Then it is easy to see that for $f: X^{n} \rightarrow \mathbb{R}$,

$$
f=\sum_{S \subseteq[n]} \widehat{f}(S) w_{S},
$$

where

$$
\widehat{f}(S)=\int f(x) w_{S}(x) d \mu^{n}(x)
$$

So here in the Fourier-Walsh expansion $f=\sum_{S \subseteq[n]} F_{S}$, we have $F_{S}=\widehat{f}(S) w_{S}$, where $\widehat{f}(S)$ is just a real number.

### 2.3 Influences

Let $(X, \mathcal{F}, \mu)$ be a probability space, and let $X^{n}$ be the product space endowed with the product probability measure $\mu^{n}$. For any $x=\left(x_{1}, \ldots, x_{n}\right) \in X^{n}$, define

$$
\begin{equation*}
s_{j}(x)=\left\{y \in X^{n}: y_{i}=x_{i} \forall i \neq j\right\}, \tag{2}
\end{equation*}
$$

or in other words $s_{j}(x)$ is the set of elements that can be obtained from $x$ by changing only the $j$-th coordinate. For $f: X^{n} \rightarrow \mathbb{R}$, we want to define the notion of the influence of the $j$-th variable as a measurement for the dependence of the value of $f$ to the value of the $j$-th coordinate. Suppose that $f$ is constant on $s_{j}(x)$ for some $x$. Then for this particular $x$, the value of $x_{j}$ is irrelevant to the value of $f$. Thus it is natural to define the influence of $x_{j}$ as the probability that $f$ is not constant on $s_{j}(x)$. This gives rise to our first notion of the influence. Next consider the function $f:[0,1]^{2} \rightarrow \mathbb{R}$ defined as $f\left(x_{1}, x_{2}\right)=10 x_{1}+x_{2}$. With the above definition both $x_{1}$ and $x_{2}$ have influences 1 . But note that changing the value of $x_{1}$ changes the value of $f$ more drastically than changing the value of $x_{2}$. The next definition of the influence captures this difference and is defined as the following. For a particular $x, s_{j}(x)$ is endowed with the probability measure $\mu$, and $\operatorname{Var}_{s_{j}(x)}[f]$ measures the variation of $f$ on $s_{j}(x)$. Thus we can define the influence of $x_{j}$ as $\mathbf{E}\left[\operatorname{Var}_{s_{j}(x)}[f]\right]$. Let us summarize the discussion in the following definition.

Definition 2.1 Let $f: X^{n} \rightarrow \mathbb{R}$ where $X^{n}$ is a product probability space with the product probability measure $\mu^{n}$.

- The influence of the $i$-th variable on $f$ is defined as

$$
\begin{equation*}
I_{f}(i)=\operatorname{Pr}\left(\left\{x: f \text { is not constant on } s_{i}(x)\right\}\right) . \tag{3}
\end{equation*}
$$

- The variational influence of the $i$-th variable on $f$ is defined as

$$
\begin{equation*}
\tilde{I}_{f}(i)=\mathbf{E}\left[\operatorname{Var}_{s_{i}(x)}[f]\right] . \tag{4}
\end{equation*}
$$

The total influence and total variational influences on $f$ are defined respectively as

$$
I_{f}=\sum_{i=1}^{n} I_{f}(i) \quad \text { and } \quad \tilde{I}_{f}=\sum_{i=1}^{n} \tilde{I}_{f}(i)
$$

Remark 2.2 Note that (3) and (4) imply some measurability conditions on $f$ which we assume to hold throughout this article. Since we are dealing with probability spaces, for our purpose, without loss of generality we might assume that $(X, \mu)$ is the Lebesgue measure on $[0,1]$. Throughout the article when a discrete set $X$ is considered as a measure space without mentioning the measure, the corresponding measure is assumed to be the uniform probability measure.

The influences are of particular interest when $f$ is the characteristic function of a set, or in other words the range of $f$ is $\{0,1\}$. Note that for a function $f: X^{n} \rightarrow\{0,1\}$, we always have $4 \tilde{I}_{f}(j) \leq I_{f}(j)$. Indeed let

$$
a_{j}(x)=\left\{\begin{array}{ll}
1 & f \text { is not constant on } s_{j}(x) \\
0 & f \text { is a constant on } s_{j}(x)
\end{array} .\right.
$$

Let $x$ be a random variable that takes values in $X^{n}$ according to $\mu^{n}$. Note that $a_{j}(x)=0$ implies $\operatorname{Var}_{s_{j}(x)}[f]=0$, and in general since $f$ is Boolean, $\operatorname{Var}_{s_{j}(x)}[f] \leq \frac{1}{4}$. Thus

$$
\begin{equation*}
I_{f}(j)=\mathbf{E}\left[a_{j}(x)\right] \geq 4 \mathbf{E}\left[\operatorname{Var}_{s_{j}(x)}[f]\right]=4 \tilde{I}_{f}(j) \tag{5}
\end{equation*}
$$

In the discrete case of the uniform measure on $\{0,1\}^{n}$, the equality holds in (5), and the two definitions differ only in a constant factor of 4 . But in the context of general Boolean functions, $I_{f}(j)$ can be much larger than $\tilde{I}_{f}(j)$. For example define $f:[0,1]^{n} \rightarrow\{0,1\}$ as $f(x)=1$ if $x_{j} \geq \epsilon$, and $f(x)=0$ otherwise. Then $I_{f}(j)=1$ while $\tilde{I}_{f}(j)=\epsilon(1-\epsilon)$.

The well-known result of [KKL88] which is known as the KKL inequality says that for every function $f:\{0,1\}^{n} \rightarrow\{0,1\}$, there exists a variable whose influence is at least $C \rho(1-\rho) \frac{\ln n}{n}$ where $\rho=\mathbf{E}[f]=\operatorname{Pr}_{x}[f(x)=1]$. Later Friedgut Fri04 noticed that the approach of KKL88] leads to an interesting result that later found many applications [DS05, KR03, DRS05, OS06:

Theorem A. For every $f:\{0,1\}^{n} \rightarrow\{0,1\}$, there exists an approximation $g:\{0,1\}^{n} \rightarrow\{0,1\}$ with $\|f-g\|_{2}^{2} \leq \epsilon$ which depends only on $e^{C I_{f} / \epsilon}$ variables.

In BKK $^{+} 92$ ], Bourgain et al. generalized the framework of KKL to the continuous domain, $f:[0,1]^{n} \rightarrow\{0,1\}$, and proved that the KKL inequality holds in this setting as well. The inequality in this case is usually called the BKKKL inequality.

Friedgut and Kalai [FK96] noticed that the proof of $\left[\mathrm{BKK}^{+} 92\right]$ can be modified to imply the following statement which is one of our main tools in this article.

Theorem B. For every $f:[0,1]^{n} \rightarrow\{0,1\}$ with $\mathbf{E}[f]=\rho$, there always exists $i \in[n]$ such that

$$
\begin{equation*}
e^{-C I_{f} / \rho(1-\rho)} \leq I_{f}(i) . \tag{6}
\end{equation*}
$$

The main tool in the proof of many results about the influences including Theorem A and Theorem B is the Bonami-Beckner inequality. This inequality was originally proved by Bonami [Bon70] and then independently by Beckner [Bec75]. It was first used to analyze discrete problems in the proof of the KKL inequality KKL88. We will use the inequality in the following form.

Theorem C.[Bonami-Beckner Inequality] Let $f:\{-1,1\}^{n} \rightarrow \mathbb{R}$ be a function with the FourierWalsh expansion $f:=\sum_{S \subseteq[n]} \widehat{f}(S)$. Then for $p \geq 2$, and $\delta=\sqrt{p-1}$,

$$
\|f\|_{p} \leq\left\|\sum_{S \subseteq[n]} \delta^{|S|} \widehat{f}(S) w_{S}\right\|_{2} ;
$$

and for $1 \leq p \leq 2$,

$$
\|f\|_{p} \geq\left\|\sum_{S \subseteq[n]} \delta^{|S|} \widehat{f}(S) w_{S}\right\|_{2}
$$

### 2.4 Decision trees

Consider a product probability space $X^{n}$ with the product probability measure $\mu^{n}$. A decision tree $T$ is a tree where every internal node $v$ is labeled with an index $i_{v} \in[n]$ such that if $u$ is the ancestor of $v$, then $i_{u} \neq i_{v}$. Children of $v$ are in a one-to-one correspondence with the elements of $X$. Moreover a value $\operatorname{val}(v) \in \mathbb{R}$ is assigned to every leaf $v$ of $T$. The depth of the decision tree is the maximum number of edges on a path from the root to a leaf. We call the immediate parents of the leaves, the leaf-parents. A decision tree is called complete if all leaves are in the same distance from the root.

Every decision tree $T$ corresponds to a function $f_{T}: X^{n} \rightarrow \mathbb{R}$ : consider $x \in X^{n}$; we start from the root and explore a path to one of the leaves in the following way: Every time that we meet an internal node $v$, we look at the value $x_{i_{v}}$ and move to the corresponding child of $v$. We continue until we reach a leaf $u$. Then $f(x)=\operatorname{val}(u)$. The traversed path from the root to the leaf is called the computing path of $x$.

There are interesting relations between the structure of a decision tree $T$, and the influences of the variables on $f_{T}$ (see for example [OSS05]). The following simple lemma shows that the total influence on $f_{T}$ is bounded by the depth of $T$.

Lemma 2.3 Consider a decision tree $T$ of depth $k$ and the function $f_{T}$. Then $I_{f_{T}} \leq k$.
Proof. Choose $x \in X^{n}$ at random according to $\mu^{n}$, and $i \in[n]$ uniformly at random. If $i$ does not appear on the computing path of $x$, then $f_{T}$ is constant on $s_{i}(x)$. The probability that $i$ appears on the computing path is at most $k / n$. Now the assertion of the lemma follows from the linearity of the expectation and the fact that there are $n$ different variables.

## 3 Main Results

### 3.1 Counter-example to Dinur-Friedgut Conjecture

The proof of BKKKL inequality and Theorem B rely on a simple reduction that reduces the problem to the case that $f$ is increasing. Similar but more complicated arguments of this sort were previously used by Talagrand in the context of isoperimetric inequalities for Banach-Space valued random variable [Tal89].

A key but simple observation in $\left[\mathrm{BKK}^{+} 92\right]$ is that in the case of increasing functions, one can without increasing the influences by a factor of more than 2 , reduce the problem further to the case where the domain of the function is $\{0,1\}^{m}$ endowed with the uniform probability measure, where $m=O(n \log n)$.

Although Theorem B provides a continuous version of the KKL inequality, a continuous version of Theorem A remained unknown. In fact Dinur and Friedgut (see [Fri04]) considered the case where $f$ is increasing, and observed that in this case the reduction of $\left[\mathrm{BKK}^{+} 92\right]$ to $\{0,1\}^{m}$ shows that $f$ essentially depends on $e^{C I_{f} / \epsilon}$ number of variables, or formally for every $\epsilon>0$ there exists a function $g:[0,1]^{n} \rightarrow\{0,1\}$ which depends on $e^{C I_{f} / \epsilon}$ number of variables and $\|f-g\|_{2}^{2} \leq \epsilon$. This provides a continuous version of Theorem A, but under the extra assumption of monotonicity. They conjectured (Conjecture 2.13 in [Fri04]) that this extra condition is unnecessary. The reason that the reduction to increasing functions cannot be used here is that with this reduction one may get a function which depends on a small number of variables while the original function cannot be approximated by any such function.

We disprove Dinur-Friedgut conjecture in the following lemma by introducing a function for which the total influence is small, but it cannot be approximated by any function which depends on a small number of variables:

Lemma 3.1 Let $k \in \mathbb{N}$ and $r \geq 2$ be an even integer, and let $\mu$ be the uniform probability measure on $X=\{0, \ldots, r-1\}$. For $n \geq r^{k}$, there exists a function $f: X^{n} \rightarrow\{0,1\}$ such that $I_{f} \leq k$ and $\|f-g\|_{2}^{2} \geq \frac{1}{4}$ for every $g: \bar{X}^{n} \rightarrow\{0,1\}$ that depends on at most $r^{k-1} / 2$ variables.

Proof. Consider a complete decision tree $T$ of depth $k$ where $i_{v} \neq i_{u}$ for every two internal nodes $u$ and $v$. Consider a leaf $w$ which is the immediate child of a node $u$, and is corresponded to $a \in\{0, \ldots, r-1\}$. We let $\operatorname{val}(w)=\left\lfloor\frac{2 a}{r}\right\rfloor$.

By Lemma 2.3 we have $I_{f_{T}} \leq k$. On the other hand consider a function $g: X^{n} \rightarrow\{0,1\}$ that depends on at most $r^{k-1} / 2$ variables. Let $L$ be the set of the leaf-parents of $T$. Note that $|L|=r^{k-1}$ which shows that for at least half of the nodes $v \in L, g$ is a constant on the children of $v$. This completes the proof because $f_{T} \neq g$ on half of the children of every such $v$.

In the first look Lemma 3.1 may sound disappointing because it shows that the continuous version of Theorem A is not true. But further inspection shows that indeed a continuous version does exist, but it is slightly different from the discrete case. Our counter-example to Dinur-Friedgut conjecture has a decision tree of constant depth. We will show that for a fixed $\epsilon$, having a constant-depth decision tree is the right reformulation of the conjecture. First we "discretize" the problem:

Remark 3.2 Let $f:[0,1]^{n} \rightarrow\{0,1\}$ be a function that satisfies the measurability conditions of Remark [2.2. Fix any $0<\delta<\frac{1}{10}$. Consider $m \in \mathbb{N}$ and subdivide $[0,1]^{n}$ into $m^{n}$ equal size disjoint cells by subdividing each one of the base intervals into $m$ intervals. For every $x \in[0,1]^{n}$ denote by $C_{x}$ the cell that contains $x$. For every $1 \leq i \leq n$, and cell $C$, let $s_{i}(C)$ denote the set of $m$ cells that are obtained by changing the cell $C$ in the $i$ th coordinate. Consider a measurable set $A \subseteq[0,1]^{n}$. We say that a cell $C$ is $(1-\delta, A)$-determined, if either $\operatorname{Pr}(A \mid C) \geq 1-\delta$ or $\operatorname{Pr}\left(A^{c} \mid C\right) \geq 1-\delta$. Since $A$ is measurable, for sufficiently large $m$, more than $(1-\delta)$ fraction of the cells will be $(1-\delta, A)$-determined. For every $1 \leq i \leq n$, let

$$
A_{i}:=\left\{x: f \text { is a constant on } s_{i}(x)\right\} .
$$

Since $A_{i}$ are measurable, one can take $m$ to be sufficiently large so that $(1-\delta)$ fraction of the cells are simultaneously $(1-\delta, f)$-determined and $\left(1-\delta, A_{i}\right)$-determined, for every $1 \leq i \leq n$.

Now we construct a "discrete" approximation $h$ of $f$, in the following way:
(i) For every cell $C, h(C)=\{1\}$, if $\mathbf{E}(f \mid C) \geq 1-\delta$. Furthermore for every such $C$ and every $1 \leq i \leq n$, if $\operatorname{Pr}\left(A_{i} \mid C\right) \geq 1-\delta$, then $h\left(C^{\prime}\right)=\{1\}$ for every $C^{\prime} \in s_{i}(C)$.
(ii) Similarly for every cell $C, h(C)=\{0\}$, if $\mathbf{E}(1-f \mid C) \geq 1-\delta$. Furthermore for every such $C$ and every $1 \leq i \leq n$, if $\operatorname{Pr}\left(A_{i} \mid C\right) \geq 1-\delta$, then $h\left(C^{\prime}\right)=\{0\}$ for every $C^{\prime} \in s_{i}(C)$.
(iii) The value of $h$ on the rest of the cells is defined arbitrarily.

First we need to verify that $h$ is well-defined. Note that for every $C^{\prime}$ in (i) we have $\mathbf{E}\left(f \mid C^{\prime}\right) \geq$ $1-2 \delta>1 / 2$, while for every $C^{\prime}$ in (ii), $\mathbf{E}\left(1-f \mid C^{\prime}\right) \geq 1-2 \delta>1 / 2$. This shows that $h$ is well-defined.

Note that $h$ is defined so that if $C_{x}$ is $(1-\delta, f)$-determined and $\operatorname{Pr}\left(A_{i} \mid C_{x}\right) \geq 1-\delta$, then $h$ is constant on $s_{i}\left(C_{x}\right)$. Thus

$$
\begin{align*}
I_{h}(i) \leq & \operatorname{Pr}\left(\left\{x: C_{x} \text { is not }(1-\delta, f) \text {-determined }\right\}\right)+\operatorname{Pr}\left(\left\{x: C_{x} \text { is not }\left(1-\delta, A_{i}\right) \text {-determined }\right\}\right) \\
& +\operatorname{Pr}\left(\left\{x: \operatorname{Pr}\left(A_{i}^{c} \mid C_{x}\right) \geq 1-\delta\right\}\right) \leq \delta+\delta+\frac{\operatorname{Pr}\left(A_{i}^{c}\right)}{1-\delta}=2 \delta+\frac{I_{f}(i)}{1-\delta} \tag{7}
\end{align*}
$$

Moreover

$$
\begin{align*}
\operatorname{Pr}[f \neq h] & \leq \operatorname{Pr}\left(\left\{x: C_{x} \text { is not }(1-\delta, f) \text {-determined }\right\}\right)+\delta \operatorname{Pr}\left(\left\{x: C_{x} \text { is }(1-\delta, f) \text {-determined }\right\}\right) \\
& \leq 2 \delta . \tag{8}
\end{align*}
$$

Inequalities (7) and (8) show that by taking $\delta$ to be arbitrarily small, we can obtain a "discrete" approximation $h$ of $f$, which is arbitrarily close to $f$ in $\ell_{2}$ norm, and its total influence is not much larger than $I_{f}$.

Theorem 3.3 Let $f:[0,1]^{n} \rightarrow\{0,1\}$ satisfy $I_{f} \leq B$. Then for every $\epsilon>0$ there exists a function $g:[0,1]^{n} \rightarrow\{0,1\}$ such that $\|f-g\|_{2}^{2} \leq \epsilon$ and $g$ has a decision tree of depth at most $e^{C B / \epsilon^{2}}$.

Proof. By Remark 3.2 we can assume that the underlying probability space is the uniform probability measure on $\{0, \ldots, r-1\}^{n}$ for a sufficiently large $r$ that depends on $f$ and $\epsilon$.

By a raw decision tree we mean the mathematical object that is obtained by removing the values $\operatorname{val}(v)$ from the leaves of a decision tree. Consider a function $f:\{0, \ldots, r-1\}^{n} \rightarrow\{0,1\}$ with $I_{f} \leq B$. For every node $v$ in a raw decision tree, $f$ induces a function $f_{v}$ of the variables that are not assigned a value through the path from the root to $v$. This function is obtained by fixing the values of the variables on the path from the root to $v$. Denote by $m(v)$ the variable with maximum influence in $f_{v}$, and note that if $v$ is in distance $k$ from the root then $\mu(v)=r^{-k}$ is the fraction of $x \in\{0, \ldots, r-1\}^{n}$ that $v$ belongs to their computing paths (i.e. $x$ agrees with the path from the root to $v$ ).

Let $C^{\prime} \geq 11$ be the constant from the exponent in (6). Start with a single node as a raw decision tree. Consider the following procedure: For every leaf $v$, we let $i_{v}:=m(v)$ and we branch on this variable, adding $r$ children to $v$. Each application of this procedure increases the depth of the tree by one. We repeatedly apply the procedure until we obtain a raw decision tree $T$ in which the fraction of the leaf-parents $v$ that satisfy $I_{f_{v}}\left(i_{v}\right) \geq e^{-18 C^{\prime} B / \epsilon^{2}}$ is at most $\epsilon / 3$. Let $S$ denote the set of the nodes $v$ that satisfy $I_{f_{v}}\left(i_{v}\right) \geq e^{-18 C^{\prime} B / \epsilon^{2}}$, and let $L_{1} \subseteq S$ be the set of the leaf-parents in $S$. We claim that the depth of $T$ is at most $e^{\left(18 C^{\prime}+4\right) B / \epsilon^{2}}$.

Suppose to the contrary that this is not true. Consider a node $u$ and note that at node $u$, the $i_{u}$ 'th variable contributes $\mu(u) \times I_{f_{u}}\left(i_{u}\right)$ to $I_{f}\left(i_{u}\right)$. Now consider another node $v \in T$ such that $i_{v}=i_{u}$. Since none of $u$ and $v$ is the ancestor of the other, the contribution of the $i_{u}{ }^{\prime}$ 'th variable to $I_{f}\left(i_{u}\right)$ at node $u$ is disjoint from its contribution at node $v$. Denoting by $h$ the depth of $T$ and by $D_{k}$ the set of the vertices in distance $k$ from its root, the above discussion

[^0]shows that
\[

$$
\begin{aligned}
I_{f} & =\sum_{v \in T} \mu(v) I_{f_{v}}\left(i_{v}\right)=\sum_{k=0}^{h-1} \sum_{v \in D_{k}} \mu(v) I_{f_{v}}\left(i_{v}\right) \geq \\
& \geq \sum_{k=0}^{h-1} \sum_{v \in D_{k} \cap S} \mu(v) I_{f_{v}}\left(i_{v}\right) \geq \sum_{k=0}^{h-2}(\epsilon / 3) e^{-18 C^{\prime} B / \epsilon^{2}} .
\end{aligned}
$$
\]

Thus $h \leq(3 B / \epsilon) e^{18 C^{\prime} B / \epsilon^{2}}+1 \leq e^{\left(18 C^{\prime}+4\right) B / \epsilon^{2}}$.
It is easy to see that for a similar reason, $\sum_{v} \mu(v) I_{f_{v}} \leq I_{f}$ where now the sum is only over all leaf-parents of $T$. This shows that the fraction of the leaf-parents $v$ that satisfy $I_{f_{v}} \geq 3 B / \epsilon$ is at most $\epsilon / 3$. Let $L_{2}$ denote the set of the leaf-parents that satisfy $I_{f_{v}} \geq 3 B / \epsilon$.

Consider a leaf-parent $v \notin L_{1} \cup L_{2}$, i.e. $I_{f_{v}}\left(m\left(f_{v}\right)\right) \leq e^{-18 C^{\prime} B / \epsilon^{2}}$ and $I_{f_{v}} \leq 3 B / \epsilon$. By Theorem B.

$$
\left(\mathbf{E}\left[f_{v}\right]\right)\left(1-\mathbf{E}\left[f_{v}\right]\right) \leq \epsilon / 6,
$$

which implies that $\min \left(\mathbf{E}\left[f_{v}\right], 1-\mathbf{E}\left[f_{v}\right]\right) \leq \epsilon / 3$. Now to every leaf $w$ with parent $u$ we assign

$$
\operatorname{val}(w)=\left\{\begin{array}{ll}
0 & \mathbf{E}\left[f_{u}\right] \leq 1 / 2 \\
1 & \mathbf{E}\left[f_{u}\right]>1 / 2
\end{array} .\right.
$$

Let $g$ be the function that is computed by this decision tree. Denote by $L$ the set of all leaf-parents and correspond to every $x$ the unique leaf-parent $l(x) \in L$ on its computing path.

$$
\begin{aligned}
\|f-g\|_{2}^{2}= & \operatorname{Pr}[f(x) \neq g(x)] \leq \operatorname{Pr}\left[l(x) \in L_{1}\right]+\operatorname{Pr}\left[l(x) \in L_{2}\right]+ \\
& \sum_{v \in L \backslash\left(L_{1} \cup L_{2}\right)} \mu(v) \min \left(\mathbf{E}\left[f_{v}\right], 1-\mathbf{E}\left[f_{v}\right]\right) \leq \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} \leq \epsilon .
\end{aligned}
$$

### 3.2 Variational Influences for Boolean functions

Our next goal is to study the relation between the influences and the variational influences. The next theorem shows that when all influences are small then the total variational influence is large. One can prove this theorem easily using Theorem 1.5 in [Tal94], but for the sake of the completeness we give a direct proof here.

Theorem 3.4 For $f: X^{n} \rightarrow \mathbb{R}$, we have

$$
\begin{equation*}
\operatorname{Var}[f] \leq 10 \sum_{i=1}^{n} \frac{\tilde{I}_{f}(i)}{\log \left(1 / I_{f}(i)\right)} \tag{9}
\end{equation*}
$$

Proof. Let $\iota_{i}(x)=f(x)-\mathbf{E}_{s_{i}(x)}[f]$, and note that $I_{f}(i) \geq \mathbf{E}\left[1_{\left[\iota_{i} \neq 0\right.}\right]$, and $\tilde{I}_{f}(i)=\left\|\iota_{i}\right\|_{2}^{2}$. Let $f=\sum_{S \subseteq[n]} F_{S}$ be the Fourier-Walsh expansion of $f$. Note that

$$
\mathbf{E}_{s_{i}(x)}\left[F_{S}\right]=\int F_{S} d \mu\left(x_{i}\right)= \begin{cases}F_{S} & i \notin S \\ 0 & i \in S\end{cases}
$$

Thus the Fourier-Walsh expansion of $\iota_{i}$ is the following.

$$
\iota_{i}=\sum_{S \subseteq[n]} F_{S}-\mathbf{E}_{s_{i}(x)}\left[\sum_{S \subseteq[n]} F_{S}\right]=\sum_{S \subseteq[n]} F_{S}-\sum_{S: i \notin S} F_{S}=\sum_{S: i \in S} F_{S} .
$$

For $1<p \leq 2$, by Hölder's inequality we have

$$
\left\|\iota_{i}\right\|_{p}=\left(\int \iota_{i}^{p}\right)^{1 / p}=\left(\int \iota_{i}^{p} 1_{\left[\iota_{i} \neq 0\right]}\right)^{1 / p} \leq\left\|\iota_{i}\right\|_{2}\left(\int 1_{\left[\iota_{i} \neq 0\right]}\right)^{(1-p / 2) / p} \leq\left\|\iota_{i}\right\|_{2} I_{f}(i)^{\frac{1}{p}-\frac{1}{2}} .
$$

On the other hand by (a generalization of) Theorem C , we have

$$
\left(\sum_{S: i \in S,|S|=k}\left\|F_{S}\right\|_{2}^{2}\right)^{1 / 2}=\left(\sum_{|S|=k}{\left.\widehat{\left(\iota_{i}\right.}\right)}_{S}^{2}\right)^{1 / 2} \leq(p-1)^{-k / 2}\left\|\iota_{i}\right\|_{p}
$$

Thus for $p=3 / 2$ we get

$$
\sum_{S: i \in S,|S| \leq k} \frac{\left\|F_{S}\right\|_{2}^{2}}{|S|}=\sum_{t=1}^{k} \frac{1}{t} \sum_{S: i \in S,|S|=t}\left\|F_{S}\right\|_{2}^{2} \leq \sum_{t=1}^{k} \frac{2^{t}}{t} \tilde{I}_{f}(i) I_{f}(i)^{1 / 3} \leq \frac{2^{k+2}}{k+1} \tilde{I}_{f}(i) I_{f}(i)^{1 / 3} .
$$

On the other hand since $\sum_{S: i \in S}\left\|F_{S}\right\|_{2}^{2}=\tilde{I}_{f}(i)$, we get

$$
\sum_{S: i \in S,|S|>k} \frac{\left\|F_{S}\right\|_{2}^{2}}{|S|} \leq \frac{\tilde{I}_{f}(i)}{k+1}
$$

and hence

$$
\sum_{S: i \in S} \frac{\left\|F_{S}\right\|_{2}^{2}}{|S|} \leq \frac{1}{k+1}\left(2^{k+2} \tilde{I}_{f}(i) I_{f}(i)^{1 / 3}+\tilde{I}_{f}(i)\right) .
$$

Letting $k=\frac{\log 1 / I_{f}(i)}{3}$ we get

$$
\begin{equation*}
\sum_{S: i \in S} \frac{\left\|F_{S}\right\|_{2}^{2}}{|S|} \leq \frac{10 \tilde{I}_{f}(i)}{\log 1 / I_{f}(i)} \tag{10}
\end{equation*}
$$

Summing (10) over all $i \in[n]$, we get (9).
Let us next investigate the properties of the functions that satisfy $\tilde{I}_{f} \leq B$ for some constant $B$. As it is already noticed by Dinur and Friedgut (see [Fri04]) a hypercontractivity argument similar to Fri98 implies that every function $f:\{0,1, \ldots, r-1\}^{n} \rightarrow\{0,1\}$ with $I_{f} \leq B$ essentially depends on $r^{C B / \epsilon}$ number of variables. In fact such an argument implies more:

Lemma 3.5 Let $f:\{0,1, \ldots, r-1\}^{n} \rightarrow\{0,1\}$ satisfy $\tilde{I}_{f} \leq B$. Then for every $\epsilon>0$, there exists a function $g:\{0,1, \ldots, r-1\}^{n} \rightarrow\{0,1\}$ which depends on $r^{C B / \epsilon}$ number of variables, and $\|f-g\|_{2}^{2} \leq \epsilon$.

Lemma 3.1 shows that Lemma 3.5 is sharp up to the constant in the exponent. In the more general setting of functions $f:[0,1]^{n} \rightarrow\{0,1\}$, bounding the total variational influence implies less structure on the function. The following example shows that Theorem 3.3 does not need to hold when one replaces the bound $I_{f} \leq B$ with $\tilde{I}_{f} \leq B$.

Example 3.6 Let $f:[0,1]^{n} \rightarrow\{0,1\}$ be defined as $f(x)=0$ if and only if $x \leq\left(1-\frac{1}{n}, \ldots, 1-\frac{1}{n}\right)$. It is easy to see that $\mathbf{E}[f] \approx 1 / e$ and $\tilde{I}_{f} \leq 2$. Note that $f$ cannot be approximated by a function with a decision tree of depth $o(n)$.

The following theorem which is implied by Bourgain's proof for Proposition 1 in the appendix of [Fri99] shows that bounded variational influence implies some weak structure on the function.
Theorem 3.7 Let $f:[0,1]^{n} \rightarrow\{0,1\}$ satisfy $\tilde{I}_{f} \leq B$. Then there exists a set $J \subseteq[n]$ of size at most $10 B$ and an assignment of values to the variables in $J$ such that conditioning on this partial assignment, the expected value of $f$ changes by at least $3^{-500 B^{2}}$.

To see that why Bourgain's proof implies Theorem 3.7 note that according to Equation (2.18) in the appendix of [Fri99] if $\tilde{I}_{f} \leq B$, then

$$
\begin{equation*}
\mathbf{E}\left[\max _{J:|J| \leq 10 B}\left|\mathbf{E}_{J}[f](x)-\mathbf{E}[f]\right|\right] \geq 3^{-500 B^{2}} \tag{11}
\end{equation*}
$$

This immediately implies Theorem [3.7. Inspired by Conjecture 1.5 in [Fri99], we conjecture that Theorem 3.7 can be improved to the following:
Conjecture 3.8 There exists a function $k: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that the following holds. Let $B, \epsilon>0$ be two constants. For every $f:[0,1]^{n} \rightarrow\{0,1\}$ that satisfies $\tilde{I}_{f} \leq B$, there exists a set $J \subseteq[n]$ of size $k(B, \epsilon)$ and an assignment of values to the variables in $J$ such that conditioning on this partial assignment, the expected value of $f$ is either less than $\epsilon$ or greater than $1-\epsilon$.

### 3.3 Subcubes of increasing sets

The following theorem proved in KKL88] is a straightforward corollary of the KKL inequality.
Theorem D. There exists a function $k: \mathbb{R}^{2} \rightarrow \mathbb{R}^{+}$such that the following holds. For every increasing function $f:\{0,1\}^{n} \rightarrow\{0,1\}$ with $\operatorname{Var}[f]=\rho$ and every $\epsilon>0$, there exists a set of coordinates $J \subseteq[n]$ such that $|J| \leq k(\rho, \epsilon) n / \log n$ and $\mathbf{E}[f \mid J=1] \geq 1-\epsilon$.

Although it is wrongfully claimed in $\mathrm{BKK}^{+} 92$ that one can use Theorem $B$ to prove that the same statement holds for functions $f:[0,1]^{n} \rightarrow\{0,1\}$, as it is noticed by Friedgut in [Fri04], the function in Example 3.6 shows that this is not true.

Friedgut Fri04] suggested the following conjecture as the continuous version of Theorem D .
Conjecture 3.9 There exists a function $k: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that for every $\epsilon>0, \lim _{n \rightarrow \infty} \frac{k(\epsilon, n)}{n}=$ 0 , and the following holds. Let $f:[0,1]^{n} \rightarrow\{0,1\}$ be an increasing function. There exists a set of coordinates $J \subseteq[n]$ such that $|J| \leq k(\epsilon, n)$ and either $\mathbf{E}[f(x) \mid J=1] \geq 1-\epsilon$ or $\mathbf{E}[f(x) \mid J=0] \leq \epsilon$.

The next proposition shows that Conjecture 3.8 implies Conjecture 3.9.
Proposition 3.10 If Conjecture 3.8 is true, then there exists a function $k: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that for every $\epsilon>0, \lim _{n \rightarrow \infty} \frac{k(\epsilon, n)}{n}=0$, and the following holds. Let $f:[0,1]^{n} \rightarrow\{0,1\}$ be an increasing function. Then there exists a set of coordinates $J \subseteq[n]$ such that $|J| \leq k(\epsilon, n)$ and either $\mathbf{E}[f(x) \mid J=1] \geq 1-\epsilon$ or $\mathbf{E}[f(x) \mid J=0] \leq \epsilon$.

Proof. For a set $J \subseteq[n]$, let $\left.f\right|_{J=1}:[0,1]^{n} \rightarrow\{0,1\}$ be defined as $\left.f\right|_{J=1}: x \mapsto f(y)$ where

$$
y_{i}= \begin{cases}1 & i \in J \\ x_{i} & i \notin J\end{cases}
$$

Fix some $B>0$, and let $J_{0}:=\emptyset$. For $i \geq 0$, inductively obtain $J_{i+1}$ from $J_{i}$ as in the following. We will use the notation $f_{i}=\left.f\right|_{J_{i}=1}$.

If $\tilde{I}_{f_{i}} \geq B$, then there exists a variable $j \in[n]$ such that $\tilde{I}_{f_{i}}(j) \geq B / n$. Let $J_{i+1}=J_{i} \cup\{j\}$. Note that for every $y \in A_{i}:=\left\{x: \mathbf{E}_{s_{j}(x)}\left[f_{i}\right] \neq 0\right\}$, we have $f_{i+1}(y)=1$. Thus $\mathbf{E}\left[f_{i+1}\right] \geq$ $\mathbf{E}\left[1_{\left[x \in A_{i}\right]}\right]$ and furthermore we have $\mathbf{E}\left[f_{i}\right]=\mathbf{E}\left[\mathbf{E}_{s_{j}(x)}\left[f_{i}\right]\right]$. Hence

$$
\begin{aligned}
\mathbf{E}\left[f_{i+1}\right]-\mathbf{E}\left[f_{i}\right] & \geq \mathbf{E}\left[1_{\left[x \in A_{i}\right]}\right]-\mathbf{E}\left[\mathbf{E}_{s_{j}(x)}\left[f_{i}\right]\right]=\mathbf{E}\left[1_{\left[x \in A_{i}\right]}-\mathbf{E}_{s_{j}(x)}\left[f_{i}\right]\right] \\
& \geq \mathbf{E}\left[\left(\mathbf{E}_{s_{j}(x)}\left[f_{i}\right]\right)\left(1-\mathbf{E}_{s_{j}(x)}\left[f_{i}\right]\right)\right]=\mathbf{E}\left[\operatorname{Var}_{s_{j}(x)}\left[f_{i}\right]\right]=\tilde{I}_{f_{i}}(j) \geq B / n .
\end{aligned}
$$

Continuing in this manner we find a set $J_{k}$ for some $k \leq n / B$ such that $\left|J_{k}\right| \leq n / B$ and either $E\left[f \mid J_{k}=1\right] \geq 1-\epsilon$ or $\tilde{I}_{f_{k}} \leq B$. But if $\tilde{I}_{f_{k}} \leq B$, then assuming Conjecture 3.8 we can find a set $J^{\prime}$ of size $\kappa(B, \epsilon)$ such that $\mathbf{E}\left[f_{k} \mid J^{\prime}=0\right] \leq \epsilon$ or $\mathbf{E}\left[f_{k} \mid J^{\prime}=1\right] \geq 1-\epsilon$. Note that $J=J_{k} \cup J^{\prime}$ is of size at most $n / B+\kappa(B, \epsilon)$ and $\mathbf{E}\left[f_{k} \mid J=0\right] \leq \epsilon$ or $\mathbf{E}\left[f_{k} \mid J=1\right] \geq 1-\epsilon$. Now taking $B$ such that $\kappa(B, \epsilon) \approx n / B$ completes the proof.

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[^0]:    ${ }^{1}$ We can assume that $C^{\prime} \geq 1$ as (6) remains valid if one increases the constant in the exponent.

