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Distance preserving Ramsey graphs

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DISTANCE PRESERVING RAMSEY GRAPHS

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Abstract. We prove the following metric Ramsey theorem. For any connected graph G endowed with a linear order on its vertex set there exists a graph R such that in every coloring of the t-tuples of vertices of R it is possible to find a copy $\mathsf{G} \subset R$ satisfying:

- $\operatorname{dist}_{\mathsf{G}}(x, y) = \operatorname{dist}_R(x, y)$ for every $x, y \in V(\mathsf{G})$;
- the color of a t-tuple in G is a function of the graph-distance metric induced by the t-tuple.

1. Introduction

In [2], [3] and [12, 13] the following extension of the Ramsey Theorem was proved.

Theorem 1. *For any graph G there exists a graph R with the property that in any 2-coloring of the edges of R there exists an induced copy $\mathsf{G} \subset R$ (i.e. $\mathsf{G} \in \binom{R}{G}_{\text{ind}}$) which is monochromatic.*

Notice that for an induced copy $\mathsf{G} \in \binom{R}{G}_{\text{ind}}$ and for an isomorphism $\phi\colon V(G) \to V(\mathsf{G}) \subset V(R)$, we have

$$\operatorname{dist}_G(x, y) = \operatorname{dist}_R\big(\phi(x), \phi(y)\big) \tag{1}$$

whenever $\operatorname{dist}_G(x, y) \le 2$. A question, whether the above restriction on distances is necessary was answered by Nešetřil and the second author [8] (see Remark 5 below). In [8] it is shown that Theorem 1 may be strengthened by obtaining a monochromatic *isometric* copy of G in R (*i.e.* (1) holds for all $x, y \in V(\mathsf{G})$) instead of just an induced copy.

Another possible generalization of Theorem 1 deals with partitioning (coloring) graphs other than edges (K_2). Such an extension was obtained in [1] and [7]. In this paper we will considered a joint extension of both [1, 7] and [8].

Remark 2 (Ordered graphs)**.** Similarly as in [1, 7], in this paper every graph is implicitly assumed to have a total order in its vertex-set. ******* All maps considered here are assumed to be *monotone*, that is $\phi(u) < \phi(v)$ whenever $u < v$. ******* In particular, all graph isomorphisms are unique.

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Definition 3. For graphs G and H, the graph G is a subgraph of H (we write $G \subset H$) if $V(G) \subset V(H)$, $E(G) \subset E(H)$ and the order $<_G$ in $V(G)$ respects the order $<_H$ in $V(H)$, that is, for every $u, v \in V(G)$ we have $u <_G v$ iff $u <_H v$.

We denote by $\binom{H}{G}$ the set of all subgraphs of H which are isomorphic to G.

Theorem 4 (Main Result)**.** *Let $t \in \mathbb{N}$ and G be a connected graph.*

Then there exists a graph R with the following properties: for every 2-coloring of $\binom{V(R)}{t}$ there exists $\mathsf{G} \in \binom{R}{G}$ such that

- $\mathrm{dist}_{\mathsf{G}}(x, y) = \mathrm{dist}_R(x, y)$ *for all* $x, y \in V(\mathsf{G})$*;*
- *the color of any $S = \{v_1, v_2, \dots, v_t\} \subset V(\mathsf{G})$, with $v_1 < v_2 < \cdots < v_t$, depends only on $\big(\mathrm{dist}_{\mathsf{G}}(v_i, v_j)\big)_{1 \le i < j \le t}$, namely, the color of S is a function of the metric induced by S.*

Remark 5. The particular case $t = 2$ of Theorem 4 implies that for any graph G it is possible to find some graph R in which every coloring of the pairs in $\binom{V(R)}{2}$ yields a metric copy $\mathsf{G} \in \binom{R}{G}$ in which the color of $\{x, y\} \in \binom{V(\mathsf{G})}{2}$ is a function of $\mathrm{dist}_{\mathsf{G}}(x, y)$. (In particular, the edges of $E(\mathsf{G})$ are monochromatic.) This special case $t = 2$ was stated in [8].

Definition 6. A discrete *metric* ρ on the ordered set $[t] = \{1, 2, \dots, t\}$ is a symmetric function $\rho\colon [t]^2 \to \mathbb{N} \cup \{\infty\}$ satisfying the triangle inequality:

$$\rho(i, j) + \rho(j, k) \ge \rho(i, k).$$

Let $\ell \in \mathbb{N}$ be fixed. For a graph H and a set $S = \{v_1, v_2, \dots, v_t\} \subset V(H)$ with $v_1 < v_2 < \cdots < v_t$ we say that S is *ρ_ℓ-metric with respect to H* if for all $1 \le i < j \le t$

- $\mathrm{dist}_H(v_i, v_j) = \rho(i, j)$ whenever $\rho(i, j) \le \ell$;
- $\mathrm{dist}_H(v_i, v_j) \ge \ell$ whenever $\rho(i, j) > \ell$.

A set S as above is called a (ρ_ℓ, t)-tuple. We denote by $\binom{H}{\rho_\ell}$ the family of all (ρ_ℓ, t)-tuples of H.

Definition 7. A graph G naturally induces a metric $\rho(G)$ over its vertices by defining the distance between pairs of vertices as the length of a shortest path connecting them (when the pair is not connected, their distance is ∞). For a pair of graphs $G \subset R$, the graph G is said to be *ℓ-metric in R* if $V(G)$ is $\rho(G)_\ell$-metric with respect to R. Namely, G is ℓ-metric in R if no pair of vertices in G admits a shortcut in R of length smaller than ℓ. For instance, G is 2-metric in R iff it is an induced subgraph of R. A graph G is said to be *metric in R* if it is ℓ-metric for all ℓ (namely, $\mathrm{dist}_G(x, y) = \mathrm{dist}_R(x, y)$ for every $x, y \in V(G)$).

For $A, B \subset V(G)$ we will write $A \prec B$ if $\max(A) < \min(B)$.

Definition 8. Let G be a graph and $q \ge 2$. Suppose that G admits a vertex partition $V(G) = V_1^q(G) \cup \cdots \cup V_q^q(G)$ in which all edges of G are *crossing*, that is, no edge intersects the same class in more than one vertex. Furthermore suppose that $V_1^q(G) \prec V_2^q \prec \cdots \prec V_q^q(G)$. Such a graph G is called a *q-partite* graph.

If G and H are q-partite graphs, a *partite embedding* is an injective monotone map $\phi\colon V(G) \to V(H)$ which is edge-preserving ($\phi(e) \in E(H)$ for all $e \in E(G)$) and satisfies $\phi(V_j^q(G)) \subset V_j^q(H)$ for all $j = 1, \ldots, q$.

Definition 9. We will use the following notation.

- Let $\phi\colon V(G) \to V(H)$ be an embedding of G into H. Then we set
$$\phi(G) = \big(\phi(V(G)), \{\phi(e) \,:\, e \in E(G)\}\big) \subset H.$$
- For q-partite graphs G and H we denote by $\binom{H}{G}_{\mathrm{Part}(q)}$ the set of all subgraphs $\phi(G)$ of H where $\phi\colon V(G) \to V(H)$ is a partite embedding.
- For a graph G it will be convenient to use G (typeset in a sans-serif font) to denote an isomorphic copy of G.
- Suppose that G is a graph and $\mathcal{I}$ is a hypergraph with vertex set $V(\mathcal{I}) \subset V(G)$. For G with $\sigma\colon V(G) \to V(\mathsf{G})$ being the monotone isomorphism of G into G, let $\mathcal{I}_{\mathsf{G}}$ denote the hypergraph $\sigma(\mathcal{I})$ with vertex set $\sigma(V(\mathcal{I})) \subset V(\mathsf{G})$ and edges $\{\sigma(I) \,:\, I \in \mathcal{I}\}$.

Lemma 10 below is a technical result which will be used in the proof of our main result, Theorem 4.

Lemma 10 (Partite Lemma)**.** *Let $\ell, t, q \in \mathbb{N}$, $t \leq q$, and ρ be a metric on $[t]$.*

Suppose that G is a q-partite graph with $V(G) = V_1^q(G) \cup \cdots \cup V_q^q(G)$ and, for some $1 \leq j_1 < j_2 < \cdots < j_t \leq q$, $\mathcal{I} \subseteq \binom{G}{\rho_\ell}$ is a t-partite t-uniform hypergraph with classes $\{V_{j_i}^q(G)\}_{i=1}^t$.

Then there exists a q-partite graph R and $\mathcal{F} \subset \binom{R}{G}_{\mathrm{Part}(q)}$ satisfying the following properties.

(1) For any 2-coloring of $\binom{V(R)}{t}$ there exists $\mathsf{G} \in \mathcal{F}$ such that every (ρ_ℓ, t)-tuple in $\mathcal{I}_{\mathsf{G}} \subset \binom{\mathsf{G}}{\rho_\ell}$ is monochromatic.

(2) Every $\mathsf{G} \in \mathcal{F}$ is ℓ-metric in R.

Remark 11. Note that in Condition *(1)* the only relevant colored t-tuples of $V(R)$ are those in $\bigcup_{\mathsf{G} \in \mathcal{F}} \mathcal{I}_{\mathsf{G}}$.

The proof of Lemma 10 uses the *partite construction* method, which was introduced in [9] and has been a successful tool for proving the existence of several Ramsey structures such as metric spaces [6], systems of sets [11], Steiner systems [10] etc.

Using an extra round of the partite construction we will extend Lemma 10 and establish a result guaranteeing the whole family $\binom{H}{\rho}$ being monochromatic rather than only a t-partite t-uniform family $\mathcal{I} \subset \binom{H}{\rho}$. Namely, we prove the following lemma.

Lemma 12. *Let $t \in \mathbb{N}$, ρ be a metric on $[t]$ and H be a connected graph.*

There exists a graph R such that for every 2-coloring of $\binom{V(R)}{t}$ there exists a metric $\mathsf{H} \subset R$ such that $\binom{\mathsf{H}}{\rho}$ is monochromatic.

A sketch of the proof of Lemma 12 will be given in Section 4. By repeated applications of Lemma 12, we obtain Theorem 4.

Proof of Theorem 4. Let $\mathcal{M} = \{\rho^1, \dots, \rho^m\}$ be the set of all metrics induced by t vertices of G. Apply Lemma 12 to $R_0 = G$ and ρ^1 to obtain a graph R_1. After R_i is constructed, $1 \le i \le m-1$, obtain R_{i+1} by applying Lemma 12 to R_i and ρ^{i+1}.

We claim that $R = R_m$ satisfies the conditions of Theorem 4. Indeed, given any 2-coloring of $\binom{V(R)}{t}$, we can find a metric copy R^{m-1} of R_{m-1} in which every (ρ^m, t)-tuple in $\binom{\mathsf{R}^{m-1}}{\rho^m}$ is colored by c_m. Iterating this argument yields a sequence $\mathsf{R}^0 \subset \mathsf{R}^1 \subset \cdots \subset \mathsf{R}^{m-1} \subset R$ such that $\mathsf{R}^i \cong R_i$ is metric in R^{i+1} and every (ρ^{i+1}, t)-tuple in $\binom{\mathsf{R}^i}{\rho^{i+1}}$ has the same color c_{i+1}. The graph $\mathsf{G} = \mathsf{R}^0 \cong G$ is metric in R and is such that $\binom{\mathsf{G}}{\rho}$ is monochromatic for every ρ. $\square$

2. Proof of Lemma 10

Our proof will use a double induction argument. The main induction is over ℓ. In order to carry on the induction we need to prove a slightly stronger statement (see the box below). For each $\ell \ge 2$ we have a graph R_ℓ and a family $\mathcal{F}_\ell \subset \binom{R_\ell}{G}_{\mathrm{Part}(q)}$. Lemma 23 in Section 3, which is a straightforward adaptation of the result of [10], shows that the base case holds ($\ell = 2$).

Induction over ℓ – Hypothesis for R_ℓ and $\mathcal{F}_\ell$
Lemma 10 holds for ℓ, namely, R_ℓ and $\mathcal{F}_\ell \subset \binom{R_\ell}{G}_{\mathrm{Part}(q)}$ satisfy conditions *(1)*, *(2)*.
In addition,

(A) If $\mathsf{G}_1, \mathsf{G}_2 \in \mathcal{F}_\ell$ and $u \in V(\mathsf{G}_1) \cap V(\mathsf{G}_2)$ then there are (ρ_ℓ, t)-tuples $I^j \in \mathcal{I}_{\mathsf{G}_j}$, $j = 1, 2$, such that $u \in I^1 \cap I^2$.

(B) If $\mathsf{G}_1, \mathsf{G}_2 \in \mathcal{F}_\ell$ are distinct and $u, v \in V(\mathsf{G}_1) \cap V(\mathsf{G}_2)$ then either
  - (B1) there exist (ρ_ℓ, t)-tuples $I^j \in \mathcal{I}_{\mathsf{G}_j}$, $j = 1, 2$, such that $\{u, v\} \subset I^1 \cap I^2$ or
  - (B2) if $\sigma_j \colon V(\mathsf{G}_j) \to V(G)$, $j = 1, 2$, are the isomorphisms of $\mathsf{G}_1, \mathsf{G}_2$ into G then $\sigma_1(u) = \sigma_2(u)$ and $\sigma_1(v) = \sigma_2(v)$.

Suppose now that the induction hypothesis holds for $\ell \ge 2$. We will show that it also holds for $\ell + 1$.

Let G be the given q-partite graph and let $\mathcal{I} \subset \binom{G}{\rho_{\ell+1}} \subset \binom{G}{\rho_\ell}$ be a t-partite t-uniform hypergraph with classes $\{V^q_{j_i}(G)\}_{i=1}^t$. We may assume without loss

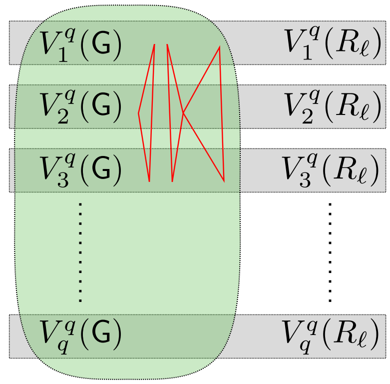

FIGURE 1. An illustration of R_ℓ and $\mathsf{G} \in \mathcal{F}_\ell$. Here we assume $t = 3$, $j_1 = 1$, $j_2 = 2$ and $j_3 = 3$. The triples in $\mathcal{I}_{\mathsf{G}}$ are represented by the crossing triangles.

of generality that $t < |V(G)|$ since otherwise G itself would trivially satisfy the conditions of the lemma. Let the q-partite graph R_ℓ and $\mathcal{F}_\ell \subset \binom{R_\ell}{G}_{\mathrm{Part}(q)}$ be obtained from our induction hypothesis.

Consider the family

$$\bigcup_{\mathsf{G} \in \mathcal{F}_\ell} \mathcal{I}_{\mathsf{G}} = \{I_1, I_2, \ldots, I_m\}. \tag{2}$$

This family is a t-partite t-uniform hypergraph with partition $\{V_{j_i}^q(R_\ell)\}_{i=1}^t$ (see Figure 1).

We will construct a sequence of $|V(R_\ell)|$-partite graphs $P_0, P_1, \ldots, P_m$, which we will call *pictures*, and families $\mathcal{F}(P_k) \subset \binom{P_k}{G}_{\mathrm{Part}(q)}$, $k = 0, 1, \ldots, m$. We will then show that $R_{\ell+1} = P_m$ and $\mathcal{F}_{\ell+1} = \mathcal{F}(P_m)$ satisfy conditions *(1)*, *(2)*, (A) and (B). This will establish the induction step and conclude the proof of Lemma 10.

Let us start by constructing P_0 (see Figure 2). For convenience, let $r_\ell = |V(R_\ell)|$. For each $u \in V(R_\ell)$, let

$$V_u^{r_\ell}(P_0) = \{(u, \mathsf{G}) \,:\, \mathsf{G} \in \mathcal{F}_\ell, V(\mathsf{G}) \ni u\}. \tag{3}$$

Recalling the total order on $V(R_\ell)$ we may assume in fact that $V(R_\ell) = \{1, 2, \ldots, r_\ell\}$. We then impose a total order in $V(P_0)$ so that $V_j^{r_\ell}(P_0)$, $j = 1, \ldots, r_\ell$, satisfies $V_j^{r_\ell}(P_0) \prec V_{j+1}^{r_\ell}(P_0)$ for all j.

The edges of P_0 are of the form $\{(u, \mathsf{G}), (w, \mathsf{G})\}$, where $uw \in E(\mathsf{G})$, $\mathsf{G} \in \mathcal{F}_\ell$. Notice that the r_ℓ-partition of P_0 given by (3) is indeed such that every edge of P_0 is crossing. We set $\mathcal{F}(P_0)$ to be the set of copies of G in correspondence with $\mathcal{F}_\ell$. In particular, $|\mathcal{F}(P_0)| = |\mathcal{F}_\ell|$. Moreover, the projection $\pi_0(u, G) = u$ defines a monotone homomorphism from P_0 to R_ℓ.

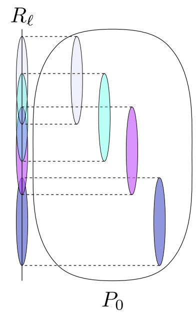

FIGURE 2. The graph P_0 is a disjoint union of copies of G where each copy is projected by π_0 into a copy of G in $\mathcal{F}_\ell$.

Assuming that the hypothesis hold for some $\ell \geq 2$ we will now induce on k.

Induction over k – Hypothesis on Pictures

(I) The picture P_k is r_ℓ-partite with classes $V_j^{r_\ell}(P_k)$, $j = 1, \ldots, r_\ell$. The projection map $\pi_k \colon V(P_k) \to V(R_\ell)$ given by $\pi_k(x) = j$ iff $x \in V_j^{r_\ell}(P_k)$ is a homomorphism of P_k into R_ℓ. Moreover, $\pi_k(\mathsf{G}) \in \mathcal{F}_\ell$ for every $\mathsf{G} \in \mathcal{F}(P_k)$.

(II) The family $\mathcal{F}(P_k)$ is contained in $\binom{P_k}{G}_{\mathrm{Part}(q)}$.

(III) The family $\mathcal{F}(P_k)$ satisfies conditions (A), (B).

(IV) Every $\mathsf{G} \in \mathcal{F}(P_k)$ is $(\ell+1)$-metric in P_k.

Claim 13. *The graph P_0 satisfies the induction hypothesis for pictures.*

Since the copies of G in P_0 are vertex-disjoint (and thus metric) and are projected by π_0 into copies of G in R_ℓ it is clear that (I) and (II) hold and that $\mathcal{F}(P_0)$ satisfies conditions (A) and (B). It remains to check (III), namely, that $\mathcal{F}(P_0)$ is contained in $\binom{P_0}{G}_{\mathrm{Part}(q)}$.

We now observe that the q-partition of $V(P_0)$ may be expressed in terms of π_0 as

$$V_j^q(P_0) = \pi_0^{-1}(V_j^q(R_\ell)) = \bigcup_{u \in V_j^q(R_\ell)} V_u^{r_\ell}(P_0)$$

for $j = 1, \ldots, q$. For every $\mathsf{G} \in \mathcal{F}(P_0)$, we have $\mathsf{G}' = \pi_0(\mathsf{G}) \in \mathcal{F}_\ell$. Let $\sigma \colon V(G) \to V(\mathsf{G}')$ be the partite isomorphism between G and G' guaranteed by the induction hypothesis. Then $\pi_0^{-1} \circ \sigma \colon V(G) \to V(\mathsf{G})$ is a partite isomorphism of G into G by our choice of $V_j^q(P_0)$, $j = 1, \ldots, q$.

Hence P_0 satisfies the induction hypothesis for pictures and Claim 13 is proved.

Suppose that P_k, $\mathcal{F}(P_k)$, and π_k, $k \geq 0$, are constructed and satisfy the induction hypothesis. Since every $\mathsf{G} \in \mathcal{F}(P_k)$ is $(\ell+1)$-metric in P_k, it follows

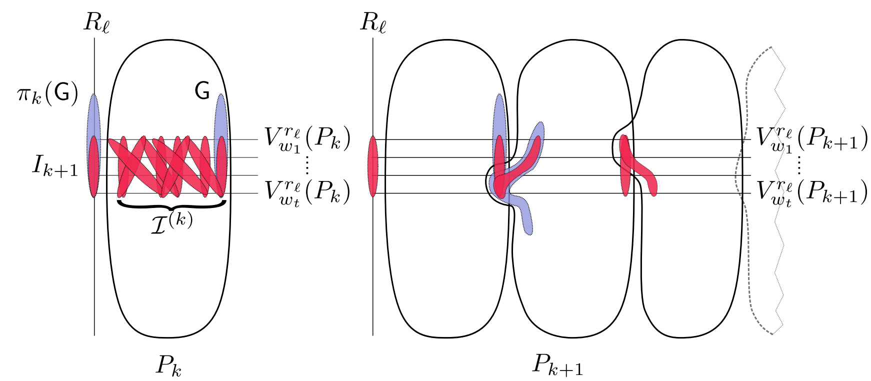

FIGURE 3. The picture P_{k+1} is obtained from picture P_k through the induction hypothesis over ℓ. The vertices in R_ℓ are not vertically ordered to simplify the figure.

that $\mathcal{I}_{\mathsf{G}} \subset \binom{\mathsf{G}}{\rho_{\ell+1}} \subset \binom{P_k}{\rho_{\ell+1}}$ for every $\mathsf{G} \in \mathcal{F}(P_k)$. For $k = 0, 1, \dots, m-1$, define

$$\mathcal{I}^{(k)} = \Big\{ I \in \bigcup_{\mathsf{G}\in\mathcal{F}(P_k)} \mathcal{I}_{\mathsf{G}} \,:\, \pi_k(I) = I_{k+1} \Big\} \subset \binom{P_k}{\rho_{\ell+1}}, \tag{4}$$

where the $(\rho_{\ell+1}, t)$-tuple I_{k+1} is defined as the $(k+1)$th tuple in (2).

Observe that by construction, $\mathcal{I}^{(k)}$ is a t-partite t-uniform hypergraph. Indeed, every tuple in $\mathcal{I}^{(k)}$ is crossing with respect to the t-tuple of sets $\{\pi_k^{-1}(u) = V_u^{r_\ell}(P_k)\}_{u\in I_{k+1}}$. To construct P_{k+1} we invoke our induction assumption over ℓ with

- r_ℓ in place of q;
- P_k in place of G;
- $\mathcal{I}^{(k)}$ in place of $\mathcal{I}$.

We then obtain the graph P_{k+1} and a family $\widehat{\mathcal{F}}_{k+1} \subset \binom{P_{k+1}}{P_k}_{\mathrm{Part}(r_\ell)}$ satisfying conditions *(1)*, *(2)*, (A) and (B). More specifically, the following holds

$(1)_{k+1}$ For every coloring of $\binom{V(P_{k+1})}{t}$ there exists $\mathsf{P} \in \widehat{\mathcal{F}}_{k+1}$ such that $\mathcal{I}^{(k)}_{\mathsf{P}}$ is monochromatic (and t-partite with respect to $\{V^{r_\ell}_{w_i}(P_{k+1})\}_{i=1}^t$).

$(2)_{k+1}$ Every $\mathsf{P} \in \widehat{\mathcal{F}}_{k+1}$ is ℓ-metric in P_{k+1}.

$(\mathrm{A})_{k+1}$ If $\mathsf{P}^1, \mathsf{P}^2 \in \widehat{\mathcal{F}}_{k+1}$ are distinct and $u \in V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$ then there are tuples $I^j_* \in \mathcal{I}^{(k)}_{\mathsf{P}^j}$, $j = 1, 2$, such that $u \in I^1_* \cap I^2_*$.

$(\mathrm{B})_{k+1}$ If $\mathsf{P}^1, \mathsf{P}^2 \in \widehat{\mathcal{F}}_{k+1}$ are distinct and $u, v \in V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$ then either

  $(\mathrm{B1})_{k+1}$ there exist tuples $I^j_* \in \mathcal{I}^{(k)}_{\mathsf{P}^j}$, $j = 1, 2$, such that $\{u, v\} \subset I^1_* \cap I^2_*$ or

  $(\mathrm{B2})_{k+1}$ if $\phi_j \colon V(\mathsf{P}^j) \to V(P_k)$, $j = 1, 2$, are the isomorphisms of $\mathsf{P}^1, \mathsf{P}^2$ into P_k then $\phi_1(u) = \phi_2(u)$ and $\phi_1(v) = \phi_2(v)$.

The projection $\pi_{k+1} \colon V(P_{k+1}) \to V(R_\ell)$ is naturally defined in terms of the partition $\{V^{r_\ell}_j(P_{k+1})\}_{j=1}^{r_\ell}$ (this partition is given by the induction hypothesis over ℓ). More concretely, $\pi_{k+1}(u) = j$ iff $u \in V_j^{r_\ell}(P_{k+1})$. For any $\mathsf{P} \in \widehat{\mathcal{F}}_{k+1}$ with isomorphism $\phi\colon V(P_k) \to V(\mathsf{P})$ the following diagram commutes:

$$\begin{tikzcd} P_k \arrow[r, leftrightarrow, "\phi"] \arrow[d, "\pi_k"'] & \mathsf{P} \arrow[r, "\imath"] & P_{k+1} \arrow[dll, "\pi_{k+1}"] \\ R_\ell & & \end{tikzcd} \tag{5}$$

Indeed, because ϕ is a partite embedding, we have $\phi\big(V_j^{r_\ell}(P_k)\big) \subset V_j^{r_\ell}(P_{k+1})$ for all $j = 1, \ldots, r_\ell$. Hence, for $u \in V(P_k)$, $\pi_k(u) = j$ iff $u \in V_j^{r_\ell}(P_k)$ iff $\phi(u) \in V_j^{r_\ell}(P_{k+1})$ iff $\pi_{k+1} \circ \phi(u) = j$. This shows that $\pi_k = \pi_{k+1} \circ \phi$ and thus the diagram commutes.

The graph P_{k+1} may also be viewed as q-partite with partition given by the classes

$$V_j^q(P_{k+1}) = \pi_{k+1}^{-1}\big(V_j^q(R_\ell)\big) = \bigcup_{u \in V_j^q(R_\ell)} V_u^{r_\ell}(P_{k+1}), \quad j = 1, \ldots, q. \tag{6}$$

Notice that because $V_1^q(R_\ell) \prec V_2^q(R_\ell) \prec \cdots \prec V_q^q(R_\ell)$ and $V_1^{r_\ell}(P_{k+1}) \prec \cdots \prec V_{r_\ell}^{r_\ell}(P_{k+1})$ we also have $V_1^q(P_{k+1}) \prec \cdots \prec V_q^q(P_{k+1})$. Also observe that the r_ℓ-partition of P_{k+1} is a refinement of its q-partition.

We will now start the proof of the induction step over k.

Claim 14. *Condition* (I) *holds for* P_{k+1}.

We will start by showing that the projection map π_{k+1} is a homomorphism of P_{k+1} into R_ℓ.

By construction, the r_ℓ-partite graph P_{k+1} has a partition with classes $V_j^{r_\ell}(P_{k+1}) = \pi_{k+1}^{-1}(j)$, $j = 1, \ldots, r_\ell$, such that for every $\mathsf{P} \in \widehat{\mathcal{F}}_{k+1} \subset \binom{P_{k+1}}{P_k}_{\mathrm{Part}(r_\ell)}$ the (unique monotone) isomorphism $\phi\colon V(P_k) \to V(\mathsf{P})$ satisfies $\phi(V_j^{r_\ell}(P_k)) \subset V_j^{r_\ell}(P_{k+1})$.

We assume without loss of generality that every edge in P_{k+1} is contained in some copy $\mathsf{P} \in \widehat{\mathcal{F}}_{k+1}$. Indeed, otherwise we could delete such an edge without affecting the essential properties of P_{k+1} (distances may only increase after an edge is deleted). Since the edges of P must be crossing with respect to $\{V_j^{r_\ell}(P_{k+1})\}_{j=1}^{r_\ell}$, it follows that the projection π_{k+1} is a homomorphism of P_{k+1} into R_ℓ.

For any $\mathsf{P} \in \widehat{\mathcal{F}}_{k+1}$, given the (unique monotone) isomorphism $\phi\colon V(P_k) \to V(\mathsf{P})$, set

$$\mathcal{F}(\mathsf{P}) = \{\phi(\mathsf{G}) \;:\; \mathsf{G} \in \mathcal{F}(P_k)\}.$$

Define

$$\mathcal{F}(P_{k+1}) = \bigcup_{\mathsf{P} \in \widehat{\mathcal{F}}_{k+1}} \mathcal{F}(\mathsf{P}). \tag{7}$$

Observe that there is a rich structure of copies of G in P_{k+1} which is inherited by the many overlapping copies of P_k in P_{k+1}.

Now we will prove that $\pi_{k+1}(\mathsf{G}) \in \mathcal{F}_\ell$ for every $\mathsf{G} \in \mathcal{F}(P_{k+1})$. For $\mathsf{G} \in \mathcal{F}(\mathsf{P})$, $\mathsf{P} \in \widehat{\mathcal{F}}_{k+1}$, and the isomorphism $\phi\colon V(P_k) \to V(\mathsf{P})$ we have $\phi^{-1}(\mathsf{G}) \in \mathcal{F}(P_k)$ and, by the induction hypothesis, $\pi_k\big(\phi^{-1}(\mathsf{G})\big) \in \mathcal{F}_\ell$. The fact that $\mathsf{P} \in \binom{P_{k+1}}{P_k}_{\mathrm{Part}(r_\ell)}$ implies that $\phi(V_j^{r_\ell}(P_k)) = V_j^{r_\ell}(\mathsf{P}) \subset V_j^{r_\ell}(P_{k+1})$, for $j = 1, \ldots, r_\ell$. Consequently, we have $\pi_{k+1}|_{V(\mathsf{G})} = \pi_k \circ \phi^{-1}|_{V(\mathsf{G})}$ (see the diagram (5)) and it follows that $\pi_{k+1}(\mathsf{G}) \in \mathcal{F}_\ell$ for every $\mathsf{G} \in \mathcal{F}(P_{k+1})$. This concludes the proof that (I) holds.

Claim 15. *Condition* (II) *holds for P_{k+1}, namely, $\mathcal{F}(P_{k+1}) \subset \binom{P_{k+1}}{G}_{\mathrm{Part}(q)}$.*

To prove that (II) holds consider the q-partition described in (6) in terms of π_{k+1}^{-1} by $V_j^q(P_{k+1}) = \pi_{k+1}^{-1}(V_j^q(R_\ell))$, $j = 1, \ldots, q$. Notice that every edge of P_{k+1} is crossing with respect to this partition since for every j, $V_j^q(R_\ell)$ is an independent set in R_ℓ and the projection π_{k+1} is a homomorphism from P_{k+1} to R_ℓ. We use the fact that every $\mathsf{G} \in \mathcal{F}(P_{k+1})$ is such that $\mathsf{G}' = \pi_{k+1}(\mathsf{G}) \in \mathcal{F}_\ell$ and $\mathcal{F}_\ell \subset \binom{R_\ell}{G}_{\mathrm{Part}(q)}$. Namely, the isomorphism $\sigma\colon V(G) \to V(\mathsf{G}')$ is partite, meaning that $\sigma(V_j^q(G)) \subset V_j^q(R_\ell)$ for all $j = 1, \ldots, q$. It follows that the composition $\pi_{k+1}^{-1} \circ \sigma\colon V(G) \to V(\mathsf{G})$ is a partite isomorphism of G into G establishing that $\mathsf{G} \in \binom{P_{k+1}}{G}_{\mathrm{Part}(q)}$.

Claim 16. *If $\mathsf{P}^1, \mathsf{P}^2 \in \widehat{\mathcal{F}}_{k+1}$ are distinct and $u \in V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$ then $\pi_{k+1}(u) \in I_{k+1}$. Consequently, for each $\mathsf{G} \in \mathcal{F}(P_{k+1})$ there is a unique $\mathsf{P} \in \widehat{\mathcal{F}}_{k+1}$ such that $\mathsf{G} \subset \mathsf{P}$.*

From Condition $(\mathrm{A})_{k+1}$ there exist $I_*^j \in \mathcal{I}_{\mathsf{P}^j}^{(k)}$, $j = 1, 2$, such that $u \in I_*^1 \cap I_*^2$. From Diagram (5) we conclude that the isomorphism $\phi_1\colon V(P_k) \to V(\mathsf{P}^1)$ satisfies $\pi_k = \pi_{k+1} \circ \phi_1$. Because $I^1 = \phi_1^{-1}(I_*^1) \in \mathcal{I}^{(k)}$, we have

$$\pi_{k+1}(I_*^1) = \pi_{k+1} \circ \phi_1(I^1) = \pi_k(I^1) \overset{(4)}{=} I_{k+1}.$$

Consequently, $\pi_{k+1}(u) \in I_{k+1}$.

Since each $\mathsf{G} \in \mathcal{F}(P_{k+1})$ is mapped by π_{k+1} onto a member of $\mathcal{F}_\ell$, the projection must be one-to-one over $V(\mathsf{G})$. Therefore $\big|\pi_{k+1}\big(V(\mathsf{G})\big)\big| = |V(G)| > t$ and thus $\pi_{k+1}\big(V(\mathsf{G})\big) \not\subset I_{k+1}$. It follows that $V(\mathsf{G}) \not\subset V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$.

Claim 17. *Condition* (III) *holds for P_{k+1}, namely, $\mathcal{F}(P_{k+1})$ satisfies the intersection conditions* (A) *and* (B)*.*

Let $\mathsf{G}_1, \mathsf{G}_2 \in \mathcal{F}(P_{k+1})$ be distinct and arbitrary. By Claim 16 there are unique $\mathsf{P}^1, \mathsf{P}^2 \in \widehat{\mathcal{F}}_{k+1}$ such that $\mathsf{G}_j \subset \mathsf{P}^j$, $j = 1, 2$. If $\mathsf{P}^1 = \mathsf{P}^2$ then the induction hypothesis over $\mathsf{P}^1 = \mathsf{P}^2 \cong P_k$ implies that both conditions (A) and (B) hold for G_1 and G_2. Hence let us suppose that $\mathsf{P}^1 \neq \mathsf{P}^2$.

Proof of (A)**.** By the assumption $(\mathrm{A})_{k+1}$, it follows that for any $u \in V(\mathsf{G}_1) \cap V(\mathsf{G}_2) \subset V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$ there exist edges $I_*^j \in \mathcal{I}_{\mathsf{P}^j}^{(k)}$, $j = 1, 2$, such that $u \in I^1_* \cap I^2_*$. Let $\mathsf{G}'_j \in \mathcal{F}(\mathsf{P}^j)$ be such that $I^j_* \in \mathcal{I}_{\mathsf{G}'_j}$. For each $j = 1, 2$ we will obtain $I^j \in \mathcal{I}_{\mathsf{G}_j}$ with $u \in I^1 \cap I^2$.

First we show that there exists $I^1 \in \mathcal{I}_{\mathsf{G}_1}$ such that $u \in I^1$. If $\mathsf{G}_1 = \mathsf{G}'_1$, take $I^1 = I^1_*$; otherwise $u \in V(\mathsf{G}_1) \cap V(\mathsf{G}'_1)$ and the induction hypothesis (A) over $\mathsf{P}^1 \cong P_k$ implies that there exists $I^1 \in \mathcal{I}_{\mathsf{G}_1}$ such that $u \in I^1 \cap I^1_*$. Similarly we find $I^2 \in \mathcal{I}_{\mathsf{G}_2}$ such that $u \in I^2$ and therefore the condition (A) holds for $\mathcal{F}(P_{k+1})$.

Proof of (B)**.** Suppose that there are two distinct $u, v \in V(\mathsf{G}_1) \cap V(\mathsf{G}_2) \subset V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$. Then either (B1)$_{k+1}$ or (B2)$_{k+1}$ holds.

In case (B1)$_{k+1}$ holds we will show that (B1) holds. Consider the tuples $I^j_* \in \mathcal{I}^{(k)}_{\mathsf{P}^j}$, $j = 1, 2$ such that $u, v \in I^1_* \cap I^2_*$. Let $\mathsf{G}'_j \in \mathcal{F}(\mathsf{P}^j)$ be such that $I^j_* \in \mathcal{I}_{\mathsf{G}'_j}$, $j = 1, 2$.

First we will show that there exists $I^1 \in \mathcal{I}_{\mathsf{G}_1}$ such that $u, v \in I^1$. If $\mathsf{G}'_1 = \mathsf{G}_1$, set $I^1 = I^1_*$. Otherwise, observe that $u, v \in V(\mathsf{G}_1) \cap V(\mathsf{G}'_1)$ and $\mathsf{G}_1, \mathsf{G}'_1 \in \mathcal{F}(\mathsf{P}^1)$. We may now use the induction hypothesis on P_k which states that Condition (B) holds for $\mathcal{F}(P_k)$. In particular, if there is no $I^1 \in \mathcal{I}_{\mathsf{G}_1}$ satisfying $u, v \in I^1 \cap I^1_*$ then the isomorphisms σ_1, σ'_1 from $\mathsf{G}_1, \mathsf{G}'_1$ to G are such that $\sigma_1(u) = \sigma'_1(u)$ and $\sigma_1(v) = \sigma'_1(v)$. However, this means that $I^1 = \sigma_1^{-1} \circ \sigma'_1(I^1_*) \in \mathcal{I}_{\mathsf{G}_1}$ satisfies $u, v \in I^1$. Similarly we obtain $I^2 \in \mathcal{I}_{\mathsf{G}_2}$ such that $u, v \in I^2$ and thus establish that (B1) holds.

In case (B2)$_{k+1}$ holds we will show that either (B2) or (B1) hold. Consider the isomorphisms $\phi_j \colon V(\mathsf{P}^j) \to V(P_k)$, $j = 1, 2$ (which satisfy $\phi_1(u) = \phi_2(u)$ and $\phi_1(v) = \phi_2(v)$). Let $\mathsf{G}'_j = \phi_j(\mathsf{G}_j) \in \mathcal{F}(P_k)$, $j = 1, 2$. If $\mathsf{G}'_1 = \mathsf{G}'_2$ then let $\sigma \colon V(\mathsf{G}'_1) = V(\mathsf{G}'_2) \to V(G)$ be the isomorphism between $\mathsf{G}'_1 = \mathsf{G}'_2$ into G. The isomorphisms $\sigma_j \colon V(\mathsf{G}_j) \to V(G)$ are then defined by $\sigma_j = \sigma \circ \phi_j|_{V(\mathsf{G}_j)}$. Therefore

$$\sigma_1(u) = \sigma(\phi_1(u)) = \sigma(\phi_2(u)) = \sigma_2(u).$$

Similarly, $\sigma_1(v) = \sigma_2(v)$. In particular, (B2) holds.

If $\mathsf{G}'_1 \neq \mathsf{G}'_2$ then $x = \phi_1(u) = \phi_2(u)$ and $y = \phi_1(v) = \phi_2(v)$ satisfy $x, y \in V(\mathsf{G}'_1) \cap V(\mathsf{G}'_2)$. By the induction assumption over P_k either the isomorphisms $\sigma'_j \colon V(\mathsf{G}'_j) \to V(G)$ satisfy $\sigma'_1(x) = \sigma'_2(x)$ and $\sigma'_1(y) = \sigma'_2(y)$—in which case the isomorphisms $\sigma'_j \circ \phi_j|_{V(\mathsf{G}_j)} \colon V(\mathsf{G}_j) \to V(G)$, $j = 1, 2$, satisfy (B2)—or there exist $I^j_* \in \mathcal{I}_{\mathsf{G}'_j}$, $j = 1, 2$, such that $x, y \in I^1_* \cap I^2_*$. In the latter case, let $I^j = \phi_j^{-1}(I^j_*) \in \mathcal{I}_{\mathsf{G}_j}$ for $j = 1, 2$. Notice that $u, v \in I^1 \cap I^2$. Therefore condition (B1) holds.

Before showing that condition (IV) holds we will prove two auxiliary claims.

Claim 18. *Suppose that* $\mathsf{P}^1, \mathsf{P}^2 \in \widehat{\mathcal{F}}_{k+1}$, $u, v \in V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$, $d_1 = \operatorname{dist}_{\mathsf{P}^1}(u, v)$ *and* $d_2 = \operatorname{dist}_{\mathsf{P}^2}(u, v)$. *Then either* $\min\{d_1, d_2\} \geq \ell + 1$ *or* $d_1 = d_2$.

Without loss of generality assume that $\mathsf{P}^1 \neq \mathsf{P}^2$, $d_1 = \min\{d_1, d_2\} \leq \ell$, and $u \neq v$. By assumption, either Condition $(\mathrm{B1})_{k+1}$ or Condition $(\mathrm{B2})_{k+1}$ holds.

Suppose first that $(\mathrm{B2})_{k+1}$ holds, namely, the isomorphisms $\phi_j \colon V(\mathsf{P}^j) \to V(P_k)$ are such that $\phi_1(u) = \phi_2(u)$ and $\phi_1(v) = \phi_2(v)$. Hence $\phi = \phi_2 \circ \phi_1^{-1} \colon V(\mathsf{P}^1) \to V(\mathsf{P}^2)$ is an isomorphism from P^1 to P^2 satisfying $\phi(u) = u$ and $\phi(v) = v$. It follows that

$$\mathrm{dist}_{\mathsf{P}^1}(u, v) = \mathrm{dist}_{\mathsf{P}^2}(\phi(u), \phi(v)) = \mathrm{dist}_{\mathsf{P}^2}(u, v).$$

The equality in this case holds even for arbitrary distances d_1, d_2.

Suppose now that Condition $(\mathrm{B1})_{k+1}$ holds, namely, there exist tuples $I^j \in \mathcal{I}^{(k)}_{\mathsf{P}^j} \subset \binom{\mathsf{P}^j}{\rho_{\ell+1}}$, $j = 1, 2$, such that $u, v \in I^1 \cap I^2$.

Let $\mathsf{G}_j \in \mathcal{F}(\mathsf{P}^j)$ be such that $I^j \in \mathcal{I}_{\mathsf{G}_j}$ for $j = 1, 2$. By the induction hypothesis over $\mathsf{P}^j \cong P_k$, the graph G_j is $(\ell + 1)$-metric in P^j. In particular, $\mathrm{dist}_{\mathsf{P}^1}(u, v) = d_1 \leq \ell$ implies that $\mathrm{dist}_{\mathsf{G}_1}(u, v) = d_1$.

Recall that

$$\pi_{k+1}(I^1) = \pi_{k+1}(I^2) = I_{k+1} = \{w_1 < w_2 < \cdots < w_t\} \subset V(R_\ell).$$

Moreover, the I^j's are crossing with respect to the classes $V^{r_\ell}_{w_i}(P_{k+1})$, $i = 1, \ldots, t$. Consequently, there are indices $1 \leq a, b \leq t$ such that u is the ath element of I^j $(j = 1, 2)$ and v is the bth element of I^j $(j = 1, 2)$. Because $\mathrm{dist}_{\mathsf{G}_1}(u, v) = d_1 \leq \ell$ and each I^j is $\rho_{\ell+1}$-metric with respect to G_j we have

$$d_1 = \mathrm{dist}_{\mathsf{G}_1}(u, v) = \rho(a, b) = \mathrm{dist}_{\mathsf{G}_2}(a, b) \geq \mathrm{dist}_{\mathsf{P}^2}(a, b) = d_2 = \max\{d_1, d_2\}$$

and thus $d_1 = d_2$. Hence, Claim 18 follows.

Claim 19. *Suppose that $\mathsf{G}_1, \mathsf{G}_2 \in \mathcal{F}_\ell$ and there are distinct $u, v \in V(\mathsf{G}_1) \cap V(\mathsf{G}_2)$. Moreover, assume that there exists $I^1 \in \mathcal{I}_{\mathsf{G}_1}$ such that $u, v \in I^1$. Then there exists $I^2 \in \mathcal{I}_{\mathsf{G}_2}$ such that $u, v \in I^2$.*

If $\mathsf{G}_1 = \mathsf{G}_2$ then the claim is trivial so let as assume the graphs are distinct. By assumption, $\mathcal{F}_\ell$ satisfies Condition (B). If (B1) holds then the existence of I^2 is immediate.

If, on the other hand, (B2) holds, then the isomorphisms $\sigma_j \colon V(\mathsf{G}_j) \to V(G)$ satisfy $\sigma_1(u) = \sigma_2(u)$ and $\sigma_1(v) = \sigma_2(v)$. The map $\sigma = \sigma_2^{-1} \circ \sigma_1 \colon V(\mathsf{G}_1) \to V(\mathsf{G}_2)$ is clearly the isomorphism from G_1 to G_2. Since $\sigma(u) = u$ and $\sigma(v) = v$, it follows that $I^2 = \sigma(I^1) \in \mathcal{I}_{\mathsf{G}_2}$ satisfies the conditions of the claim.

Claim 20. *Condition* (IV) *holds for P_{k+1}, namely, every $\mathsf{G} \in \mathcal{F}(P_{k+1})$ is $(\ell + 1)$-metric.*

For an arbitrary $\mathsf{G} \in \mathcal{F}(P_{k+1})$ and $u, v \in V(\mathsf{G})$ we will show the following:

(i) If $\mathrm{dist}_{\mathsf{G}}(u, v) \leq \ell$ then $\mathrm{dist}_{P_{k+1}}(u, v) = \mathrm{dist}_{\mathsf{G}}(u, v)$.
(ii) If $\mathrm{dist}_{\mathsf{G}}(u, v) \geq \ell + 1$ then $\mathrm{dist}_{P_{k+1}}(u, v) \geq \ell + 1$.

The two conditions above imply that G is $(\ell+1)$-metric in P_{k+1}. Indeed, notice that when $\mathrm{dist}_{\mathsf{G}}(u,v)=\ell+1$ we have

$$\ell+1 \overset{\text{(ii)}}{\leq} \mathrm{dist}_{P_{k+1}}(u,v) \leq \mathrm{dist}_{\mathsf{G}}(u,v)=\ell+1$$

and equality holds. Therefore, for all $u,v\in V(\mathsf{G})$ we have $\mathrm{dist}_{P_{k+1}}(u,v)=\mathrm{dist}_{\mathsf{G}}(u,v)$ whenever $\mathrm{dist}_{\mathsf{G}}(u,v)\leq \ell+1$ and $\mathrm{dist}_{P_{k+1}}(u,v)\geq \ell+1$ whenever $\mathrm{dist}_{\mathsf{G}}(u,v)>\ell+1$.

We start by proving (i). Assume that $\mathrm{dist}_{\mathsf{G}}(u,v)\leq \ell$. If $\mathrm{dist}_{P_{k+1}}(u,v)<\mathrm{dist}_{\mathsf{G}}(u,v)$, consider a shortest path $\mathcal{P}(u,v)$ in P_{k+1}. The projection of this path, $\pi_{k+1}(\mathcal{P}(u,v))$, is a trail in R_ℓ starting at $x=\pi_{k+1}(u)$ and ending at $y=\pi_{k+1}(v)$. Since $\mathsf{G}'=\pi_{k+1}(\mathsf{G})\in\mathcal{F}_\ell$ and π_{k+1} is an isomorphism between G and G', it follows that $\mathrm{dist}_{\mathsf{G}'}(x,y)=\mathrm{dist}_{\mathsf{G}}(u,v)\leq\ell$. On the other hand, the trail $\pi_{k+1}(\mathcal{P}(u,v))$ shows that

$$\begin{aligned}\mathrm{dist}_{R_\ell}(x,y) &\leq |\pi_{k+1}(\mathcal{P}(u,v))| \leq |\mathcal{P}(u,v)| \\ &= \mathrm{dist}_{P_{k+1}}(u,v) < \mathrm{dist}_{\mathsf{G}}(u,v) = \mathrm{dist}_{\mathsf{G}'}(x,y).\end{aligned} \tag{8}$$

However, this contradicts the fact that G' is ℓ-metric in R_ℓ.

Now let us prove (ii). Assume that $\mathrm{dist}_{\mathsf{G}}(u,v)\geq\ell+1$. Suppose for the sake of contradiction that there exists a path $\mathcal{P}(u,v)$ in P_{k+1} with length ℓ or less. By Claim 16, there exists a unique $\mathsf{P}^1\in\widehat{\mathcal{F}}_{k+1}\subset\binom{P_{k+1}}{P_k}_{\mathrm{Part}(r_\ell)}$ such that $\mathsf{G}\subset\mathsf{P}^1$. We will show that the path $\mathcal{P}(u,v)$ satisfies the following:

(a) $\mathcal{P}(u,v)\not\subset\mathsf{P}^1$;
(b) there is no internal vertex of $\mathcal{P}(u,v)$ in $V(\mathsf{P}^1)$, in particular, $E\big(\mathcal{P}(u,v)\big)\cap E(\mathsf{P}^1)=\emptyset$;
(c) $\pi_{k+1}(u),\pi_{k+1}(v)\in I_{k+1}$;
(d) $\mathcal{P}(u,v)\not\subset\mathsf{P}^2$ for every $\mathsf{P}^2\in\widehat{\mathcal{F}}_{k+1}$;

By the induction hypothesis over the picture $\mathsf{P}^1\cong P_k$ the graph G must be $(\ell+1)$-metric in P^1 and thus

$$\mathrm{dist}_{\mathsf{P}^1}(u,v)\geq\ell+1. \tag{9}$$

In particular, (a) holds, that is, the path $\mathcal{P}(u,v)$ cannot be entirely contained in P^1.

Suppose that the path $\mathcal{P}(u,v)$ contains an internal vertex $w\in V(\mathsf{P}^1)$. Then the (non-trivial) induced subpaths $\mathcal{P}(u,w)$ and $\mathcal{P}(w,v)$ have length strictly shorter than ℓ. Our assumption that P^1 is ℓ-metric in P_{k+1} implies that $|\mathcal{P}(u,w)|\geq\mathrm{dist}_{\mathsf{P}^1}(u,w)$ and $|\mathcal{P}(w,v)|\geq\mathrm{dist}_{\mathsf{P}^1}(w,v)$. Therefore

$$\begin{aligned}|\mathcal{P}(u,v)| &= |\mathcal{P}(u,w)|+|\mathcal{P}(w,v)| \geq \mathrm{dist}_{\mathsf{P}^1}(u,w)+\mathrm{dist}_{\mathsf{P}^1}(w,v) \\ &\geq \mathrm{dist}_{\mathsf{P}^1}(u,v) \overset{(9)}{\geq} \ell+1,\end{aligned} \tag{10}$$

which contradicts the fact that $|\mathcal{P}(u,v)|\leq\ell$. Therefore (b) holds.

Because of (b) the edge of the path incident to u must be contained in some $\mathsf{P}^2\in\widehat{\mathcal{F}}_{k+1}$, $\mathsf{P}^2\neq\mathsf{P}^1$. In particular, $u\in V(\mathsf{P}^1)\cap V(\mathsf{P}^2)$. From Claim 16 we conclude that $\pi_{k+1}(v)\in I_{k+1}$ therefore establishing (c).

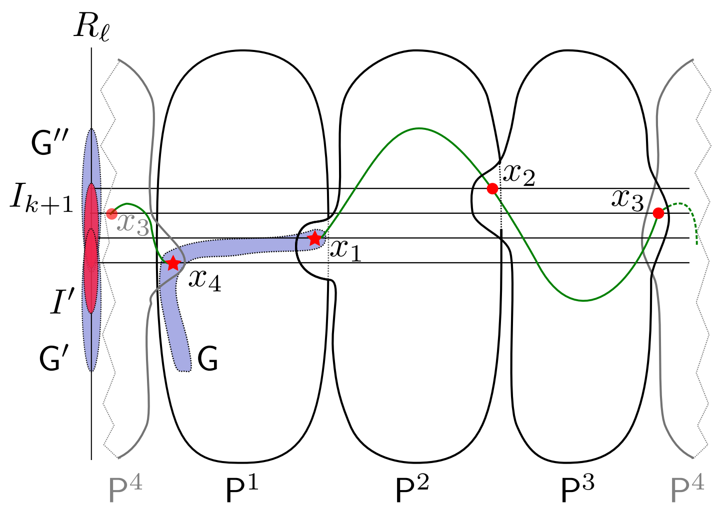

FIGURE 4. An illustration of a path $\mathcal{P}(u,v)$ and its subpaths from case (ii) of Claim 20 with $u = x_1$ and $v = x_4$. We also have $t = 4$, $a_1 = 3$, $a_2 = 1$, $a_3 = 2$ and $a_4 = 4$. The vertex x_3 is repeated because P^4 is wrapped around and effectively intersects both P^3 and P^1. Only the vertices in I_{k+1} are vertically ordered to simplify the figure.

To show that (d) is satisfied, suppose that $\mathcal{P}(u,v) \subset \mathsf{P}^2$ for some $\mathsf{P}^2 \in \widehat{\mathcal{F}}_{k+1}$, $\mathsf{P}^2 \neq \mathsf{P}^1$. Then $d_2 = \mathrm{dist}_{\mathsf{P}^2}(u,v) \le \ell$. From Claim 18 we conclude that

$$\mathrm{dist}_{\mathsf{P}^1}(u,v) = d_1 = d_2 = \ell,$$

which contradicts (9). Therefore (d) holds.

From (a)–(d) we conclude that the path $\mathcal{P}(u,v)$ can be decomposed in subpaths contained in at least two distinct copies of P_k in $\widehat{\mathcal{F}}_{k+1}$. Therefore we may find vertices $u = x_1, x_2, \dots, x_r = v$, $r \ge 3$, belonging to $\mathcal{P}(u,v)$ such that each (non-trivial) subpath $\mathcal{P}(x_j, x_{j+1})$, $j = 1, \dots, r-1$, is entirely contained in some $\mathsf{P}^{j+1} \in \widehat{\mathcal{F}}_{k+1}$, and $\mathsf{P}^{j+1} \neq \mathsf{P}^{j+2}$ for $j = 1, \dots, r-2$ (see the illustration in Figure 4).

Note that each $\mathcal{P}(x_j, x_{j+1})$ has length at most $\ell - 1$ since the sum of the lengths of each subpath equals $|\mathcal{P}(u,v)| \le \ell$. From Claim 16 we infer that $\pi_{k+1}(x_j) \in I_{k+1} = \{w_1 < w_2 < \cdots < w_t\}$ since each x_j, $2 \le j \le r-1$, is such that $x_j \in V(\mathsf{P}^j) \cap V(\mathsf{P}^{j+1})$.

For every $j = 1, \dots, r-1$, the projection $\pi_{k+1}\big(\mathcal{P}(x_j, x_{j+1})\big)$ is a trail connecting $w_{a_j} = \pi_{k+1}(x_j)$ and $w_{a_{j+1}} = \pi_{k+1}(x_{j+1})$ of length $|\mathcal{P}(x_j, x_{j+1})| \le \ell - 1$. Consequently, $\mathrm{dist}_{R_\ell}(w_{a_j}, w_{a_{j+1}}) \le \ell - 1$. Let $\mathsf{G}'' \in \mathcal{F}_\ell \subset \binom{R_\ell}{G}_{\mathrm{Part}(q)}$ be such that $I_{k+1} \in \mathcal{I}_{\mathsf{G}''} \subset \binom{\mathsf{G}''}{\rho_{\ell+1}}$. Since G'' is ℓ-metric in R_ℓ it follows that

$$\mathrm{dist}_{\mathsf{G}''}(w_{a_j}, w_{a_{j+1}}) = \mathrm{dist}_{R_\ell}(w_{a_j}, w_{a_{j+1}}) \le |\mathcal{P}(x_j, x_{j+1})| \le \ell - 1.$$

Because $I_{k+1} \in \binom{\mathsf{G}''}{\rho_{\ell+1}}$ we must have $\mathrm{dist}_{\mathsf{G}''}(w_{a_j}, w_{a_{j+1}}) = \rho(a_j, a_{j+1})$ and thus

$$\begin{aligned} |\mathcal{P}(u,v)| &= \sum_{j=1}^{r-1} |\mathcal{P}(x_j, x_{j+1})| \geq \sum_{j=1}^{r-1} \mathrm{dist}_{\mathsf{G}''}(w_{a_j}, w_{a_{j+1}}) \\ &= \sum_{j=1}^{r-1} \rho(a_j, a_{j+1}) \geq \rho(a_1, a_r), \end{aligned} \tag{11}$$

where in the last part we used the triangle inequality.

Let $\mathsf{G}' = \pi_{k+1}(\mathsf{G}) \in \mathcal{F}_\ell$. Notice that $w_{a_1} = \pi_{k+1}(u), w_{a_r} = \pi_{k+1}(v) \in V(\mathsf{G}') \cap V(\mathsf{G}'')$. From Claim 19 applied to G' and G'' we conclude that there exists $I' \in \mathcal{I}_{\mathsf{G}'}$ such that $w_{a_1}, w_{a_r} \in I' \cap I_{k+1}$. Moreover, by the induction hypothesis every graph in $\mathcal{F}_\ell$ is partite embedded into R_ℓ, that is $\mathcal{F}_\ell \subset \binom{R_\ell}{G}_{\mathrm{Part}(q)}$. This ensures that I' and I_{k+1} are crossing with respect to $\{V^q_{j_i}(\mathsf{G}') \subset V^q_{j_i}(R_\ell)\}_{i=1}^t$ and $\{V^q_{j_i}(\mathsf{G}'') \subset V^q_{j_i}(R_\ell)\}_{i=1}^t$ respectively. In particular, the a_1th element in I' is w_{a_1} and the a_rth element in I' is w_{a_r}. Because $I' \in \binom{\mathsf{G}'}{\rho_{\ell+1}}$ and $\rho(a_1, a_r) \leq \ell$, we have $\mathrm{dist}_{\mathsf{G}'}(w_{a_1}, w_{a_r}) = \rho(a_1, a_r) \leq \ell$

Since π_{k+1} is the isomorphism of G into G' we have

$$\mathrm{dist}_{\mathsf{G}}(u,v) = \mathrm{dist}_{\mathsf{G}'}(w_{a_1}, w_{a_r}) = \rho(a_1, a_r) \leq \ell,$$

which is a contradiction with the original assumption that $\mathrm{dist}_{\mathsf{G}}(u,v) \geq \ell+1$. This finishes the proof of Claim 20.

We have proved the induction step over k by establishing Claims 14, 15, 17 and 20. In order to prove that $R_{\ell+1} = P_m$ and $\mathcal{F}_{\ell+1} = \mathcal{F}(P_m)$ satisfy the induction hypothesis for $\ell + 1$, it remains to show the following claim.

Claim 21. *For every 2-coloring of $\binom{V(R_{\ell+1})}{t}$ there exists $\mathsf{G} \in \mathcal{F}_{\ell+1}$ such that every $(\rho_{\ell+1}, t)$-tuple in $\mathcal{I}_{\mathsf{G}}$ is monochromatic.*

Suppose that the t-tuples of vertices in $R_{\ell+1}$ are 2-colored. By construction (see Property $(1)_m$), there exist some $\mathsf{P}^{m-1} \in \widehat{\mathcal{F}}_m \subset \binom{R_{\ell+1}}{P_{m-1}}_{\mathrm{Part}(r_\ell)}$ such that $\mathcal{I}^{(m-1)}_{\mathsf{P}^{m-1}}$ is monochromatic (with color c_m). Similarly, we obtain $\mathsf{P}^{m-2} \in \widehat{\mathcal{F}}_{m-1} \subset \binom{\mathsf{P}^{m-1}}{P_{m-2}}_{\mathrm{Part}(r_\ell)}$ such that $\mathcal{I}^{(m-2)}_{\mathsf{P}^{m-2}}$ is monochromatic (with color c_{m-1}). Repeating the argument we obtain a sequence $\mathsf{P}^0 \subset \mathsf{P}^1 \subset \cdots \subset \mathsf{P}^{m-1}$ such that each $\mathcal{I}^{(k)}_{\mathsf{P}^k}$, $k = 0, \ldots, m-1$, is monochromatic with color c_{k+1}.

Recall that P_0 consists of disjoint copies of G which are in correspondence with members of $\mathcal{F}_\ell \subset \binom{R_\ell}{G}_{\mathrm{Part}(q)}$ by π_0 (see Figure 2). Given the isomorphism $\phi\colon V(P_0) \to V(\mathsf{P}^0)$, the map $\lambda = \pi_0 \circ \phi^{-1}$ is a projection of P^0 onto R_ℓ. We will now show that for each

$$I_k \in \bigcup_{\mathsf{G} \in \mathcal{F}_\ell} \mathcal{I}_{\mathsf{G}} \overset{(2)}{=} \{I_1, \ldots, I_m\},$$

every $I \in \bigcup_{\mathsf{G}\in\mathcal{F}(\mathsf{P}^0)} \mathcal{I}_\mathsf{G}$ with $\lambda(I) = I_k$ is colored with the same color c_k.

For any $I \in \bigcup_{\mathsf{G}\in\mathcal{F}(\mathsf{P}^0)} \mathcal{I}_\mathsf{G}$ with $\lambda(I) = I_k$ there is a unique $\bar{\mathsf{G}} \in \mathcal{F}(\mathsf{P}^0)$ such that $I \in \mathcal{I}_{\bar{\mathsf{G}}}$. By (7), we have $\bar{\mathsf{G}} \in \mathcal{F}(\mathsf{P}^0) \subset \mathcal{F}(\mathsf{P}^1) \subset \cdots \subset \mathcal{F}(\mathsf{P}^{k-1})$ and hence $I \in \bigcup_{\mathsf{G}\in\mathcal{F}(\mathsf{P}^{k-1})} \mathcal{I}_\mathsf{G}$. Because $V_j^{r_\ell}(\mathsf{P}^0) \subset V_j^{r_\ell}(\mathsf{P}^1) \subset \cdots \subset V_j^{r_\ell}(\mathsf{P}^{k-1})$ for all $j = 1, \ldots, r_\ell$ and I is crossing with respect to $\{V_j^{r_\ell}(\mathsf{P}^0)\}_{j\in I_k}$ it is obvious that I is crossing with respect to $\{V_j^{r_\ell}(\mathsf{P}^{k-1})\}_{j\in I_k}$ as well. Given the isomorphism $\phi_{k-1}\colon V(P_{k-1}) \to V(\mathsf{P}^{k-1})$ we conclude that $\pi_{k-1} \circ \phi_{k-1}^{-1}(I) = I_k$. From (4) we conclude that $\phi_{k-1}^{-1}(I) \in \mathcal{I}^{(k-1)}$ and thus $I \in \mathcal{I}^{(k-1)}_{\mathsf{P}^{k-1}}$. However, $\mathcal{I}^{(k-1)}_{\mathsf{P}^{k-1}}$ is monochromatic (with color c_k) by the definition of P^{k-1}. Consequently, the color of I is c_k.

This induces a 2-coloring on the tuples in $\bigcup_{\mathsf{G}\in\mathcal{F}_\ell} \mathcal{I}_\mathsf{G}$ by setting $\chi(I_k) = c_k$ for all $k = 0, \ldots, m-1$. By the induction hypothesis over ℓ, there exists a copy $\mathsf{G}^* \in \mathcal{F}_\ell$ such that $\mathcal{I}_{\mathsf{G}^*}$ is monochromatic (under χ). There exist a unique $\mathsf{G} \in \mathcal{F}(\mathsf{P}^0)$ such that $\lambda(\mathsf{G}) = \mathsf{G}^*$. Since the color of any $I \in \mathcal{I}_\mathsf{G}$ is equal to $\chi\big(\lambda(I)\big)$ and $\lambda(I) \in \mathcal{I}_{\mathsf{G}^*}$, it follows that $\mathcal{I}_\mathsf{G}$ is monochromatic.

The induction hypothesis for pictures applied to $R_{\ell+1} = P_m$ and $\mathcal{F}_{\ell+1} = \mathcal{F}(P_m)$ together with Claim 21 establish that the induction hypothesis holds for $\ell + 1$. Lemma 10 then follows by induction.

3. The base of the induction

Here we prove the induction base of the proof of Lemma 10. This will be done by an application of the Hales–Jewett theorem.

Suppose that $\mathcal{I}$ is a t-partite t-uniform hypergraph with vertex set V and classes $V_1, \ldots, V_t$. Let $\mathcal{I}^n$ be the set of n-tuples of elements of $\mathcal{I}$. A *combinatorial line* L in $\mathcal{I}^n$ associated with a partition $[n] = M_L \cup F_L$, $M_L \neq \emptyset$, and an $|F_L|$-tuple $(I_k^L)_{k\in F_L} \in \mathcal{I}^{F_L}$ is given by

$$L = \{(I_1, I_2, \ldots, I_n) \in \mathcal{I}^n \;:\; I_r = I_s \text{ for } r, s \in M_L \text{ and } I_k = I_k^L \text{ for } k \in F_L\}.$$

The set M_L is called the set of *moving* coordinates, while F_L is called the set of *fixed* coordinates. Notice that every combinatorial line has precisely $|\mathcal{I}|$ elements.

The Hales–Jewett theorem is stated as follows. For a proof, see for instance [4].

Theorem 22 ([5]). *For any integer $r \geq 2$ and finite set $\mathcal{I}$ there exists n such that in every r-coloring of $\mathcal{I}^n$ there exists a monochromatic line.*

For our purposes it will be useful to interpret an element $I \in \mathcal{I}$ as a vector with t coordinates where the jth coordinate is simply the unique vertex in $I \cap V_j$. In this way, an element in $\mathcal{I}^n$ may be viewed as a $t \times n$ matrix. Consequently, a line of $\mathcal{I}^n$ may be described as a collection of matrices Q_I^L, $I \in \mathcal{I}$, where the columns of Q_I^L indexed by F_L are fixed and independent of I while every column indexed by M_L is precisely I. For example, for $n = 4$,

$M_L = \{1,2\}$, $F_L = [4] \setminus M_L = \{3,4\}$ and $L = \{(I, I, I_3^L, I_4^L) \,:\, I \in \mathcal{I}\}$, the elements of L are the matrices

$$Q_I^L = \begin{bmatrix} | & | & | & | \\ I & I & I_3^L & I_4^L \\ | & | & | & | \end{bmatrix} \tag{12}$$

for all $I \in \mathcal{I}$.

We will now prove Lemma 23 which is the base of the induction in the proof of Lemma 10.

Lemma 23. *Let $t, q \in \mathbb{N}$, $t \le q$, and ρ be a metric on $[t]$.*

Suppose that G is a q-partite graph with $V(G) = V_1^q(G) \cup \cdots \cup V_q^q(G)$ and, for some $1 \le j_1 < j_2 < \cdots < j_t \le q$, $\mathcal{I} \subseteq \binom{G}{\rho_2}$ is a t-partite t-uniform hypergraph with classes $\{V_{j_i}^q(G)\}_{i=1}^t$.

Then there exists a q-partite graph R and $\mathcal{F} \subset \binom{R}{G}_{\mathrm{Part}(q)}$ satisfying the following properties.

(1) For any 2-coloring of $\binom{V(R)}{t}$ there exists $\mathsf{G} \in \mathcal{F}$ such that every (ρ_2, t)-tuple in $\mathcal{I}_{\mathsf{G}} \subset \binom{\mathsf{G}}{\rho_2}$ is monochromatic.
(2) Every $\mathsf{G} \in \mathcal{F}$ is 2-metric in R.
(3) The family $\mathcal{F}$ satisfies conditions (A) *and* (B)*.*

Remark 24. Consider a graph F_ρ with vertex set $[t]$ such that $ij \in F_\rho$ iff $\rho(x,y) = 1$. With this definition we have $\binom{G}{\rho_2} \cong \binom{G}{F_\rho}$, *i.e.*, $\binom{G}{\rho_2}$ coincides with the set of all induced copies of F_ρ in G.

Notice also that the fact that every $\mathsf{G} \in \mathcal{F}$ is 2-metric in R implies that G is an induced subgraph of R. Indeed, by the definition, for all $x, y \in V(\mathsf{G})$, when $\mathrm{dist}_R(x,y) \le 2$ we must have $\mathrm{dist}_{\mathsf{G}}(x,y) = \mathrm{dist}_R(x,y)$ and when $\mathrm{dist}_R(x,y) > 2$ we must have $\mathrm{dist}_{\mathsf{G}}(x,y) \ge 2$. In particular, $xy \in R$ iff $xy \in \mathsf{G}$.

Lemma 23 appears in [10] without explicitly stating Condition *(3)*, which is needed here for technical reasons to carry on the induction. For completeness we include here the proof of [10] modified to explicitly establish *(3)*.

Proof. Suppose that G and $\mathcal{I}$ are given as in the statement of the lemma. Let $J = \{j_1, \ldots, j_t\}$ be the set of indices with the property of the assumption, namely, $\mathcal{I}$ is a t-partite t-uniform hypergraph with classes $\{V_j^q(G)\}_{j \in J}$. Let n be given by Theorem 22 (with $r = 2$) applied to $\mathcal{I}$. Let $\{L_1, \ldots, L_N\}$ denote the set of all lines in $\mathcal{I}^n$

Let $W = \bigcup_{I \in \mathcal{I}} I$ and $W_j = V_j^q(G) \cap W$. (Notice that $W_j = \emptyset$ when $j \notin J$.) The vertex set of R is given by

$$V(R) = \big([N] \times (V(G) \setminus W)\big) \cup \bigcup_{j \in J} W_j^n.$$

The edge set of R will be defined after we prove Claim 25.

For a line L_a with fixed values $\big(I_k^a\big)_{k \in F_a}$, we view $I_k^a = \{I_{k,j}^a \in W_j\}_{j \in J}$ as a column-vector $[I_{k,j}^a]_{j \in J}$. Let us define the map $\psi_a \colon V(G) \to V(R)$ as follows:

$$\psi_a(v) = \begin{cases} (a, v) & \text{for } v \in V(G) \setminus W; \\ (v_1, v_2, \dots, v_n) & \text{for } v \in W_j, j \in J, \text{ where} \\ \quad v_k = v \text{ for } k \in M_a \text{ and } v_k = I^a_{k,j} \text{ for } k \in F_a. \end{cases} \tag{13}$$

In view of (12) and (13), for every $I = \{u_1 < u_2 < \cdots < u_t\} \in \mathcal{I}$ we have

$$Q_I^{L_a} = \psi_a(I) = \begin{bmatrix} \psi_a(u_1) \\ \psi_a(u_2) \\ \vdots \\ \psi_a(u_t) \end{bmatrix}.$$

Observe that the rows of the matrices $Q_I^{L_a}$ are seen as vertices of R.

Claim 25. *The map $\psi_a \colon V(G) \to V(R)$ is one-to-one.*

Suppose for the sake of contradiction that two distinct $u, v \in V_j^q(G)$, $1 \le j \le q$, are such that $\psi_a(u) = \psi_a(v)$. We cannot have $\psi_a(u) = (a, u)$ since that would imply $u = v$. Consequently, $u, v \in W_j$ with $j \in J$. Hence both $\psi_a(u)$ and $\psi_a(v)$ must be n-tuples such that $\psi_a(u)_k = u \ne v = \psi_a(v)_k$ for all $k \in M_a$. Therefore u cannot be distinct from v.

Set

$$E(R) = \bigcup_{a=1}^{N} E\big(\psi_a(G)\big)$$

and let $\mathcal{F} = \{\mathsf{G}_a = \psi_a(G) \,:\, a = 1, \dots, N\}$.

We now must prove that the conclusions of the lemma hold for R and $\mathcal{F}$. This will be accomplished by the following steps.

(a) Define a total order on $V(R)$ and a q-partition $V(R) = V_1^q(R) \cup V_2^q(R) \cup \cdots \cup V_q^q(R)$ such that every ψ_a is a monotone map satisfying $\psi_a(V_j^q(G)) \subset V_j^q(R)$ for every j. This order ensures that $\mathcal{F} \subset \binom{R}{G}_{\mathrm{Part}(q)}$.

(b) Establish the intersection properties of $\mathcal{F}$ described in *(3)*.

(c) Use (ii) to show that every $\mathsf{G}_a \in \mathcal{F}$ is an induced subgraph of R and thus prove *(2)*.

(d) Show that the family $\mathcal{F}$ is Ramsey in R, namely, prove *(1)*.

Proof of (a). For all j, define

$$V_j^q(R) = \big([N] \times (V_j^q(G) \setminus W)\big) \cup W_j^n. \tag{14}$$

Observe that $V(R) = V_1^q(R) \cup V_2^q(R) \cup \cdots \cup V_q^q(R)$. Moreover, it is simple to check that $\psi_a(V_j^q(G)) \subset V_j^q(R)$ for all j. Let us now define a total order on $V(R)$ for which every map ψ_a is monotone. It is enough to define the order for each $V_j^q(R)$ since we require $V_1^q(R) \prec V_2^q(R) \prec \cdots \prec V_q^q(R)$.

Let $U_j = W_j^n$ be linearly ordered using the lexicographic order in the n-tuples (recall that $W_j \subset V_j^q(G) \subset V(G)$ and $V(G)$ is also linearly ordered). We extend the linear order of U_j as follows: let $v \in V_j^q(G) \setminus W$ be the smallest element such that $\psi_1(v) = (1, v) \notin U_j$. If there is a predecessor $u \in V_j^q(G)$ of v then add $\psi_1(v)$ to U_j as a successor of $\psi_1(u)$ otherwise let $\psi_1(v)$ be the first (smallest) element of U_j.

Repeat the extension steps until $\psi_1(V_j^q(G)) \subset U_j$. Then repeat the same steps for $\psi_2, \psi_3, \ldots, \psi_N$. After the end of this procedure we have obtained a total order on $V_j^q(R)$. It remains to check that every ψ_a is monotone under this ordering.

Initially $U_j = W_j^n$ and the elements of U_j were ordered lexicographically. For arbitrary $u, v \in W_j$ we have $\psi_a(u)_k = \psi_a(v)_k$ for every $k \in F_a$. This means that the first coordinate in which $\psi_a(u)$ differs from $\psi_a(v)$ is in M_a. Since for every $k \in M_a$, we have $\psi_a(u)_k = u$, $\psi_a(v)_k = v$, it follows that $\psi_a(u) < \psi_a(v)$ in the lexicographic order.

We show that the linear order above is such that each ψ_a is monotone. Suppose that the order on $\psi_a(V_j^q(G)) \cap U_j$, $a = 1, \ldots, N$, is consistent with the order on $V_j^q(G)$ at a given step. If U_j is extended by including some element (a, v), then this extension does not affect the maps ψ_b, $b \neq a$. The placement of (a, v) in the linearly ordered set U_j is such that $\psi_a(V_j^q(G)) \cap U_j$ remains consistent with the order on $V_j^q(G)$. Since initially U_j was consistent with every map ψ_a, the statement follows by induction.

Proof of (b). Suppose that $x \in V(\mathsf{G}_a) \cap V(\mathsf{G}_b)$ with $a \neq b$. We must have $x \in W_j^n$ for some $j \in J$ since otherwise for some $v \in V(G) \setminus W$, we have $x = (a, v) = (b, v)$ which contradicts $a \neq b$. It follows therefore that $\psi_a^{-1}(x), \psi_b^{-1}(x) \in W_j$ and therefore by definition ($W_j \subset W = \bigcup_{I \in \mathcal{I}} I$) there exists $I'_a, I'_b \in \mathcal{I}$ such that $\psi_a^{-1}(x) \in I'_a$ and $\psi_b^{-1}(x) \in I'_b$. Consequently, $x \in I_a = \psi_a(I'_a) \in \mathcal{I}_{\mathsf{G}_a}$ and $x \in I_b = \psi_b(I'_b) \in \mathcal{I}_{\mathsf{G}_b}$. This establishes the intersection Condition (A) for members of $\mathcal{F}$.

Now let us prove Condition (B). Suppose that there are distinct $x, y \in V(\mathsf{G}_a) \cap V(\mathsf{G}_b)$, $a \neq b$.

Let $\big(I_k^a = [I_{k,j}^a]_{j \in J}\big)_{k \in F_a}$ and $\big(I_k^b = [I_{k,j}^b]_{j \in J}\big)_{k \in F_b}$ be the set of fixed elements in the lines L_a and L_b respectively. By (13), for $j \in J$ such that $x = (x_k)_{k=1}^n \in W_j^n$ we have $x_k = I_{k,j}^a$ for $k \in F_a$ and $x_k = I_{k,j}^b$ for $k \in F_b$.

We distinguish between two cases.

(i) $M_a \cap M_b \neq \emptyset$.
(ii) $M_a \cap M_b = \emptyset$ (then $M_a \subset F_b$ and $M_b \subset F_a$).

We have $\psi_a^{-1}(x) = x_k$ for every $k \in M_a$ and $\psi_b^{-1}(x) = x_k$ for every $k \in M_b$. If (i) holds, take $k \in M_a \cap M_b$ and observe that $\psi_a^{-1}(x) = \psi_b^{-1}(x)$ (and $\psi_a^{-1}(y) = \psi_b^{-1}(y)$). Therefore in this case Condition (B2) holds as the isomorphisms $\sigma_a = \psi_a^{-1} \colon V(\mathsf{G}_a) \to V(G)$ and $\sigma_b = \psi_b^{-1} \colon V(\mathsf{G}_b) \to V(G)$ satisfy $\sigma_a(x) = \sigma_b(x)$ and $\sigma_a(y) = \sigma_b(y)$.

If (ii) holds, we must have $M_a \subset F_b$ and $M_b \subset F_a$. Observe that for $k \in M_a \subset F_b$ there are $j, j' \in J$ such that

$$\psi_a^{-1}(x) \overset{k\in M_a}{=} x_k \overset{k\in F_b}{=} I_{k,j}^b$$

and $\psi_a^{-1}(y) = y_k = I_{k,j'}^b$. In particular, $\{\psi_a^{-1}(x), \psi_a^{-1}(y)\} = \{I_{k,j}^b, I_{k,j'}^b\} \subset I_k^b \in \mathcal{I}$ and we set $\tilde{I}_a = I_k^b$. Similarly we conclude that $\{\psi_b^{-1}(x), \psi_b^{-1}(y)\} \subset \tilde{I}_b$ for some $\tilde{I}_b \in \mathcal{I}$.

Let $I_a = \psi_a(\tilde{I}_a) \in \mathcal{I}_{\mathsf{G}_a}$ and $I_b = \psi_b(\tilde{I}_b) \in \mathcal{I}_{\mathsf{G}_b}$. Notice that $\{x, y\} \subset I_a \cap I_b$. This shows that Condition (B1) holds.

Proof of (c). Let $\mathsf{G}_a \in \mathcal{F}$ be arbitrary. To prove that G_a is an induced subgraph of R we must check that for every pair of distinct $x, y \in V(\mathsf{G}_a)$ if $x, y \in V(\mathsf{G}_b)$ for some $b \neq a$ then $\{x, y\} \in \mathsf{G}_a$ iff $\{x, y\} \in \mathsf{G}_b$. Since $x, y \in V(\mathsf{G}_a) \cap V(\mathsf{G}_b)$, we may invoke the intersection properties of $\mathcal{F}$ proved in (b).

In case Condition (B2) holds, we have $\psi_a^{-1}(x) = \psi_b^{-1}(x)$ and $\psi_a^{-1}(y) = \psi_b^{-1}(y)$. Therefore $\{x, y\} \in \mathsf{G}_a$ iff $\{\psi_a^{-1}(x), \psi_a^{-1}(y)\} = \{\psi_b^{-1}(x), \psi_b^{-1}(y)\} \in G$ iff $\{x, y\} \in \mathsf{G}_b$.

In case Condition (B1) holds, let $I_a \in \mathcal{I}_{\mathsf{G}_a}$ and $I_b \in \mathcal{I}_{\mathsf{G}_b}$ be such that $x, y \in I_a \cap I_b$. Let $j_r, j_s \in J$ $(1 \le r, s \le t)$ be such that $x \in V_{j_r}^q(R)$ and $y \in V_{j_s}^q(R)$. Because $I_a \in \binom{\mathsf{G}_a}{\rho_2}$ it follows that $\mathrm{dist}_{\mathsf{G}_a}(x, y) = \rho(r, s)$ whenever $\rho(r, s) \le 2$ and $\mathrm{dist}_{\mathsf{G}_a}(x, y) \ge 2$ whenever $\rho(r, s) > 2$. In particular, $\{x, y\} \in \mathsf{G}_a$ iff $\rho(r, s) = 1$. Similarly, $\{x, y\} \in \mathsf{G}_b$ iff $\rho(r, s) = 1$. Therefore $\{x, y\} \in \mathsf{G}_a$ iff $\{x, y\} \in \mathsf{G}_b$.

Proof of (d). We will now show that for any 2-coloring of $\binom{V(R)}{t}$ there exists $\mathsf{G} \in \mathcal{F}$ such that every t-tuple in $\mathcal{I}_{\mathsf{G}} \subset \binom{\mathsf{G}}{\rho_2}$ is monochromatic.

Consider $Q = (I_1, \ldots, I_n) \in \mathcal{I}^n$ as a $t \times n$ matrix with columns $I_1, \ldots, I_n$. The kth row of the matrix is in $V_{j_k}^q(R)$ (recall that $J = \{j_1, \ldots, j_t\}$). In particular, Q is in correspondence with a t-tuple of $V_{j_1}^q(R) \times \cdots \times V_{j_t}^q(R) \subset \binom{V(R)}{t}$. Define the color of Q as the color of the corresponding t-tuple.

By the Hales–Jewett theorem, there is a monochromatic line L_a, $a \in [N]$, in such a coloring. It follows that $\mathsf{G} = \mathsf{G}_a$ is such that $\mathcal{I}_{\mathsf{G}}$ is monochromatic. Indeed, every t-tuple $\psi_a(I) \in \mathcal{I}_{\mathsf{G}_a}$, $I \in \mathcal{I}$, corresponds to the matrix $Q_I^{L_a}$ contained in the line L_a (see (13) and the discussion that follows). $\square$

4. Sketch of Lemma 12

In this section we give a sketch of the proof of Lemma 12. Since this proof is very similar to the proof of the induction step in Lemma 10 (albeit simpler), we avoid repeating some details and instead refer the reader to parts of the proof of Lemma 10 that present similar arguments.

Let H be a given connected graph on n vertices and ρ be a metric on t elements.

Set $N = \mathrm{R}_t(n)$, where $\mathrm{R}_t(n)$ is the smallest number such that for every 2-coloring of the complete t-uniform hypergraph $K_N^{(t)}$ there exists a monochromatic $K_n^{(t)}$ as a subhypergraph.

Consider the complete graph $R_0 \cong K_N^{(2)}$ with vertex set $V(R_0) = [N]$. Clearly, for every set $S \subset V(R_0)$, $|S| = n$, there is a unique monotone injective map $\phi\colon V(H) \to S$. Since R_0 is complete, the map ϕ is trivially edge-preserving. In particular, the family $\mathcal{F}_0$ of all not necessarily induced ordered copies of H in R_0 is in correspondence with $\binom{[N]}{n}$.

Just as in the proof of Lemma 10 we construct an N-partite graph P_0 consisting of disjoint copies of H that project onto (non-induced) copies of H through π_0 (see Figure 2). Let $\mathcal{F}(P_0) = \binom{P_0}{H}$ and notice that (due to the fact that H is connected) there is a one-to-one correspondence of $\mathcal{F}(P_0)$ and $\mathcal{F}_0 \cong \binom{[N]}{m}$ through the projection π_0.

Consider the hypergraph

$$\bigcup_{\mathsf{H}\in\mathcal{F}_0} \binom{\mathsf{H}}{\rho} = \{I_1, \dots, I_m\} \subset \binom{V(R_0)}{t},$$

and set

$$\mathcal{I}^{(0)} = \left\{ I \in \bigcup_{\mathsf{H}\in\mathcal{F}(P_0)} \binom{\mathsf{H}}{\rho} \;:\; \pi_0(I) = I_1 \right\} \subset \binom{P_0}{\rho},$$

which is defined in a similar as the hypergraph in (4). Observe that the t-uniform hypergraph $\mathcal{I}^{(0)}$ is t-partite with respect to $\{V_j^N(P_0)\}_{j\in I_1}$.

Let $\ell = \max\{\mathrm{dist}_H(x,y) \;:\; x, y \in V(H)\} < \infty$. Apply Lemma 10 to the N-partite graph P_0 (instead of a q-partite G) and the family $\mathcal{I}^{(0)} \subset \binom{P_0}{\rho_\ell}$. We then obtain the Ramsey N-partite graph P_1 and $\widehat{\mathcal{F}}_1 \subset \binom{P_1}{P_0}_{\mathrm{Part}(N)}$ for which *(1)* and *(2)* hold. In particular, *(2)* ensures that every $\mathsf{P} \in \widehat{\mathcal{F}}_1$ is ℓ-metric in P_1. By our choice of ℓ, this implies that every $\mathsf{H} \in \mathcal{F}(\mathsf{P})$ is metric in P_1.

In general, we obtain P_{k+1} from P_k, $k = 0, \dots, m-1$, by applying Lemma 10 to the N-partite graph P_k and the t-partite t-uniform hypergraph

$$\mathcal{I}^{(k)} = \left\{ I \in \bigcup_{\mathsf{H}\in\mathcal{F}(P_k)} \binom{\mathsf{H}}{\rho} \;:\; \pi_k(I) = I_{k+1} \right\} \subset \binom{P_k}{\rho_\ell}.$$

The graph P_{k+1} and the family $\widehat{\mathcal{F}}_{k+1} \subset \binom{P_{k+1}}{P_k}_{\mathrm{Part}(N)}$ we obtain are such that every $\mathsf{H} \in \mathcal{F}(P_{k+1}) = \bigcup_{\mathsf{P}\in\widehat{\mathcal{F}}_{k+1}} \mathcal{F}(\mathsf{P})$ is metric in P_{k+1} and $\pi_{k+1}(\mathsf{H}) \in \mathcal{F}_0$ (where $\pi_{k+1}\colon V(P_{k+1}) \to V(R_0) = [N]$ is defined as the projection that maps every $v \in V_j^N(P_{k+1})$ to j for all $j = 1, \dots, N$).

Take $R = P_m$ and $\mathcal{F} = \mathcal{F}(P_m) \subset \binom{R}{H}$. Just as in Claim 21 one may show that in any 2-coloring of $\binom{V(R)}{t}$ there exists a copy of P_0 in R, say $\mathsf{P} = \phi(P_0) \subset R$, such that the color of a tuple $I \in \binom{\mathsf{H}}{\rho}$, $\mathsf{H} \in \mathcal{F}(\mathsf{P})$, depends only on $\pi_0 \circ \phi^{-1}(I) \in \{I_1, \ldots, I_m\}$. In particular, there is an induced 2-coloring of the tuples $I_1, I_2, \ldots, I_m \in K_N^{(t)}$. Extend this induced 2-coloring to all of $K_N^{(t)}$ arbitrarily.

By the definition of N, there must be a monochromatic $K_n^{(t)}$ in $K_N^{(t)}$. Let $S \in \binom{[N]}{n}$ be the set of vertices of this monochromatic $K_n^{(t)}$ and let $\mathsf{H}^* \in \mathcal{F}_0$ be the copy of H in correspondence with S. The graph $\mathsf{H} = \phi \circ \pi_0^{-1}(\mathsf{H}^*) \in \mathcal{F}(\mathsf{P})$ is such that $\mathsf{H} \subset R$ is metric and $\binom{\mathsf{H}}{\rho}$ is monochromatic.

References

[1] F. G. Abramson and L. A. Harrington. Models without indiscernibles. *J. Symbolic Logic*, 43(3):572–600, 1978.

[2] W. Deuber. Generalizations of Ramsey's theorem. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I*, pages 323–332. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

[3] P. Erdős, A. Hajnal, and L. Pósa. Strong embeddings of graphs into colored graphs. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I*, pages 585–595. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

[4] R. L. Graham, B. L. Rothschild, and J. H. Spencer. *Ramsey theory*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, second edition, 1990. A Wiley-Interscience Publication.

[5] A. W. Hales and R. I. Jewett. Regularity and positional games. *Trans. Amer. Math. Soc.*, 106:222–229, 1963.

[6] J. Nešetřil. Metric spaces are Ramsey. *European J. Combin.*, 28(1):457–468, 2007.

[7] J. Nešetřil and V. Rödl. Partitions of finite relational and set systems. *J. Combinatorial Theory Ser. A*, 22(3):289–312, 1977.

[8] J. Nešetřil and V. Rödl. Partition theory and its application. In *Surveys in combinatorics (Proc. Seventh British Combinatorial Conf., Cambridge, 1979)*, volume 38 of *London Math. Soc. Lecture Note Ser.*, pages 96–156. Cambridge Univ. Press, Cambridge, 1979.

[9] J. Nešetřil and V. Rödl. A short proof of the existence of highly chromatic hypergraphs without short cycles. *J. Combin. Theory Ser. B*, 27(2):225–227, 1979.

[10] J. Nešetřil and V. Rödl. Strong Ramsey theorems for Steiner systems. *Trans. Amer. Math. Soc.*, 303(1):183–192, 1987.

[11] J. Nešetřil and V. Rödl. The partite construction and Ramsey set systems. *Discrete Math.*, 75(1-3):327–334, 1989. Graph theory and combinatorics (Cambridge, 1988).

[12] V. Rödl. A generalization of Ramsey theorem and dimension of graphs. Master's thesis, Charles University, Prague, 1973.

[13] V. Rödl. A generalization of the Ramsey theorem. Graphs, Hypergraphs, Block Syst.; Proc. Symp. comb. Anal., Zielona Gora 1976, 211-219 (1976)., 1976.

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