DETACHMENTS OF HYPERGRAPHS I: THE BERGE-JOHNSON PROBLEM

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Abstract. A detachment of a hypergraph is formed by splitting each vertex into one or more subvertices, and sharing the incident edges arbitrarily among the subvertices. For a given edge-colored hypergraph $\mathscr{F}$, we prove that there exists a detachment $\mathscr{G}$ such that the degree of each vertex and the multiplicity of each edge in $\mathscr{F}$ (and each color class of $\mathscr{F}$) are shared fairly among the subvertices in $\mathscr{G}$ (and each color class of $\mathscr{G}$, respectively).

Let $(\lambda_1 \dots, \lambda_m)K^{h_1,\dots,h_m}_{p_1,\dots,p_n}$ be a hypergraph with vertex partition $\{V_1, \dots, V_n\}$, $|V_i| = p_i$ for $1 \leqslant i \leqslant n$ such that there are λ_i edges of size h_i incident with every h_i vertices, at most one vertex from each part for $1 \leqslant i \leqslant m$ (so no edge is incident with more than one vertex of a part). We use our detachment theorem to show that the obvious necessary conditions for $(\lambda_1 \dots, \lambda_m)K^{h_1,\dots,h_m}_{p_1,\dots,p_n}$ to be expressed as the union $\mathscr{G}_1 \cup \ldots \cup \mathscr{G}_k$ of k edge-disjoint factors, where for $1 \leqslant i \leqslant k$, $\mathscr{G}_i$ is r_i-regular, are also sufficient. Baranyai solved the case of $h_1 = \dots = h_m$, $\lambda_1 = \dots, \lambda_m = 1$, $p_1 = \dots = p_m$, $r_1 = \dots = r_k$. Berge and Johnson, (and later Brouwer and Tijdeman, respectively) considered (and solved, respectively) the case of $h_i = i$, $1 \leqslant i \leqslant m$, $p_1 = \dots = p_m = \lambda_1 = \dots = \lambda_m = r_1 = \dots = r_k = 1$. We also extend our result to the case where each $\mathscr{G}_i$ is almost regular.

1. Introduction

Intuitively speaking, a detachment of a hypergraph is formed by splitting each vertex into one or more subvertices, and sharing the incident edges arbitrarily among the subvertices. As the main result of this paper (see Theorem 4.1), we prove that for a given edge-colored hypergraph $\mathscr{F}$, there exists a detachment $\mathscr{G}$ such that the degree of each vertex and the multiplicity of each edge in $\mathscr{F}$ (and each color class of $\mathscr{F}$) are shared fairly among the subvertices in $\mathscr{G}$ (and each color class of $\mathscr{G}$, respectively). This result is not only interesting by itself and generalizes various graph theoretic results (see for example [1, 10, 12, 14, 15, 17, 18, 19]), but also is used to obtain extensions of existing results on edge-decompositions of hypergraphs by Bermond, Baranyai [2, 3], Berge and Johnson [4, 13], and Brouwer and Tijdeman [5, 6].

Given a set N of n elements, Berge and Johnson [4, 13] addressed the question of when do there exist disjoint partitions of N, each partition containing only subsets of h or fewer elements, such that every subset of N having h or fewer elements is in exactly one partition. Here we state the problem in a more general setting with the hypergraph theoretic notation.

Let $(\lambda_1 \dots, \lambda_m)K^{h_1,\dots,h_m}_{p_1,\dots,p_n}$ be a hypergraph with vertex partition $\{V_1, \dots, V_n\}$, $|V_i| = p_i$ for $1 \leqslant i \leqslant n$ such that there are λ_i edges of size h_i incident with every h_i vertices, at most one vertex from each part for $1 \leqslant i \leqslant m$ (so no edge is incident with more than one vertex of a part). We use our detachment theorem to show that the obvious necessary conditions for $(\lambda_1 \dots, \lambda_m)K^{h_1,\dots,h_m}_{p_1,\dots,p_n}$ to be expressed as the union $\mathscr{G}_1 \cup \ldots \cup \mathscr{G}_k$ of k edge-disjoint factors,

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where for $1 \leqslant i \leqslant k$, $\mathscr{G}_i$ is r_i-regular, are also sufficient. Baranyai [2, 3] solved the case of $h_1 = \dots = h_m$, $\lambda_1 = \dots, \lambda_m = 1$, $p_1 = \dots = p_m$, $r_1 = \dots = r_k$. Berge and Johnson [4, 13], (and later Brouwer and Tijdeman [5, 6], respectively) considered (and solved, respectively) the case of $h_i = i$, $1 \leqslant i \leqslant m$, $p_1 = \dots = p_m = \lambda_1 = \dots = \lambda_m = r_1 = \dots = r_k = 1$. We also extend our result to the case where each $\mathscr{G}_i$ is almost regular.

In the next two sections, we give more precise definitions along with terminology. In Section 4, we state our main result, followed by the proof in Section 5. In the last section, we show the usefulness of the main result on decompositions of various classes of hypergraphs. We defer the applications of the main result in solving embedding problems to a future paper.

2. Terminology and precise definitions

If $x, y \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers), then $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the integers such that $x - 1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1$, and $x \approx y$ means $\lfloor y \rfloor \leqslant x \leqslant \lceil y \rceil$. We observe that the relation $\approx$ is transitive (but not symmetric) and for $x, y \in \mathbb{R}$, and $n \in \mathbb{N}$ ($\mathbb{N}$ is the set of positive integers), $x \approx y$ implies $x/n \approx y/n$. These properties of $\approx$ will be used in Section 5 without further explanation. For a multiset A and $u \in A$, let $\mu_A(u)$ denote the multiplicity of u in A, and let $|A| = \sum_{u \in A} \mu_A(u)$. For multisets $A_1, \dots, A_n$, we define $A = \bigcup_{i=1}^n A_i$ by $\mu_A(u) = \sum_{i=1}^n \mu_{A_i}(u)$. We may use abbreviations such as $\{u^r\}$ for $\{\underbrace{u, \dots, u}_{r}\}$ — for example $\{u^2, v, w^2\} \cup \{u, w^2\} = \{u^3, v, w^4\}$.

For the purpose of this paper, a *hypergraph* $\mathscr{G}$ is an ordered quintuple $(V(\mathscr{G}), E(\mathscr{G}), H(\mathscr{G}), \psi, \phi)$ where $V(\mathscr{G}), E(\mathscr{G}), H(\mathscr{G})$ are disjoint finite sets, $\psi : H(\mathscr{G}) \to V(\mathscr{G})$ is a function and $\phi : H(\mathscr{G}) \to E(\mathscr{G})$ is a surjection. Elements of $V(\mathscr{G}), E(\mathscr{G}), H(\mathscr{G})$ are called *vertices*, *edges* and *hinges* of $\mathscr{G}$, respectively. A vertex v (edge e, respectively) and hinge h are said to be *incident* with each other if $\psi(h) = v$ ($\phi(h) = e$, respectively). A hinge h is said to *attach* the edge $\phi(h)$ to the vertex $\psi(h)$. In this manner, the vertex $\psi(h)$ and the edge $\phi(h)$ are said to be *incident* with each other. If $e \in E(\mathscr{G})$, and e is incident with n hinges $h_1, \dots, h_n$ for some $n \in \mathbb{N}$, then the edge e is said to *join* (not necessarily distinct) vertices $\psi(h_1), \dots, \psi(h_n)$. If $v \in V(\mathscr{G})$, then the number of hinges incident with v (i.e. $|\psi^{-1}(v)|$) is called the *degree* of v and is denoted by $d(v)$. The number of (distinct) vertices incident with an edge e, denoted by $|e|$, is called the *size* of e. If for all edges e of $\mathscr{G}$, $|e| \leqslant 2$ and $|\phi^{-1}(e)| = 2$, then $\mathscr{G}$ is a *graph*.

Thus a hypergraph, in the sense of our definition, is a generalization of a hypergraph as it is usually defined. In fact, if for every edge e, $|e| = |\phi^{-1}(e)|$, then our definition is essentially the same as the usual definition. Here for convenience, we imagine each edge of a hypergraph to be attached to the vertices which it joins by in-between objects called hinges. Readers from a graph theory background may think of this as a bipartite multigraph with vertex bipartition $\{V, E\}$, in which the hinges form the edges. A hypergraph may be drawn as a set of points representing the vertices. A hyperedge is represented by a simple closed curve enclosing its incident vertices. A hinge is represented by a small line attached to the vertex incident with it (see Figure 1).

The set of hinges of $\mathscr{G}$ which are incident with a vertex v (and an edge e, respectively), is denoted by $H(v)$ ($H(v, e)$, respectively). Thus if $v \in V(\mathscr{G})$, then $H(v) = \psi^{-1}(v)$, and $|H(v)|$ is the degree $d(v)$ of v. If U is a multi-subset of $V(\mathscr{G})$, and $u \in V(\mathscr{G})$, let $E(U)$ denote the set of edges e with $|\phi^{-1}(e)| = |U|$ joining vertices in U. More precisely, $E(U) = \{e \in E(\mathscr{G}) |$ for all $v \in V(\mathscr{G}), |H(v, e)| = \mu_U(v)\}$. For $U_1, \dots, U_n \subset V$ where for $1 \leqslant i \leqslant n$ each U_i

is a multiset, let $E(U_1,\dots,U_n)$ denote $E(\bigcup_{i=1}^n U_i)$. We write $m(U)$ for $|E(U)|$ and call it the *multiplicity* of U. For simplicity, $E(u^r,U)$ denotes $E(\{u^r\},U)$, and $m(u_1^{m_1},\dots,u_r^{m_r})$ denotes $m(\{u_1^{m_1},\dots,u_r^{m_r}\})$. The set of hinges that are incident with u and an edge in $E(u^r,U)$ is denoted by $H(u^r,U)$.

Example 2.1. Let $\mathscr{G}=(V,E,H,\psi,\phi)$, with $V=\{v_1,v_2,v_3,v_4,v_5\}, E=\{e_1,e_2,e_3\}, H=\{h_i, 1\leqslant i\leqslant 7\}$, such that $\psi(h_1)=\psi(h_2)=v_1, \psi(h_3)=v_2, \psi(h_4)=\psi(h_5)=v_3, \psi(h_6)=v_4, \psi(h_7)=v_5$ and $\phi(h_1)=\phi(h_2)=\phi(h_3)=\phi(h_4)=e_1, \phi(h_5)=\phi(h_6)=e_2, \phi(h_7)=e_3$. We have:

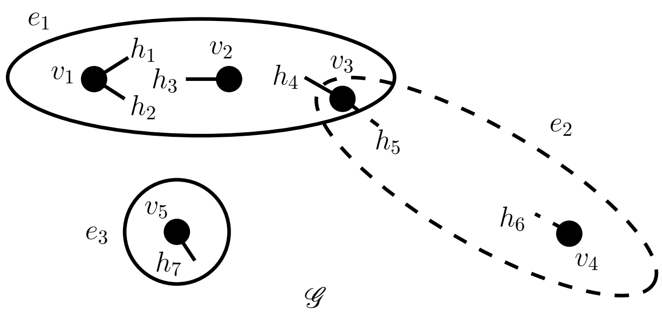

FIGURE 1. Representation of a hypergraph $\mathscr{G}$

- $|e_1|=3, |e_2|=2, |e_3|=1$,
- $d(v_1)=d(v_3)=2, d(v_2)=d(v_4)=d(v_5)=1$,
- $H(v_1)=\{h_1,h_2\}, H(v_2)=\{h_3\}, H(v_3)=\{h_4,h_5\}$,
- $H(v_3,e_1)=\{h_4\}, H(v_3,e_2)=\{h_5\}, H(v_3,e_3)=\varnothing$,
- $E(\{v_1,v_2,v_3\})=\varnothing, E(\{v_1^2,v_2,v_3\})=E(v_1^2,\{v_2,v_3\})=\{e_1\}$,
- $m(v_1,v_2,v_3)=0, m(v_1^2,v_2,v_3)=1$,
- $H(v_1^2,\{v_2,v_3\})=\{h_1,h_2\}, H(v_1,\{v_2,v_3\})=\varnothing, H(v_3,\{v_1^2,v_2\})=\{h_4\}$.

A *k-edge-coloring* of $\mathscr{G}$ is a mapping $f: E(\mathscr{G})\to C$, where C is a set of k *colors* (often we use $C=\{1,\dots,k\}$), and the edges of one color form a *color class*. The sub-hypergraph of $\mathscr{G}$ induced by the color class j is denoted by $\mathscr{G}(j)$. To avoid ambiguity, subscripts may be used to indicate the hypergraph in which hypergraph-theoretic notation should be interpreted — for example, $d_{\mathscr{G}}(v)$, $E_{\mathscr{G}}(v^2,w)$, $H_{\mathscr{G}}(v)$.

3. Amalgamations and detachments

If $\mathscr{F}=(V,E,H,\psi,\phi)$ is a hypergraph and Ψ is a function from V onto a set W, then we shall say that the hypergraph $\mathscr{G}=(W,E,H,\Psi\circ\psi,\phi)$ is an *amalgamation* of $\mathscr{F}$ and that $\mathscr{F}$ is a *detachment* of $\mathscr{G}$. Associated with Ψ is the *number function* $g: W\to\mathbb{N}$ defined by $g(w)=|\Psi^{-1}(w)|$, for each $w\in W$; being more specific, we may also say that $\mathscr{F}$ is a *g-detachment* of $\mathscr{G}$. Intuitively speaking, a g-detachment of $\mathscr{G}$ is obtained by splitting each $u\in V(\mathscr{G})$ into $g(u)$ vertices. Thus $\mathscr{F}$ and $\mathscr{G}$ have the same edges and hinges, and each vertex v of $\mathscr{G}$ is obtained by identifying those vertices of $\mathscr{F}$ which belong to the set $\Psi^{-1}(v)$. In this process, a hinge incident with a vertex u and an edge e in $\mathscr{F}$ becomes incident with the vertex $\Psi(u)$ and the edge e in $\mathscr{G}$.

There are quite a lot of other papers on amalgamations and some highlights include [7, 8, 9, 10, 12, 14, 18, 19].

4. Main Result

A function $g : V(\mathscr{G}) \to \mathbb{N}$ is said to be *simple* if

$$|H(v,e)| \leqslant g(v) \text{ for } v \in V(\mathscr{G}), e \in E(\mathscr{G}).$$

A hypergraph $\mathscr{G}$ is said to be *simple* if $g : V(\mathscr{G}) \to \mathbb{N}$ with $g(v) = 1$ for $v \in V(\mathscr{G})$ is simple. It is clear that for a hypergraph $\mathscr{F}$ and a function $g : V(\mathscr{F}) \to \mathbb{N}$, there exists a simple g-detachment if and only if g is simple.

Theorem 4.1. *Let $\mathscr{F}$ be a k-edge-colored hypergraph and let $g : V(\mathscr{F}) \to \mathbb{N}$ be a simple function. Then there exists a simple g-detachment $\mathscr{G}$ (possibly with multiple edges) of $\mathscr{F}$ with amalgamation function $\Psi : V(\mathscr{G}) \to V(\mathscr{F})$, g being the number function associated with Ψ, such that:*

- (A1) $d_{\mathscr{G}}(v) \approx d_{\mathscr{F}}(u)/g(u)$ *for each* $u \in V(\mathscr{F})$ *and each* $v \in \Psi^{-1}(u)$*;*
- (A2) $d_{\mathscr{G}(j)}(v) \approx d_{\mathscr{F}(j)}(u)/g(u)$ *for each* $u \in V(\mathscr{F})$*, each* $v \in \Psi^{-1}(u)$ *and* $1 \leqslant j \leqslant k$*;*
- (A3) $m_{\mathscr{G}}(U_1, \dots, U_r) \approx m_{\mathscr{F}}(u_1^{m_1}, \dots, u_r^{m_r})/\Pi_{i=1}^r \binom{g(u_i)}{m_i}$ *for distinct* $u_1, \dots, u_r \in V(\mathscr{F})$ *and* $U_i \subset \Psi^{-1}(u_i)$ *with* $|U_i| = m_i \leqslant g(u_i)$ *for* $1 \leqslant i \leqslant r$*;*
- (A4) $m_{\mathscr{G}(j)}(U_1, \dots, U_r) \approx m_{\mathscr{F}(j)}(u_1^{m_1}, \dots, u_r^{m_r})/\Pi_{i=1}^r \binom{g(u_i)}{m_i}$ *for distinct* $u_1, \dots, u_r \in V(\mathscr{F})$ *and* $U_i \subset \Psi^{-1}(u_i)$ *with* $|U_i| = m_i \leqslant g(u_i)$ *for* $1 \leqslant i \leqslant r$ *and* $1 \leqslant j \leqslant k$*.*

A family $\mathscr{A}$ of sets is *laminar* if, for every pair A, B of sets belonging to $\mathscr{A}$, either $A \subset B$, or $B \subset A$, or $A \cap B = \varnothing$. To prove the main result, we need the following lemma:

Lemma 4.2. (Nash-Williams [18, Lemma 2]) *If $\mathscr{A}, \mathscr{B}$ are two laminar families of subsets of a finite set S, and $n \in \mathbb{N}$, then there exist a subset A of S such that for every $P \in \mathscr{A} \cup \mathscr{B}$, $|A \cap P| \approx |P|/n$.*

5. proof of Theorem 4.1

5.1. **Inductive construction of $\mathscr{G}$.** Let $\mathscr{F} = (V, E, H, \psi, \phi)$. Let $n = \sum_{v \in V}(g(v) - 1)$. Initially we let $\mathscr{F}_0 = \mathscr{F}$ and $g_0 = g$, and we let Φ_0 be the identity function from V into V. Now assume that $\mathscr{F}_0 = (V_0, E_0, H_0, \psi_0, \phi_0), \dots, \mathscr{F}_i = (V_i, E_i, H_i, \psi_i, \phi_i)$ and $\Phi_0, \dots, \Phi_i$ have been defined for some $i \geqslant 0$. Also assume that the simple functions $g_0 : V_0 \to \mathbb{N}, \dots, g_i : V_i \to \mathbb{N}$ have been defined for some $i \geqslant 0$. Let $\Psi_i = \Phi_0 \dots \Phi_i$. If $i = n$, we terminate the construction, letting $\mathscr{G} = \mathscr{F}_n$ and $\Psi = \Psi_n$.

If $i < n$, we can select a vertex α of $\mathscr{F}_i$ such that $g_i(\alpha) \geqslant 2$. As we will see, $\mathscr{F}_{i+1}$ is formed from $\mathscr{F}_i$ by splitting off a vertex v_{i+1} from α so that we end up with α and v_{i+1}. Let

$$\begin{aligned} \mathscr{A}_i &= \{H_{\mathscr{F}_i}(\alpha)\} \\ &\bigcup \{H_{\mathscr{F}_i(1)}(\alpha), \dots, H_{\mathscr{F}_i(k)}(\alpha)\} \\ &\bigcup \{H_{\mathscr{F}_i(j)}(\alpha, e) : e \in E_{\mathscr{F}_i(j)}(\alpha), 1 \leqslant j \leqslant k\}, \end{aligned} \tag{1}$$

and let

$$\begin{aligned} \mathscr{B}_i &= \{H_{\mathscr{F}_i}(\alpha^t, U) : t \geqslant 1, U \subset V_i \backslash \{\alpha\}\} \\ &\bigcup \{H_{\mathscr{F}_i(j)}(\alpha^t, U) : t \geqslant 1, U \subset V_i \backslash \{\alpha\}, 1 \leqslant j \leqslant k\}. \end{aligned} \tag{2}$$

It is easy to see that both $\mathscr{A}_i$ and $\mathscr{B}_i$ are laminar families of subsets of $H(\mathscr{F}_i, \alpha)$. Therefore, by Lemma 4.2, there exists a subset Z_i of $H(\mathscr{F}_i, \alpha)$ such that

$$|Z_i \cap P| \approx |P|/g_i(\alpha), \text{ for every } P \in \mathscr{A}_i \cup \mathscr{B}_i. \tag{3}$$

Let v_{i+1} be a vertex which does not belong to V_i and let $V_{i+1} = V_i \cup \{v_{i+1}\}$. Let Φ_{i+1} be the function from V_{i+1} onto V_i such that $\Phi_{i+1}(v) = v$ for every $v \in V_i$ and $\Phi_{i+1}(v_{i+1}) = \alpha$. Let $\mathscr{F}_{i+1}$ be the detachment of $\mathscr{F}_i$ under Φ_{i+1} such that $V(\mathscr{F}_{i+1}) = V_{i+1}$, and

$$H_{\mathscr{F}_{i+1}}(v_{i+1}) = Z_i, H_{\mathscr{F}_{i+1}}(\alpha) = H_{\mathscr{F}_i}(\alpha)\backslash Z_i. \tag{4}$$

In fact, $\mathscr{F}_{i+1}$ is obtained from $\mathscr{F}_i$ by splitting α into two vertices α and v_{i+1} in such a way that hinges which were incident with α in $\mathscr{F}_i$ become incident in $\mathscr{F}_{i+1}$ with α or v_{i+1} according as they do not or do belong to Z_i, respectively. Obviously, Ψ_i is an amalgamation function from $\mathscr{F}_{i+1}$ into $\mathscr{F}_i$. Let g_{i+1} be the function from V_{i+1} into $\mathbb{N}$, such that $g_{i+1}(v_{i+1}) = 1, g_{i+1}(\alpha) = g_i(\alpha)-1$, and $g_{i+1}(v) = g_i(v)$ for every $v \in V_i\backslash\{\alpha\}$. This finishes the construction of $\mathscr{F}_{i+1}$.

5.2. **Relations between $\mathscr{F}_{i+1}$ and $\mathscr{F}_i$.** The hypergraph $\mathscr{F}_{i+1}$, satisfies the following conditions:

- (B1) $d_{\mathscr{F}_{i+1}}(\alpha) \approx d_{\mathscr{F}_i}(\alpha)g_{i+1}(\alpha)/g_i(\alpha)$;
- (B2) $d_{\mathscr{F}_{i+1}}(v_{i+1}) \approx d_{\mathscr{F}_i}(\alpha)/g_i(\alpha)$;
- (B3) $m_{\mathscr{F}_{i+1}}(v_{i+1}^s, \alpha^t, U) = 0$ for $s \geqslant 2$, and $t \geqslant 0$;
- (B4) $m_{\mathscr{F}_{i+1}}(\alpha^t, U) \approx m_{\mathscr{F}_i}(\alpha^t, U)(g_i(\alpha) - t)/g_i(\alpha)$ for each $U \subset V_i\backslash\{\alpha\}$, and $g_i(\alpha) \geqslant t \geqslant 1$;
- (B5) $m_{\mathscr{F}_{i+1}}(\alpha^t, v_{i+1}, U) \approx (t+1)m_{\mathscr{F}_i}(\alpha^{t+1}, U)/g_i(\alpha)$ for each $U \subset V_i\backslash\{\alpha\}$, and $t \geqslant 0$.

Proof. Since $H_{\mathscr{F}_i}(\alpha) \in \mathscr{A}_i$, from (4) it follows that

$$\begin{aligned} d_{\mathscr{F}_{i+1}}(v_{i+1}) &= |H_{\mathscr{F}_{i+1}}(v_{i+1})| = |Z_i| = |Z_i \cap H_{\mathscr{F}_i}(\alpha)| \\ &\approx |H_{\mathscr{F}_i}(\alpha)|/g_i(\alpha) = d_{\mathscr{F}_i}(\alpha)/g_i(\alpha), \\ d_{\mathscr{F}_{i+1}}(\alpha) &= |H_{\mathscr{F}_{i+1}}(\alpha)| = |H_{\mathscr{F}_i}(\alpha)| - |Z_i| \\ &\approx d_{\mathscr{F}_i}(\alpha) - d_{\mathscr{F}_i}(\alpha)/g_i(\alpha) = (g_i(\alpha)-1)d_{\mathscr{F}_i}(\alpha)/g_i(\alpha) \\ &= d_{\mathscr{F}_i}(\alpha)g_{i+1}(\alpha)/g_i(\alpha). \end{aligned}$$

This proves (B1) and (B2).

If $t \geqslant 1, U \subset V_i\backslash\{\alpha\}$, and $e \in E_{\mathscr{F}_i}(\alpha^t, U)$, then for some j, $1 \leqslant j \leqslant k$, $H_{\mathscr{F}_i(j)}(\alpha, e) \in \mathscr{A}_i$, so

$$\left|Z_i \cap H_{\mathscr{F}_i(j)}(\alpha, e)\right| \approx |H_{\mathscr{F}_i(j)}(\alpha, e)|/g_i(\alpha) = t/g_i(\alpha) \leqslant 1,$$

where the inequality implies from the fact that g_i is simple. Therefore either $|Z_i \cap H_{\mathscr{F}_i(j)}(\alpha, e)| = 1$ and consequently $e \in E_{\mathscr{F}_{i+1}}(\alpha^{t-1}, v_{i+1}, U)$ or $Z_i \cap H_{\mathscr{F}_i(j)}(\alpha, e) = \varnothing$ and consequently $e \in E_{\mathscr{F}_{i+1}}(\alpha^t, U)$. Therefore

$$m_{\mathscr{F}_{i+1}}(v_{i+1}^s, \alpha^r, U) = 0,$$

for $r \geqslant 1$, and $s \geqslant 2$. This proves (B3). Moreover, since $H_{\mathscr{F}_i}(\alpha^t, U) \in \mathscr{B}_i$, we have

$$\begin{aligned} m_{\mathscr{F}_{i+1}}(\alpha^{t-1}, v_{i+1}, U) &= |Z_i \cap H_{\mathscr{F}_i}(\alpha^t, U)| \approx |H_{\mathscr{F}_i}(\alpha^t, U)|/g_i(\alpha) = tm_{\mathscr{F}_i}(\alpha^t, U)/g_i(\alpha), \\ m_{\mathscr{F}_{i+1}}(\alpha^t, U) &\approx m_{\mathscr{F}_i}(\alpha^t, U) - |H_{\mathscr{F}_i}(\alpha^t, U)|/g_i(\alpha) = m_{\mathscr{F}_i}(\alpha^t, U) - tm_{\mathscr{F}_i}(\alpha^t, U)/g_i(\alpha) \\ &= m_{\mathscr{F}_i}(\alpha^t, U)(g_i(\alpha) - t)/g_i(\alpha). \end{aligned}$$

This proves (B4) and (B5). □

Let us fix $j \in \{1, \dots, k\}$. It is enough to replace $\mathscr{F}_i$ with $\mathscr{F}_i(j)$ in the statement and the proof of (B1)–(B5) to obtain companion conditions, say (C1)–(C5) for each color class.

5.3. **Relations between $\mathscr{F}_i$ and $\mathscr{F}$.** Recall that $\Psi_i = \Phi_0 \dots \Phi_i$, that $\Phi_0 : V \to V$, and that $\Phi_i : V_i \to V_{i-1}$ for $i > 0$. Therefore $\Psi_i : V_i \to V$ and thus $\Psi_i^{-1} : V \to V_i$. Now we use (B1)–(B5) to prove that the hypergraph $\mathscr{F}_i$ satisfies the following conditions for $0 \leqslant i \leqslant n$:

(D1) $d_{\mathscr{F}_i}(v)/g_i(v) \approx d_{\mathscr{F}}(u)/g(u)$ for each $u \in V$ and each $v \in \Psi_i^{-1}(u)$;

(D2) $m_{\mathscr{F}_i}(u_1^{a_1}, U_1, \dots, u_r^{a_r}, U_r)/\Pi_{j=1}^r \binom{g_i(u_j)}{a_j} \approx m_{\mathscr{F}}(u_1^{m_1}, \dots, u_r^{m_r})/\Pi_{j=1}^r \binom{g(u_j)}{m_j}$ for distinct vertices $u_1, \dots, u_r \in V$, $a_j \geqslant 0$, $U_j \subset \Psi_i^{-1}(u_j)\backslash\{u_j\}$ with $1 \leqslant m_j = a_j + |U_j| \leqslant g(u_j)$, $1 \leqslant j \leqslant r$ if $g_i(u_j) \geqslant a_j$, $1 \leqslant j \leqslant r$.

Proof. The proof is by induction. Recall that $\mathscr{F}_0 = \mathscr{F}$, and $g_0(u) = g(u)$ for $u \in V$. Thus, (D1) and (D2) are trivial for $i = 0$. Now we will show that if $\mathscr{F}_i$ satisfies the conditions (D1) and (D2) for some $i < n$, then $\mathscr{F}_{i+1}$ satisfies these conditions by replacing i with $i+1$; we denote the corresponding conditions for $\mathscr{F}_{i+1}$ by (D1)′ and (D2)′.

Let $u \in V$. If $g_{i+1}(u) = g_i(u)$, then (D1)′ is obviously true. So we just check (D1)′ in the case where $u = \alpha$. By (B1) and (D1) we have $d_{\mathscr{F}_{i+1}}(\alpha)/g_{i+1}(\alpha) \approx d_{\mathscr{F}_i}(\alpha)/g_i(\alpha) \approx d_{\mathscr{F}}(\alpha)/g(\alpha)$. Moreover, from (B2) and (D1) it follows that $d_{\mathscr{F}_{i+1}}(v_{i+1}) \approx d_{\mathscr{F}_i}(\alpha)/g_i(\alpha) \approx d_{\mathscr{F}}(\alpha)/g(\alpha)$. Since in forming $\mathscr{F}_{i+1}$ no edge is detached from v_r for each $v_r \in \Psi_i^{-1}(\alpha)\backslash\{\alpha\}$, we have $d_{\mathscr{F}_{i+1}}(v_r) = d_{\mathscr{F}_i}(v_r)$. Therefore $d_{\mathscr{F}_{i+1}}(v_r) = d_{\mathscr{F}_i}(v_r) \approx d_{\mathscr{F}}(\alpha)/g(\alpha)$ for each $v_r \in \Psi_i^{-1}(\alpha)\backslash\{\alpha\}$. This proves (D1)′. Let $u_1, \dots, u_r$ be distinct vertices in V. If $g_{i+1}(u_j) = g_i(u_j)$ for $1 \leqslant j \leqslant r$, then (D2)′ is clearly true. Therefore, in order to prove (D2)′, without loss of generality we may assume that $g_{i+1}(u_1) = g_i(u_1) - 1$ (so $\alpha = u_1$ and $v_{i+1} \in \Psi_i^{-1}(u_1)$). First, note that for integers a, b we always have $(a-b)\binom{a}{b} = a\binom{a-1}{b} = (b+1)\binom{a}{b+1}$. If $v_{i+1} \notin U_1$, we have

$$
\begin{aligned}
\frac{m_{\mathscr{F}_{i+1}}(u_1^{a_1}, U_1, \dots, u_r^{a_r}, U_r)}{\Pi_{j=1}^r \binom{g_{i+1}(u_j)}{a_j}} &\overset{\text{(B4)}}{\approx} \frac{m_{\mathscr{F}_i}(u_1^{a_1}, U_1, \dots, u_r^{a_r}, U_r)(g_i(u_1) - a_1)/g_i(u_1)}{\binom{g_i(u_1)-1}{a_1}\Pi_{j=2}^r \binom{g_i(u_j)}{a_j}} \\
&= \frac{m_{\mathscr{F}_i}(u_1^{a_1}, U_1, \dots, u_r^{a_r}, U_r)(g_i(u_1) - a_1)/g_i(u_1)}{(g_i(u_1) - a_1)/g_i(u_1)\binom{g_i(u_1)}{a_1}\Pi_{j=2}^r \binom{g_i(u_j)}{a_j}} \\
&= \frac{m_{\mathscr{F}_i}(u_1^{a_1}, U_1, \dots, u_r^{a_r}, U_r)}{\Pi_{j=1}^r \binom{g_i(u_j)}{a_j}} \\
&\overset{\text{(D2)}}{\approx} \frac{m_{\mathscr{F}}(u_1^{m_1}, \dots, u_r^{m_r})}{\Pi_{j=1}^r \binom{g(u_j)}{m_j}}.
\end{aligned}
$$

If $v_{i+1} \in U_1$, we have

$$
\begin{aligned}
\frac{m_{\mathscr{F}_{i+1}}(u_1^{a_1}, U_1, \dots, u_r^{a_r}, U_r)}{\Pi_{j=1}^r \binom{g_{i+1}(u_j)}{a_j}} &\overset{\text{(B5)}}{\approx} \frac{m_{\mathscr{F}_i}(u_1^{a_1+1}, U_1\backslash\{v_{i+1}\}, \dots, u_r^{a_r}, U_r)(a_1 + 1)/g_i(u_1)}{\binom{g_i(u_1)-1}{a_1}\Pi_{j=2}^r \binom{g_i(u_j)}{a_j}} \\
&= \frac{m_{\mathscr{F}_i}(u_1^{a_1+1}, U_1\backslash\{v_{i+1}\}, \dots, u_r^{a_r}, U_r)}{g_i(u_1)/(a_1 + 1)\binom{g_i(u_1)-1}{a_1}\Pi_{j=2}^r \binom{g_i(u_j)}{a_j}} \\
&= \frac{m_{\mathscr{F}_i}(u_1^{a_1+1}, U_1\backslash\{v_{i+1}\}, \dots, u_r^{a_r}, U_r)}{\binom{g_i(u_1)}{a_1+1}\Pi_{j=2}^r \binom{g_i(u_j)}{a_j}} \\
&\overset{\text{(D2)}}{\approx} \frac{m_{\mathscr{F}}(u_1^{m_1}, \dots, u_r^{m_r})}{\Pi_{j=1}^r \binom{g(u_j)}{m_j}}.
\end{aligned}
$$

This proves (D2)$'$. $\square$

Let us fix $j \in \{1, \dots, k\}$. It is enough to replace $\mathscr{F}$ with $\mathscr{F}(j)$, $\mathscr{F}_i$ with $\mathscr{F}_i(j)$, $\mathscr{F}_{i+1}$ with $\mathscr{F}_{i+1}(j)$, and (Bi) with (Ci) for $i = 1, 2, 4, 5$, in the statement and the proof of (D1) and (D2) to obtain companion conditions, say (E1) and (E2) for each color class.

5.4. $\mathscr{G}$ **satisfies (A1)–(A4).** Recall that $\mathscr{G} = \mathscr{F}_n$ and $g_n(u) = 1$ for every $u \in V$, therefore when $i = n$, (D1) implies (A1). Moreover, if we let $i = n$ in (D2), we have $a_j \in \{0, 1\}$ for $1 \leqslant j \leqslant r$ and thus $\Pi_{j=1}^r \binom{g_i(u_j)}{a_j} = \Pi_{j=1}^r \binom{1}{a_j} = 1$. This proves (A3). By a similar argument, one can prove (A2) and (A4), and this completes the proof of Theorem 4.1. $\square$

6. Corollaries

For a matrix A, let A_j denote the j^{th} column of A, and let $s(A)$ denote the sum of all the elements of A. Let $R = [r_1 \dots r_k]^T$ (or $R^T = [r_i]_{1\times k}$), $\Lambda = [\lambda_1 \dots \lambda_m]^T$ and $H = [h_1 \dots h_m]^T$ be three column vectors with $r_i, \lambda_i \in \mathbb{N}$, and $h_i \in \{1, \dots, n\}$ for $1 \leqslant i \leqslant m$, such that $h_1 \dots, h_m$ are distinct. Let ΛK_n^H denote a hypergraph with vertex set V, $|V| = n$, such that there are λ_i edges of size h_i incident with every h_i vertices for $1 \leqslant i \leqslant m$. A hypergraph $\mathscr{G}$ is said to be *k-regular* if every vertex has degree k. A *k-factor* of $\mathscr{G}$ is a k-regular spanning sub-hypergraph of $\mathscr{G}$. An *R-factorization* is a partition (decomposition) $\{F_1, \dots, F_k\}$ of $E(\mathscr{G})$ in which F_i is an r_i-factor for $1 \leqslant i \leqslant k$. Notice that ΛK_n^H is $\sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1}$-regular. We show that the obvious necessary conditions for the existence of an R-factorization of ΛK_n^H, are also sufficient.

Theorem 6.1. *ΛK_n^H is R-factorizable if and only if $s(R) = \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1}$, and there exists a non-negative integer matrix $A = [a_{ij}]_{k\times m}$ such that $AH = nR$, and $s(A_j) = \lambda_j \binom{n}{h_j}$ for $1 \leqslant j \leqslant m$.*

Proof. To prove the necessity, suppose that ΛK_n^H is R-factorizable. Since each r_i-factor is an r_i-regular spanning sub-hypergraph for $1 \leqslant i \leqslant k$, and ΛK_n^H is $\sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1}$-regular, we must have $s(R) = \sum_{i=1}^k r_i = \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1}$. Let a_{ij} be the number of edges (counting multiplicities) of size h_j contributing to the i^{th} factor for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$. Since for $1 \leqslant j \leqslant m$, each edge of size h_j contributes h_j to the the sum of the degrees of the vertices in an r_i-factor for $1 \leqslant i \leqslant k$, we must have $\sum_{j=1}^m a_{ij} h_j = nr_i$ for $1 \leqslant i \leqslant k$ and $\sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j}$ for $1 \leqslant j \leqslant m$.

To prove the sufficiency, let $\mathscr{F}$ be a hypergraph consisting of a single vertex v with $m_{\mathscr{F}}(v^{h_j}) = \lambda_j \binom{n}{h_j}$ for $1 \leqslant j \leqslant m$. Note that $\mathscr{F}$ is an amalgamation of ΛK_n^H. Now we color the edges of $\mathscr{F}$ so that $m_{\mathscr{F}(i)}(v^{h_j}) = a_{ij}$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$. This can be done, because:

$$\sum_{i=1}^k m_{\mathscr{F}(i)}(v^{h_j}) = \sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j} = m_{\mathscr{F}}(v^{h_j}) \quad \text{for } 1 \leqslant j \leqslant m.$$

Moreover,

$$d_{\mathscr{F}(i)}(v) = \sum_{j=1}^m a_{ij} h_j = nr_i \quad \text{for } 1 \leqslant i \leqslant k.$$

Let $g : V(\mathscr{F}) \to \mathbb{N}$ be a function so that $g(v) = n$. Since for $1 \leqslant i \leqslant m$, $h_i \leqslant n$, g is simple. By Theorem 4.1, there exists a simple g-detachment $\mathscr{G}$ of $\mathscr{F}$ with n vertices, say $v_1, \dots, v_n$

such that by (A2), $d_{\mathscr{G}(i)}(v_j) \approx d_{\mathscr{F}(i)}(v)/g(v) = nr_i/n = r_i$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$, and by (A3), for each $U \subset \{v_1, \dots, v_n\}$ with $|U| = h_j$, $m_{\mathscr{G}}(U) \approx m_{\mathscr{F}}(v^{h_j})/\binom{n}{h_j} = \lambda_j \binom{n}{h_j}/\binom{n}{h_j} = \lambda_j$ for $1 \leqslant j \leqslant m$. Therefore $\mathscr{G} \cong \Lambda K_n^H$, and the i^{th} color class induces an r_i-factor for $1 \leqslant i \leqslant k$. $\square$

In particular, if $m = 1$, $h := h_1$, $\lambda_1 = 1$, $r := r_1 = \cdots = r_k$, then Theorem 6.1 implies Baranyai's theorem: the complete h-uniform hypergraph K_n^h is r-factorizable if and only if $h \mid rn$ and $r \mid \binom{n-1}{h-1}$.

Now let $h_i \geqslant 2$ for $1 \leqslant i \leqslant m$, and let $\Lambda K^H_{p_1,\dots,p_n}$ be a hypergraph with vertex partition $\{V_1, \dots, V_n\}$, $|V_i| = p_i$ for $1 \leqslant i \leqslant n$ such that there are λ_i edges of size h_i incident with every h_i vertices, at most one vertex from each part for $1 \leqslant i \leqslant m$ (so no edge is incident with more than one vertex of a part). If $p_1 = \cdots = p_n := p$, we denote $\Lambda K^H_{p_1,\dots,p_n}$ by $\Lambda K^H_{n\times p}$.

Theorem 6.2. *$\Lambda K^H_{p_1,\dots,p_n}$ is R-factorizable if and only if $p_1 = \cdots = p_n := p$, $s(R) = \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1}$, and there exists a non-negative integer matrix $A = [a_{ij}]_{k\times m}$ such that $AH = npR$, and $s(A_j) = \lambda_j \binom{n}{h_j} p^{h_j}$ for $1 \leqslant j \leqslant m$.*

Proof. To prove the necessity, suppose that $\Lambda K^H_{p_1,\dots,p_n}$ is R-factorizable (so it is regular). Let u and v be two vertices from two different parts, say a^{th} and b^{th} parts, respectively. Since $d(u) = d(v)$, we have

$$\sum_{1\leqslant j\leqslant m} \lambda_j \sum_{\substack{1\leqslant i_1<\cdots<i_{h_j-1}\leqslant n\\ a\notin\{i_1,\dots,i_{h_j-1}\}}} p_{i_1}\dots p_{i_{h_j-1}} = \sum_{1\leqslant j\leqslant m} \lambda_j \sum_{\substack{1\leqslant i_1<\cdots<i_{h_j-1}\leqslant n\\ b\notin\{i_1,\dots,i_{h_j-1}\}}} p_{i_1}\dots p_{i_{h_j-1}} \iff$$

$$\sum_{1\leqslant j\leqslant m} \lambda_j \Big(\sum_{\substack{1\leqslant i_1<\cdots<i_{h_j-1}\leqslant n\\ a\notin\{i_1,\dots,i_{h_j-1}\}}} p_{i_1}\dots p_{i_{h_j-1}} - \sum_{\substack{1\leqslant i_1<\cdots<i_{h_j-1}\leqslant n\\ b\notin\{i_1,\dots,i_{h_j-1}\}}} p_{i_1}\dots p_{i_{h_j-1}} \Big) = 0 \iff$$

$$\sum_{1\leqslant j\leqslant m} \lambda_j \Big(p_b \sum_{1\leqslant i_1<\cdots<i_{h_j-2}\leqslant n} p_{i_1}\dots p_{i_{h_j-2}} - p_a \sum_{1\leqslant i_1<\cdots<i_{h_j-2}\leqslant n} p_{i_1}\dots p_{i_{h_j-2}} \Big) = 0 \iff$$

$$(p_b - p_a) \sum_{1\leqslant j\leqslant m} \lambda_j \sum_{1\leqslant i_1<\cdots<i_{h_j-2}\leqslant n} p_{i_1}\dots p_{i_{h_j-2}} = 0 \iff$$

$$p_b = p_a.$$

Therefore, $p_1 = \cdots = p_n := p$. So $\Lambda K^H_{n\times p}$ is $\sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1}$-regular, and we must have $s(R) = \sum_{i=1}^k r_i = \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1}$. Moreover, there must exist non-negative integers a_{ij}, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$, such that $\sum_{j=1}^m a_{ij} h_j = npr_i$ for $1 \leqslant i \leqslant k$ and $\sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j} p^{h_j}$ for $1 \leqslant j \leqslant m$. We note that a_{ij} is in fact the number of edges (counting multiplicities) of size h_j contributing to the i^{th} factor.

To prove the sufficiency, let $\Lambda^p = [p^{h_i}\lambda_i]^T_{1\times m}$, and let $\mathscr{F} = \Lambda^p K_n^H$ with vertex set $V = \{v_1, \dots, v_n\}$. Notice that $\mathscr{F}$ is an amalgamation of $\Lambda K^H_{n\times p}$. By Theorem 6.1, $\mathscr{F}$ is pR-factorizable. Therefore, we can color the edges of $\mathscr{F}$ so that

$$d_{\mathscr{F}(i)}(v) = pr_i \text{ for } v \in V, 1 \leqslant i \leqslant k.$$

Let $g : V \to \mathbb{N}$ be a function so that $g(v) = p$ for $v \in V$. Since $p \geqslant 1$, g is simple. By Theorem 4.1, there exists a simple g-detachment $\mathscr{G}$ of $\mathscr{F}$ with np vertices, say v_i is detached to $v_{i1}, \dots, v_{ip}$ for $1 \leqslant i \leqslant n$, such that by (A2), $d_{\mathscr{G}(i)}(v_{ab}) \approx d_{\mathscr{F}(i)}(v_a)/g(v_a) = pr_i/p = r_i$ for $1 \leqslant i \leqslant k$, $1 \leqslant a \leqslant n$, $1 \leqslant b \leqslant p$, and by (A3), $m_{\mathscr{G}}(v_{a_1b_1}, \dots, v_{a_{h_j}b_{h_j}}) \approx m_{\mathscr{F}}(v_{a_1}, \dots, v_{a_{h_j}})/p^{h_j} =$

$p^{h_j}\lambda_j/p^{h_j} = \lambda_j$ for $1 \leqslant j \leqslant m$, $1 \leqslant a_1 < \cdots < a_{h_j} \leqslant n$, $1 \leqslant b_1, \dots, b_{h_j} \leqslant p$. Therefore $\mathscr{G} \cong \Lambda K^H_{n\times p}$, and the i^{th} color class induces an r_i-factor for $1 \leqslant i \leqslant k$. □

In particular, if $m = 1$, $h := h_1$, $\lambda_1 = 1$, $r := r_1 = \cdots = r_k$, then Theorem 6.2 implies another one of Baranyai's theorems: the complete h-uniform n-partite hypergraph $K^h_{n\times p}$ is r-factorizable if and only if $h \mid npr$ and $r \mid \binom{n-1}{h-1}p^{h-1}$.

Let $J^T_k = [1 \dots 1]_{1\times k}$. For two column vectors $Q = [q_1 \dots q_k]^T$, $R = [r_1 \dots r_k]^T$, if $q_i \leqslant r_i$ for $1 \leqslant i \leqslant k$, we say that $Q \leqslant R$. For a hypergraph $\mathscr{G}$, a (q, r)-factor is a spanning sub-hypergraph in which

$$q \leqslant d(v) \leqslant r \text{ for each } v \in V(\mathscr{G}).$$

A (Q, R)-*factorization* is a partition $\{F_1, \dots, F_k\}$ of $E(\mathscr{G})$ in which F_i is a (q_i, r_i)-factor for $1 \leqslant i \leqslant k$. An *almost k-factor* of $\mathscr{G}$ is $(k-1, k)$-factor. An *almost R-factorization* is an $(R - J_k, R)$-factorization. The proof of the following theorems are very similar to those of Theorem 6.1 and 6.2.

Theorem 6.3. *ΛK^H_n is (Q, R)-factorizable if and only if $s(Q) \leqslant \sum_{i=1}^m \lambda_i\binom{n-1}{h_i-1} \leqslant s(R)$, and there exists a non-negative integer matrix $A = [a_{ij}]_{k\times m}$ such that $nQ \leqslant AH \leqslant nR$, and $s(A_j) = \lambda_j\binom{n}{h_j}$ for $1 \leqslant j \leqslant m$.*

Proof. To prove the necessity, suppose that ΛK^H_n is (Q, R)-factorizable. Since ΛK^H_n is $\sum_{i=1}^m \lambda_i\binom{n-1}{h_i-1}$-regular, we must have $s(Q) = \sum_{i=1}^k q_i \leqslant \sum_{i=1}^m \lambda_i\binom{n-1}{h_i-1} \leqslant \sum_{i=1}^k r_i = s(R)$. Since for $1 \leqslant j \leqslant m$, each edge of size h_j contributes h_j to the the sum of the degrees of the vertices in (q_i, r_i)-factor for $1 \leqslant i \leqslant k$, there must exist non-negative integers a_{ij}, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$, such that $nq_i \leqslant \sum_{j=1}^m a_{ij}h_j \leqslant nr_i$ for $1 \leqslant i \leqslant k$ and $\sum_{i=1}^k a_{ij} = \lambda_j\binom{n}{h_j}$ for $1 \leqslant j \leqslant m$.

To prove the sufficiency, let $\mathscr{F}$ be a hypergraph consisting of a single vertex v with $m_{\mathscr{F}}(v^{h_j}) = \lambda_j\binom{n}{h_j}$ for $1 \leqslant j \leqslant m$. Note that $\mathscr{F}$ is an amalgamation of ΛK^H_n. Now we color the edges of $\mathscr{F}$ so that $m_{\mathscr{F}(i)}(v^{h_j}) = a_{ij}$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$. This can be done, because:

$$\sum_{i=1}^k m_{\mathscr{F}(i)}(v^{h_j}) = \sum_{i=1}^k a_{ij} = \lambda_j\binom{n}{h_j} = m_{\mathscr{F}}(v^{h_j}) \quad \text{for } 1 \leqslant j \leqslant m.$$

Moreover,

$$nq_i \leqslant d_{\mathscr{F}(i)}(v) = \sum_{j=1}^m a_{ij}h_j \leqslant nr_i \quad \text{for } 1 \leqslant i \leqslant k.$$

Let $g : V(\mathscr{F}) \to \mathbb{N}$ be a function so that $g(v) = n$. Since for $1 \leqslant i \leqslant m$, $h_i \leqslant n$, g is simple. By Theorem 4.1, there exists a simple g-detachment $\mathscr{G}$ of $\mathscr{F}$ with n vertices, say $v_1, \dots, v_n$ such that by (A2), $q_i = nq_i/n \leqslant d_{\mathscr{G}(i)}(v_j) \leqslant nr_i/n = r_i$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$, and by (A3), for each $U \subset \{v_1, \dots, v_n\}$ with $|U| = h_j$, $m_{\mathscr{G}}(U) \approx m_{\mathscr{F}}(v^{h_j})/\binom{n}{h_j} = \lambda_j\binom{n}{h_j}/\binom{n}{h_j} = \lambda_j$ for $1 \leqslant j \leqslant m$. Therefore $\mathscr{G} \cong \Lambda K^H_n$, and the i^{th} color class induces a (q_i, r_i)-factor for $1 \leqslant i \leqslant k$. □

Theorem 6.4. *ΛK^H_n is almost R-factorizable if and only if $s(R) - k \leqslant \sum_{i=1}^m \lambda_i\binom{n-1}{h_i-1} \leqslant s(R)$, and there exists a non-negative integer matrix $A = [a_{ij}]_{k\times m}$ such that $n(R - J_k) \leqslant AH \leqslant nR$, and $s(A_j) = \lambda_j\binom{n}{h_j}$ for $1 \leqslant j \leqslant m$.*

Proof. It is enough to take $Q = R - J_k$ in Theorem 6.3. □

Theorem 6.5. *$\Lambda K^H_{n\times p}$ is (Q,R)-factorizable if and only if $s(Q) \leqslant \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1} \leqslant s(R)$, and there exists a non-negative integer matrix $A = [a_{ij}]_{k\times m}$ such that $npQ \leqslant AH \leqslant npR$, and $s(A_j) = \lambda_j \binom{n}{h_j} p^{h_j}$ for $1 \leqslant j \leqslant m$.*

Proof. To prove the necessity, suppose that $\Lambda K^H_{n\times p}$ is (Q,R)-factorizable. Since $\Lambda K^H_{n\times p}$ is $\sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1}$-regular, we must have $s(Q) = \sum_{i=1}^k q_i \leqslant \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1} \leqslant \sum_{i=1}^k r_i = s(R)$. Moreover, there must exist non-negative integers a_{ij}, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$, such that $npq_i \leqslant \sum_{j=1}^m a_{ij} h_j \leqslant npr_i$ for $1 \leqslant i \leqslant k$ and $\sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j} p^{h_j}$ for $1 \leqslant j \leqslant m$.

To prove the sufficiency, let $\Lambda^p = [p^{h_i}\lambda_i]^T_{1\times m}$, and let $\mathscr{F} = \Lambda^p K^H_n$ with vertex set $V = \{v_1, \dots, v_n\}$. Notice that $\mathscr{F}$ is an amalgamation of $\Lambda K^H_{n\times p}$. By Theorem 6.3, $\mathscr{F}$ is (pQ, pR)-factorizable. Therefore, we can color the edges of $\mathscr{F}$ so that

$$pq_i \leqslant d_{\mathscr{F}(i)}(v) \leqslant pr_i \text{ for } v \in V, 1 \leqslant i \leqslant k.$$

Let $g : V \to \mathbb{N}$ be a function so that $g(v) = p$ for $v \in V$. Since $p \geqslant 1$, g is simple. By Theorem 4.1, there exists a simple g-detachment $\mathscr{G}$ of $\mathscr{F}$ with np vertices, say v_i is detached to $v_{i1}, \dots, v_{ip}$ for $1 \leqslant i \leqslant n$, such that by (A2), $q_i = pq_i/p \leqslant d_{\mathscr{G}(i)}(v_{ab}) \leqslant pr_i/p = r_i$ for $1 \leqslant i \leqslant k$, $1 \leqslant a \leqslant n$, $1 \leqslant b \leqslant p$, and by (A3), $m_{\mathscr{G}}(v_{a_1b_1}, \dots, v_{a_{h_j}b_{h_j}}) \approx m_{\mathscr{F}}(v_{a_1}, \dots, v_{a_{h_j}})/p^{h_j} = p^{h_j}\lambda_j/p^{h_j} = \lambda_j$ for $1 \leqslant j \leqslant m$, $1 \leqslant a_1 < \cdots < a_{h_j} \leqslant n$, $1 \leqslant b_1, \dots, b_{h_j} \leqslant p$. Therefore $\mathscr{G} \cong \Lambda K^H_{n\times p}$, and the i^{th} color class induces a (p_i, r_i)-factor for $1 \leqslant i \leqslant k$. □

Theorem 6.6. *$\Lambda K^H_{n\times p}$ is almost R-factorizable if and only if $s(R)-k \leqslant \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1} \leqslant s(R)$, and there exists a non-negative integer matrix $A = [a_{ij}]_{k\times m}$ such that $np(R - J_k) \leqslant AH \leqslant npR$, and $s(A_j) = \lambda_j \binom{n}{h_j} p^{h_j}$ for $1 \leqslant j \leqslant m$.*

Proof. It is enough to take $Q = R - J_k$ in Theorem 6.5. □

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