# Almost isoperimetric subsets of the discrete cube

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#### Abstract

We show that a set  $A \subset \{0,1\}^n$  with edge-boundary of size at most

 $|A|(\log_2(2^n/|A|) + \epsilon)$ 

can be made into a subcube by at most  $(2\epsilon/\log_2(1/\epsilon))|A|$  additions and deletions, provided  $\epsilon$  is less than an absolute constant.

We deduce that if  $A \subset \{0, 1\}^n$  has size  $2^t$  for some  $t \in \mathbb{N}$ , and A cannot be made into a subcube by fewer than  $\delta|A|$  additions and deletions, then the edge-boundary of A has size at least

 $|A|\log_2(2^n/|A|) + |A|\delta \log_2(1/\delta) = 2^t(n - t + \delta \log_2(1/\delta)),$ 

provided  $\delta$  is less than an absolute constant. This is sharp whenever  $\delta = 1/2^j$  for some  $j \in \{1, 2, ..., t\}$ .

#### 1 Introduction

We work in the *n*-dimensional discrete cube  $\{0,1\}^n$ , the set of all 0-1 vectors of length *n*. This may be identified with  $\mathcal{P}([n])$ , the set of all subsets of [n] = $\{1,2,\ldots,n\}$ , by identifying a set  $x \subset [n]$  with its characteristic vector  $\chi_x$  in the usual way. A *d*-dimensional subcube of  $\{0,1\}^n$  is a set of the form

 $\{x \in \{0,1\}^n : x_{i_1} = a_1, x_{i_2} = a_2, \dots, x_{i_{n-d}} = a_{n-d}\},\$ 

where  $i_1 < i_2 < \ldots < i_{n-d}$  are coordinates, and  $a_1, a_2, \ldots$  and  $a_{n-d}$  are fixed elements of  $\{0, 1\}$ . The coordinates  $i_1, i_2, \ldots, i_{n-d}$  are called the *fixed* coordinates; the other coordinates are called the *moving* coordinates, and n-d is called the *codimension* of the subcube.

Consider the graph  $Q_n$  with vertex-set  $\{0,1\}^n$ , where we join two 0-1 vectors if they differ in exactly one coordinate; this graph is called the *n*-dimensional hypercube. Given a set  $A \subset \{0,1\}^n$ , the edge-boundary of A is defined to be the set of all edges of  $Q_n$  joining a point in A to a point not in A. We write  $\partial A$  for the edge-boundary of A.

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For  $1 \leq k \leq 2^n$ , let  $C_{n,k}$  be the first k elements of the binary ordering on  $\mathcal{P}([n])$ , defined by

$$x < y \Leftrightarrow \max(x\Delta y) \in y.$$

The edge-isoperimetric inequality of Harper [6], Lindsey [12], Bernstein [2] and Hart [7] states that among all subsets of  $\{0,1\}^n$  of size k,  $C_{n,k}$  has the smallest possible edge-boundary.

A slightly weaker form is as follows:

$$|\partial A| \ge |A| \log_2(2^n/|A|) \quad \forall A \subset \{0,1\}^n; \tag{1}$$

equality holds if and only if A is a subcube. We call  $|\partial A|/|A|$  the average outdegree of A; (1) says that the average out-degree of A is at least  $\log_2(2^n/|A|)$ (which is the average out-degree of a subcube of size |A|, when |A| is a power of 2). Writing  $p = |A|/2^n$  for the measure of the set A, we may rewrite (1) as:

$$|\partial A| \ge 2^n p \log_2(1/p) \quad \forall A \subset \{0, 1\}^n.$$

Hence, if  $|A| = 2^{n-1}$ ,  $|\partial A| \ge 2^{n-1}$ , and equality holds only if A is a codimension-1 subcube, in which case the edge-boundary consists of all the edges in one direction.

It is natural to ask whether it is always possible to find a direction in which there are many boundary edges. For  $i \in [n]$ , we write

$$A_i^+ = \{x \setminus \{i\} : x \in A, \ i \in x\} \subset \mathcal{P}([n] \setminus \{i\}),$$

and

$$A_i^- = \{ x \in A : i \notin x \} \subset \mathcal{P}([n] \setminus \{i\});$$

 $A^+_i$  and  $A^-_i$  are called the upper and  $lower \, i\text{-}sections$  of A, respectively. We write

$$\partial_i A = |A_i^+ \Delta A_i^-|$$

for the number of edges of the boundary of A in direction i. The *influence* of the coordinate i on the set A is defined to be

$$\beta_i = |A_i^+ \Delta A_i^-| / 2^{n-1},$$

i.e. the fraction of direction-*i* edges of  $Q_n$  which belong to  $\partial A$ . This is simply the probability that if  $S \subset \mathcal{P}([n])$  is chosen uniformly at random, A contains exactly one of S and  $S\Delta\{i\}$ .

exactly one of S and  $S\Delta\{i\}$ . Clearly, we have  $\sum_{i=1}^{n} \beta_i = |\partial A|/2^{n-1}$ . The quantity  $\sum_{i=1}^{n} \beta_i$  is sometimes called the *total influence*.

Ben-Or and Linial [1] conjectured that for any set  $A \subset \{0,1\}^n$  with  $|A| = 2^{n-1}$ , there exists a coordinate with influence at least  $\Omega(\frac{\log_2 n}{n})$ . This was proved by Kahn, Kalai and Linial; it follows from the celebrated KKL Theorem:

**Theorem 1** (Kahn, Kalai, Linial [9]). If  $A \subset \{0,1\}^n$  with measure p, then

$$\sum_{i=1}^{n} \beta_i^2 \ge C p^2 (1-p)^2 (\ln n)^2 / n,$$

where C > 0 is an absolute constant.

**Corollary 2.** If  $A \subset \{0,1\}^n$  with measure p, then there exists a coordinate  $i \in [n]$  with

$$\beta_i \ge C' p(1-p)(\ln n)/n,$$

where C' > 0 is an absolute constant.

Corollary 2 is sharp up to the value of the absolute constant C', as can be seen from the 'tribes' construction of Ben-Or and Linial [1]. Let n = kl, and split [n] into l 'tribes' of size k. Let A be the set of all 0-1 vectors which are identically 0 on at least one tribe. Observe that

$$|A| = (1 - (1 - 2^{-k})^l)2^n,$$
$$|\partial A| = n2^{n-k}(1 - 2^{-k})^{l-1},$$

and

$$\beta_i = 2^{-(k-1)} (1 - 2^{-k})^{l-1} \quad \forall i \in [n].$$

Let  $k = 2^j$  for some  $j \in \mathbb{N}$ , and let  $l = 2^k/k$ , so that  $n = 2^k = 2^{2^j}$ ; then

$$1 - p = (1 - 2^{-k})^{l} = (1 - 2^{-k})^{2^{k}/k} = 1 - 1/k + O(1/k^{2}),$$

and

$$\beta_i = \frac{2(1-p)}{n(1-2^{-k})} = \frac{2(1-1/k+O(1/k^2))}{n} \quad \forall i \in [n],$$

 $\mathbf{SO}$ 

$$\frac{\beta_i}{p(1-p)\ln(n)/n} = \frac{2(1-1/k+O(1/k^2))}{(1/k-O(1/k^2)(1-O(1/k))k\ln 2)} = \frac{2}{\ln 2}(1+O(1/k)).$$

The best possible values of the constants C and C' (in Theorem 1 and Corollary 2 respectively) remain unknown. Falik and Samorodnitsky [3] have shown that one can take C = 4, and therefore C' = 2.

Kahn, Kalai and Linial's proof of Theorem 1 is one of the first instances of Fourier analysis on  $\{0, 1\}^n$  being used to prove a purely combinatorial result; Fourier analysis has since become a very important tool in both probabilistic and extremal combinatorics. More recently, Falik and Samorodnitsky [3] gave an entirely combinatorial proof of Theorem 1; a similar proof was found independently by Rossignol [13].

In [4], Friedgut considers the problem of determining the structure of subsets of  $\{0,1\}^n$  with edge-boundary of size at most  $K2^{n-1}$ , or equivalently, with total influence at most K, where K is a constant (or a slowly-growing function of n). Using the Fourier-analytic machinery of [9], Friedgut proved the following.

**Theorem 3** (Friedgut's 'Junta' theorem). Let  $A \subset \{0,1\}^n$ , and suppose that  $|\partial A| \leq K2^{n-1}$ . Then there exists  $B \subset \{0,1\}^n$  such that  $|A\Delta B| \leq \epsilon 2^n$ , and B is a  $\lfloor 2^{C_0K/\epsilon} \rfloor$ -junta, where  $C_0$  is an absolute constant.

Here, if  $B \subset \{0,1\}^n$ , and  $j \in \mathbb{N}$ , we say that B is a *j*-junta if there exists a set of coordinates  $J \subset [n]$  such that  $|J| \leq j$ , and the event  $\{x \in B\}$  depends only upon the values  $(x_j)_{j \in J}$ . The condition in italics is of course equivalent to saying that B is a union of subcubes which all have J as their set of fixed coordinates, or that the characteristic function of B depends only upon the coordinates in J.

Freidgut's theorem is sharp up to the value of the absolute constant  $C_0$ , as can be seen by taking the set A to be a product of a subcube of codimension  $\lfloor \log_2(1/(6\epsilon)) \rfloor$ , with a set defined by the 'tribes' construction above.

In [3], Falik and Samorodnitsky use influence-based methods to obtain several other results on subsets of  $\{0,1\}^n$  with small edge-boundary.

In this paper, we will investigate the structure of subsets  $A \subset \{0,1\}^n$  whose the edge-boundary has size somewhat closer to  $|A| \log_2(2^n/|A|)$ . In particular, we will try to determine how small the edge-boundary must be, to guarantee that A is close in structure to a single subcube. This question has already been investigated by several researchers. Using the techniques of Fourier analysis, Friedgut, Kalai and Naor [5] proved that if  $A \subset \{0,1\}^n$  with  $|A| = 2^{n-1}$  and  $|\partial A| \leq 2^{n-1}(1+\epsilon)$ , then A can be made into a codimension-1 subcube by at most  $K\epsilon 2^{n-1}$  additions and deletions, where K is an absolute constant. Bollobás, Leader and Riordan [11] conjectured that for any  $N \in \mathbb{N}$ , there exists a constant  $K_N$  depending on N such that any  $A \subset \{0,1\}^n$  with  $|A| = 2^{n-N}$  and

$$|\partial A| \le (1+\epsilon)|A|\log_2(2^n/|A|)$$

can be made into a codimension-N subcube by at most  $K_N \epsilon 2^{n-N}$  additions and deletions. They proved this for N = 2 and N = 3, also using the techniques of Fourier analysis. We remark that  $K_N$  must necessarily depend on N. Indeed, as was observed by Samorodnitsky [14], a variant of the 'tribes' construction of Ben-Or and Linial provides an example of a (small) set A satisfying

$$|\partial A| \le (1+\epsilon)|A|\log_2(2^n/|A|),$$

and yet requiring at least (1 - o(1))|A| additions and deletions to make it into a subcube. As above, let n = kl, split [n] into l 'tribes' of size k, and let A be the set of all 0-1 vectors which are identically 0 on at least one tribe. Fix an integer s. Let  $k = 2^j$ , and let  $l = 2^{k/2^s}/k = 2^{2^{j-s}-j}$ , so that  $n = 2^{k/2^s} = 2^{2^{j-s}}$ . Let  $j \to \infty$ . Then

$$1 - p = (1 - 2^{-k})^{l} = 1 - l2^{-k} + O((l2^{-k})^{2}) \ge 1 - l2^{-k},$$

 $\mathbf{SO}$ 

$$p \le l2^{-k}$$

and therefore

$$\log_2(1/p) \ge k - \log_2 l = (1 - 2^{-s})k + \log_2 k.$$

Note that

$$|\partial A| = n2^{n-k}(1-2^{-k})^{l-1} = \frac{n2^{n-k}(1-p)}{1-2^{-k}} = n2^{n-k}(1+O(l2^{-k})).$$

Hence,

$$\begin{aligned} \frac{|\partial A|}{|A|\log_2(2^n/|A|)} &\leq \frac{n2^{n-k}(1+O(l2^{-k}))}{(l2^{-k}(1-O(l2^{-k})))((1-2^{-s})k+\log_2k)2^n} \\ &= \frac{kl(1+O(l2^{-k}))}{l((1-2^{-s})k+\log_2k)} \\ &= \frac{1+O(l2^{-k})}{1-2^{-s}+(\log_2k)/k} \\ &< \frac{1}{1-2^{-s}}, \end{aligned}$$

provided j is sufficiently large depending on s. For any  $\epsilon > 0$ , this can clearly be made  $\leq 1 + \epsilon$  by choosing s to be sufficiently large depending on  $\epsilon$ . However, A is a union of l codimension-k subcubes with disjoint sets of fixed coordinates, and therefore requires at least (1 - o(1))|A| additions and deletions to make it into a subcube.

Samorodnitsky [14] conjectured that given any  $\delta > 0$ , there exists an a > 0 such that any  $A \subset \{0,1\}^n$  with

$$|\partial A| \le (1 + a/n)|A|\log_2(2^n/|A|)$$

can be made into a subcube by at most  $\delta|A|$  additions and deletions. Making use of a result of Keevash [10] on the structure of *r*-uniform hypergraphs with small shadows, he proved that any  $A \subset \{0, 1\}^n$  with

$$|\partial A| \le (1+n^{-4})|A|\log_2(2^n/|A|)$$

can be made into a subcube by at most o(|A|) additions and deletions.

It turns out that the correct condition to ensure that A is close to a subcube is that  $|\partial A|/|A|$ , the average out-degree of A, is close to  $\log_2(2^n/|A|)$ . Our first main result (Theorem 8) implies that if  $A \subset \{0,1\}^n$  has edge-boundary of size at most

$$|A|(\log_2(2^n/|A|) + \epsilon), \tag{2}$$

where  $\epsilon$  is less than an absolute constant, then it can be made into a subcube by at most

$$(1 + O(1/\log_2(1/\epsilon)))\frac{\epsilon}{\log_2(1/\epsilon)}|A| \le \frac{2\epsilon}{\log_2(1/\epsilon)}|A|$$

additions and deletions. This proves the above conjecture of Bollobás, Leader and Riordan, and also that of Samorodnitsky.

We then prove Theorem 9, which states that if  $A \subset \{0,1\}^n$  has size  $2^t$  for some  $t \in \mathbb{N}$ , and edge-boundary of size at most

$$|A|(\log_2(2^n/|A|) + \epsilon) = 2^t(n - t + \epsilon),$$

where  $\epsilon$  is less than an absolute constant, then it can be made into a *t*-dimensional subcube by at most  $\delta_1(\epsilon)|A|$  additions and deletions, where  $\delta_1(\epsilon)$  is the unique root of

$$x\log_2(1/x) = \epsilon$$

in (0, 1/e). It follows that if  $A \subset \{0, 1\}^n$  has size  $2^t$  for some  $t \in \mathbb{N}$ , and cannot be made into a subcube by fewer than  $\delta|A|$  additions and deletions, then

$$|\partial A| \ge |A| \log_2(2^n/|A|) + |A| \delta \log_2(1/\delta) = 2^t (n - t + \delta \log_2(1/\delta)),$$

provided  $\delta$  is less than an absolute constant. This is sharp whenever  $\delta = 1/2^{j}$  for some  $j \in \{1, 2, \dots, t\}$ .

Our first aim is to prove a 'rough' stability result (Theorem 7), stating that if A is 'almost isoperimetric', in the sense that the average out-degree of  $\partial A$  is not too far above  $\log_2(2^n/|A|)$ , then A can be made into a subcube by a small number of additions and deletions. Influence-based methods play a crucial role in our proof. Indeed, it will turn out that a set  $A \subset \{0,1\}^n$  satisfying (2) must have each influence either very small or very large. We will use the following theorem of Talagrand [16]:

**Theorem 4** (Talagrand). Suppose  $A \subset \{0,1\}^n$  with measure

$$\frac{|A|}{2^n} = p;$$

then its influences satisfy:

$$\sum_{i=1}^{n} \beta_i / \log_2(1/\beta_i) \ge Kp(1-p),$$

where K > 0 is an absolute constant.

This implies that if all the influences are small, the edge-boundary must be very large. This will help to show that there must be a coordinate, i say, of very large influence. It will follow that one of the *i*-sections of A is very small. An inductive argument will enable us to complete the proof.

### 2 Main results

We first prove a sequence of results on the rough structure of subsets of  $\{0, 1\}^n$  with small edge-boundary. If  $A \subset \{0, 1\}^n$ , and  $i \in [n]$ , we define

$$\gamma_i = \frac{\min\{|A_i^+|, |A_i^-|\}}{|A|}.$$

(Observe that we always have  $\gamma_i \leq 1/2$ .) We first show that if  $A \subset \{0, 1\}^n$  has small edge-boundary, then for each  $i \in [n]$ , either one of the *i*-sections of A is very small, or else the upper and lower *i*-sections of A have very similar sizes.

**Lemma 5.** Let  $A \subset \{0,1\}^n$  with

$$|\partial A| = |A|(\log_2(2^n/|A|) + \epsilon_0). \tag{3}$$

Then for each  $i \in [n]$ , either

- 1.  $\gamma_i \leq \epsilon_0 / (5(\log_2 5 2)), \text{ or }$
- 2.  $1/2 \epsilon_0 < \gamma_i \le 1/2$ .

*Proof.* Let  $A \subset \{0,1\}^n$  satisfying the hypothesis of the lemma. Write

$$p = \frac{|A|}{2^n}$$

for the measure of A; then

$$|\partial A| = 2^n p(\log_2(1/p) + \epsilon_0).$$

Fix  $i \in [n]$ . Without loss of generality, we may assume that  $|A_i^+| \leq |A_i^-|$ , so

$$\gamma_i = \frac{|A_i^+|}{|A|}.$$

Write  $\gamma = \gamma_i$ . Let

$$p^+ = \frac{|A_i^+|}{2^{n-1}}, \ p^- = \frac{|A_i^-|}{2^{n-1}};$$

note that

$$p^+ = 2\gamma p, \ p^- = 2(1-\gamma)p.$$

Define  $\epsilon^+, \epsilon^-$  by

$$|\partial A_i^+| = |A_i^+|(\log_2(2^{n-1}/|A_i^+|) + \epsilon^+), \quad |\partial A_i^-| = |A_i^-|(\log_2(2^{n-1}/|A_i^-|) + \epsilon^-).$$

Observe that

$$\begin{split} |\partial A| &= |\partial A_i^+| + |\partial A_i^-| + |A_i^+ \Delta A_i^-| \\ &= |A_i^+|(\log_2(2^{n-1}/|A_i^+|) + \epsilon^+) + |A_i^-|(\log_2(2^{n-1}/|A^-|) + \epsilon^-) + |A_i^+ \Delta A_i^-| \\ &= \gamma |A|\log_2(2^n/(2\gamma|A|)) + (1-\gamma)|A|(\log_2(2^n/(2(1-\gamma)|A|) + \epsilon^+|A_i^+| + \epsilon^-|A_i^-| \\ &+ |A_i^+ \Delta A_i^-| \\ &= |A|\log_2(2^n/|A|) - (1-H_2(\gamma))|A| + \epsilon^+|A_i^+| + \epsilon^-|A_i^-| + |A_i^+ \Delta A_i^-| \quad (4) \\ &\geq |A|\log_2(2^n/|A|) - (1-H_2(\gamma))|A| + \epsilon^+|A_i^+| + \epsilon^-|A_i^-| + ||A_i^+| - |A_i^-|| \\ &= |A|\log_2(2^n/|A|) - (1-H_2(\gamma))|A| + \epsilon^+|A_i^+| + \epsilon^-|A_i^-| + ||A_i^+| - |A_i^-|| \\ &= |A|\log_2(2^n/|A|) - (1-H_2(\gamma))|A| + \epsilon^+|A_i^+| + \epsilon^-|A_i^-| \\ &= |A|\log_2(2^n/|A|) + (H_2(\gamma) - 2\gamma)|A| + \epsilon^+|A_i^+| + \epsilon^-|A_i^-| \\ &= |A|\log_2(2^n/|A|) + F(\gamma)|A| + \epsilon^+|A_i^+| + \epsilon^-|A_i^-|, \end{split}$$

where  $H_2: [0,1] \to \mathbb{R}$  denotes the *binary entropy* function,

$$H_2(\gamma) := \gamma \log_2(1/\gamma) + (1-\gamma) \log_2(1/(1-\gamma)),$$

and

$$F(\gamma) := H_2(\gamma) - 2\gamma.$$

Hence, (3) implies that

$$\gamma \epsilon^+ + (1 - \gamma)\epsilon^- + F(\gamma) \le \epsilon_0.$$
(5)

Therefore, crudely,

$$F(\gamma) \leq \epsilon_0.$$

The function F is concave on [0, 1/2], and attains its maximum at  $\gamma = 1/5$ , where it takes the value  $\log_2 5 - 2$ . Hence, for  $\gamma \le 1/5$ ,

$$F(\gamma) \ge 5(\log_2 5 - 2)\gamma,$$

whereas for  $1/5 \leq \gamma \leq 1/2$ ,

$$F(1/2 - \eta) \ge \frac{10}{3}(\log_2 5 - 2)\eta > \eta$$

Hence, for each  $i \in [n]$ , either

- 1.  $\gamma_i \le \epsilon_0 / (5(\log_2 5 2)), \text{ or }$
- 2.  $1/2 \epsilon_0 < \gamma_i \le 1/2$ ,

proving the lemma.

*Remark* 1. We can of course rephrase the conclusion of Lemma 5 in terms of influences. Let  $A \subset \{0,1\}^n$  satisfying (7). Observe that if case 1 occurs for  $i \in [n]$ , then

$$\beta_i \ge (1 - 2\gamma_i)|A|/2^{n-1} = 2(1 - 2\gamma_i)p \ge 2\left(1 - 2\frac{\epsilon_0}{5(\log_2 5 - 2)}\right)p, \quad (6)$$

—the ith influence is 'large'.

If, on the other hand, case 2 occurs, then by (4), we have

$$|A_i^+ \Delta A_i^-| \le |\partial A| - |A| \log_2(2^n/|A|) + (1 - H_2(\gamma_i))|A| = (\epsilon_0 + 1 - H_2(\gamma_i))|A|.$$

Since  $H_2$  is concave, with  $H_2(1/2) = 1$ , we have

$$1 - H_2(1/2 - \eta) \le 2\eta \ (0 \le \eta \le 1/2),$$

and therefore

$$|A_i^+ \Delta A_i^-| < 3\epsilon_0 |A|$$

i.e.

$$\beta_i < 6\epsilon_0 p,$$

—the ith influence is 'small'.

We now show that if the edge-boundary of A is sufficiently small, then case 1 in Lemma 5 must occur for some  $i \in [n]$ .

**Lemma 6.** There exists an absolute constant c > 0 such that the following holds. If  $\epsilon \leq c$ , and  $A \subset \{0,1\}^n$  with measure

$$\frac{|A|}{2^n} \le 1 - \epsilon,$$

$$|\partial A| \le |A|(\log_2(2^n/|A|) + \epsilon); \tag{7}$$

(7)

and

then case 1 must occur for some 
$$i \in [n]$$
, i.e.  $\gamma_i \leq \epsilon/(5(\log_2 5 - 2))$  for some  $i \in [n]$ .

*Proof.* We can easily prove the lemma for sets with measure  $p \in [1/2, 7/8]$ . Suppose  $A \subset \{0,1\}^n$  has measure  $p \in [1/2,7/8]$  and satisfies (7). Suppose for a contradiction that case 2 occurs for every  $i \in [n]$ . Then by Remark 1,  $\beta_i < 6\epsilon p$ for every  $i \in [n]$ , and therefore by Theorem 4,

$$\sum_{i=1}^{n} \beta_i > Kp(1-p) \log_2\left(\frac{1}{6\epsilon p}\right).$$

The right-hand side is at least

$$2p(\log_2(1/p) + \epsilon)$$

provided

$$\frac{K}{8}\log_2\left(\frac{1}{6\epsilon}\right) \ge 2(1+\epsilon),$$

which holds for all  $\epsilon \leq c := 2^{-32K}/6$ . This contradicts (7), proving the lemma for  $p \in [1/2, 7/8]$ .

Now observe that any set  $A \subset \{0,1\}^n$  with measure  $p \in [7/8, 1-\epsilon]$  has

$$|\partial A| > |A|(\log_2(2^n/|A|) + \epsilon), \tag{8}$$

To see this, just apply the edge-isoperimetric inequality (1) to  $A^c$ :

$$|\partial A| = |\partial (A^c)| \ge 2^n (1-p) \log_2(1/(1-p)).$$

It is easily checked that

$$2^{n}(1-p)\log_{2}(1/(1-p)) > 2^{n}p(\log_{2}(1/p)+1-p) \quad \forall p \ge 7/8,$$

so (8) holds for all  $p \in [7/8, 1-\epsilon]$ . Hence, any set  $A \subset \{0, 1\}^n$  satisfying (7) must have measure p < 7/8.

It remains to prove the lemma for all sets of measure  $p \leq 1/2$ . Suppose A has measure  $p \leq 1/2$  and satisfies (7). Suppose for a contradiction that case 2 occurs for every  $i \in [n]$ .

Fix any  $i \in [n]$ . Without loss of generality, we may assume that  $|A_i^+| \leq |A_i^-|$ , so that

$$\gamma_i = \frac{|A_i^+|}{|A|}.$$

Write  $\gamma = \gamma_i$ . Define  $\epsilon^+$  and  $\epsilon^-$  as in the proof of Lemma 5. By (5), we have

$$\gamma \epsilon^+ + (1 - \gamma)\epsilon^- + F(\gamma) \le \epsilon.$$

Hence, crudely,

$$\gamma \epsilon^+ + (1 - \gamma) \epsilon^- \le \epsilon,$$

so either  $\epsilon^+ \leq \epsilon$  or  $\epsilon^- \leq \epsilon$ . If  $\epsilon^+ \leq \epsilon$ , then let  $A' = A_i^+$ . The set A' is a subset of  $\mathcal{P}([n] \setminus \{i\})$  of measure  $p' := 2\gamma p \in ((1 - 2\epsilon)p, p) \subset [0, 1/2]$ , satisfying the conditions of the lemma.

If  $\epsilon^- \leq \epsilon$ , then let  $A' = A_i^-$ ; the set A' is a subset of  $\mathcal{P}([n] \setminus \{i\})$  of measure  $p' := 2(1-\gamma)p < 2(1/2+\epsilon)p \le 1/2+\epsilon < 7/8$ , satisfying the conditions of the lemma.

If A' has case 1 occurring for some j, then by (6),

$$\begin{array}{lll} \beta'_j & \geq & 2\left(1-2\frac{\epsilon}{5(\log_2 5-2)}\right)p'\\ & \geq & 2\left(1-2\frac{\epsilon}{5(\log_2 5-2)}\right)(1-2\epsilon)p\\ & > & 2(1-2\epsilon)^2p, \end{array}$$

and therefore

$$\beta_j > (1 - 2\epsilon)^2 p > 6\epsilon p,$$

contradicting our assumption that A has case 2 occurring for every  $i \in [n]$ . Therefore, A' also has case 2 occurring for every coordinate. Hence, it must have measure p' < 1/2, by the above argument for sets of measure in [1/2, 7/8]. Repeat the same argument for A', and continue; we obtain a sequence of set systems  $(A^{(l)})$  on ground sets of sizes n-l, all with measure < 1/2, satisfying the conditions of the lemma, and with case 2 occurring for every coordinate. Stop at the minimum M such that  $A^{(M)} = \emptyset$ ; clearly,  $M \leq n-1$ . Then  $A^{(M-1)}$ has one of its *j*-sections empty for some j, so case 1 must occur for this j, a contradiction. This proves the lemma. 

We can now prove a rough stability result for subsets of  $\{0,1\}^n$  with small edge-boundary:

**Theorem 7.** There exists an absolute constant c > 0 such that if  $A \subset \{0,1\}^n$ with

$$|\partial A| \le |A| \log_2(2^n/|A|) + \epsilon |A|$$

for some  $\epsilon \leq c$ , then

$$|A\Delta C|/|A| < 3\epsilon$$

for some subcube C.

*Proof.* Let c be the constant in Lemma 6. Let  $A \subset \{0,1\}^n$  be such that

$$|\partial A| \le |A| \log_2(2^n/|A|) + \epsilon |A|,$$

for some  $\epsilon \leq c$ . Let  $\epsilon_0 \leq \epsilon$  be such that

$$|\partial A| = |A|(\log_2(2^n/|A|) + \epsilon_0).$$

By Lemma 6, there exists  $i \in [n]$  with case 1 occurring, i.e. with

$$\gamma_i \le \epsilon / (5(\log_2 5 - 2)).$$

Without loss of generality, we may assume that i = n, and that  $|A_n^+| \le |A_n^-|$ .

In keeping with our earlier notation, we write  $\gamma = \gamma_n = |A_n^+|/|A|$ . To avoid confusion, we now write  $B^{(0)} = A$ ,  $p^{(0)} = p$ ,  $\epsilon^{(0)} = \epsilon_0$ , and  $\gamma^{(0)} = \gamma$ . Let  $B^{(1)} = A_n^- \subset \mathcal{P}([n-1])$ , let  $p^{(1)} = p_n^-$ , and let  $\epsilon^{(1)} = \epsilon_n^-$ .

By (5), we have

$$(1 - \gamma^{(0)})\epsilon^{(1)} + F(\gamma^{(0)}) \le \epsilon^{(0)}.$$

Since  $F(\gamma^{(0)}) \ge 5(\log_2 5 - 2)\gamma^{(0)}$ , we have

$$(1 - \gamma^{(0)})\epsilon^{(1)} + 5(\log_2 5 - 2)\gamma^{(0)} \le \epsilon^{(0)};$$

it follows that  $\epsilon^{(1)} \leq \epsilon \leq c$ . Hence,  $B^{(1)} \subset \mathcal{P}([n-1])$  also satisfies the hypothesis of Theorem 7 (with n replaced by n-1). Its measure  $p^{(1)}$  satisfies

$$p^{(1)} = 2(1 - \gamma^{(0)})p^{(0)}$$
  

$$\geq 2\left(1 - \frac{\epsilon^{(0)}}{5(\log_2 5 - 2)}\right)p^{(0)}$$
  

$$\geq 2(1 - \epsilon^{(0)})p^{(0)}$$
  

$$\geq 2(1 - c)p^{(0)}.$$

Repeat the same argument for  $B^{(1)}$ . We obtain a sequence of set systems  $(B^{(k)})$  on ground sets of sizes n-k, satisfying the hypotheses of Theorem 7 with  $\epsilon$  replaced by  $\epsilon^{(k)} \leq \epsilon_0 \leq c$ , with measures  $p^{(k)}$  satisfying

$$p^{(k+1)} > 2(1 - \epsilon^{(k)})p^{(k)} \quad \forall k \ge 0,$$

and with

$$(1 - \gamma^{(k)})\epsilon^{(k+1)} + F(\gamma^{(k)}) \le \epsilon^{(k)} \quad \forall k \ge 0.$$
 (9)

Without loss of generality, we may assume that  $B^{(k)} \subset \mathcal{P}([n-k])$ .

We may continue this process until we produce a set system  $B^{(N)}$  at stage N, for which  $p^{(N)} > 1 - \epsilon_0$ , at which point we can no longer apply Lemma 6. We must now show that A is close to  $\mathcal{P}([n-N])$ . Observe that

$$|A \setminus B^{(N)}| = \sum_{k=0}^{N-1} \gamma^{(k)} p^{(k)} 2^{n-k}$$
  
=  $\sum_{k=0}^{N-1} 2^k \left( \prod_{j < k} (1 - \gamma^{(j)}) \right) \gamma^{(k)} p_0 2^{n-k}$   
=  $\sum_{k=0}^{N-1} \left( \prod_{j < k} (1 - \gamma^{(j)}) \right) \gamma^{(k)} p_0 2^n$   
=  $\sum_{k=0}^{N-1} \left( \prod_{j < k} (1 - \gamma^{(j)}) \right) \gamma^{(k)} |A|.$ 

By repeatedly applying the inequality (9), we obtain

$$\sum_{k=0}^{N-1} \left( \prod_{j < k} (1 - \gamma^{(j)}) \right) F(\gamma^{(k)}) + \left( \prod_{j=0}^{N-1} (1 - \gamma^{(j)}) \right) \epsilon_N \le \epsilon_0,$$

so certainly,

$$\sum_{k=0}^{N-1} \left( \prod_{j < k} (1 - \gamma^{(j)}) \right) F(\gamma^{(k)}) \le \epsilon_0.$$

Since  $F(\gamma^{(k)}) \ge 5(\log_2 5 - 2)\gamma^{(k)}$   $(0 \le k \le N - 1)$ , it follows that

$$\sum_{k=0}^{N-1} \left( \prod_{j < k} (1 - \gamma^{(j)}) \right) \gamma^{(k)} \le \frac{\epsilon_0}{5(\log_2 5 - 2)}.$$

Hence,

$$|A \setminus B^{(N)}| \le \frac{\epsilon_0}{5(\log_2 5 - 2)} |A| < \epsilon_0 |A|.$$

Let  $C = \mathcal{P}([n - N])$ , a codimension-N subcube. Then

$$|A \setminus C| = |A \setminus B^{(N)}| < \epsilon_0 |A|.$$
(10)

Since  $p^{(N)} > 1 - \epsilon_0$ , we have

$$|C \setminus A| < \epsilon_0 |C|. \tag{11}$$

Hence,

$$|C| < \frac{1}{1-\epsilon_0}|A|,$$

and therefore

$$|C \setminus A| < \frac{\epsilon_0}{1 - \epsilon_0} |A| < 2\epsilon_0 |A|.$$

Combining this with (10) yields:

$$|A\Delta C| < 3\epsilon_0 |A|,\tag{12}$$

proving Theorem 7.

We may use this rough stability result to obtain a more precise one:

**Theorem 8.** There exists an absolute constant c > 0 such that if  $A \subset \{0, 1\}^n$  with

 $|\partial A| \le |A| \log_2(2^n/|A|) + \epsilon |A|,$ 

for some  $\epsilon \leq c$ , then

$$|A\Delta C| < \delta_0(\epsilon)|A|$$

for some subcube C, where  $\delta_0(\epsilon)$  is the smallest positive solution of

$$x\log_2(1/x) - 3x = \epsilon.$$

Proof. Write

$$\partial A| = |A|(\log_2(2^n/|A|) + \epsilon_0), \tag{13}$$

where  $0 \le \epsilon_0 \le \epsilon$ . Choose a subcube C such that  $|A\Delta C|$  is minimal, and let  $\delta = |A\Delta C|/|A|$ . By Theorem 7,  $\delta < 3\epsilon_0 \le 3c < 1/2$ .

Without loss of generality, we may assume that  $C = \mathcal{P}([n-N])$ . Let  $B = C \setminus A$  and let  $D = A \setminus C$ ; then

$$|B| + |D| < 3\epsilon_0 |A|.$$

Since every point of D is adjacent to at most one point of C, the number of edges in  $\partial A$  between points of  $A \cap C$  and points of  $\{0,1\}^n \setminus C$  is at least

$$N(2^{n-N} - |B|) - |D|.$$

The number of edges in  $\partial A$  between points of C is at least

$$|B|\log_2(2^{n-N}/|B|).$$

Finally, the number of edges of the cube in  $\partial D$  is at least

$$|D|\log_2(2^n/|D|)$$

and the number of edges of the cube between points in D and points in C is at most |D|, so the number of edges of the cube between points of D and points of  $(\{0,1\}^n \setminus C) \setminus A$  is at least

$$|D|(\log_2(2^n/|D|)-1).$$

It follows that

$$\begin{aligned} |\partial A| &\geq N(2^{n-N} - |B|) - |D| + |B| \log_2(2^{n-N}/|B|) + |D|(\log_2(2^n/|D|) - 1) \\ &= N2^{n-N} + (\log_2(2^{n-N}/|B|) - N)|B| + (\log_2(2^n/|D|) - 2)|D| \\ &= N(|A| - |D| + |B|) + (\log_2(2^{n-N}/|B|) - N)|B| \\ &+ (\log_2(2^n/|D|) - 2)|D| \\ &= N|A| + |B|(\log_2(2^n/|B|) - N) + |D|(\log_2(2^n/|D|) - N - 2). \end{aligned}$$
(14)

Write  $|B| = \phi |A|$  and  $|D| = \psi |A|$ . Then  $\delta = \psi + \phi$ . Note that

$$N = \log_2\left(\frac{2^n}{|A| - |D| + |B|}\right) = \log_2\left(\frac{2^n}{|A|}\right) - \log_2(1 - \psi + \phi).$$

Hence, we obtain:

$$\begin{split} |\partial A| &\geq |A| \log_2(2^n/|A|) - |A| \log_2(1 - \psi + \phi) \\ &+ \phi |A| (\log_2(1/\phi) + \log_2(1 - \psi + \phi)) \\ &+ \psi |A| (\log_2(1/\psi) - 2 + \log_2(1 - \psi + \phi)) \\ &= |A| \log_2(2^n/|A|) + \\ &|A| (\phi \log_2(1/\phi) + \psi \log_2(1/\psi) - 2\psi + (\psi + \phi - 1) \log_2(1 - \psi + \phi)) \\ &> |A| \log_2(2^n/|A|) + |A| (\psi \log_2(1/\psi) + \phi \log_2(1/\phi) - 3\psi - 3\phi), \end{split}$$

where the last inequality follows from the fact that  $\psi, \phi < 1/2$ . Observe that the function

$$\begin{array}{rcl} h:(0,1] & \to & \mathbb{R}; \\ & x & \mapsto & x \log_2(1/x) \end{array}$$

is concave, and therefore

$$\psi \log_2(1/\psi) + \phi \log_2(1/\phi) \ge (\psi + \phi) \log_2(1/(\psi + \phi)).$$

We obtain:

$$|\partial A| > |A| \log_2(2^n/|A|) + |A|((\psi + \phi) \log_2(1/(\psi + \phi)) - 3(\psi + \phi)).$$

Hence, by (13),

$$(\psi+\phi)\log_2(1/(\psi+\phi)) - 3(\psi+\phi) < \epsilon_0,$$

i.e.,

$$\delta(\log_2(1/\delta) - 3) < \epsilon_0.$$

It is easy to check that the function

$$g: (0,1] \to \mathbb{R};$$
  
$$x \mapsto x \log_2(1/x) - 3x$$

is strictly increasing between 0 and  $2^{-(3+1/\ln(2))}$ ; provided  $3c \leq 2^{-(3+1/\ln(2))}$ , it follows that  $\delta < \delta_0(\epsilon)$ , where  $\delta_0(\epsilon)$  is the smallest positive solution of

$$x\log_2(1/x) - 3x = \epsilon,$$

proving Theorem 8.

Remark 2. Observe that

$$\delta_0(\epsilon) = (1 + O(1/\log_2(1/\epsilon)))\frac{\epsilon}{\log_2(1/\epsilon)} \le \frac{2\epsilon}{\log_2(1/\epsilon)}.$$

Similarly, we may obtain an exact stability result for set systems whose size is a power of 2:

**Theorem 9.** There exists an absolute constant c > 0 such that if  $A \subset \{0,1\}^n$  with size  $|A| = 2^{n-N}$  for some  $N \in \mathbb{N}$ , and with edge-boundary

$$|\partial A| \le |A| \log_2(2^n/|A|) + \epsilon |A|,$$

where  $\epsilon \leq c$ , then there exists a codimension-N subcube C such that

$$|A\Delta C| \le \delta_1(\epsilon)|A|,$$

where  $\delta_1(\epsilon)$  is the unique root of the equation

$$x\log_2(1/x) = \epsilon$$

in (0, 1/e).

Proof. Write

$$|\partial A| = |A|(\log_2(2^n/|A|) + \epsilon_0).$$
(15)

where  $0 \le \epsilon_0 \le \epsilon$ . Choose a subcube C such that  $|A\Delta C|$  is minimal, and let  $\delta = |A\Delta C|/|A|$ . By Theorem 7,  $\delta < 3\epsilon_0 \le 3c < 1/2$ .

Suppose C has codimension N'. Note that if  $N \neq N'$ , then |A| and |C| would differ by a factor of at least 2, so

$$|A\Delta C|/|A| \ge ||A| - |C||/|A| \ge 1/2,$$

a contradiction. Hence, N' = N, i.e. |C| = |A|. Let  $B = C \setminus A$ ; then  $|A \setminus C| = |C \setminus A| = |B|$ . From (14), we have

$$\begin{aligned} |\partial A| &\geq |A| \log_2(2^n/|A|) + |B| (\log_2(2^n/|B|) - N) + |B| (\log_2(2^n/|B|) - N - 2) \\ &= |A| \log_2(2^n/|A|) + 2|B| \log_2(2^n/|B|) - 2|B| \log_2(2^n/|A|) - 2|B| \\ &= |A| \log_2(2^n/|A|) + |A| \delta \log_2(1/\delta). \end{aligned}$$

It follows that

$$\delta \log_2(1/\delta) \le \epsilon.$$

Observe that the function

$$\begin{array}{rcl} h:(0,1] & \to & \mathbb{R}; \\ & x & \mapsto & x \log_2(1/x) \end{array}$$

has

$$h'(x) = -\frac{1}{\ln 2}(1 + \ln x)$$

and is therefore strictly increasing between 0 and 1/e, where it attains its maximum of  $1/(e \ln 2)$ , and strictly decreasing between 1/e and 1. Since  $\delta < 3\epsilon \leq 3c < 1/e$ , it follows that  $\delta \leq \delta_1(\epsilon)$ , where  $\delta_1(\epsilon)$  is the unique root of the equation

$$x \log_2(1/x) = \epsilon$$

in (0, 1/e), proving the theorem.

The following is an immediate consequence of Theorem 9:

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**Corollary 10.** If  $A \subset \{0,1\}^n$  has size  $2^t$  for some  $t \in \mathbb{N}$ , and cannot be made into a subcube by fewer than  $\delta|A|$  additions and deletions, then its edge-boundary satisfies

$$|\partial A| \ge |A| \log_2(2^n/|A|) + |A| \max\{\delta \log_2(1/\delta), c\} = 2^t (n - t + \max\{\delta \log_2(1/\delta), c\}),$$

where c > 0 is an absolute constant. There exists an absolute constant c' > 0 such that if  $\delta \leq c'$ , then

$$|\partial A| \ge |A| \log_2(2^n/|A|) + |A| \delta \log_2(1/\delta) = 2^t (n - t + \delta \log_2(1/\delta)).$$

Remark 3. Observe that all we need from Theorem 7 to prove Theorem 9 is that

$$\delta = |A\Delta C|/|A| < 1/e.$$

If we just knew that  $\delta < 1/2$ , we could still deduce from the above argument that  $\delta \log_2(1/\delta) \leq \epsilon$ .

Remark 4. Observe that Theorem 9 is best possible, apart from the restriction  $\epsilon \leq c$ . To see this, let  $C = \mathcal{P}([n - N])$ , a codimension-*N*-subcube, where  $1 \leq N \leq n - 1$ . Let  $2 \leq M \leq n - N$ , and delete from *C* the codimension-(N + M) subcube

$$B = \{ x \cup \{ n - N \} : x \in \mathcal{P}([n - N - M]) \}.$$

Now add on the codimension-(N + M) subcube

$$D = \{x \cup \{n\} : x \in \mathcal{P}([n - N - M])\}.$$

The resulting family  $A = (C \setminus B) \cup D$  has

$$|A\Delta C|/|A| = 2^{-(M-1)} \le 1/2;$$

it is easy to check that all other subcubes  $C' \neq C$  have

$$|A\Delta C'| > |A\Delta C|.$$

Hence,

$$\delta := \min\{|A\Delta C'|: C' \text{ is a subcube}\}/|A| = |A\Delta C|/|A| = 2^{-(M-1)}.$$

Observe that we have equality in (14) for A, and therefore

 $|\partial A| = |A| \log_2(2^n/|A|) + |A|\delta \log_2(1/\delta).$ 

## 3 Conclusion and Open Problems

Consider the function

$$\begin{split} f(\delta) &= & \inf\{\frac{|\partial A| - |A| \log_2(2^n/|A|)}{|A|}: \ n \in \mathbb{N}, \ A \subset \{0,1\}^n, \\ & |A| \text{ is a power of } 2, \ |A\Delta C| \geq \delta |A| \text{ for all subcubes } C\}. \end{split}$$

We have shown that  $f(\delta) = \max(\delta \log_2(1/\delta), c)$  when  $\delta = 1/2^j$  for some  $j \in \mathbb{N}$ , where c > 0 is an absolute constant, implying that  $f(2^{-j}) = j2^{-j}$  for  $j \in \mathbb{N}$ sufficiently large. We conjecture that the restriction on j could be removed:

**Conjecture 11.** For any  $j \in \mathbb{N}$ ,

$$f(2^{-j}) = j2^{-j}.$$

As observed above, the function

$$\begin{array}{rcl} h:(0,1]&\to&\mathbb{R};\\ &x&\mapsto&x\log_2(1/x) \end{array}$$

is strictly decreasing between 1/e and 1, whereas f is clearly an non-decreasing function of  $\delta$ . It would be interesting to determine the behaviour of  $f(\delta)$  for  $1/2 < \delta \leq 1$ .

We also conjecture that Talagrand's Theorem (Theorem 4) holds with K = 2. This was independently conjectured by Samorodnitsky [14]. It would be best possible, as can be seen by taking A to be a t-dimensional subcube; then n - t influences are  $2^{-(n-t-1)}$ , and the rest are zero, so

$$\sum_{i=0}^{n} \beta_i / \log_2(1/\beta_i) = \frac{(n-t)2^{-(n-t-1)}}{n-t-1}$$

Hence,

$$\frac{1}{p(1-p)} \sum_{i=0}^{n} \beta_i / \log_2(1/\beta_i) = \frac{2(n-t)}{(n-t-1)(1-2^{-(n-t)})} \to 2 \quad \text{as } n \to \infty.$$

Knowing this would obviously weaken the upper bound on  $\epsilon$  required to prove Theorem 7, though it would not result in a proof of Conjecture 11.

It would be interesting to determine the structure of subsets  $A \subset \{0,1\}^n$  satisfying

$$|\partial A| \le L|A|\log_2(2^n/|A|) \tag{16}$$

for L a fixed positive constant. It is easy to check that

$$k\log_2(2^n/k) \le |\partial C_{n,k}| \le 2k\log_2(2^n/k) \quad \forall k \le 2^{n-1},$$

so when  $|A| \leq 2^{n-1}$ , condition (16) is equivalent to saying that the edgeboundary of A is within a constant factor of the minimum. Regarding this case, Kahn and Kalai [8] make the following conjecture.

**Conjecture 12** (Kahn, Kalai). For any L > 0, there exist L' > 0 and  $\delta > 0$  such that the following holds. If  $A \subset \{0,1\}^n$  is monotone increasing, with measure  $p = \frac{|A|}{2^n} \leq 1/2$ , and with edge-boundary satisfying

$$|\partial A| \le L|A|\log_2(2^n/|A|),$$

then there exists a subcube  $C \subset \{0,1\}^n$  with codimension at most  $L' \log_2(1/p)$ and all fixed coordinates equal to 1, such that

$$\frac{|A \cap C|}{|C|} \ge (1+\delta)p.$$

We believe Conjecture 12 to be true for non-monotone sets as well, if one allows the subcube C to have fixed 0's as well as fixed 1's:

**Conjecture 13.** For any L > 0, there exist L' > 0 and  $\delta > 0$  such that the following holds. If  $A \subset \{0,1\}^n$  has measure  $p = \frac{|A|}{2^n} \leq 1/2$  and has edge-boundary satisfying

$$|\partial A| \le L|A| \log_2(2^n/|A|),$$

then there exists a subcube  $C \subset \{0,1\}^n$  with codimension at most  $L' \log_2(1/p)$ , such that

$$\frac{|A \cap C|}{|C|} \ge (1+\delta)p.$$

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