# Almost isoperimetric subsets of the discrete cube 

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#### Abstract

We show that a set $A \subset\{0,1\}^{n}$ with edge-boundary of size at most $$
|A|\left(\log _{2}\left(2^{n} /|A|\right)+\epsilon\right)
$$ can be made into a subcube by at most $\left(2 \epsilon / \log _{2}(1 / \epsilon)\right)|A|$ additions and deletions, provided $\epsilon$ is less than an absolute constant.

We deduce that if $A \subset\{0,1\}^{n}$ has size $2^{t}$ for some $t \in \mathbb{N}$, and $A$ cannot be made into a subcube by fewer than $\delta|A|$ additions and deletions, then the edge-boundary of $A$ has size at least $$
|A| \log _{2}\left(2^{n} /|A|\right)+|A| \delta \log _{2}(1 / \delta)=2^{t}\left(n-t+\delta \log _{2}(1 / \delta)\right)
$$ provided $\delta$ is less than an absolute constant. This is sharp whenever $\delta=1 / 2^{j}$ for some $j \in\{1,2, \ldots, t\}$.


## 1 Introduction

We work in the $n$-dimensional discrete cube $\{0,1\}^{n}$, the set of all $0-1$ vectors of length $n$. This may be identified with $\mathcal{P}([n])$, the set of all subsets of $[n]=$ $\{1,2, \ldots, n\}$, by identifying a set $x \subset[n]$ with its characteristic vector $\chi_{x}$ in the usual way. A $d$-dimensional subcube of $\{0,1\}^{n}$ is a set of the form

$$
\left\{x \in\{0,1\}^{n}: x_{i_{1}}=a_{1}, x_{i_{2}}=a_{2}, \ldots, x_{i_{n-d}}=a_{n-d}\right\}
$$

where $i_{1}<i_{2}<\ldots<i_{n-d}$ are coordinates, and $a_{1}, a_{2}, \ldots$ and $a_{n-d}$ are fixed elements of $\{0,1\}$. The coordinates $i_{1}, i_{2}, \ldots, i_{n-d}$ are called the fixed coordinates; the other coordinates are called the moving coordinates, and $n-d$ is called the codimension of the subcube.

Consider the graph $Q_{n}$ with vertex-set $\{0,1\}^{n}$, where we join two $0-1$ vectors if they differ in exactly one coordinate; this graph is called the $n$-dimensional hypercube. Given a set $A \subset\{0,1\}^{n}$, the edge-boundary of $A$ is defined to be the set of all edges of $Q_{n}$ joining a point in $A$ to a point not in $A$. We write $\partial A$ for the edge-boundary of $A$.

[^0]For $1 \leq k \leq 2^{n}$, let $C_{n, k}$ be the first $k$ elements of the binary ordering on $\mathcal{P}([n])$, defined by

$$
x<y \Leftrightarrow \max (x \Delta y) \in y
$$

The edge-isoperimetric inequality of Harper [6, Lindsey [12], Bernstein [2] and Hart [7] states that among all subsets of $\{0,1\}^{n}$ of size $k, C_{n, k}$ has the smallest possible edge-boundary.

A slightly weaker form is as follows:

$$
\begin{equation*}
|\partial A| \geq|A| \log _{2}\left(2^{n} /|A|\right) \quad \forall A \subset\{0,1\}^{n} \tag{1}
\end{equation*}
$$

equality holds if and only if $A$ is a subcube. We call $|\partial A| /|A|$ the average outdegree of $A$; (1) says that the average out-degree of $A$ is at least $\log _{2}\left(2^{n} /|A|\right)$ (which is the average out-degree of a subcube of size $|A|$, when $|A|$ is a power of 2). Writing $p=|A| / 2^{n}$ for the measure of the set $A$, we may rewrite (1) as:

$$
|\partial A| \geq 2^{n} p \log _{2}(1 / p) \quad \forall A \subset\{0,1\}^{n}
$$

Hence, if $|A|=2^{n-1},|\partial A| \geq 2^{n-1}$, and equality holds only if $A$ is a codimension-1 subcube, in which case the edge-boundary consists of all the edges in one direction.

It is natural to ask whether it is always possible to find a direction in which there are many boundary edges. For $i \in[n]$, we write

$$
A_{i}^{+}=\{x \backslash\{i\}: x \in A, i \in x\} \subset \mathcal{P}([n] \backslash\{i\})
$$

and

$$
A_{i}^{-}=\{x \in A: i \notin x\} \subset \mathcal{P}([n] \backslash\{i\}) ;
$$

$A_{i}^{+}$and $A_{i}^{-}$are called the upper and lower $i$-sections of $A$, respectively. We write

$$
\partial_{i} A=\left|A_{i}^{+} \Delta A_{i}^{-}\right|
$$

for the number of edges of the boundary of $A$ in direction $i$. The influence of the coordinate $i$ on the set $A$ is defined to be

$$
\beta_{i}=\left|A_{i}^{+} \Delta A_{i}^{-}\right| / 2^{n-1}
$$

i.e. the fraction of direction- $i$ edges of $Q_{n}$ which belong to $\partial A$. This is simply the probability that if $S \subset \mathcal{P}([n])$ is chosen uniformly at random, $A$ contains exactly one of $S$ and $S \Delta\{i\}$.

Clearly, we have $\sum_{i=1}^{n} \beta_{i}=|\partial A| / 2^{n-1}$. The quantity $\sum_{i=1}^{n} \beta_{i}$ is sometimes called the total influence.

Ben-Or and Linial [1] conjectured that for any set $A \subset\{0,1\}^{n}$ with $|A|=$ $2^{n-1}$, there exists a coordinate with influence at least $\Omega\left(\frac{\log _{2} n}{n}\right)$. This was proved by Kahn, Kalai and Linial; it follows from the celebrated KKL Theorem:

Theorem 1 (Kahn, Kalai, Linial [9]). If $A \subset\{0,1\}^{n}$ with measure $p$, then

$$
\sum_{i=1}^{n} \beta_{i}^{2} \geq C p^{2}(1-p)^{2}(\ln n)^{2} / n
$$

where $C>0$ is an absolute constant.
Corollary 2. If $A \subset\{0,1\}^{n}$ with measure $p$, then there exists a coordinate $i \in[n]$ with

$$
\beta_{i} \geq C^{\prime} p(1-p)(\ln n) / n
$$

where $C^{\prime}>0$ is an absolute constant.
Corollary 2 is sharp up to the value of the absolute constant $C^{\prime}$, as can be seen from the 'tribes' construction of Ben-Or and Linial [1]. Let $n=k l$, and split [ $n$ ] into $l$ 'tribes' of size $k$. Let $A$ be the set of all $0-1$ vectors which are identically 0 on at least one tribe. Observe that

$$
\begin{gathered}
|A|=\left(1-\left(1-2^{-k}\right)^{l}\right) 2^{n} \\
|\partial A|=n 2^{n-k}\left(1-2^{-k}\right)^{l-1}
\end{gathered}
$$

and

$$
\beta_{i}=2^{-(k-1)}\left(1-2^{-k}\right)^{l-1} \quad \forall i \in[n] .
$$

Let $k=2^{j}$ for some $j \in \mathbb{N}$, and let $l=2^{k} / k$, so that $n=2^{k}=2^{2^{j}}$; then

$$
1-p=\left(1-2^{-k}\right)^{l}=\left(1-2^{-k}\right)^{2^{k} / k}=1-1 / k+O\left(1 / k^{2}\right),
$$

and

$$
\beta_{i}=\frac{2(1-p)}{n\left(1-2^{-k}\right)}=\frac{2\left(1-1 / k+O\left(1 / k^{2}\right)\right)}{n} \quad \forall i \in[n]
$$

so

$$
\frac{\beta_{i}}{p(1-p) \ln (n) / n}=\frac{2\left(1-1 / k+O\left(1 / k^{2}\right)\right)}{\left(1 / k-O\left(1 / k^{2}\right)(1-O(1 / k)) k \ln 2\right.}=\frac{2}{\ln 2}(1+O(1 / k))
$$

The best possible values of the constants $C$ and $C^{\prime}$ (in Theorem 1 and Corollary 2 respectively) remain unknown. Falik and Samorodnitsky [3] have shown that one can take $C=4$, and therefore $C^{\prime}=2$.

Kahn, Kalai and Linial's proof of Theorem 1 is one of the first instances of Fourier analysis on $\{0,1\}^{n}$ being used to prove a purely combinatorial result; Fourier analysis has since become a very important tool in both probabilistic and extremal combinatorics. More recently, Falik and Samorodnitsky [3] gave an entirely combinatorial proof of Theorem a similar proof was found independently by Rossignol 13.

In 4], Friedgut considers the problem of determining the structure of subsets of $\{0,1\}^{n}$ with edge-boundary of size at most $K 2^{n-1}$, or equivalently, with total influence at most $K$, where $K$ is a constant (or a slowly-growing function of $n$ ). Using the Fourier-analytic machinery of [9, Friedgut proved the following.

Theorem 3 (Friedgut's 'Junta' theorem). Let $A \subset\{0,1\}^{n}$, and suppose that $|\partial A| \leq K 2^{n-1}$. Then there exists $B \subset\{0,1\}^{n}$ such that $|A \Delta B| \leq \epsilon 2^{n}$, and $B$ is a $\left\lfloor 2^{C_{0} K / \epsilon}\right\rfloor$-junta, where $C_{0}$ is an absolute constant.

Here, if $B \subset\{0,1\}^{n}$, and $j \in \mathbb{N}$, we say that $B$ is a $j$-junta if there exists a set of coordinates $J \subset[n]$ such that $|J| \leq j$, and the event $\{x \in B\}$ depends only upon the values $\left(x_{j}\right)_{j \in J}$. The condition in italics is of course equivalent to saying that $B$ is a union of subcubes which all have $J$ as their set of fixed coordinates, or that the characteristic function of $B$ depends only upon the cooordinates in $J$.

Freidgut's theorem is sharp up to the value of the absolute constant $C_{0}$, as can be seen by taking the set $A$ to be a product of a subcube of codimension $\left\lfloor\log _{2}(1 /(6 \epsilon))\right\rfloor$, with a set defined by the 'tribes' construction above.

In [3], Falik and Samorodnitsky use influence-based methods to obtain several other results on subsets of $\{0,1\}^{n}$ with small edge-boundary.

In this paper, we will investigate the structure of subsets $A \subset\{0,1\}^{n}$ whose the edge-boundary has size somewhat closer to $|A| \log _{2}\left(2^{n} /|A|\right)$. In particular, we will try to determine how small the edge-boundary must be, to guarantee that $A$ is close in structure to a single subcube. This question has already been investigated by several researchers. Using the techniques of Fourier analysis, Friedgut, Kalai and Naor [5] proved that if $A \subset\{0,1\}^{n}$ with $|A|=2^{n-1}$ and $|\partial A| \leq 2^{n-1}(1+\epsilon)$, then $A$ can be made into a codimension- 1 subcube by at most $K \epsilon 2^{n-1}$ additions and deletions, where $K$ is an absolute constant. Bollobás, Leader and Riordan [11] conjectured that for any $N \in \mathbb{N}$, there exists a constant $K_{N}$ depending on $N$ such that any $A \subset\{0,1\}^{n}$ with $|A|=2^{n-N}$ and

$$
|\partial A| \leq(1+\epsilon)|A| \log _{2}\left(2^{n} /|A|\right)
$$

can be made into a codimension- $N$ subcube by at most $K_{N} \epsilon 2^{n-N}$ additions and deletions. They proved this for $N=2$ and $N=3$, also using the techniques of Fourier analysis. We remark that $K_{N}$ must necessarily depend on $N$. Indeed, as was observed by Samorodnitsky [14, a variant of the 'tribes' construction of Ben-Or and Linial provides an example of a (small) set $A$ satisfying

$$
|\partial A| \leq(1+\epsilon)|A| \log _{2}\left(2^{n} /|A|\right)
$$

and yet requiring at least $(1-o(1))|A|$ additions and deletions to make it into a subcube. As above, let $n=k l$, split $[n]$ into $l$ 'tribes' of size $k$, and let $A$ be the set of all $0-1$ vectors which are identically 0 on at least one tribe. Fix an integer $s$. Let $k=2^{j}$, and let $l=2^{k / 2^{s}} / k=2^{2^{j-s}-j}$, so that $n=2^{k / 2^{s}}=2^{2^{j-s}}$. Let $j \rightarrow \infty$. Then

$$
1-p=\left(1-2^{-k}\right)^{l}=1-l 2^{-k}+O\left(\left(l 2^{-k}\right)^{2}\right) \geq 1-l 2^{-k}
$$

so

$$
p \leq l 2^{-k}
$$

and therefore

$$
\log _{2}(1 / p) \geq k-\log _{2} l=\left(1-2^{-s}\right) k+\log _{2} k
$$

Note that

$$
|\partial A|=n 2^{n-k}\left(1-2^{-k}\right)^{l-1}=\frac{n 2^{n-k}(1-p)}{1-2^{-k}}=n 2^{n-k}\left(1+O\left(l 2^{-k}\right)\right)
$$

Hence,

$$
\begin{aligned}
\frac{|\partial A|}{|A| \log _{2}\left(2^{n} /|A|\right)} & \leq \frac{n 2^{n-k}\left(1+O\left(l 2^{-k}\right)\right)}{\left(l 2^{-k}\left(1-O\left(l 2^{-k}\right)\right)\right)\left(\left(1-2^{-s}\right) k+\log _{2} k\right) 2^{n}} \\
& =\frac{k l\left(1+O\left(l 2^{-k}\right)\right)}{l\left(\left(1-2^{-s}\right) k+\log _{2} k\right)} \\
& =\frac{1+O\left(l 2^{-k}\right)}{1-2^{-s}+\left(\log _{2} k\right) / k} \\
& <\frac{1}{1-2^{-s}},
\end{aligned}
$$

provided $j$ is sufficiently large depending on $s$. For any $\epsilon>0$, this can clearly be made $\leq 1+\epsilon$ by choosing $s$ to be sufficiently large depending on $\epsilon$. However, $A$ is a union of $l$ codimension- $k$ subcubes with disjoint sets of fixed coordinates, and therefore requires at least $(1-o(1))|A|$ additions and deletions to make it into a subcube.

Samorodnitsky [14] conjectured that given any $\delta>0$, there exists an $a>0$ such that any $A \subset\{0,1\}^{n}$ with

$$
|\partial A| \leq(1+a / n)|A| \log _{2}\left(2^{n} /|A|\right)
$$

can be made into a subcube by at most $\delta|A|$ additions and deletions. Making use of a result of Keevash [10] on the structure of $r$-uniform hypergraphs with small shadows, he proved that any $A \subset\{0,1\}^{n}$ with

$$
|\partial A| \leq\left(1+n^{-4}\right)|A| \log _{2}\left(2^{n} /|A|\right)
$$

can be made into a subcube by at most $o(|A|)$ additions and deletions.
It turns out that the correct condition to ensure that $A$ is close to a subcube is that $|\partial A| /|A|$, the average out-degree of $A$, is close to $\log _{2}\left(2^{n} /|A|\right)$. Our first main result (Theorem 8) implies that if $A \subset\{0,1\}^{n}$ has edge-boundary of size at most

$$
\begin{equation*}
|A|\left(\log _{2}\left(2^{n} /|A|\right)+\epsilon\right) \tag{2}
\end{equation*}
$$

where $\epsilon$ is less than an absolute constant, then it can be made into a subcube by at most

$$
\left(1+O\left(1 / \log _{2}(1 / \epsilon)\right)\right) \frac{\epsilon}{\log _{2}(1 / \epsilon)}|A| \leq \frac{2 \epsilon}{\log _{2}(1 / \epsilon)}|A|
$$

additions and deletions. This proves the above conjecture of Bollobás, Leader and Riordan, and also that of Samorodnitsky.

We then prove Theorem 9] which states that if $A \subset\{0,1\}^{n}$ has size $2^{t}$ for some $t \in \mathbb{N}$, and edge-boundary of size at most

$$
|A|\left(\log _{2}\left(2^{n} /|A|\right)+\epsilon\right)=2^{t}(n-t+\epsilon)
$$

where $\epsilon$ is less than an absolute constant, then it can be made into a $t$-dimensional subcube by at most $\delta_{1}(\epsilon)|A|$ additions and deletions, where $\delta_{1}(\epsilon)$ is the unique root of

$$
x \log _{2}(1 / x)=\epsilon
$$

in $(0,1 / e)$. It follows that if $A \subset\{0,1\}^{n}$ has size $2^{t}$ for some $t \in \mathbb{N}$, and cannot be made into a subcube by fewer than $\delta|A|$ additions and deletions, then

$$
|\partial A| \geq|A| \log _{2}\left(2^{n} /|A|\right)+|A| \delta \log _{2}(1 / \delta)=2^{t}\left(n-t+\delta \log _{2}(1 / \delta)\right)
$$

provided $\delta$ is less than an absolute constant. This is sharp whenever $\delta=1 / 2^{j}$ for some $j \in\{1,2, \ldots, t\}$.

Our first aim is to prove a 'rough' stability result (Theorem 7), stating that if $A$ is 'almost isoperimetric', in the sense that the average out-degree of $\partial A$ is not too far above $\log _{2}\left(2^{n} /|A|\right)$, then $A$ can be made into a subcube by a small number of additions and deletions. Influence-based methods play a crucial role in our proof. Indeed, it will turn out that a set $A \subset\{0,1\}^{n}$ satisfying (2) must have each influence either very small or very large. We will use the following theorem of Talagrand [16]:

Theorem 4 (Talagrand). Suppose $A \subset\{0,1\}^{n}$ with measure

$$
\frac{|A|}{2^{n}}=p
$$

then its influences satisfy:

$$
\sum_{i=1}^{n} \beta_{i} / \log _{2}\left(1 / \beta_{i}\right) \geq K p(1-p)
$$

where $K>0$ is an absolute constant.
This implies that if all the influences are small, the edge-boundary must be very large. This will help to show that there must be a coordinate, $i$ say, of very large influence. It will follow that one of the $i$-sections of $A$ is very small. An inductive argument will enable us to complete the proof.

## 2 Main results

We first prove a sequence of results on the rough structure of subsets of $\{0,1\}^{n}$ with small edge-boundary. If $A \subset\{0,1\}^{n}$, and $i \in[n]$, we define

$$
\gamma_{i}=\frac{\min \left\{\left|A_{i}^{+}\right|,\left|A_{i}^{-}\right|\right\}}{|A|}
$$

(Observe that we always have $\gamma_{i} \leq 1 / 2$.) We first show that if $A \subset\{0,1\}^{n}$ has small edge-boundary, then for each $i \in[n]$, either one of the $i$-sections of $A$ is very small, or else the upper and lower $i$-sections of $A$ have very similar sizes.

Lemma 5. Let $A \subset\{0,1\}^{n}$ with

$$
\begin{equation*}
|\partial A|=|A|\left(\log _{2}\left(2^{n} /|A|\right)+\epsilon_{0}\right) \tag{3}
\end{equation*}
$$

Then for each $i \in[n]$, either

1. $\gamma_{i} \leq \epsilon_{0} /\left(5\left(\log _{2} 5-2\right)\right)$, or
2. $1 / 2-\epsilon_{0}<\gamma_{i} \leq 1 / 2$.

Proof. Let $A \subset\{0,1\}^{n}$ satisfying the hypothesis of the lemma. Write

$$
p=\frac{|A|}{2^{n}}
$$

for the measure of $A$; then

$$
|\partial A|=2^{n} p\left(\log _{2}(1 / p)+\epsilon_{0}\right)
$$

Fix $i \in[n]$. Without loss of generality, we may assume that $\left|A_{i}^{+}\right| \leq\left|A_{i}^{-}\right|$, so

$$
\gamma_{i}=\frac{\left|A_{i}^{+}\right|}{|A|}
$$

Write $\gamma=\gamma_{i}$. Let

$$
p^{+}=\frac{\left|A_{i}^{+}\right|}{2^{n-1}}, p^{-}=\frac{\left|A_{i}^{-}\right|}{2^{n-1}}
$$

note that

$$
p^{+}=2 \gamma p, p^{-}=2(1-\gamma) p
$$

Define $\epsilon^{+}, \epsilon^{-}$by

$$
\left|\partial A_{i}^{+}\right|=\left|A_{i}^{+}\right|\left(\log _{2}\left(2^{n-1} /\left|A_{i}^{+}\right|\right)+\epsilon^{+}\right), \quad\left|\partial A_{i}^{-}\right|=\left|A_{i}^{-}\right|\left(\log _{2}\left(2^{n-1} /\left|A_{i}^{-}\right|\right)+\epsilon^{-}\right)
$$

Observe that

$$
\begin{align*}
|\partial A|= & \left|\partial A_{i}^{+}\right|+\left|\partial A_{i}^{-}\right|+\left|A_{i}^{+} \Delta A_{i}^{-}\right| \\
= & \left|A_{i}^{+}\right|\left(\log _{2}\left(2^{n-1} /\left|A_{i}^{+}\right|\right)+\epsilon^{+}\right)+\left|A_{i}^{-}\right|\left(\log _{2}\left(2^{n-1} /\left|A^{-}\right|\right)+\epsilon^{-}\right)+\left|A_{i}^{+} \Delta A_{i}^{-}\right| \\
= & \gamma|A| \log _{2}\left(2^{n} /(2 \gamma|A|)\right)+(1-\gamma)|A|\left(\operatorname { l o g } _ { 2 } \left(2^{n} /(2(1-\gamma)|A|)+\epsilon^{+}\left|A_{i}^{+}\right|+\epsilon^{-}\left|A_{i}^{-}\right|\right.\right. \\
& +\left|A_{i}^{+} \Delta A_{i}^{-}\right| \\
= & |A| \log _{2}\left(2^{n} /|A|\right)-\left(1-H_{2}(\gamma)\right)|A|+\epsilon^{+}\left|A_{i}^{+}\right|+\epsilon^{-}\left|A_{i}^{-}\right|+\left|A_{i}^{+} \Delta A_{i}^{-}\right|  \tag{4}\\
\geq & |A| \log _{2}\left(2^{n} /|A|\right)-\left(1-H_{2}(\gamma)\right)|A|+\epsilon^{+}\left|A_{i}^{+}\right|+\epsilon^{-}\left|A_{i}^{-}\right|+\left|\left|A_{i}^{+}\right|-\left|A_{i}^{-}\right|\right| \\
= & |A| \log _{2}\left(2^{n} /|A|\right)-\left(1-H_{2}(\gamma)\right)|A|+\epsilon^{+}\left|A_{i}^{+}\right|+\epsilon^{-}\left|A_{i}^{-}\right|+(1-2 \gamma)|A| \\
= & |A| \log _{2}\left(2^{n} /|A|\right)+\left(H_{2}(\gamma)-2 \gamma\right)|A|+\epsilon^{+}\left|A_{i}^{+}\right|+\epsilon^{-}\left|A_{i}^{-}\right| \\
= & |A| \log _{2}\left(2^{n} /|A|\right)+F(\gamma)|A|+\epsilon^{+}\left|A_{i}^{+}\right|+\epsilon^{-}\left|A_{i}^{-}\right|
\end{align*}
$$

where $H_{2}:[0,1] \rightarrow \mathbb{R}$ denotes the binary entropy function,

$$
H_{2}(\gamma):=\gamma \log _{2}(1 / \gamma)+(1-\gamma) \log _{2}(1 /(1-\gamma))
$$

and

$$
F(\gamma):=H_{2}(\gamma)-2 \gamma
$$

Hence, (3) implies that

$$
\begin{equation*}
\gamma \epsilon^{+}+(1-\gamma) \epsilon^{-}+F(\gamma) \leq \epsilon_{0} \tag{5}
\end{equation*}
$$

Therefore, crudely,

$$
F(\gamma) \leq \epsilon_{0}
$$

The function $F$ is concave on $[0,1 / 2]$, and attains its maximum at $\gamma=1 / 5$, where it takes the value $\log _{2} 5-2$. Hence, for $\gamma \leq 1 / 5$,

$$
F(\gamma) \geq 5\left(\log _{2} 5-2\right) \gamma
$$

whereas for $1 / 5 \leq \gamma \leq 1 / 2$,

$$
F(1 / 2-\eta) \geq \frac{10}{3}\left(\log _{2} 5-2\right) \eta>\eta
$$

Hence, for each $i \in[n]$, either

1. $\gamma_{i} \leq \epsilon_{0} /\left(5\left(\log _{2} 5-2\right)\right)$, or
2. $1 / 2-\epsilon_{0}<\gamma_{i} \leq 1 / 2$,
proving the lemma.
Remark 1. We can of course rephrase the conclusion of Lemma 5 in terms of influences. Let $A \subset\{0,1\}^{n}$ satisfying (77). Observe that if case 1 occurs for $i \in[n]$, then

$$
\begin{equation*}
\beta_{i} \geq\left(1-2 \gamma_{i}\right)|A| / 2^{n-1}=2\left(1-2 \gamma_{i}\right) p \geq 2\left(1-2 \frac{\epsilon_{0}}{5\left(\log _{2} 5-2\right)}\right) p \tag{6}
\end{equation*}
$$

-the $i$ th influence is 'large'.
If, on the other hand, case 2 occurs, then by (4), we have

$$
\left|A_{i}^{+} \Delta A_{i}^{-}\right| \leq|\partial A|-|A| \log _{2}\left(2^{n} /|A|\right)+\left(1-H_{2}\left(\gamma_{i}\right)\right)|A|=\left(\epsilon_{0}+1-H_{2}\left(\gamma_{i}\right)\right)|A| .
$$

Since $H_{2}$ is concave, with $H_{2}(1 / 2)=1$, we have

$$
1-H_{2}(1 / 2-\eta) \leq 2 \eta(0 \leq \eta \leq 1 / 2)
$$

and therefore

$$
\left|A_{i}^{+} \Delta A_{i}^{-}\right|<3 \epsilon_{0}|A|
$$

i.e.

$$
\beta_{i}<6 \epsilon_{0} p
$$

-the $i$ th influence is 'small'.

We now show that if the edge-boundary of $A$ is sufficiently small, then case 1 in Lemma 5 must occur for some $i \in[n]$.

Lemma 6. There exists an absolute constant $c>0$ such that the following holds. If $\epsilon \leq c$, and $A \subset\{0,1\}^{n}$ with measure

$$
\frac{|A|}{2^{n}} \leq 1-\epsilon
$$

and

$$
\begin{equation*}
|\partial A| \leq|A|\left(\log _{2}\left(2^{n} /|A|\right)+\epsilon\right) \tag{7}
\end{equation*}
$$

then case 1 must occur for some $i \in[n]$, i.e. $\gamma_{i} \leq \epsilon /\left(5\left(\log _{2} 5-2\right)\right.$ ) for some $i \in[n]$.

Proof. We can easily prove the lemma for sets with measure $p \in[1 / 2,7 / 8]$. Suppose $A \subset\{0,1\}^{n}$ has measure $p \in[1 / 2,7 / 8]$ and satisfies (7). Suppose for a contradiction that case 2 occurs for every $i \in[n]$. Then by Remark $1 \beta_{i}<6 \epsilon p$ for every $i \in[n]$, and therefore by Theorem 4

$$
\sum_{i=1}^{n} \beta_{i}>K p(1-p) \log _{2}\left(\frac{1}{6 \epsilon p}\right)
$$

The right-hand side is at least

$$
2 p\left(\log _{2}(1 / p)+\epsilon\right)
$$

provided

$$
\frac{K}{8} \log _{2}\left(\frac{1}{6 \epsilon}\right) \geq 2(1+\epsilon)
$$

which holds for all $\epsilon \leq c:=2^{-32 K} / 6$. This contradicts (7), proving the lemma for $p \in[1 / 2,7 / 8]$.

Now observe that any set $A \subset\{0,1\}^{n}$ with measure $p \in[7 / 8,1-\epsilon]$ has

$$
\begin{equation*}
|\partial A|>|A|\left(\log _{2}\left(2^{n} /|A|\right)+\epsilon\right) \tag{8}
\end{equation*}
$$

To see this, just apply the edge-isoperimetric inequality (1) to $A^{c}$ :

$$
|\partial A|=\left|\partial\left(A^{c}\right)\right| \geq 2^{n}(1-p) \log _{2}(1 /(1-p))
$$

It is easily checked that

$$
2^{n}(1-p) \log _{2}(1 /(1-p))>2^{n} p\left(\log _{2}(1 / p)+1-p\right) \quad \forall p \geq 7 / 8
$$

so (8) holds for all $p \in[7 / 8,1-\epsilon]$. Hence, any set $A \subset\{0,1\}^{n}$ satisfying (77) must have measure $p \leq 7 / 8$.

It remains to prove the lemma for all sets of measure $p \leq 1 / 2$. Suppose $A$ has measure $p \leq 1 / 2$ and satisfies (7). Suppose for a contradiction that case 2 occurs for every $i \in[n]$.

Fix any $i \in[n]$. Without loss of generality, we may assume that $\left|A_{i}^{+}\right| \leq\left|A_{i}^{-}\right|$, so that

$$
\gamma_{i}=\frac{\left|A_{i}^{+}\right|}{|A|}
$$

Write $\gamma=\gamma_{i}$. Define $\epsilon^{+}$and $\epsilon^{-}$as in the proof of Lemma 5. By (5), we have

$$
\gamma \epsilon^{+}+(1-\gamma) \epsilon^{-}+F(\gamma) \leq \epsilon
$$

Hence, crudely,

$$
\gamma \epsilon^{+}+(1-\gamma) \epsilon^{-} \leq \epsilon,
$$

so either $\epsilon^{+} \leq \epsilon$ or $\epsilon^{-} \leq \epsilon$.
If $\epsilon^{+} \leq \epsilon$, then let $A^{\prime}=A_{i}^{+}$. The set $A^{\prime}$ is a subset of $\mathcal{P}([n] \backslash\{i\})$ of measure $p^{\prime}:=2 \gamma p \in((1-2 \epsilon) p, p) \subset[0,1 / 2]$, satisfying the conditions of the lemma.

If $\epsilon^{-} \leq \epsilon$, then let $A^{\prime}=A_{i}^{-}$; the set $A^{\prime}$ is a subset of $\mathcal{P}([n] \backslash\{i\})$ of measure $p^{\prime}:=2(1-\gamma) p<2(1 / 2+\epsilon) p \leq 1 / 2+\epsilon<7 / 8$, satisfying the conditions of the lemma.

If $A^{\prime}$ has case 1 occurring for some $j$, then by (6),

$$
\begin{aligned}
\beta_{j}^{\prime} & \geq 2\left(1-2 \frac{\epsilon}{5\left(\log _{2} 5-2\right)}\right) p^{\prime} \\
& \geq 2\left(1-2 \frac{\epsilon}{5\left(\log _{2} 5-2\right)}\right)(1-2 \epsilon) p \\
& >2(1-2 \epsilon)^{2} p
\end{aligned}
$$

and therefore

$$
\beta_{j}>(1-2 \epsilon)^{2} p>6 \epsilon p
$$

contradicting our assumption that $A$ has case 2 occurring for every $i \in[n]$. Therefore, $A^{\prime}$ also has case 2 occurring for every coordinate. Hence, it must have measure $p^{\prime}<1 / 2$, by the above argument for sets of measure in $[1 / 2,7 / 8]$. Repeat the same argument for $A^{\prime}$, and continue; we obtain a sequence of set systems $\left(A^{(l)}\right)$ on ground sets of sizes $n-l$, all with measure $<1 / 2$, satisfying the conditions of the lemma, and with case 2 occurring for every coordinate. Stop at the minimum $M$ such that $A^{(M)}=\emptyset$; clearly, $M \leq n-1$. Then $A^{(M-1)}$ has one of its $j$-sections empty for some $j$, so case 1 must occur for this $j$, a contradiction. This proves the lemma.

We can now prove a rough stability result for subsets of $\{0,1\}^{n}$ with small edge-boundary:

Theorem 7. There exists an absolute constant $c>0$ such that if $A \subset\{0,1\}^{n}$ with

$$
|\partial A| \leq|A| \log _{2}\left(2^{n} /|A|\right)+\epsilon|A|
$$

for some $\epsilon \leq c$, then

$$
|A \Delta C| /|A|<3 \epsilon
$$

for some subcube $C$.

Proof. Let $c$ be the constant in Lemma66. Let $A \subset\{0,1\}^{n}$ be such that

$$
|\partial A| \leq|A| \log _{2}\left(2^{n} /|A|\right)+\epsilon|A|
$$

for some $\epsilon \leq c$. Let $\epsilon_{0} \leq \epsilon$ be such that

$$
|\partial A|=|A|\left(\log _{2}\left(2^{n} /|A|\right)+\epsilon_{0}\right)
$$

By Lemma 6, there exists $i \in[n]$ with case 1 occurring, i.e. with

$$
\gamma_{i} \leq \epsilon /\left(5\left(\log _{2} 5-2\right)\right)
$$

Without loss of generality, we may assume that $i=n$, and that $\left|A_{n}^{+}\right| \leq\left|A_{n}^{-}\right|$. In keeping with our earlier notation, we write $\gamma=\gamma_{n}=\left|A_{n}^{+}\right| /|A|$.

To avoid confusion, we now write $B^{(0)}=A, p^{(0)}=p, \epsilon^{(0)}=\epsilon_{0}$, and $\gamma^{(0)}=\gamma$. Let $B^{(1)}=A_{n}^{-} \subset \mathcal{P}([n-1])$, let $p^{(1)}=p_{n}^{-}$, and let $\epsilon^{(1)}=\epsilon_{n}^{-}$.

By (5), we have

$$
\left(1-\gamma^{(0)}\right) \epsilon^{(1)}+F\left(\gamma^{(0)}\right) \leq \epsilon^{(0)}
$$

Since $F\left(\gamma^{(0)}\right) \geq 5\left(\log _{2} 5-2\right) \gamma^{(0)}$, we have

$$
\left(1-\gamma^{(0)}\right) \epsilon^{(1)}+5\left(\log _{2} 5-2\right) \gamma^{(0)} \leq \epsilon^{(0)}
$$

it follows that $\epsilon^{(1)} \leq \epsilon \leq c$. Hence, $B^{(1)} \subset \mathcal{P}([n-1])$ also satisfies the hypothesis of Theorem 7 (with $n$ replaced by $n-1$ ). Its measure $p^{(1)}$ satisfies

$$
\begin{aligned}
p^{(1)} & =2\left(1-\gamma^{(0)}\right) p^{(0)} \\
& \geq 2\left(1-\frac{\epsilon^{(0)}}{5\left(\log _{2} 5-2\right)}\right) p^{(0)} \\
& >2\left(1-\epsilon^{(0)}\right) p^{(0)} \\
& \geq 2(1-c) p^{(0)}
\end{aligned}
$$

Repeat the same argument for $B^{(1)}$. We obtain a sequence of set systems $\left(B^{(k)}\right)$ on ground sets of sizes $n-k$, satisfying the hypotheses of Theorem 7 with $\epsilon$ replaced by $\epsilon^{(k)} \leq \epsilon_{0} \leq c$, with measures $p^{(k)}$ satisfying

$$
p^{(k+1)}>2\left(1-\epsilon^{(k)}\right) p^{(k)} \quad \forall k \geq 0
$$

and with

$$
\begin{equation*}
\left(1-\gamma^{(k)}\right) \epsilon^{(k+1)}+F\left(\gamma^{(k)}\right) \leq \epsilon^{(k)} \quad \forall k \geq 0 \tag{9}
\end{equation*}
$$

Without loss of generality, we may assume that $B^{(k)} \subset \mathcal{P}([n-k])$.
We may continue this process until we produce a set system $B^{(N)}$ at stage $N$, for which $p^{(N)}>1-\epsilon_{0}$, at which point we can no longer apply Lemma 6

We must now show that $A$ is close to $\mathcal{P}([n-N])$. Observe that

$$
\begin{aligned}
\left|A \backslash B^{(N)}\right| & =\sum_{k=0}^{N-1} \gamma^{(k)} p^{(k)} 2^{n-k} \\
& =\sum_{k=0}^{N-1} 2^{k}\left(\prod_{j<k}\left(1-\gamma^{(j)}\right)\right) \gamma^{(k)} p_{0} 2^{n-k} \\
& =\sum_{k=0}^{N-1}\left(\prod_{j<k}\left(1-\gamma^{(j)}\right)\right) \gamma^{(k)} p_{0} 2^{n} \\
& =\sum_{k=0}^{N-1}\left(\prod_{j<k}\left(1-\gamma^{(j)}\right)\right) \gamma^{(k)}|A|
\end{aligned}
$$

By repeatedly applying the inequality (9), we obtain

$$
\sum_{k=0}^{N-1}\left(\prod_{j<k}\left(1-\gamma^{(j)}\right)\right) F\left(\gamma^{(k)}\right)+\left(\prod_{j=0}^{N-1}\left(1-\gamma^{(j)}\right)\right) \epsilon_{N} \leq \epsilon_{0}
$$

so certainly,

$$
\sum_{k=0}^{N-1}\left(\prod_{j<k}\left(1-\gamma^{(j)}\right)\right) F\left(\gamma^{(k)}\right) \leq \epsilon_{0}
$$

Since $F\left(\gamma^{(k)}\right) \geq 5\left(\log _{2} 5-2\right) \gamma^{(k)}(0 \leq k \leq N-1)$, it follows that

$$
\sum_{k=0}^{N-1}\left(\prod_{j<k}\left(1-\gamma^{(j)}\right)\right) \gamma^{(k)} \leq \frac{\epsilon_{0}}{5\left(\log _{2} 5-2\right)}
$$

Hence,

$$
\left|A \backslash B^{(N)}\right| \leq \frac{\epsilon_{0}}{5\left(\log _{2} 5-2\right)}|A|<\epsilon_{0}|A|
$$

Let $C=\mathcal{P}([n-N])$, a codimension- $N$ subcube. Then

$$
\begin{equation*}
|A \backslash C|=\left|A \backslash B^{(N)}\right|<\epsilon_{0}|A| \tag{10}
\end{equation*}
$$

Since $p^{(N)}>1-\epsilon_{0}$, we have

$$
\begin{equation*}
|C \backslash A|<\epsilon_{0}|C| \tag{11}
\end{equation*}
$$

Hence,

$$
|C|<\frac{1}{1-\epsilon_{0}}|A|
$$

and therefore

$$
|C \backslash A|<\frac{\epsilon_{0}}{1-\epsilon_{0}}|A|<2 \epsilon_{0}|A|
$$

Combining this with (10) yields:

$$
\begin{equation*}
|A \Delta C|<3 \epsilon_{0}|A| \tag{12}
\end{equation*}
$$

proving Theorem 7
We may use this rough stability result to obtain a more precise one:
Theorem 8. There exists an absolute constant $c>0$ such that if $A \subset\{0,1\}^{n}$ with

$$
|\partial A| \leq|A| \log _{2}\left(2^{n} /|A|\right)+\epsilon|A|
$$

for some $\epsilon \leq c$, then

$$
|A \Delta C|<\delta_{0}(\epsilon)|A|
$$

for some subcube $C$, where $\delta_{0}(\epsilon)$ is the smallest positive solution of

$$
x \log _{2}(1 / x)-3 x=\epsilon
$$

Proof. Write

$$
\begin{equation*}
|\partial A|=|A|\left(\log _{2}\left(2^{n} /|A|\right)+\epsilon_{0}\right) \tag{13}
\end{equation*}
$$

where $0 \leq \epsilon_{0} \leq \epsilon$. Choose a subcube $C$ such that $|A \Delta C|$ is minimal, and let $\delta=|A \Delta C| /|A|$. By Theorem [7] $\delta<3 \epsilon_{0} \leq 3 c<1 / 2$.

Without loss of generality, we may assume that $C=\mathcal{P}([n-N])$. Let $B=$ $C \backslash A$ and let $D=A \backslash C$; then

$$
|B|+|D|<3 \epsilon_{0}|A|
$$

Since every point of $D$ is adjacent to at most one point of $C$, the number of edges in $\partial A$ between points of $A \cap C$ and points of $\{0,1\}^{n} \backslash C$ is at least

$$
N\left(2^{n-N}-|B|\right)-|D|
$$

The number of edges in $\partial A$ between points of $C$ is at least

$$
|B| \log _{2}\left(2^{n-N} /|B|\right)
$$

Finally, the number of edges of the cube in $\partial D$ is at least

$$
|D| \log _{2}\left(2^{n} /|D|\right)
$$

and the number of edges of the cube between points in $D$ and points in $C$ is at most $|D|$, so the number of edges of the cube between points of $D$ and points of $\left(\{0,1\}^{n} \backslash C\right) \backslash A$ is at least

$$
|D|\left(\log _{2}\left(2^{n} /|D|\right)-1\right)
$$

It follows that

$$
\begin{align*}
|\partial A| \geq & N\left(2^{n-N}-|B|\right)-|D|+|B| \log _{2}\left(2^{n-N} /|B|\right)+|D|\left(\log _{2}\left(2^{n} /|D|\right)-1\right) \\
= & N 2^{n-N}+\left(\log _{2}\left(2^{n-N} /|B|\right)-N\right)|B|+\left(\log _{2}\left(2^{n} /|D|\right)-2\right)|D| \\
= & N(|A|-|D|+|B|)+\left(\log _{2}\left(2^{n-N} /|B|\right)-N\right)|B| \\
& +\left(\log _{2}\left(2^{n} /|D|\right)-2\right)|D| \\
= & N|A|+|B|\left(\log _{2}\left(2^{n} /|B|\right)-N\right)+|D|\left(\log _{2}\left(2^{n} /|D|\right)-N-2\right) \tag{14}
\end{align*}
$$

Write $|B|=\phi|A|$ and $|D|=\psi|A|$. Then $\delta=\psi+\phi$. Note that

$$
N=\log _{2}\left(\frac{2^{n}}{|A|-|D|+|B|}\right)=\log _{2}\left(\frac{2^{n}}{|A|}\right)-\log _{2}(1-\psi+\phi)
$$

Hence, we obtain:

$$
\begin{aligned}
|\partial A| \geq & |A| \log _{2}\left(2^{n} /|A|\right)-|A| \log _{2}(1-\psi+\phi) \\
& +\phi|A|\left(\log _{2}(1 / \phi)+\log _{2}(1-\psi+\phi)\right) \\
& +\psi|A|\left(\log _{2}(1 / \psi)-2+\log _{2}(1-\psi+\phi)\right) \\
= & |A| \log _{2}\left(2^{n} /|A|\right)+ \\
& |A|\left(\phi \log _{2}(1 / \phi)+\psi \log _{2}(1 / \psi)-2 \psi+(\psi+\phi-1) \log _{2}(1-\psi+\phi)\right) \\
> & |A| \log _{2}\left(2^{n} /|A|\right)+|A|\left(\psi \log _{2}(1 / \psi)+\phi \log _{2}(1 / \phi)-3 \psi-3 \phi\right)
\end{aligned}
$$

where the last inequality follows from the fact that $\psi, \phi<1 / 2$. Observe that the function

$$
\begin{aligned}
h:(0,1] & \rightarrow \mathbb{R} ; \\
x & \mapsto x \log _{2}(1 / x)
\end{aligned}
$$

is concave, and therefore

$$
\psi \log _{2}(1 / \psi)+\phi \log _{2}(1 / \phi) \geq(\psi+\phi) \log _{2}(1 /(\psi+\phi))
$$

We obtain:

$$
|\partial A|>|A| \log _{2}\left(2^{n} /|A|\right)+|A|\left((\psi+\phi) \log _{2}(1 /(\psi+\phi))-3(\psi+\phi)\right)
$$

Hence, by (13),

$$
(\psi+\phi) \log _{2}(1 /(\psi+\phi))-3(\psi+\phi)<\epsilon_{0}
$$

i.e.,

$$
\delta\left(\log _{2}(1 / \delta)-3\right)<\epsilon_{0}
$$

It is easy to check that the function

$$
\begin{aligned}
g:(0,1] & \rightarrow \mathbb{R} \\
x & \mapsto x \log _{2}(1 / x)-3 x
\end{aligned}
$$

is strictly increasing between 0 and $2^{-(3+1 / \ln (2))}$; provided $3 c \leq 2^{-(3+1 / \ln (2))}$, it follows that $\delta<\delta_{0}(\epsilon)$, where $\delta_{0}(\epsilon)$ is the smallest positive solution of

$$
x \log _{2}(1 / x)-3 x=\epsilon
$$

proving Theorem 8 .
Remark 2. Observe that

$$
\delta_{0}(\epsilon)=\left(1+O\left(1 / \log _{2}(1 / \epsilon)\right)\right) \frac{\epsilon}{\log _{2}(1 / \epsilon)} \leq \frac{2 \epsilon}{\log _{2}(1 / \epsilon)} .
$$

Similarly, we may obtain an exact stability result for set systems whose size is a power of 2 :

Theorem 9. There exists an absolute constant $c>0$ such that if $A \subset\{0,1\}^{n}$ with size $|A|=2^{n-N}$ for some $N \in \mathbb{N}$, and with edge-boundary

$$
|\partial A| \leq|A| \log _{2}\left(2^{n} /|A|\right)+\epsilon|A|
$$

where $\epsilon \leq c$, then there exists a codimension- $N$ subcube $C$ such that

$$
|A \Delta C| \leq \delta_{1}(\epsilon)|A|
$$

where $\delta_{1}(\epsilon)$ is the unique root of the equation

$$
x \log _{2}(1 / x)=\epsilon
$$

in $(0,1 / e)$.
Proof. Write

$$
\begin{equation*}
|\partial A|=|A|\left(\log _{2}\left(2^{n} /|A|\right)+\epsilon_{0}\right) \tag{15}
\end{equation*}
$$

where $0 \leq \epsilon_{0} \leq \epsilon$. Choose a subcube $C$ such that $|A \Delta C|$ is minimal, and let $\delta=|A \Delta C| /|A|$. By Theorem $7, \delta<3 \epsilon_{0} \leq 3 c<1 / 2$.

Suppose $C$ has codimension $N^{\prime}$. Note that if $N \neq N^{\prime}$, then $|A|$ and $|C|$ would differ by a factor of at least 2 , so

$$
|A \Delta C| /|A| \geq||A|-|C|| /|A| \geq 1 / 2
$$

a contradiction. Hence, $N^{\prime}=N$, i.e. $|C|=|A|$.
Let $B=C \backslash A$; then $|A \backslash C|=|C \backslash A|=|B|$. From (14), we have

$$
\begin{aligned}
|\partial A| & \geq|A| \log _{2}\left(2^{n} /|A|\right)+|B|\left(\log _{2}\left(2^{n} /|B|\right)-N\right)+|B|\left(\log _{2}\left(2^{n} /|B|\right)-N-2\right) \\
& =|A| \log _{2}\left(2^{n} /|A|\right)+2|B| \log _{2}\left(2^{n} /|B|\right)-2|B| \log _{2}\left(2^{n} /|A|\right)-2|B| \\
& =|A| \log _{2}\left(2^{n} /|A|\right)+|A| \delta \log _{2}(1 / \delta)
\end{aligned}
$$

It follows that

$$
\delta \log _{2}(1 / \delta) \leq \epsilon
$$

Observe that the function

$$
\begin{aligned}
h:(0,1] & \rightarrow \mathbb{R} ; \\
x & \mapsto x \log _{2}(1 / x)
\end{aligned}
$$

has

$$
h^{\prime}(x)=-\frac{1}{\ln 2}(1+\ln x)
$$

and is therefore strictly increasing between 0 and $1 / e$, where it attains its maximum of $1 /(e \ln 2)$, and strictly decreasing between $1 / e$ and 1 . Since $\delta<3 \epsilon \leq$ $3 c<1 / e$, it follows that $\delta \leq \delta_{1}(\epsilon)$, where $\delta_{1}(\epsilon)$ is the unique root of the equation

$$
x \log _{2}(1 / x)=\epsilon
$$

in $(0,1 / e)$, proving the theorem.
The following is an immediate consequence of Theorem 9
Corollary 10. If $A \subset\{0,1\}^{n}$ has size $2^{t}$ for some $t \in \mathbb{N}$, and cannot be made into a subcube by fewer than $\delta|A|$ additions and deletions, then its edge-boundary satisfies
$|\partial A| \geq|A| \log _{2}\left(2^{n} /|A|\right)+|A| \max \left\{\delta \log _{2}(1 / \delta), c\right\}=2^{t}\left(n-t+\max \left\{\delta \log _{2}(1 / \delta), c\right\}\right)$,
where $c>0$ is an absolute constant. There exists an absolute constant $c^{\prime}>0$ such that if $\delta \leq c^{\prime}$, then

$$
|\partial A| \geq|A| \log _{2}\left(2^{n} /|A|\right)+|A| \delta \log _{2}(1 / \delta)=2^{t}\left(n-t+\delta \log _{2}(1 / \delta)\right)
$$

Remark 3. Observe that all we need from Theorem 7 to prove Theorem 9 is that

$$
\delta=|A \Delta C| /|A|<1 / e
$$

If we just knew that $\delta<1 / 2$, we could still deduce from the above argument that $\delta \log _{2}(1 / \delta) \leq \epsilon$.
Remark 4. Observe that Theorem 9 is best possible, apart from the restriction $\epsilon \leq c$. To see this, let $C=\mathcal{P}([n-N])$, a codimension- $N$-subcube, where $1 \leq N \leq n-1$. Let $2 \leq M \leq n-N$, and delete from $C$ the codimension$(N+M)$ subcube

$$
B=\{x \cup\{n-N\}: x \in \mathcal{P}([n-N-M])\} .
$$

Now add on the codimension- $(N+M)$ subcube

$$
D=\{x \cup\{n\}: x \in \mathcal{P}([n-N-M])\} .
$$

The resulting family $A=(C \backslash B) \cup D$ has

$$
|A \Delta C| /|A|=2^{-(M-1)} \leq 1 / 2
$$

it is easy to check that all other subcubes $C^{\prime} \neq C$ have

$$
\left|A \Delta C^{\prime}\right|>|A \Delta C|
$$

Hence,

$$
\delta:=\min \left\{\left|A \Delta C^{\prime}\right|: C^{\prime} \text { is a subcube }\right\} /|A|=|A \Delta C| /|A|=2^{-(M-1)} .
$$

Observe that we have equality in (14) for $A$, and therefore

$$
|\partial A|=|A| \log _{2}\left(2^{n} /|A|\right)+|A| \delta \log _{2}(1 / \delta)
$$

## 3 Conclusion and Open Problems

Consider the function

$$
f(\delta)=\inf \left\{\frac{|\partial A|-|A| \log _{2}\left(2^{n} /|A|\right)}{|A|}: n \in \mathbb{N}, A \subset\{0,1\}^{n}\right.
$$

$|A|$ is a power of $2,|A \Delta C| \geq \delta|A|$ for all subcubes $C\}$.
We have shown that $f(\delta)=\max \left(\delta \log _{2}(1 / \delta), c\right)$ when $\delta=1 / 2^{j}$ for some $j \in \mathbb{N}$, where $c>0$ is an absolute constant, implying that $f\left(2^{-j}\right)=j 2^{-j}$ for $j \in \mathbb{N}$ sufficiently large. We conjecture that the restriction on $j$ could be removed:

Conjecture 11. For any $j \in \mathbb{N}$,

$$
f\left(2^{-j}\right)=j 2^{-j}
$$

As observed above, the function

$$
\begin{aligned}
h:(0,1] & \rightarrow \mathbb{R} ; \\
x & \mapsto x \log _{2}(1 / x)
\end{aligned}
$$

is strictly decreasing between $1 / e$ and 1 , whereas $f$ is clearly an non-decreasing function of $\delta$. It would be interesting to determine the behaviour of $f(\delta)$ for $1 / 2<\delta \leq 1$.

We also conjecture that Talagrand's Theorem (Theorem (4) holds with $K=$ 2. This was independently conjectured by Samorodnitsky [14]. It would be best possible, as can be seen by taking $A$ to be a $t$-dimensional subcube; then $n-t$ influences are $2^{-(n-t-1)}$, and the rest are zero, so

$$
\sum_{i=0}^{n} \beta_{i} / \log _{2}\left(1 / \beta_{i}\right)=\frac{(n-t) 2^{-(n-t-1)}}{n-t-1}
$$

Hence,

$$
\frac{1}{p(1-p)} \sum_{i=0}^{n} \beta_{i} / \log _{2}\left(1 / \beta_{i}\right)=\frac{2(n-t)}{(n-t-1)\left(1-2^{-(n-t)}\right)} \rightarrow 2 \quad \text { as } n \rightarrow \infty
$$

Knowing this would obviously weaken the upper bound on $\epsilon$ required to prove Theorem 7, though it would not result in a proof of Conjecture 11.

It would be interesting to determine the structure of subsets $A \subset\{0,1\}^{n}$ satisfying

$$
\begin{equation*}
|\partial A| \leq L|A| \log _{2}\left(2^{n} /|A|\right) \tag{16}
\end{equation*}
$$

for $L$ a fixed positive constant. It is easy to check that

$$
k \log _{2}\left(2^{n} / k\right) \leq\left|\partial C_{n, k}\right| \leq 2 k \log _{2}\left(2^{n} / k\right) \quad \forall k \leq 2^{n-1}
$$

so when $|A| \leq 2^{n-1}$, condition (16) is equivalent to saying that the edgeboundary of $A$ is within a constant factor of the minimum. Regarding this case, Kahn and Kalai 8 make the following conjecture.

Conjecture 12 (Kahn, Kalai). For any $L>0$, there exist $L^{\prime}>0$ and $\delta>0$ such that the following holds. If $A \subset\{0,1\}^{n}$ is monotone increasing, with measure $p=\frac{|A|}{2^{n}} \leq 1 / 2$, and with edge-boundary satisfying

$$
|\partial A| \leq L|A| \log _{2}\left(2^{n} /|A|\right)
$$

then there exists a subcube $C \subset\{0,1\}^{n}$ with codimension at most $L^{\prime} \log _{2}(1 / p)$ and all fixed coordinates equal to 1 , such that

$$
\frac{|A \cap C|}{|C|} \geq(1+\delta) p
$$

We believe Conjecture 12 to be true for non-monotone sets as well, if one allows the subcube $C$ to have fixed 0's as well as fixed 1's:

Conjecture 13. For any $L>0$, there exist $L^{\prime}>0$ and $\delta>0$ such that the following holds. If $A \subset\{0,1\}^{n}$ has measure $p=\frac{|A|}{2^{n}} \leq 1 / 2$ and has edgeboundary satisfying

$$
|\partial A| \leq L|A| \log _{2}\left(2^{n} /|A|\right)
$$

then there exists a subcube $C \subset\{0,1\}^{n}$ with codimension at most $L^{\prime} \log _{2}(1 / p)$, such that

$$
\frac{|A \cap C|}{|C|} \geq(1+\delta) p
$$

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