

DISTRIBUTIONS OF ORDER PATTERNS OF INTERVAL MAPS

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ABSTRACT. A permutation σ describing the relative orders of the first n iterates of a point x under a self-map f of the interval $I = [0, 1]$ is called an *order pattern*. For fixed f and n , measuring the points $x \in I$ (according to Lebesgue measure) that generate the order pattern σ gives a probability distribution $\mu_n(f)$ on the set of length n permutations. We study the distributions that arise this way for various classes of functions f .

Our main results treat the class of measure preserving functions. We obtain an exact description of the set of realizable distributions in this case: for each n this set is a union of open faces of the polytope of flows on a certain digraph, and a simple combinatorial criterion determines which faces are included. We also show that for general f , apart from an obvious compatibility condition, there is no restriction on the sequence $\{\mu_n(f)\}_{n=1,2,\dots}$.

In addition, we give a necessary condition for f to have *finite exclusion type*, i.e., for there to be finitely many order patterns that generate all order patterns not realized by f . Using entropy we show that if f is piecewise continuous, piecewise monotone, and either ergodic or with points of arbitrarily high period, then f cannot have finite exclusion type. This generalizes results of S. Elizalde.

Given a function $f : [0, 1] \rightarrow [0, 1]$, it is natural to examine properties of the sequence of iterates of f beginning at some point $x \in [0, 1]$:

$$x, f(x), f^2(x) \dots$$

The *order pattern* for a sequence of distinct reals y_1, y_2, \dots, y_n is the permutation $\sigma \in S_n$ that ranks the elements in increasing order; specifically, $y_i < y_j$ if and only if $\sigma(i) < \sigma(j)$. A number of authors have explored the relationship between functions f and the set of order patterns realized by the iterates of f . Work of C. Bandt, G. Keller, B. Pompe, J. M. Amigó, M. Kennel, and M. Misiurewicz [1, 2, 3, 9] relates the number of distinct order patterns arising from a function f to the entropy of f . S. Elizalde and others [4, 5, 6] have examined which and how many order patterns do not arise for particular functions and classes of functions.

Here we take a slightly broader view and investigate the collection of distributions of order patterns that particular classes of functions achieve. Specifically, if $I = [0, 1]$ is equipped with Lebesgue measure and f is almost aperiodic (meaning that the set of points with finite orbit has measure zero) then f induces a probability distribution $\mu_n(f)$ on S_n in a natural way:

$$\mu_n(f)(\sigma) = \mu_{\text{Leb}}\{x \mid \text{Order}(x, f(x), \dots, f^{n-1}(x)) = \sigma\}.$$

We shall focus on the functions μ_n as well as the function μ which maps f to the sequence $(\mu_1(f), \mu_2(f), \dots)$.

Throughout the paper we consider functions with the property that almost all orbits are infinite:

$$\mathcal{A} = \{f : I \rightarrow I \mid \mu_{\text{Leb}}(I_{ap}) = 1\}$$

where I_{ap} is the set of aperiodic points, i.e., points with infinite orbit. We address the following natural questions: if $\mathcal{C} \subset \mathcal{A}$ is a collection of functions, then

Question 1. *What is $\mu_n(\mathcal{C})$?*

Question 2. *What is $\mu(\mathcal{C})$?*

We begin by answering both questions for the class $\mathcal{C} = \mathcal{A}$. For any f , the distributions $\mu_n(f)$, $n = 2, 3, \dots$ must satisfy a certain compatibility condition. In Theorem 1.1 we show that this is the only constraint on what is realizable for arbitrary $f \in \mathcal{A}$: that is, for any sequence $\{\mu_n\}_{n \geq 1}$ of compatible distributions on S_n , there is a function $f \in \mathcal{A}$ which simultaneously satisfies $\mu_n(f) = \mu_n$.

We then turn our attention to the class of measure preserving functions,

$$\mathcal{C} = \mathcal{A}^{\text{mp}} = \{f \in \mathcal{A} \mid f \text{ preserves } \mu_{\text{Leb}}\}.$$

Our main theorem (Theorem 5.8) provides a complete answer to Question 1 for $\mathcal{C} = \mathcal{A}^{\text{mp}}$. It is easy to see that the conclusion of Theorem 1.1 cannot hold for $\mathcal{C} = \mathcal{A}^{\text{mp}}$; in fact we observe that $\mu_n(\mathcal{A}^{\text{mp}})$ is contained in a polytope P_n consisting of all (normalized) flows on a certain digraph, which we call a *permutation digraph*. We then show that $\mu_n(\mathcal{A}^{\text{mp}})$ is a union of open faces of P_n including the top-dimensional face, and we give a combinatorial criterion for determining whether or not a given open face of P_n is contained in $\mu_n(\mathcal{A}^{\text{mp}})$.

To prove the main theorem we introduce the fundamental notion of *drift*. Naively, if one wants to construct $f \in \mathcal{A}^{\text{mp}}$ realizing a given distribution μ , one might chop the interval into several subintervals and define f to permute the intervals to produce the desired frequencies. Problems soon arise, however: for example if we want half the mass of the interval to have iterates with order pattern (132) and the other half (213) then we quickly realize that this is impossible, because f^2 would move all the mass to the right, which is impossible for a measure preserving function. This is the essence of drift, and the upshot of Theorem 5.8 is that this is the only obstruction: a face of P_n either has drift or not, and the faces contained in $\mu_n(\mathcal{A}^{\text{mp}})$ are exactly those without drift.

Finally, we discuss the relationship between the entropy of f and a property we call *finite exclusion type*. The latter is equivalent to f having finitely many *basic forbidden patterns*, in the language introduced by Amigó, Elizalde, and Kennel [4]; these properties mean that there are finitely many fixed patterns such that every permutation either arises as an order pattern of iterates of f or contains one of the forbidden patterns. A function with finite entropy can realize at most exponentially many permutations of length N (see [1]), but using the notion of drift we show that quite often, a function with finite exclusion type must realize a super-exponential number of permutations. In particular, if either f is continuous and has points of arbitrarily large period or f is ergodic, then f cannot have both finite entropy and finite exclusion type; see Corollary 6.7. This generalizes results from [5].

The paper is organized as follows. We introduce some language and give our result for $\mathcal{C} = \mathcal{A}$ in Section 1, although we defer the proof to Section 7. Sections 2-4 develop the combinatorial ideas required for our main theorem, including several preliminary results about permutation digraphs and drift. The main theorem is

stated and proved in Section 5. Our discussion of entropy and finite exclusion type makes up Section 6, and Section 7 contains the proof of Theorem 1.1. We close with some open questions in Section 8.

ACKNOWLEDGMENTS

The authors thank Sergi Elizalde for suggesting this line of research and for several useful conversations along the way. Thanks also to Julie Landau for her hospitality and kick serve.

1. GENERALITIES

In this section we introduce some language and notation which will be used throughout the paper, and we state our first result, Theorem 1.1, which says that if no restriction is placed on f , then one can always find f realizing a given compatible sequence of permutation distributions.

Order patterns. For a positive integer n we denote $\{1, \dots, n\}$ by $[n]$ and the group of bijections of $[n]$ by S_n .

Let g be an injective map from a finite totally ordered set $X = \{x_1, \dots, x_n\}$ (where $x_1 < x_2 < \dots < x_n$) to a totally ordered set Y . Let $y_i = g(x_i)$. We define the *order pattern* $\text{Order}(g)$ to be the unique permutation $\sigma \in S_n$ satisfying $y_i < y_j$ if and only if $\sigma(i) < \sigma(j)$. Equivalently $y_{\sigma^{-1}(1)} < \dots < y_{\sigma^{-1}(n)}$. Note that if $\sigma \in S_n$ then $\text{Order}(\sigma) = \sigma$. The *order pattern* of an n -tuple of distinct real numbers (x_1, \dots, x_n) is $\text{Order}(x_1, \dots, x_n) = \text{Order}(g)$ where $g: [n] \rightarrow \mathbb{R}$ takes i to x_i .

There is a restriction map $\rho: S_{n+1} \rightarrow S_n$ given by $\rho(\sigma) = \text{Order}(\sigma|_{[n]})$. Using this we define S_∞ as $\{(\sigma_1, \sigma_2, \dots) \mid \sigma_i \in S_i, \rho(\sigma_{i+1}) = \sigma_i \forall i \geq 1\}$, which is equal to the inverse limit of the maps $\rho: S_{n+1} \rightarrow S_n$. Let $S = \cup_{n=1}^\infty S_n$. The set S is graded by n and we use notation like $S_{\geq n}$ to mean $\cup_{j=n}^\infty S_j$.

Distributions. Next, let Δ_n be the space of probability distributions on S_n . Note that Δ_n is the standard simplex in $\mathbb{R}^{S_n} \cong \mathbb{R}^{n!}$. We denote by χ_σ the vertex of Δ_n which has mass 1 at $\sigma \in S_n$ and 0 elsewhere.

If $\mu \in \Delta_n$ and $\mu' \in \Delta_{n+1}$ we say μ and μ' are *compatible* if $\mu(\sigma) = \sum_{\rho(\sigma')=\sigma} \mu'(\sigma')$.

Then $\Delta_\infty = \{(\mu_1, \mu_2, \dots) \mid \mu_i \in \Delta_i, \mu_{i+1} \text{ and } \mu_i \text{ are compatible } \forall i \geq 1\}$, and $\Delta = \cup_{n=1}^\infty \Delta_n$. As an example, the uniform distributions from each Δ_n form a compatible sequence, hence an element of Δ_∞ .

Induced distributions. For $f \in \mathcal{A}$ and $x \in I_{ap}$ let $\sigma^f(x) = (\sigma_1^f(x), \sigma_2^f(x), \dots) \in S_\infty$ where

$$\sigma_n^f(x) = \text{Order}(x, f(x), \dots, f^{n-1}(x)).$$

Let $\mu_n: \mathcal{A} \rightarrow \Delta_n$ be the map taking a function $f \in \mathcal{A}$ to the distribution defined by

$$\mu_n(f)(\sigma) = \mu_{\text{Leb}}\{x \mid \sigma_n^f(x) = \sigma\}.$$

Note that for any f and n , the distributions $\mu_n(f)$ and $\mu_{n+1}(f)$ are compatible; thus we may define $\mu: \mathcal{A} \rightarrow \Delta_\infty$ by $\mu(f) = (\mu_1(f), \mu_2(f), \dots) \in \Delta_\infty$.

We can now state our first result.

Theorem 1.1. *For every $\mu = (\mu_1, \mu_2, \dots) \in \Delta_\infty$ there exists a function $f \in \mathcal{A}$ with $\mu(f) = \mu$. That is, $\mu(\mathcal{A}) = \Delta_\infty$.*

The proof is constructive, a little involved, and unnecessary for the results that follow. Therefore we defer the proof to Section 7.

Convexity. Before we end this section we make an observation about convexity. Suppose $\mathcal{C} \subset \mathcal{A}$ is a collection of functions such that whenever $f, g \in \mathcal{C}$ and $t \in [0, 1]$, the function

$$h(x) = \begin{cases} tf(\frac{x}{t}) & \text{if } x < t \\ t + (1-t)g(\frac{x-t}{1-t}) & \text{if } t < x \leq 1 \end{cases}$$

is also in \mathcal{C} . Then $\mu_n(\mathcal{C})$ is a convex subset of Δ_n . This is because h is the “block sum” of f scaled by t and g scaled by $1-t$, and so for all n , $\mu_n(h) = t\mu_n(f) + (1-t)\mu_n(g)$.

This will usually hold if \mathcal{C} has “piecewise” in the title, such as piecewise continuous functions, piecewise monotone functions, etc. It also holds for (aperiodic) measure preserving functions.

2. DIGRAPHS

The next several sections develop the language used in the remainder of the paper. We begin with digraphs.

A *digraph* is a quadruple $G = (VG, EG, h, t)$ with VG the vertex set, EG the edge set, and h and t the head and tail maps from EG to VG .

Recall that $\rho : S_{n+1} \rightarrow S_n$ is defined by $\rho(\sigma) = \text{Order}(\sigma|_{[n]})$. Similarly define $\rho' : S_{n+1} \rightarrow S_n$ by $\rho'(\sigma) = \text{Order}(\sigma|_{[2, n+1]})$.

Definition 2.1. For $n \geq 1$ let G_n denote the *permutation digraph* $(S_n, S_{n+1}, \rho, \rho')$. The digraphs G_2 and G_3 are shown in Figure 1.

Paths. A *path* of length ℓ (where $0 \leq \ell < \infty$) in a digraph is an alternating sequence $p = (v_0, e_1, v_1, e_2, \dots, v_\ell)$ with $v_i \in VG$ and $e_i \in EG$ such that $h(e_i) = v_{i-1}$ and $t(e_i) = v_i$. A path of length ∞ is $p = (v_0, e_1, v_1, \dots)$ such that each finite initial segment ending with a vertex is a (finite) path. We write $\text{Path}_\ell(G)$ for the set of all paths of length ℓ in G and $\text{Path}(G)$ for the set of all paths in G . Note that $S_n = \text{Path}_0(G_n)$. To define specific paths we sometimes abuse notation slightly by thinking of v_i and e_i as functions from $\text{Path}_{\geq i}(G)$ to VG and EG .

For example, if p is a path of finite length ℓ and q is any path with $v_0(q) = v_\ell(p)$ then the *concatenation* pq of p and q has $v_i(pq) = v_i(p)$ and $e_i(pq) = e_i(p)$ for $i \leq \ell$ and $v_i(pq) = v_{i-\ell}(q)$ and $e_i(pq) = e_{i-\ell}(q)$ for $i > \ell$.

A digraph is *strongly connected* if there are paths connecting every ordered pair of vertices. A (finite) path is *embedded* if all $\ell + 1$ vertices are distinct, except possibly $v_0 = v_\ell$. A *loop* is a finite path with $v_0 = v_\ell$.

Projections. For each $n < \infty$ we define

$$\pi_n : S_\infty \cup \left(\bigcup_{m \geq n} \text{Path}(G_m) \right) \rightarrow \text{Path}(G_n)$$

as follows. First, π_n is the identity on $\text{Path}(G_n)$. if $\sigma \in \text{Path}_0(G_{n+1}) = S_{n+1}$, let $\pi_{n+1, n}(\sigma)$ be the path $(\rho(v), v, \rho'(v)) \in \text{Path}_1(G_n)$. If $p = (v_0, e_0, \dots, v_\ell) \in \text{Path}_\ell(G_{n+1})$ then let $\pi_{n+1, n}(p)$ be the concatenation $\pi_{n+1, n}(v_0)\pi_{n+1, n}(v_1) \cdots \pi_{n+1, n}(v_\ell)$. (The result is an infinite path if $\ell = \infty$; otherwise the result is a path of length $\ell + 1$.) Thus $\pi_{n+1, n} : \text{Path}(G_{n+1}) \rightarrow \text{Path}(G_n)$. Let $\pi_{m, n} = \pi_{n+1, n} \circ \cdots \circ$

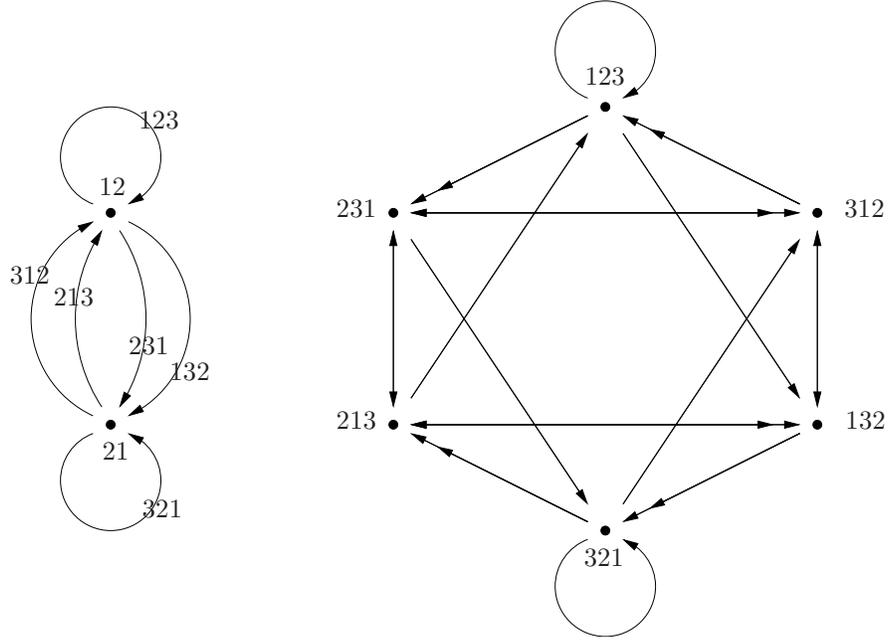


FIGURE 1. The digraphs G_2 and G_3 . The edges of G_2 are shown with labels; the edges of G_3 are abbreviated and the labels omitted. For instance two directed edges go from 231 to 312, with labels 2413 and 3412. An edge labeled 4231 goes in the reverse direction.

$\pi_{m,m-1} : \text{Path}(G_m) \rightarrow \text{Path}(G_n)$ and let π_n be the union of the functions $\pi_{m,n}$ on $\bigcup_{m \geq n} \text{Path}(G_m)$.

Finally, extend π_n further by defining $\pi_n(\sigma)$ for $\sigma = (\sigma_1, \sigma_2, \dots) \in S_\infty$ to be the infinite path whose initial subpath of length ℓ is equal to $\pi_n(\sigma_{\ell+n})$.

Note: if $p \in \text{Path}_\ell(G_m)$ then $\pi_n(p) \in \text{Path}_{\ell+m-n}(G_n)$.

Lifts. The next lemma says that any path (of length > 0) on G_n can be lifted to G_{n+1} (where it becomes shorter if its length is finite). Note however that the edges of the lift are not determined; only the vertices are determined, because the edges of p do not appear in the definition of $\pi_{n+1,n}(p)$. The ambiguity in the lifting process will play an important role later.

Lemma 2.2 (Path lifting). *The map $\pi_{m,n}$ is surjective. The image of $\pi_n|_{S_\infty}$ is $\text{Path}_\infty(G_n)$.*

Proof. For the first part, it suffices to show that $\pi_{n+1,n}$ is surjective, as $\pi_{m,n}$ is a composition of maps of this form. If $p \in \text{Path}(G_n)$ then each edge of p is a vertex of G_{n+1} . We only need to show that if $e, e' \in EG_n = VG_{n+1}$ with $t(e) = h(e')$ then there is an edge $f \in EG_{n+1} = S_{n+2}$ with $h(f) = e$ and $t(f) = e'$. Extend the function $e : [n+1] \rightarrow [n+1]$ to $\bar{e} : [n+2] \rightarrow \mathbb{R}$ by defining $e(n+2)$ such that $e' = \text{Order}(e|_{[2,n+2]})$. Then $f = \text{Order}(e) \in S_{n+2}$ is the desired edge.

For the second part, if $p \in \text{Path}_\infty(G_n)$ then set $p = p_0$ and for each $i > 0$ let $p_i \in \text{Path}_\infty(G_{n+i})$ be a lift of p_{i-1} . Then for $m \geq n$ let σ_m be the initial vertex of

p_{m-n} , and note that $\sigma = (\sigma_1, \sigma_2, \dots) \in S_\infty \cap (\pi_n)^{-1}(p)$. \square

Example 2.3. Consider the infinite path $p \in \text{Path}_\infty(G_3)$ that begins at the vertex (12) and traverses the edges (132) followed by (312) repeatedly. Then p projects to the path $q = \pi_2(p) \in \text{Path}_\infty(G_2)$ which traverses the loop labeled (12) and then the loop labeled (21) and then repeats. There are infinitely many paths other than p in $\pi_3((\pi_2)^{-1}(q))$, since the vertices must alternate between (12) and (21) but there are two choices for each edge. By contrast, at the next step, $\pi_4((\pi_3)^{-1}(p))$ is the singleton consisting of the infinite path on G_4 that starts at the vertex (312) and traverses the edges (1423) and (4132) repeatedly. In fact $(\pi_3)^{-1}(p) \subseteq S_\infty$ is already a singleton, being the compatible sequence $(\sigma_1, \sigma_2, \dots)$ where σ_n is the permutation $(1, n, 2, n-1, \dots)$.

3. THE POSET OF A PATH

Given a path $p = (v_0, e_1, v_1, \dots, v_\ell)$ on G_n , consider the set

$$\overline{Q}_p = (\{v\} \times [0, \ell] \times [n]) \cup (\{e\} \times [1, \ell] \times [n+1]).$$

This set is (in 1-1 correspondence with) the disjoint union of the domains of all the permutations v_i and e_i . They are ‘‘patched together’’ by the equivalence \sim generated by

- (i) $(v, a, c) \sim (e, a+1, c)$
- (ii) $(v, a, c) \sim (e, a, c+1)$.

The equivalence class of (v, a, c) in $Q_p = \overline{Q}_p / \sim$ will be denoted by $x_{a+c}(p)$, or x_{a+c} if the path p is understood; note that (a) this is well-defined and (b) every element of Q_p is equal to x_i for some $1 \leq i \leq \ell + n := m$. By (a), if $x_i = x_j$ for $1 \leq i, j \leq m$ then $i = j$, and so $Q_p = \{x_1, \dots, x_m\}$.

The set Q_p is easy to visualize, but let us first define a partial ordering on it.

Consider the relation \leq on Q_p generated by

- (iii) $[(v, a, c)] \leq [(v, a, d)]$ if $v_a(c) \leq v_a(d)$
- (iv) $[(e, a, c)] \leq [(e, a, d)]$ if $e_a(c) \leq e_a(d)$

and extended by transitivity.

We will show in a moment that \leq is a partial ordering on Q_p . The point of Q_p is to keep track of all order relationships which necessarily hold among $\sigma(1), \dots, \sigma(m)$, if σ is a permutation in $\pi_n^{-1}(p)$.

Example 3.1. Consider the path p of length 5 in G_3 with edges $e_1 = (2134)$, $e_2 = (1342)$, $e_3 = (2314)$, $e_4 = (3241)$, $e_5 = (2314)$. This is a loop based at (213). Attempts to construct real numbers z_1, \dots, z_8 such that $\text{Order}(z_i, z_{i+1}, z_{i+2}, z_{i+3}) = e_i$ quickly lead one to draw pictures like Figure 2. The top picture is a plot of the desired y 's. Note that y_5 could be perturbed to be larger or smaller than y_1 , and similarly for y_7 and y_2 . The dotted lines indicate the duration of the influence of y_i on future y_j 's. This information is abstracted in the middle picture, in which the dots are the elements of \overline{Q}_p and equivalent elements are joined by an arc. Each arc is an element of Q_p . The bottom picture shows the partial ordering: an edge pointing from x_i to x_j indicates that $x_i \leq x_j$.

Lemma 3.2. *Let p be a path on G_n , and let $\tilde{p} \in \pi_n^{-1}(p)$. If $x_i(p) \leq x_j(p)$ then $x_i(\tilde{p}) \leq x_j(\tilde{p})$.*

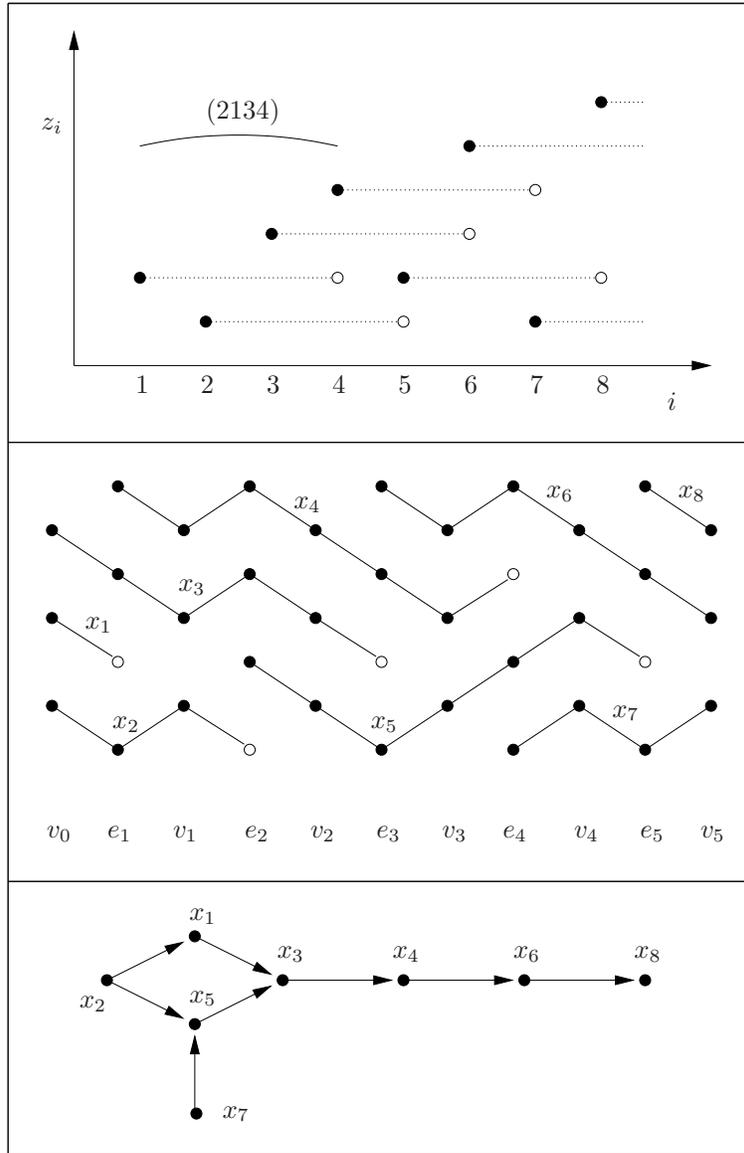


FIGURE 2. The path p is a loop of length 5 in G_3 traversing in this order the edges 2134, 1342, 2314, 3241, 2314. Figure (a) is a plot of a sequence z_i (for $i = 1, \dots, 8$) which maps to p under π_3 . The elements of \overline{Q}_p , shown in (b) as dots, are the intersections of the above plot with vertical lines at $i = 3, 3.5, 4, \dots, 8$. The elements $(v, 0, 3), (v, 0, 1), (v, 0, 2)$ of \overline{Q}_p make up the leftmost column of dots (read from top to bottom). Equivalent elements are joined by an arc. Figure (c) depicts the poset Q_p whose elements are the arcs in either of the previous pictures.

Proof. We may assume the length ℓ is not zero. It suffices to prove for $\tilde{p} \in \text{Path}(G_{n+1})$, as we can lift multiple times.

The hypothesis $x_i(p) \leq x_j(p)$ implies there is a sequence

$$(1) \quad x_i \ni (y_0, a_0, c_0) \sim (y_1, a_1, c_1) \leq (y_2, a_2, c_2) \sim \cdots \leq (y_r, a_r, c_r) \in x_j$$

in \overline{Q}_p , where each step is one of the types (i)-(iv). In this sequence, if one of the inequalities has $y_i = y_{i+1} = v$ then $a_i = a_{i+1}$ and using (i) and (ii) we can replace y_i and y_{i+1} by e and either increase a_i and a_{i+1} by 1 (if $a_i \neq \ell$) or increase c_i and c_{i+1} by 1 (if $a_i = 0$). Since $\rho(e_i) = v_{i-1}$ and $\rho'(e_i) = v_i$, the inequality is preserved in either case. Thus we obtain a new sequence (1) with each $y_i = e$.

Now note that the elements (e, a, c) of \overline{Q}_p are in one-one correspondence with the elements (v, a, c) of $\overline{Q}_{\tilde{p}}$. Thus if we now switch every e to a v and subtract 1 from each a , we obtain a sequence in $\overline{Q}_{\tilde{p}}$ showing $x_i(\tilde{p}) \leq x_j(\tilde{p})$. \square

Corollary 3.3. *The relation \leq is a partial order on Q_p .*

Proof. The relation is reflexive and transitive by definition. We must show that if $x_i \leq x_j$ and $x_j \leq x_i$ then $i = j$. Lift p to a path $v_0 = \sigma \in \text{Path}_0(G_m) = S_m$ (where $m = \ell + n$). There are no equivalences in \overline{Q}_σ , and $x_i \leq x_j$ in the poset Q_σ if and only if $\sigma(i) \leq \sigma(j)$. By Lemma 3.2, $x_i \leq x_j$ and $x_j \leq x_i$ in Q_σ , so $i = j$. \square

Remark 3.4. Note that for any $1 \leq i \leq \ell$, the elements $x_i, x_{i+1}, \dots, x_{i+n}$ are totally ordered in Q_p .

Let $\psi(v) = 0$ and $\psi(e) = -1/2$. If $x_i \leq x_j$ in Q_p then there is a sequence (1) with each $(y_k, a_k, c_k) \in \overline{Q}_p$. Call such a sequence *monotonic* if the function $\psi(y_k) + a_k$ is monotonic in k .

Lemma 3.5. *If $x_i \leq x_j$ in Q_p then there is a monotonic sequence of the form (1).*

Proof. Choose a sequence of the form (1); one exists by definition. Note that only rules (1) and (2) change ψ , and that ψ increases (by $1/2$) if either of these rules is applied by replacing the left side with the right. Suppose the given sequence is not monotonic. Specifically suppose that ψ increases and later decreases; the other case is virtually identical. Choosing an innermost such backtrack, we find a subsequence of one of the following two forms:

- (i) $(v, a, c) \sim (e, a+1, c) \leq (e, a+1, d) \sim (v, a, d)$
- (ii) $(e, a, c+1) \sim (v, a, c) \leq (v, a, d) \sim (e, a, d+1)$

In case (i), we have $e_{a+1}(c) \leq e_{a+1}(d)$. But $v_a = \rho(e_{a+1})$ so we also have $v_a(c) \leq v_a(d)$. Thus we can delete the middle two terms of (i) and eliminate the backtracking.

Case (ii) is similar: $v_a(c) \leq v_a(d)$ but this time $v_a = \rho'(e_a)$. Now it follows that $e_a(c+1) \leq e_a(d+1)$ so again we can eliminate the backtracking. \square

Referring again to Figure 2, Lemma 3.5 says that it is very easy to determine whether $x_i \leq x_j$. If $i < j$, one just sees whether it is possible to connect the right endpoint of x_i to any point above x_j with a path that passes the vertical line test. If not, then $x_i \leq x_j$.

Lemma 3.6. *Let p, q be paths on G_n of lengths ℓ, ℓ' such that the concatenation pq is defined. For $1 \leq i, j \leq \ell' + n$, if $x_i(q) \leq x_j(q)$ then $x_{\ell+i}(pq) \leq x_{\ell+j}(pq)$.*

Proof. Choose a monotone sequence of the form (1), and add ℓ to the second coordinate of each term. The new sequence proves the result. \square

Remark 3.7. Let p be a path of length ℓ on G_n , and let $m = n + \ell$. A choice of lift $\sigma \in \pi_{m,n}^{-1}(p)$ amounts to a choice of extension of \leq to a total order on $\{x_1, \dots, x_m\}$. That this can be done is well-known; the process is sometimes called a “topological sort.” In particular, for a subset $\{i_1, \dots, i_k\} \subset [m]$ of indices, if the x_{i_j} are pairwise incomparable in Q_p then for any permutation $\nu \in S_k$ there is an extension of \leq to a total order on $\{x_1, \dots, x_m\}$ satisfying $x_{i_{\nu(1)}} < \dots < x_{i_{\nu(k)}}$. In the terminology of lifts this becomes the following statement, which bears on the discussion of entropy in a later section.

Corollary 3.8. *Let p be a path of length ℓ on G_n , and let $m = \ell + n$. If the elements $\{x_{i_1}, \dots, x_{i_k}\}$ of Q_p are pairwise incomparable, then for any permutation $\nu \in S_k$ there is a lift $\sigma \in S_m \subset \pi_n^{-1}(p)$ such that $\sigma(i_{\nu(1)}) < \dots < \sigma(i_{\nu(k)})$. In particular $|\pi_n^{-1}(p) \cap S_m| \geq k!$.*

As a special case of this we also note the following.

Corollary 3.9. *Let p be a path of length ℓ on G_n . Then $x_i \leq x_j$ if and only if $\sigma(i) \leq \sigma(j)$ for every $\sigma \in S_m \cap \pi_n^{-1}(p)$ (where necessarily $m = \ell + n$).*

4. DRIFT

If γ is a loop of length $\ell < \infty$ on G_n , then the elements x_1, \dots, x_n of Q_p are totally ordered, as are the elements $x_{\ell+1}, \dots, x_{\ell+n}$, and if we set $y_i = x_{\ell+i}$, then we have $x_i \leq x_j$ if and only if $y_i \leq y_j$, for all $i, j \in [n]$. The notion of drift is based on how the x_i compare to the y_j , as measured by the following two functions. Let $\langle n \rangle$ be the totally ordered set $[n] \cup \{-\infty, \infty\}$ (with $-\infty < 1$ and $n < \infty$), and for $i \in \langle n \rangle$ define

$$\text{Max}_\gamma(i) = \begin{cases} j & \text{if } x_i \leq y_j, \text{ and for } k \in [n], x_i \leq y_k \text{ implies } y_j \leq y_k \\ \infty & \text{if } i = \infty \text{ or } i \in [n] \text{ and there is no } j \text{ such that } x_i \leq y_j \\ -\infty & \text{if } i = -\infty \end{cases}$$

$$\text{Min}_\gamma(i) = \begin{cases} j & \text{if } x_i \geq y_j, \text{ and for } k \in [n], x_i \geq y_k \text{ implies } y_j \geq y_k \\ -\infty & \text{if } i = -\infty \text{ or } i \in [n] \text{ and there is no } j \text{ such that } x_i \geq y_j \\ \infty & \text{if } i = \infty \end{cases}$$

Lemma 4.1. *If $x_i \leq x_j$ then $\text{Max}_\gamma(i) \leq \text{Max}_\gamma(j)$ and $\text{Min}_\gamma(i) \leq \text{Min}_\gamma(j)$.*

Proof. This is immediate from the definitions, and from the fact that $x_i \leq x_j$ if and only if $y_i \leq y_j$. \square

Lemma 4.2. *Suppose p and q are finite paths such that pq is a path. Then $\text{Max}_{pq} = \text{Max}_q \text{Max}_p$ and $\text{Min}_{pq} = \text{Min}_q \text{Min}_p$.*

Proof. We give the verification for Max. For Min, flip the argument upside down.

Let ℓ be the length of p and ℓ' the length of q .

It is clear that the two functions are equal on $\pm\infty$. Let $i \in [n]$, let $j = \text{Max}_p(i)$. If $j = \infty$ then by Lemma 3.5 it is impossible to have $x_i \leq x_{\ell+\ell'+k}$ for any $k \in [n]$, so

$\text{Max}_{pq}(i) = \infty$. We may therefore assume $j \neq \infty$, and let $k = \text{Max}_q(j)$. If $k = \infty$, then again by Lemma 3.5 it is impossible to have $x_i \leq x_{\ell+\ell'+k'}$ for any $k' \in [n]$, so $\text{Max}_{pq}(i) = \infty$. Thus we may assume $k \neq \infty$. We want to show that $\text{Max}_{pq}(i) = k$.

In the poset Q_{pq} , we have $x_i \leq x_{\ell+j} \leq x_{\ell+\ell'+k}$. Also, if $x_i \leq x_{\ell+\ell'+k'}$ then there is a monotonic sequence in \overline{Q}_{pq} showing $x_i \leq x_{\ell+\ell'+k'}$. This sequence must contain a point of the form (v, ℓ, j') , so $x_i \leq x_{\ell+j'}$. By definition of j , we have $x_{\ell+j} \leq x_{\ell+j'}$, hence $x_{\ell+j} \leq x_{\ell+\ell'+k'}$. By definition of k we now have $x_{\ell+\ell'+k} \leq x_{\ell+\ell'+k'}$. Thus $k = \text{Max}_{pq}(i)$, as desired. \square

Let γ be a loop of length ℓ on G_n . For $i, j \in [n]$ let

$$\text{Drift}_\gamma(i, j) = \begin{cases} + & \text{if } x_i \leq y_j \\ - & \text{if } x_i \geq y_j \\ 0 & \text{otherwise.} \end{cases}$$

We will write $\text{Drift}_\gamma(i)$ for $\text{Drift}_\gamma(i, i)$.

Definition 4.3. A loop γ is *partially driftless* if $\text{Drift}_\gamma(i) = 0$ for some $i \in [n]$.

A loop γ is *driftless* if $\text{Drift}_\gamma(i) = 0$ for all $i \in [n]$.

A loop γ is *totally driftless* if $\text{Drift}_\gamma(i, j) = 0$ for all $i, j \in [n]$.

Thus γ is totally driftless if and only if $\text{Max}_\gamma(i) = \infty$ and $\text{Min}_\gamma(i) = -\infty$ for all $i \in [n]$, and there is a similar description of driftless and partially driftless loops.

Example 4.4. The loop p in Figure 2 is partially driftless. In Q_p , we have $x_1 \leq x_6 = y_1$ and $x_3 \leq x_8 = y_3$, so $\text{Drift}_p(1) = \text{Drift}_p(3) = +$. However x_2 and $x_7 = y_2$ are incomparable, so $\text{Drift}_p(2) = 0$. Note that the number z_6 is necessarily greater than z_1 , but z_7 can be chosen to be greater than or less than z_2 .

Lemma 4.5. *Let β and γ be (partially driftless) loops based at v , with $\text{Drift}_\beta(j) = \text{Drift}_\gamma(j) = 0$. Then $\text{Drift}_{\beta\gamma}(j) = 0$.*

Proof. Let ℓ be the length of β . Suppose $x_j(\beta\gamma) \leq y_j(\beta\gamma)$. Then there is a monotonic sequence (1) proving this. In the sequence there must be a representative of x_i for some $\ell + 1 \leq i \leq \ell + n$. Now $x_i = x_{\ell+j}$ or $x_i \leq x_{\ell+j}$ would contradict $\text{Drift}_\beta(j) = 0$, and $x_i \geq x_{\ell+j}$ would contradict $\text{Drift}_\gamma(j) = 0$. By Remark 3.4 these are the only possibilities. Thus $\text{Drift}_{\beta\gamma}(j) \neq +$. Similarly $\text{Drift}_{\beta\gamma}(j) \neq -$. \square

Lemma 4.6. *Let β and γ be loops on G_n based at the vertex v . If γ is totally driftless then $\beta\gamma$ is totally driftless.*

Proof. Suppose not; then without loss of generality there exists $i \in [n]$ with $\text{Max}_{\beta\gamma}(i) = j < \infty$. Thus $x_i(\beta\gamma) \leq y_j(\beta\gamma)$ and by Lemma 3.5 there is a monotonic sequence proving this inequality. This sequence must contain (v, ℓ, k) for some k , where ℓ is the length of β . Starting there, the remainder of the sequence (in combination with Lemma 3.6) shows that $\text{Max}_\gamma(k) \leq j < \infty$, a contradiction. \square

Lemma 4.7. *Cyclic permutations of driftless loops are driftless.*

Proof. Let $\gamma = (v_0, e_1, \dots, v_\ell)$ be a driftless loop and let γ_k be a cyclic permutation of γ starting at v_k . Suppose $x_i(\gamma_k) \leq x_{\ell+i}(\gamma_k)$. Fix a monotonic sequence showing

this inequality, and add ℓ to the second coordinate of each element to obtain a new sequence, and concatenate the original sequence with the new one. This longer sequence shows $x_i(\gamma_k^2) \leq x_{2\ell+i}(\gamma_k^2)$ but it contains a subsequence showing for some j that $\text{Drift}_\gamma(j) \neq 0$. \square

Definition 4.8. A *face subgraph* of G_n is a subgraph H such that every edge of H is contained in a loop in H . Equivalently H is a face subgraph if each connected component of H is strongly connected.

Definition 4.9. A strongly connected subgraph $H \subseteq G_n$ *drifts* if there exist $v \in VH$, $j \in [n]$ and $\epsilon \in \{+, -\}$ such that for every loop γ in H based at v , $\text{Drift}_\gamma(j) = \epsilon$. Otherwise H is *driftless*.

A face subgraph $H \subseteq G_n$ *drifts* if any of its connected components drifts; otherwise H is driftless.

Proposition 4.10. *Let H be a strongly connected subgraph of G_n . The following are equivalent:*

- (1) H is driftless;
- (2) there exists a totally driftless loop γ with support contained in H ;
- (3) there exists a totally driftless loop γ with support equal to H .

Proof. The last two statements are equivalent by Lemma 4.6: if γ is a totally driftless loop with support contained in H , and β is any loop with support equal to H , then $\beta\gamma$ is a totally driftless loop with support equal to H .

Statement (3) easily implies statement (1): for fixed v, j, ϵ let γ_v be a cyclic permutation of γ which starts at v . By Lemma 4.7 $\text{Drift}_{\gamma_v}(j) = 0 \neq \epsilon$.

Last, we show (1) implies (2). Let γ_0 be a loop based at σ and supported in H . Let $i = \sigma^{-1}(1)$ and $j = \sigma^{-1}(n)$, so that $x_i \leq x_k \leq x_j$ for all $k \in [n]$.

Suppose $\text{Max}_{\gamma_0}(i) = k \neq \infty$. As H is driftless, we may pick a loop γ_1 based at v and supported in H such that $\text{Drift}_{\gamma_1}(k) \neq +$. $\text{Max}_{\gamma_0\gamma_1}(i) > k$. We can continue this process until we have a loop β_0 with $\text{Max}_{\beta_0}(i) = \infty$.

Then, similarly, we concatenate loops on to the end of β_0 to create a loop β with $\text{Min}_\beta(j) = -\infty$. Note that $\text{Max}_\beta(i) = \infty$ (by Lemma 4.2). Now, by Lemma 4.1 β is totally driftless, with support contained in H . \square

Corollary 4.11. *If K and H are strongly connected, $K \subseteq H$, and K is driftless, then H is driftless.*

5. MEASURE PRESERVING FUNCTIONS

In this section we analyze the distributions of order patterns arising from (almost aperiodic) measure preserving functions

$$\mathcal{A}^{\text{mp}} = \{f \in \mathcal{A} \mid \mu_{\text{Leb}}(f^{-1}(S)) = \mu_{\text{Leb}}(S) \text{ for all measurable sets } S\}.$$

Our main theorem is that the image $\mu_n(\mathcal{A}^{\text{mp}})$ is a union of open faces of a polytope $P_n \subset \Delta_n$ of dimension $n! - (n-1)!$, and that there is an easily checkable combinatorial criterion for determining whether a particular face of P_n is in the image.

Remark 5.1. For most of these results it is not essential that μ_{Leb} be the measure preserved by f . That is, given a function $f \in \mathcal{A}$ one could choose an invariant measure λ and proceed with this section, everywhere replacing μ_{Leb} with λ . For some steps it may be necessary to assume λ has no atoms.

We start by observing that Theorem 1.1 would not hold if \mathcal{A} were replaced by \mathcal{A}^{mp} . If $\sigma \in S_n$ let $\delta_\sigma \in \Delta_n$ denote the distribution whose value is 1 on σ and 0 elsewhere.

Lemma 5.2. *If $J \subseteq I$ has positive measure and $f : J \rightarrow J$ is aperiodic and measure preserving then both $J_+ = \{x \in J \mid f(x) > x\}$ and $J_- = \{x \in J \mid f(x) < x\}$ have positive measure.*

In particular, there is no $f \in \mathcal{A}^{\text{mp}}$ such that $\mu_2(f) = \delta_{(12)}$ or $\delta_{(21)}$.

Proof. Suppose $\mu_{\text{Leb}}(J_-) = 0$, i.e., $f(x) > x$ for almost all $x \in J$. Then there is some ϵ such that $\mu_{\text{Leb}}\{x \mid f(x) - x > \epsilon\} > 0$, hence $\int_J f(x) - x > 0$. But f measure preserving implies $\int_J f(x) - x = 0$. Similarly for $\mu_{\text{Leb}}(J_+)$. \square

Note that $\delta_{(12)}$ and $\delta_{(21)}$ are in the closure of $\mu_2(\mathcal{A}^{\text{mp}})$ since $\mu_2(f_\epsilon)$ can be made arbitrarily close to these distributions by choosing $f_\epsilon(x) = x + \epsilon \pmod 1$.

The flow polytope P_n . Lemma 5.2 notwithstanding, there is a much more serious reason for the failure of Theorem 1.1 in the measure preserving category. For $f \in \mathcal{A}^{\text{mp}}$, there is an additional set of constraints on $\mu(f)$ beyond compatibility of the measures $\mu_n(f)$. Namely, the order pattern of (fx, f^2x, \dots) must be distributed in the same way as the order pattern of (x, fx, f^2x, \dots) . More precisely, if $I_\sigma = \{x \in I \mid \sigma_n^f(x) = \sigma\}$ then $\mu_n(f)(\sigma) = \mu_{\text{Leb}}(I_\sigma) = \mu_{\text{Leb}}(f^{-1}(I_\sigma)) = \mu_{\text{Leb}}(\{x \in I \mid \sigma_n^f(f(x)) = \sigma\})$. Thus if $f \in \mathcal{A}^{\text{mp}}$ we necessarily have

$$(2) \quad \mu_n(f)(\sigma) = \sum_{\rho'(\sigma')=\sigma} \mu_{n+1}(f)(\sigma').$$

(Recall that $\rho'(\sigma) = \text{Order}(\sigma|_{[2, n+1]})$.)

The functions ρ and ρ' , now thought of as maps $S_n \rightarrow S_{n-1}$, induce maps $\rho_*, \rho'_* : \Delta_n \rightarrow \Delta_{n-1}$. Explicitly, for $\mu \in \Delta_n$,

$$\begin{aligned} \rho_*(\mu)(\sigma) &= \sum_{\sigma' \in \rho^{-1}(\sigma)} \mu(\sigma') \\ \rho'_*(\mu)(\sigma) &= \sum_{\sigma' \in \rho'^{-1}(\sigma)} \mu(\sigma'). \end{aligned}$$

Thus by (2) and compatibility, $\rho_*(\mu(f)) = \rho'_*(\mu(f))$ for $f \in \mathcal{A}^{\text{mp}}$.

Definition 5.3. Set $P_n = \{\mu \in \Delta_n \mid \rho_*(\mu) = \rho'_*(\mu)\}$.

As each condition (2) is linear, P_n is a polytope contained in the simplex Δ_n , and $P_n \cap \partial\Delta_n = \partial P_n$. We have already proved the following lemma.

Lemma 5.4. *If $f \in \mathcal{A}^{\text{mp}}$ then $\mu_n(f) \in P_n$.*

Example 5.5. The polytope P_2 is all of Δ_2 ; this is a line segment connecting $\chi_{(12)}$ to $\chi_{(21)}$. The preimage of a point $a\chi_{(12)} + (1-a)\chi_{(21)} \in \text{Int}(P_2)$ under the map ρ_* is a 3-dimensional square pyramid with apex $a\chi_{(123)} + (1-a)\chi_{(321)}$. If $0 < a < 1/2$ the vertices of the square base are $a(\chi_\sigma + \chi_\tau) + (1-2a)\chi_{(321)}$ where

$\sigma \in \{(132), (231)\}$ and $\tau \in \{(213), (312)\}$, whereas if $1/2 < a < 1$ then the vertices are $(1-a)(\chi_\sigma + \chi_\tau) + 2a\chi_{(321)}$ with the same choices for σ and τ . If $a = 1/2$ then the square base is a (2-dimensional) face of P_3 ; it corresponds to the face subgraph $H \subset G_2$ consisting of all the edges except the loops (123) and (321).

The entire polytope P_3 is 4-dimensional; it resembles a suspension of the (middle) square pyramid, except that the apex of the pyramid lies on the segment connecting the suspension points $\chi_{(123)}$ and $\chi_{(321)}$, so that P_3 has six vertices rather than seven. See Figure 3 (in which ρ_* projects vertically).

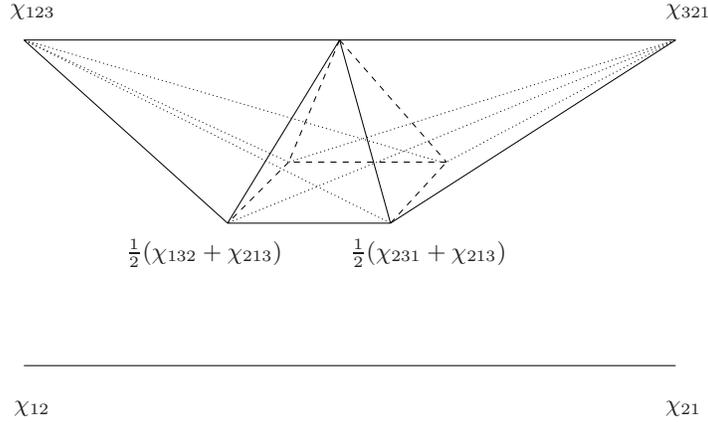


FIGURE 3. The polytopes P_2 (below) and P_3 (above). Fibers of the (vertical) projection are 3-dimensional square pyramids; the preimage of the midpoint of P_2 is shown. The whole polytope P_3 is the join of an interval and a square.

Dictionary between P_n and G_{n-1} . Before we get to the main theorem we establish several connections between P_n and G_{n-1} .

An *edge weighting* on a digraph G is a map $\phi : EG \rightarrow [0, 1]$ such that $\sum \phi(e) = 1$. A *flow* on G is an edge weighting ϕ such that for every $v \in VG$,

$$\sum_{\{e|h(e)=v\}} \phi(e) = \sum_{\{e|t(e)=v\}} \phi(e).$$

Note that the set of all edge weightings on G_{n-1} is exactly Δ_n , and the set of all flows on G_{n-1} is exactly P_n .

A flow supported on an embedded loop in G_{n-1} is a vertex of P_n . The set of all flows supported on a face subgraph $H \subset G_{n-1}$ is a face F_H of P_n . The assignment $H \mapsto F_H$ is an inclusion-preserving bijection between the set of face subgraphs of G_{n-1} and the set of faces of P_n . The dimension of F_H is one less than the rank of the first homology of H . In particular, if $H = G_{n-1}$ then $F_H = P_n$ has dimension $n! - (n-1)!$.

If two face subgraphs $H, K \subset G_{n-1}$ are disjoint, then $F_{H \cup K} = F_H * F_K$ where $*$ denotes the join.

Example 5.6. By counting the face subgraphs of various ranks in, say, G_2 , one determines the number and structure of faces of P_3 of each dimension. It is instructive to compare this with the earlier description of P_3 given in Example 5.5.

Remark 5.7. The dimension of Δ_n is $n! - 1$, and the conditions (2) impose $(n - 1)!$ additional linear constraints. These constraints are obviously independent, since their sum is zero; the fact that P_n has dimension $n! - (n - 1)!$ shows that the constraints are otherwise linearly independent.

Realizable faces. Here is our main theorem, which we prove after a sequence of lemmas.

Theorem 5.8. (1) *The set $\mu_n(\mathcal{A}^{\text{mp}})$ is a union of open faces of P_n .*

- (2) *Let F be a face of P_n and let H be the corresponding face subgraph of G_{n-1} , so that $F = F_H$. Then $\text{Int}(F) \subset \mu_n(\mathcal{A}^{\text{mp}})$ if and only if H is driftless.*
(3) *The closure of $\mu_n(\mathcal{A}^{\text{mp}})$ is P_n .*

Example 5.9. The set $\mu_2(\mathcal{A}^{\text{mp}})$ is equal to the interior of P_2 . The set $\mu_3(\mathcal{A}^{\text{mp}})$ consists of $\text{Int}(P_3)$ (which is 4-dimensional) together with all six of its open 3-dimensional facets, nine of its thirteen open 2-dimensional faces (including the square face), and two of its thirteen open edges. None of the six vertices of P_3 is in $\mu_3(\mathcal{A}^{\text{mp}})$.

There are sometimes vertices of P_n in $\mu_n(\mathcal{A}^{\text{mp}})$. For example the embedded loop in G_4 with edges (23451), (34512), (45132), (41325), (13254), (31542), (15423), (54123), (51234) is driftless, as is easily seen by computing its poset Q . Hence by Theorem 5.8 the corresponding vertex of P_5 is realizable.

Lemma 5.10. *Let γ be a driftless loop in G_{n-1} . Then there is $f \in \mathcal{A}^{\text{mp}}$ such that $\mu_n(f)$ equals the counting measure induced on EG_{n-1} by γ . In particular $\mu_n(f)$ is in the interior of the face F_H , where H is the (edge) support of γ .*

Proof. Lift γ to a permutation $\sigma \in S_m$. Let ϕ be a measure preserving ergodic function $I \rightarrow I$.

We build the *permutation function* corresponding to σ : for $\sigma \in S_\ell$, set

$$\bar{f}_\sigma(x) = x + \frac{\sigma(i+2) - \sigma(i+1)}{\ell} \quad \text{where } i = \lfloor nx \rfloor.$$

Finally, let f_σ equal \bar{f}_σ composed with a scaled down version of ϕ on the interval $[0, 1/m]$. Now f_σ has the desired property. \square

Lemma 5.11 (Balayage). *Let H be a connected face subgraph of G_n . Then H is driftless if and only if $\text{Int}(F_H) \cap \mu_n(\mathcal{A}^{\text{mp}}) \neq \emptyset$.*

Proof. Assume H driftless. By Lemma 4.10 there is a totally driftless loop γ with support H . By Lemma 5.10 there is $f \in \mathcal{A}^{\text{mp}}$ with $\mu_n(f) \in \text{Int}(F_H)$.

Conversely, assume H drifts. Let $f \in \mathcal{A}^{\text{mp}}$, and suppose that $\mu_n(f) \in \text{Int}(F_H)$. Using the drift, we will construct from f a positive measure subset of I and a measure preserving function g such that either $g(x) > x$ for all x or $g(x) < x$ for all x . This will contradict Lemma 5.2.

Let $J = \{x \in I_{\text{ap}} \mid \forall u \in VG_n, |\{i \mid \sigma_n^f(f^i(x)) = u\}| \in \{0, \infty\}\}$. Note that $\mu_{\text{Leb}}(J) = 1$, since $J \subseteq f^{-1}(J)$ and $I_{\text{ap}} = \cup_i f^{-i}(J)$. Let $v \in VH$, $j \in [n]$, $\epsilon \in \{+, -\}$ be as asserted in the definition of drift. Set $J_v = \{x \in J \mid \sigma_n^f(x) = v\}$ and $J_{v,j} = J \cap f^{(j-1)}(J_v)$. Note $\mu_{\text{Leb}}(J_{v,j}) \geq \mu_{\text{Leb}}(J_v)$ are positive by hypothesis. For $x \in J_v$ let $i(x)$ be the smallest $j > 0$ such that $f^j(x) \in J_v$. For $y \in J_{v,j}$ we write $y = f^{j-1}(x)$ with $x \in J_v$, and now define $g : J_{v,j} \rightarrow J_{v,j}$ by $g(y) = f^{i(x)}(y)$.

Note that g is measure preserving. To see this consider $A \subseteq J_{v,j}$ measurable and write $B_r = f^{-r}(A) - \cup_{i \in [0, r-1]} f^{-i}(J_{v,j}) \subseteq \cup_i f^{-i}(J_{v,j}) - \cup_{i < n} f^{-i}(J_{v,j})$ a sequence with measure decreasing to 0. Write $A_r = B_r \cap J_{v,j}$. Note that $g^{-1}(A) = \cup_r A_r$ is a disjoint decomposition and for every n there is $\mu_{\text{Leb}}(A) = \mu_{\text{Leb}}(\cup_{r < n} A_r) + \mu_{\text{Leb}}(B_n)$ so that $\mu_{\text{Leb}}(A) = \mu_{\text{Leb}}(g^{-1}(A))$.

Now if $\epsilon = +$, then $g(y) > y$ for all $y \in J_{v,j}$, and if $\epsilon = -$, then $g(y) < y$ for all $y \in J_{v,j}$. Either case contradicts Lemma 5.2. \square

Lemma 5.12. *For any face subgraph H of G_n , $\text{Int}(F_H) \cap \mu_n(\mathcal{A}^{\text{mp}}) \neq \emptyset$ if and only if $\text{Int}(F_K) \cap \mu_n(\mathcal{A}^{\text{mp}}) \neq \emptyset$ for every connected component K of H .*

Proof. Suppose $f \in \mathcal{A}^{\text{mp}}$ and $\mu_n(f) = \mu \in \text{Int}(F_H)$. Let K be a connected component of H , and let $I_K = \{x \in I \mid \sigma_n^f(x) \in VK\}$. Note that $\mu_{\text{Leb}}(I_K) \neq 0$; defining g to be a scaled up version of $f|_{I_K}$ so that $g : I \rightarrow I$, we have $\mu_n(g) \in \text{Int}(F_K)$.

The converse implication follows from the convexity of $\mu_n(\mathcal{A}^{\text{mp}})$. \square

We now prove Theorem 5.8.

Proof of main theorem. To prove (1), let $\text{Int}(F_H)$ denote the open face F_H . We will show that if $\text{Int}(F_H) \cap \mu_n(\mathcal{A}^{\text{mp}})$ is nonempty then for each vertex of F_H there are points of $\text{Int}(F_H) \cap \mu_n(\mathcal{A}^{\text{mp}})$ arbitrarily close to v . By convexity of $\mu_n(\mathcal{A}^{\text{mp}})$ it follows that $\text{Int}(F_H) \subset \mu_n(\mathcal{A}^{\text{mp}})$, thus proving (1).

Suppose $\text{Int}(F_H) \cap \mu_n(\mathcal{A}^{\text{mp}})$ is nonempty. If H is connected, then by Lemma 5.11 H is driftless, and by Lemma 4.10 there is a totally driftless loop γ with support H . Let v be a vertex of F_H and let β be an embedded loop in H such that $v = F_\beta$. By Lemma 3.6 the loop $\beta^N \gamma$ is driftless, so by Lemma 5.10 there is $f \in \mathcal{A}^{\text{mp}}$ with $\mu_n(f)$ equal to the counting measure on the loop $\beta^N \gamma$. As N grows this sequence of measures approaches v .

If H is not connected, then by Lemma 5.12, for each connected component K of H there is $f_K \in \mathcal{A}^{\text{mp}}$ with $\mu(f_K) \in \text{Int}(F_K)$. We apply the argument from the previous paragraph to each face F_K , obtaining points of $\mu_n(\mathcal{A}^{\text{mp}})$ close to the vertices of F_K . As each vertex of F_H is a vertex of one of the F_K 's, we are done.

As for (2), by (1) we know that $\text{Int}(H) \subset \mu_n(\mathcal{A}^{\text{mp}})$ if and only if $\text{Int}(H) \cap \mu_n(\mathcal{A}^{\text{mp}}) \neq \emptyset$. If H is connected, Lemma 5.11 finishes it. If H is not connected, then for any connected component K of H we have $\text{Int}(F_K) \cap \mu_n(\mathcal{A}^{\text{mp}}) \neq \emptyset$ if and only if K is driftless. So by Lemma 5.12, $\text{Int}(H) \subset \mu_n(\mathcal{A}^{\text{mp}})$ if and only if each K is driftless, i.e., if and only if H is driftless.

To prove (3), it suffices to show that $\text{Int}(P_n) \subset \mu_n(\mathcal{A}^{\text{mp}})$. This is easy: as $\mu_n(\mathcal{A}^{\text{mp}})$ is not empty, there must exist a (connected) driftless face subgraph. By Corollary 4.11, the whole graph G_{n-1} is driftless. Since $\text{Int}(P_n) = F_{G_{n-1}}$, the result is implied by (2). \square

Corollary 5.13. *For each n , there exists $f \in \mathcal{A}^{\text{mp}}$ such that $\mu_n(f)$ is uniform on S_n .*

Remark 5.14. We have answered Question 1 for $\mathcal{C} = \mathcal{A}^{\text{mp}}$. However Question 2 remains open. In particular, we do not know if there is $f \in \mathcal{A}^{\text{mp}}$ such that $\mu_n(f)$ is uniform for all n . See Section 8.

6. ENTROPY AND FINITE EXCLUSION TYPE

In this section we change our focus from the distribution $\mu_n(f)$ to a coarser statistic, namely the number of permutations of length n realized by f . We relate two notions about a continuous piecewise monotone function f : finite entropy and finite exclusion type. The basic idea is that these two concepts imply opposite things for the number of length N permutations realized by iterates of f as N gets large. Roughly speaking, finite entropy implies that the number of permutations realized by f grows (at most) exponentially in the length. On the other hand, finite exclusion type means that the only restrictions on the permutations realized by f are given by looking at permutations of a fixed finite length. Often, this will imply that the number of realizable permutations in S_N grows super-exponentially in N .

Define $\sigma_n(f)$ to be the image of σ_n^f in S_n .

Continuous functions and entropy. For (piecewise) continuous functions, several classical definitions of the topological entropy $h(f)$ are possible. The reader is referred to [7] for details. A new notion of entropy called *topological permutation entropy* has been studied recently by several people; the following combines Theorem 1 of [1] with Theorem 2.1 of [9].

Theorem 6.1. *If $f : I \rightarrow I$ is piecewise continuous and piecewise monotone then $h(f) = \lim_{n \rightarrow \infty} \frac{1}{n-1} \log(|\sigma_n(f)|)$ and $h(f)$ is finite.*

Finite exclusion type.

Definition 6.2. A function $f \in \mathcal{A}$ has *exclusion type n* if there exists $H \subseteq G_n$ such that $\sigma_m(f) = \pi_{m,n}^{-1}(\text{Path}_{m-n} H)$ for all $m \geq n$ and *finite exclusion type* if it has exclusion type n for some n .

Note that this says not only that every path in G_n realized by f is supported on H , but also that every lift of every path supported on H is realized by f . A condition equivalent to finite exclusion type is that there are finitely many *basic forbidden patterns* for f , in the language of [5]. This means that there are finitely many permutations $\sigma_1, \dots, \sigma_k$ such that any permutation σ (of any length m) either occurs as $\sigma_m^f(x)$ for some x or else satisfies $\text{Order}(\sigma|_J) = \sigma_i$ for some interval $J \subset [m]$ and some i . Elizalde has proposed the problem of characterizing those functions which have finite exclusion type. We will give a necessary condition.

Theorem 6.3. *Suppose f has finite exclusion type n , and let $H \subset G_n$ be the associated subgraph. If H contains a partially driftless loop then $|\sigma_N(f)|$ grows super-exponentially; i.e., for any $c \in \mathbb{R}$, we have*

$$|\sigma_N(f)| > c^N \quad \text{for sufficiently large } N.$$

Proof. Let γ be a loop on H with $\text{Drift}_\gamma(j) = 0$ for some particular $j \in [n]$. Let ℓ be the length of γ , and set $m_k = k\ell + n$. By the hypothesis of finite exclusion type we have

$$\sigma_{m_k}(f) = \pi_{m_k,n}^{-1}(\text{Path}_{k\ell}(H)) \supset \pi_{m_k,n}^{-1}(\gamma^k).$$

Now since $\text{Drift}_\gamma(j) = 0$, the i elements $x_j, x_{\ell+j}, \dots, x_{(k-1)\ell+j}$ of the poset Q_{γ^k} are pairwise incomparable, by Lemma 4.5. Thus the number of lifts of γ^k to S_{m_k} is at least $k!$, by Corollary 3.8. Therefore $|\sigma_{k\ell+n}(f)| \geq k!$ for all k and the result follows. \square

Remark 6.4. Elizalde and Liu [6] have shown that there is no piecewise monotonic function $f : I \rightarrow I$ of finite exclusion type with associated graph $H \subset G_2$ where $EH = \{(123), (321), (213), (312)\}$. This does not follow from the preceding theorem, as this H contains no partially driftless loop.

For a given function f , denote by $H_n(f)$ the subgraph of G_n with edge set $\sigma_{n+1}(f)$.

Theorem 6.5. *If $f : I \rightarrow I$ is ergodic then for every n , $H_n(f)$ contains a partially driftless loop.*

Proof. Consider the graph H with vertex set $VH = VG_n$, edge set $EH = \{(x, N) \in I \times \mathbb{N} \mid N > n, \forall 1 \leq m \leq N+n, d(f^m(x), x) \geq d(f^N(x), x)\}$ and head and tail maps the restrictions to the initial and final segments of $\sigma_N^f(x)$. Note that any directed cycle in H yields a partially driftless loop in $H_n(f)$ and that H has finitely many vertices so it suffices to construct an infinite path in H .

Consider $J = \{x \in I \mid \forall y \in I, \epsilon > 0, \lim_{n \rightarrow \infty} \frac{|\{m < n \mid d(f^m(x), y) < \epsilon\}|}{n} \in (\frac{\epsilon}{2}, 4\epsilon)\}$. By the compactness of I and ergodicity of f , $\mu_{\text{Leb}}(J) = \mu_{\text{Leb}}(I) = 1$. Since $f(J) \subseteq J$ there will be an infinite path in H if $J \subseteq \pi_1(EH)$; this is shown next. For any $x \in J$ choose $\epsilon < \frac{1}{4n}$ so that if $1 \leq m \leq n$ then $d(f^m(x), x) > \epsilon$. Choose $r > n$ with $d(f^r(x), x) < \epsilon$. Choose $N > r > n$ with $d(f^m(x), x) \geq d(f^N(x), x)$ for every $1 \leq m \leq N+n$ (so that $(x, N) \in EH$). Such an N exists since there is always eventually another sequence of length n avoiding the ϵ ball around x . \square

Theorem 6.6. *If $f : I \rightarrow I$ is piecewise continuous and if x_0 is a periodic point of period $p > n$ such that f is continuous at every iterate of x_0 , then $H_n(f)$ contains a partially driftless loop.*

Proof. Using continuity, choose $\epsilon > 0$ so that for any $x \in I_{ap}$ within ϵ of x_0 , the balls $B_\epsilon(f^i(x))$ are pairwise disjoint for $i = 0, \dots, p-1$ and the iterates satisfy $|f^i(x) - f^{p+i}(x)| < \epsilon$ for $0 \leq i \leq n-1$. Then the image in G_n of $\sigma_{p+n}^f(x)$ is a partially driftless loop. \square

Corollary 6.7. *If $f : I \rightarrow I$ is piecewise continuous and piecewise monotonic and either*

- *f is ergodic on a subinterval of I , or*
- *f has arbitrarily large finite orbits on which it is continuous,*

then f does not have finite exclusion type.

Recall that by Sarkovskii's Theorem [8], a continuous function has points of arbitrarily large period as long as there is a periodic point whose period is not a power of 2.

7. PROOF OF THEOREM 1.1

We now give the promised proof of Theorem 1.1. Given $\mu = (\mu_1, \mu_2, \dots)$ we will construct f with $\boldsymbol{\mu}(f) = \mu$. Our construction will involve several layers of Cantor sets, and the resulting functions will be nowhere near continuous or measure preserving.

Recall that $\rho(\sigma) = \text{Order}(\sigma|_{[n-1]})$ if $\sigma \in S_n$.

Lemma 7.1. *Given $\mu \in \Delta_\infty$, there exist intervals $\{I_\sigma \subset (\frac{1}{4}, \frac{3}{4}]\}_{\sigma \in \cup_n S_n}$, open at the left endpoint and closed at the right endpoint, with the properties that:*

- (i) $I_\sigma \cap I_\tau = \emptyset$ for all $\sigma \neq \tau \in S_n$,
- (ii) $I_\sigma \subset I_{\rho(\sigma)}$,
- (iii) $\mu_{Leb}(I_\sigma) = \frac{1}{2}\mu_n(\sigma)$ for all $\sigma \in S_n$,
- (iv) for each n , $\cup_{\sigma \in S_n} I_\sigma = (\frac{1}{4}, \frac{3}{4}]$.

Proof. We define the I_σ inductively as follows. First set $I_{(1)} = (\frac{1}{4}, \frac{3}{4}]$. Now let $n > 1$ and assume that intervals I_τ have been constructed for all $\tau \in S_{n-1}$. Since $\mu_{n-1}(\tau) = \sum_{\sigma \in S_n} \mu_n(\sigma)$, summed over all $\sigma \in S_n$ such that $\rho(\sigma) = \tau$, we may subdivide each I_τ into half-open intervals I_σ of length $\frac{1}{2}\mu_n(\sigma)$. \square

Lemma 7.2. *There exist disjoint intervals $\{J_\sigma\}_{\sigma \in \cup_n S_n}$ such that for all compatible sequences $(\sigma_1, \dots, \sigma_n)$, and any (x_1, \dots, x_n) with $x_i \in J_{\sigma_i}$, $\text{Order}(x_1, \dots, x_n) = \sigma_n$.*

Proof. Again the construction is inductive. Suppose that the J_σ have been constructed for $\sigma \in \cup_{n=1}^k S_n$, and assume further that gaps of positive lengths exist between these intervals and at both endpoints. Order the permutations in $\sigma \in S_{k+1}$ arbitrarily, and for each such σ , let J_σ be an arbitrary open interval disjoint from the previously chosen intervals and with positive length gaps away from them, subject to the further condition that J_σ should lie in the correct gap as determined by the value of $\sigma(k+1)$. \square

Proof of Theorem 1.1. Let $\mu = (\mu_1, \mu_2, \dots) \in \Delta_\infty$ be given; we will construct a function $f \in \mathcal{A}$ with $\boldsymbol{\mu}(f) = \mu$. The construction proceeds in a sequence of steps.

Step 1. Let C denote the (usual) Cantor set in $[0, 1]$. By applying an order preserving transformation we can assume that the $\{J_\sigma\}$ given by Lemma 7.2 have the additional properties that $J_1 = [\frac{1}{4}, \frac{3}{4}]$ and $J_\sigma \subset [\frac{1}{8}, \frac{7}{8}]$ for all permutations σ . For each permutation σ choose an order preserving injection $\phi_\sigma : C \rightarrow J_\sigma$ with $\mu_{Leb}(\phi_\sigma(C)) = 0$. Let I_σ be as in Lemma 7.1. Finally choose $\beta : [\frac{1}{4}, \frac{3}{4}] \rightarrow C$ to be an order preserving bijection.

We define the function f_1 on a subset of $[0, 1]$ recursively, as follows.

- First, on $(\frac{1}{4}, \frac{3}{4}] = I_{(1)} = I_{(12)} \cup I_{(21)}$: for each $\sigma \in S_2$, if $x \in I_\sigma$ then set $f_1(x) = \phi_\sigma \beta(x)$. Thus for $\sigma \in S_2$, we have $f_1(I_\sigma) \subset J_\sigma$.
- Next, assuming f_1 is defined on $f_1^{i-1}(I_{(1)})$, we define f_1 on $f_1^i(I_{(1)})$ as follows. Notice that $f_1^i(I_{(1)}) = \sqcup_{\sigma \in S_{i+2}} f_1^i(I_\sigma)$. For all $\sigma \in S_{i+2}$ and for all $x \in f_1^i(I_\sigma)$ define $f_1(x) = \phi_\sigma(\phi_{\rho(\sigma)}^{-1}(x))$. Thus we have $f_1^i(I_\sigma) \subset \phi_\sigma(C) \subset J_\sigma$.

We have now defined every power of f_1 on $(\frac{1}{4}, \frac{3}{4}]$; note that the the domain of f_1 , which we will call D , is $(\frac{1}{4}, \frac{3}{4}]$ union a measure zero set. The purpose of this construction is that for any $\sigma \in S_n$ and $x \in I_\sigma$, we now have $\sigma_n^{f_1}(x) = \sigma$. We set $f = f_1$ on D .

Step m , $m \geq 2$. Denote by K_m the measure zero set in $(0, 2^{-m}] \cup (1 - 2^{-m}, 1]$ for which $f(x)$ has already been defined. Define $g_m : (0, 2^{-m}] \cup (1 - 2^{-m}, 1] \rightarrow (0, 1]$ to be the map

$$g_m(x) = \begin{cases} 2^{m-1}x & \text{for } x \in (0, 2^{-m}], \\ 1 - 2^{m-1}(1-x) & \text{for } x \in (1 - 2^{-m}, 1]. \end{cases}$$

For all $x \in g_m^{-1}(D) \setminus K_m$, define $f(x) = g_m^{-1}(f_1(g_m(x)))$. Note that if $\sigma \in S_n$ and $x \in g_m^{-1}(I_\sigma) \setminus K_m$, we now have

$$(3) \quad \sigma_n^f(x) = \sigma.$$

After step m , the domain of f includes the interval $(2^{-m}, 1 - 2^{-m}]$, so the iterative process defines f on $(0, 1)$. It remains to show that $\mu_n(f) = \mu_n$ for all n .

For $\sigma \in S_n$, define $\bar{I}_\sigma = I_\sigma \cup (\cup_{i \geq 2} g_i^{-1}(I_\sigma))$. By (3) we have $\sigma_n^f(x) = \sigma$ for all $x \in \bar{I}_\sigma \setminus (\cup_m K_m)$. Thus $\mu_n(f)(\sigma) = |\bar{I}_\sigma \setminus (\cup_m K_m)|$. Since $(\cup_m K_m)$ is of measure zero,

$$|\bar{I}_\sigma \setminus (\cup_m K_m)| = |\bar{I}_\sigma| = \frac{1}{2}\mu_n(\sigma) + \sum_{i \geq 2} 2^{-(i-1)}\left(\frac{1}{2}\mu_n(\sigma)\right) = \mu_n(\sigma).$$

We conclude that $\mu_n(f) = \mu_n$ for each n , and $\mu(f) = \mu$. This completes the proof. \square

8. OPEN QUESTIONS

Many interesting open questions remain about the relationship between functions and their distributions of order patterns.

Measure preserving functions. The bulk of the work presented here focused on the class of measure preserving functions; however to date we have been unable to answer Question 2 for this class.

Question 3. *What is $\mu(\mathcal{A}^{\text{mp}})$?*

There is an infinite version of the polytope, P_∞ , which consists of compatible sequences (μ_1, μ_2, \dots) with $\mu_i \in P_i$. We do not know if the ‘‘interior’’ of P_∞ is realizable by some $f \in \mathcal{A}^{\text{mp}}$ (where the meaning of ‘‘interior’’ depends on the topology on P_∞), or if there is a drift condition for faces. One concrete question is this:

Question 4. *Is there $f \in \mathcal{A}^{\text{mp}}$ with $\mu_n(f)$ uniform for all n ?*

Corollary 5.13 asserts that such an f exists for any particular n , and of course by Theorem 1.1 there is $f \in \mathcal{A}$ that works for all n . Yet there is no piecewise monotonic $f \in \mathcal{A}$ that works for all n , because such an f would have finite entropy (by [1], or Theorem 6.1) hence $|\sigma_N(f)|$ would grow at most exponentially in N . Note that such a function might be desirable as a random number generator, since from the point of view of order patterns, its iterates would look perfectly random.

In a somewhat different direction, if λ is a reasonably nice measure on I then the results of Section 5 hold with $\mathcal{C} = \mathcal{A}^{\text{mp}}$ replaced by the collection $\mathcal{C} = \mathcal{A}^\lambda$ of functions which preserve λ . (See Remark 5.1.)

Question 5. *Are there measures λ for which $\mu_n(\mathcal{A}^\lambda) \neq \mu_n(\mathcal{A}^{\text{mp}})$?*

Other functions. Returning to the broader Questions 1 and 2, there are several interesting classes of functions \mathcal{C} to study, such as (piecewise) continuous functions, polynomials, etc. For example, if $\mathcal{C} = \mathcal{A}^{\text{pc}}$ is the collection of piecewise continuous functions, then it is easy to see that the only vertices of Δ_n contained in $\mu_n(\mathcal{A}^{\text{pc}})$ are $\chi_{(12\dots n)}$ and $\chi_{(n\dots 21)}$.

Question 6. *Is the closure of $\mu_n(\mathcal{A}^{\text{pc}})$ equal to Δ_n ?*

Question 7. *Is there a drift criterion which applies to \mathcal{A}^{pc} ?*

Finally, it would be natural to study the extent to which the distributions $\mu_n(f)$ determine f , for f in a given class \mathcal{C} .

Question 8. *For $\mu \in \mu_n(\mathcal{C})$, what is $\mu_n^{-1}(\mu) \cap \mathcal{C}$?*

Question 9. *For $\mu = (\mu_1, \mu_2, \dots) \in \mu(\mathcal{C})$, what is $\mu^{-1}(\mu) \cap \mathcal{C}$?*

These questions are in a sense converse to Questions 1 and 2.

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