

# Lower Bounds for the Cop Number when the Robber is Fast

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## Abstract

We consider a variant of the Cops and Robbers game where the robber can move  $t$  edges at a time, and show that in this variant, the cop number of a  $d$ -regular graph with girth larger than  $2t + 2$  is  $\Omega(d^t)$ . By the known upper bounds on the order of cages, this implies that the cop number of a connected  $n$ -vertex graph can be as large as  $\Omega(n^{2/3})$  if  $t \geq 2$ , and  $\Omega(n^{4/5})$  if  $t \geq 4$ . This improves the  $\Omega(n^{\frac{t-3}{t-2}})$  lower bound of Frieze, Krivelevich, and Loh (Variations on Cops and Robbers, preprint, 2010) when  $2 \leq t \leq 6$ . We also conjecture a general upper bound  $O(n^{t/t+1})$  for the cop number in this variant, generalizing Meyniel's conjecture.

## 1 Introduction

The game of Cops and Robbers, introduced by Nowakowski and Winkler [10] and independently by Quilliot [11], is a perfect information game played on a finite graph  $G$ . There are two players, a set of cops and a robber. Initially, the cops are placed onto vertices of their choice in  $G$  (where more than one cop can be placed at a vertex). Then the robber, being fully aware of the cops placement, positions herself in one of the vertices of  $G$ . Then the cops and the robber move in alternate rounds, with the cops moving first; however, players are permitted to remain stationary on their turn if they wish. The players use the edges of  $G$  to move from vertex to vertex. The cops win and the game ends if eventually a cop steps into the vertex currently occupied by the robber; otherwise, i.e., if the robber can elude the cops indefinitely, the robber wins. The parameter of interest is the *cop number* of  $G$ , which is defined as the minimum number of cops needed to ensure that the cops can win. We will assume that the graph  $G$  is simple and connected, because deleting multiple edges or loops does not affect the set of possible moves of the players, and the cop number of a disconnected graph obviously equals the sum of the cop numbers for each connected component.

For a survey of results on the cop number and related search parameters, see the survey by Hahn [7]. The most well known open question in this area is Meyniel's conjecture, published by Frankl in [5]. It states that for every graph  $G$  on  $n$  vertices,  $O(\sqrt{n})$  cops are enough to win. This is asymptotically tight, i.e. for every  $n$  there exists an  $n$ -vertex graph with cop number  $\Omega(\sqrt{n})$ . The best upper bound found so far is  $n2^{-(1-o(1))\sqrt{\log_2 n}}$  (see [6, 9, 12] for several proofs).

Here we consider the variant where in each move, the robber can take any path of length at most  $t$  from her current position, but she is not allowed to pass through a vertex occupied by a cop. The parameter  $t$  is called the *speed* of the robber. This variant was first considered by Fomin, Golovach, Kratochvíl, Nisse, and Suchan [4], who proved that computing the cop number is NP-hard for every  $t$ . Next, Frieze, Krivelevich, and Loh [6] showed that the cop number of an  $n$ -vertex graph can be as large as  $\Omega(n^{\frac{t-3}{t-2}})$ . They also asked whether there exist graphs with cop number  $\omega(\sqrt{n})$  if  $t = 2$ . We give a positive answer to this question, proving the existence of graphs with cop number  $\Omega(n^{2/3})$  for  $t \geq 2$ , and graphs with cop number  $\Omega(n^{4/5})$  for  $t \geq 4$ . This improves their bound  $\Omega(n^{\frac{t-3}{t-2}})$  when  $2 \leq t \leq 6$ . In Section 2 the lower bounds are proved, and in Section 3 a conjecture is proposed, predicting the asymptotic value of cop number in this general setting.

## 2 The lower bounds

**Lemma 1.** *Let  $t, d$  be positive integers with  $t \leq d + 1$ ,  $G$  be a  $(d + 1)$ -regular graph with girth larger than  $2t + 2$  and  $\alpha \in (0, 1)$  be such that  $\alpha d^t$  is an integer. Assume that the robber has speed  $t$ . Then the cop number of  $G$  is at least  $\frac{\alpha(1-\alpha)d^{2t}}{2(t+2)(d+1)^t}$ .*

*Proof.* Let us first define a few terms. A cop *controls* a vertex  $v$  if the cop is on  $v$  or on an adjacent vertex. A cop controls a path if it controls a vertex of the path. The cops control a path if there is a cop controlling it. A vertex  $r$  is *safe* if there exists a set  $S$  of vertices of size  $\alpha d^t$  such that for each  $s \in S$ , there is an  $(r, s)$ -path of length  $t$  not controlled by the cops.

Assume that there are less than  $\frac{\alpha(1-\alpha)d^{2t}}{2(t+2)(d+1)^t}$  cops in the game, and we will show that the robber can elude forever. We may assume that the cops all start in one vertex  $u$ , and the robber starts in a vertex  $v$  at distance  $t + 1$  from  $u$ . Let  $N$  be the set of vertices at distance  $t$  from  $v$ . Then by the girth condition, the cops control only one vertex from  $N$ , and since  $|N| > d^t$ ,  $v$  is a safe vertex. Hence we just need to show that if the robber is in a safe vertex before the cops move, then she can move to a safe vertex after the cops move.

Assume that the robber is in a safe vertex  $r$  after her last move. Then by definition there exists a set  $S$  of vertices of size  $\alpha d^t$  such that for each  $s \in S$ , there is an  $(r, s)$ -path of length  $t$  not controlled by the cops. Let  $U$  be the set of all vertices of these paths. Now, look at the situation after the cops move. There is no cop in  $U$ , thus the robber can move to any of the vertices in  $S$  in her turn, and it suffices to prove that there is a safe vertex in  $S$ . Note that the girth of the graph is larger than  $2t + 2$ , so  $S$  is an independent set and no vertex outside  $U$  is adjacent to two distinct vertices of  $S$ . By an *escaping path* we mean a path of length  $t$  with its first vertex in  $S$  and second vertex not in  $U$ . Clearly every  $s \in S$  is the starting vertex of exactly  $d^t$  escaping paths.

**Claim.** After the cops move, each cop controls at most  $(t + 2)(d + 1)^t$  escaping paths.

*Proof.* We first prove that every vertex  $v$  is on at most  $t(d + 1)^{t-1} + (d + 1)^t$  escaping paths, and if  $v \notin S$  then  $v$  is on at most  $t(d + 1)^{t-1}$  escaping paths. Let  $u_1 u_2 u_3 \dots u_{t+1}$  be an escaping path with  $u_1 \in S$  and  $u_2 \notin U$  such that  $v$  is its  $i$ -th vertex, i.e.  $v = u_i$ . Assume first that  $i \neq 1$ . Note that by definition we have  $u_2 \notin U$ , so  $u_1$  is determined uniquely by  $u_2$ . There are (at most)  $d + 1$  choices for each of  $u_{i-1}, \dots, u_2$ ,

and for each of  $u_{i+1}, u_{i+2}, \dots, u_{t+1}$ . Consequently, for each  $2 \leq i \leq t+1$ ,  $v$  is the  $i$ -th vertex of at most  $(d+1)^{t-1}$  escaping paths, so if  $v \notin S$  then  $v$  is on at most  $t(d+1)^{t-1}$  escaping paths. If  $i = 1$  then  $v \in S$  and there are at most  $d+1$  choices for each of  $u_2, u_3, \dots, u_{t+1}$ , thus each  $v \in S$  is the first vertex of at most  $(d+1)^t$  escaping paths. This shows that  $v$  is on at most  $t(d+1)^{t-1} + (d+1)^t$  escaping paths.

Since the robber was in a safe vertex before the cops move, no cop is in  $U$  at this moment. Hence, each cop can control at most one vertex from  $S$ , through which he can control at most  $(d+1)^t + t(d+1)^{t-1}$  escaping paths. Through every other vertex he can control at most  $t(d+1)^{t-1}$  escaping paths, and he controls  $d+2$  vertices in total. Therefore he controls no more than  $(d+1)^t + (d+2)t(d+1)^{t-1} \leq (t+2)(d+1)^t$  escaping paths.  $\square$

Now, since there are less than  $\frac{\alpha(1-\alpha)d^{2t}}{2(t+2)(d+1)^t}$  cops in the game, the cops control less than  $\alpha(1-\alpha)d^{2t}/2$  of the escaping paths. Since  $S$  has  $\alpha d^t$  vertices, and each path has two endpoints, there must be an  $s \in S$  such that at most  $(1-\alpha)d^t$  escaping paths starting from  $s$  are controlled. Consequently, there are  $\alpha d^t$  uncontrolled escaping paths starting from  $s$ . Note that girth of  $G$  is larger than  $2t$  so the other endpoints of these paths are distinct. Hence  $s$  is safe by definition and the robber moves to  $s$ .  $\square$

**Corollary 1.** *Let  $t$  be some fixed positive integer denoting the speed of the robber. If  $G$  is a  $d$ -regular graph (where  $d \geq \max\{3, t\}$ ) with girth larger than  $2t+2$ , then the cop number of  $G$  is  $\Omega(d^t)$ .*

In order to use Corollary 1 to prove interesting lower bounds for the cop number, one should look at vertex-minimal regular graphs with large girth, known as *cages*. Here are two useful results on cages (see [3] for a survey):

**Theorem 1** ([8]). *Let  $g \geq 5$ , and  $d \geq 3$  be an odd prime power. Then there exists a  $d$ -regular graph of girth  $g$  with at most  $2d^{1+\frac{3}{4}g-a}$  vertices, where  $a = 4, 11/4, 7/2, 13/4$  for  $g \equiv 0, 1, 2, 3 \pmod{4}$ , respectively.*

**Theorem 2** ([2]). *Let  $d \geq 3$  be a prime power. Then there exists a  $d$ -regular graph with girth 12 and at most  $2d^5$  vertices.*

**Theorem 3.** *Let  $t$  be some fixed positive integer denoting the speed of the robber.*

(a) *If  $t \geq 2$  then for every  $n$  there exists an  $n$ -vertex graph with cop number  $\Omega(n^{2/3})$ .*

(b) *If  $t \geq 4$  then for every  $n$  there exists an  $n$ -vertex graph with cop number  $\Omega(n^{4/5})$ .*

*Proof.* (a) As the cop number will not decrease when the speed of the robber is increased, we just need to show the proposition for  $t = 2$ . Let  $n \geq 54$  and  $d$  be the largest prime number such that  $2d^3 \leq n$ . Since there exists a prime between  $d$  and  $2d$ , we have  $n < 2(2d)^3$  so  $d = \Theta(n^{1/3})$ . By Theorem 1, there exists a  $d$ -regular graph  $H$  of girth 7 with at most  $2d^3$  vertices. By Corollary 1 the cop number of  $H$  is  $\Omega(d^2) = \Omega(n^{2/3})$ . Let  $G$  be the graph formed by joining some vertex of  $H$  to an endpoint of a disjoint path with  $n - |V(H)|$  vertices. It is easy to check that the cop number of  $G$  equals the cop number of  $H$ , which is  $\Omega(n^{2/3})$ .

(b) Again we just need to show the proposition for  $t = 4$ . Let  $n \geq 486$  and  $d$  be the largest prime number such that  $2d^5 \leq n$ . A similar argument shows that  $d = \Theta(n^{1/5})$ . By Theorem 2, there exists a  $d$ -regular graph  $H$  of girth 12 with at most  $2d^5$  vertices. By Corollary 1 the cop number of  $H$  is  $\Omega(d^4) = \Omega(n^{4/5})$ . Let  $G$  be the graph formed by joining some vertex of  $H$  to an endpoint of a

disjoint path with  $n - |V(H)|$  vertices. Then the cop number of  $G$  equals the cop number of  $H$ , which is  $\Omega(n^{4/5})$ .  $\square$

### 3 Concluding remarks

Let  $f_t(n)$  be the maximum possible cop number of a connected  $n$ -vertex graph assuming the robber has speed  $t$ . It is well-known (and also follows from Corollary 1 and Theorem 1 with  $g = 5$ ) that  $f_1(n) = \Omega(\sqrt{n})$ . Meyniel conjectured that indeed  $f_1(n) = \Theta(\sqrt{n})$ . Frieze, Krivelevich, and Loh [6] showed that  $f_t(n) = \Omega(n^{\frac{t-3}{t-2}})$  if  $t \geq 3$ . In this note we proved that  $f_2(n) = \Omega(n^{2/3})$  and  $f_4(n) = \Omega(n^{4/5})$ . A natural question is that of the asymptotic behavior of  $f_t(n)$ .

Notice that if  $G$  is a  $d$ -regular graph with girth larger than  $2t + 2$ , then Moore's bound gives  $d = O(n^{1/t+1})$ . Hence Corollary 1 cannot give a better bound than  $f_t(n) = \Omega(n^{t/t+1})$ . Generalizing Meyniel's conjecture, we conjecture that this is actually the asymptotic behavior of  $f_t(n)$ .

**Conjecture.** For every fixed  $t$  we have  $f_t(n) = \Theta(n^{t/t+1})$ .

Proving better upper bounds on the order of cages would imply that the conjecture is tight. Specifically, if for a fixed  $t$ , and infinitely many  $d$ , there exists a  $d$ -regular graph with girth larger than  $2t + 2$  on  $O(d^{t+1})$  vertices, then  $f_t(n) = \Omega(n^{t/t+1})$  (see Corollary 1).

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**Addendum.** Alon and the author [1] have recently extended the result of this note, and proved that  $f_t(n) = \Omega(n^{t/t+1})$  for every fixed positive integer  $t$ .

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