

Towards a weighted version of the Hajnal-Szemerédi Theorem

József Balogh ^{*} Graeme Kemkes [†] Choongbum Lee [‡] Stephen J. Young [§]

Abstract

For a positive integer $r \geq 2$, a K_r -factor of a graph is a collection of vertex-disjoint copies of K_r which covers all the vertices of the given graph. The celebrated theorem of Hajnal and Szemerédi asserts that every graph on n vertices with minimum degree at least $(1 - \frac{1}{r})n$ contains a K_r -factor. In this note, we propose investigating the relation between minimum degree and existence of perfect K_r -packing for edge-weighted graphs. The main question we study is the following. Suppose that a positive integer $r \geq 2$ and a real $t \in [0, 1]$ is given. What is the minimum weighted degree of K_n that guarantees the existence of a K_r -factor such that every factor has total edge weight at least $t\binom{r}{2}$? We provide some lower and upper bounds and make a conjecture on the asymptotics of the threshold as n goes to infinity. This is the long version of a “problem paper” in *Combinatorics, Probability and Computing*.

1 Introduction

Many results in graph theory study the relation between the minimum degree of a given graph and its spanning subgraphs. For example, Dirac’s theorem asserts that a graph on n vertices with minimum degree at least $\lceil \frac{n}{2} \rceil$ contains a Hamilton cycle. Hajnal and Szemerédi [3] proved that every graph on $n \in r\mathbb{Z}$ vertices with minimum degree at least $(1 - \frac{1}{r})n$ contains a spanning subgraph consisting of $\frac{n}{r}$ vertex-disjoint copies of K_r (we call such a subgraph a K_r -factor).

In this note we propose investigating this relation in edge-weighted graphs. As a concrete problem, we study the particular case when the spanning subgraph is the graph formed by vertex-disjoint copies of K_r (in other words, we would like to extend the Hajnal-Szemerédi theorem to edge-weighted graphs). Suppose we equip the complete graph K_n with edge weights $w: E(K_n) \rightarrow [0, 1]$. For a given weighted graph and vertex v we let $\deg_w(v)$ denote the weighted degree of the vertex v . Let $\delta_w(G)$ be the minimum weighted degree of the graph G . The main question can be

^{*}Department of Mathematical Sciences, University of Illinois, Urbana, IL, 61801, Supported in part by NSF CAREER Grant DMS-0745185, UIUC Campus Research Board Grants 09072 and 11067, OTKA Grant K76099, and TAMOP-4.2.1/B-09/1/KONV-2010-0005 project. Work was partly done while at Department of Mathematics, University of California, San Diego La Jolla, CA, 92093.

[†]Department of Mathematics, Ryerson University Toronto, ON, M5B 2K3

[‡]Department of Mathematics, UCLA, Los Angeles, CA, 90095. Email: choongbum.lee@gmail.com. Research supported in part by a Samsung Scholarship.

[§]Department of Mathematics, University of California, San Diego, La Jolla, CA, 92093

formulated as the following: How large must $\delta_w(K_n)$ be to guarantee that there exists a K_r -factor such that every factor has total edge weight at least $t \binom{r}{2}$ for some given $t \in [0, 1]$?

More formally, for $n \in r\mathbb{Z}$ let $\mathcal{W}(r, t, n)$ be the collection of edge weightings on K_n such that every K_r -factor has a clique with weight strictly smaller than $t \binom{r}{2}$. We then define

$$\delta(r, t, n) = \sup_{w \in \mathcal{W}(r, t, n)} \delta_w(K_n) \quad \text{and} \quad \delta(r, t) = \limsup_{n \rightarrow \infty} \frac{\delta(r, t, n)}{n}.$$

The main open question that we raise is the following.

Question 1. Determine the value of $\delta(r, t)$ for all r and t .

Let $\mathcal{W}^*(r, t, n)$ be the collection of edge weightings of K_n such that every K_r -factor has a clique with weight at most $t \binom{r}{2}$ (instead of strictly smaller than $t \binom{r}{2}$), and define the functions $\delta^*(r, t, n)$ and $\delta^*(r, t)$ accordingly.

Proposition 1.1. *For all r, t , and n , $\delta(r, t, n) = \delta^*(r, t, n)$. Therefore, $\delta(r, t) = \delta^*(r, t)$.*

Proof. The inequality $\delta(r, t, n) \leq \delta^*(r, t, n)$ easily follows from the definition. Noting that the complement of $\mathcal{W}^*(r, t, n)$ is open in the set of all real-valued edge weightings, the set $\mathcal{W}^*(r, t, n)$ is compact. Thus there is a weight function $w \in \mathcal{W}^*(r, t, n)$ so that $\delta_w(K_n) = \delta^*(r, t, n)$. Let $\varepsilon < 1$ be an arbitrary positive real, and let w' be the weight function obtained from w by multiplying $1 - \varepsilon$ to all the weights. One can easily see that $w' \in \mathcal{W}(r, t, n)$, and thus $\delta(r, t, n) \geq (1 - \varepsilon)\delta^*(r, t, n)$. Thus as ε tends to 0, we see that $\delta(r, t, n) \geq \delta^*(r, t, n)$. This concludes the proof. \square

The proposition above shows that if an edge-weighting of K_n has minimum degree greater than $\delta(r, t, n)$, then there exists a K_r -factor such that every copy of K_r has weight greater than $t \binom{r}{2}$. Therefore, the Hajnal-Szemerédi theorem in fact is a special case of our problem when $t = (\binom{r}{2} - 1) / \binom{r}{2}$ where we only consider the integer weights $\{0, 1\}$. Thus we believe that the following special case is an important and interesting instance of the problem corresponding to the Hajnal-Szemerédi theorem for $r = 3$ (which has been first proved by Corrádi and Hajnal [1]).

Question 2. What is the value of $\delta(3, \frac{2}{3})$?

In the rest of our note we describe our partial results toward answering Question 1.

2 Lower bound

It is not too difficult to deduce the bound $\delta(r, t) \geq (1 - 1/r)t$ from the graph showing the sharpness of the Hajnal-Szemerédi theorem. Our first proposition provides a better lower bound to this function.

Proposition 2.1. *The following holds for every integer $r \geq 2$ and real $t \in (0, 1]$:*

$$\delta(r, t) \geq \frac{1}{r} + \left(1 - \frac{1}{r}\right)t.$$

Proof. Let $n \in r\mathbb{Z}$ with $n > r$ and let $k = \frac{n}{r}$. Let A be an arbitrary set of $k - 1$ vertices and let B be the remaining $k(r - 1) + 1$ vertices. Consider the weight function w that assigns weight t to edges whose endpoints are both in B and weight 1 to all other edges. By the cardinality of A , we see that every K_r -factor must contain a clique that lies entirely within B . Since our weight function gives weight at most $t\binom{r}{2}$ to this clique, we see that $w \in \mathcal{W}^*(r, t, n)$. Further, $\delta_w(K_n) = \min\{n - 1, k - 1 + t(n - k)\}$. But we have that

$$k - 1 + t(n - k) = \left(\frac{1}{r} - \frac{1}{n} + t\left(1 - \frac{1}{r}\right)\right)n.$$

Therefore by Proposition 1.1, we have $\delta(r, t, n) \geq \left(\frac{1}{r} - \frac{1}{n} + t\left(1 - \frac{1}{r}\right)\right)n$ and $\delta(r, t) \geq \frac{1}{r} + \left(1 - \frac{1}{r}\right)t$. \square

Proposition 2.1 illustrates the fundamental difference between the minimum degree threshold for containing a K_r -factor in graphs and edge-weighted graphs. For example, when $r = 3$, we see that $\delta(3, 2/3) \geq 7/9$, while the corresponding function for graphs has value $2/3$ by the Hajnal-Szemerédi theorem. This difference suggests that we indeed need some new ideas and techniques to solve our problem.

3 Upper bound

Next, we establish an upper bound on $\delta(r, t)$. To do so, it is helpful to consider the graph induced by the edges of heavy weights in a given edge-weighted graph. Thus, given an edge-weighted graph G_w , we denote by $G_w(t)$ the subgraph of K_n consisting of edges of weight at least t . For $r = 2$, it is easy to establish the correct value of the function $\delta(2, t)$.

Observation 1. For every $t \in (0, 1]$ we have $\delta(2, t) = \frac{1+t}{2}$.

Proof. The lower bound on $\delta(2, t)$ follows from Proposition 2.1, and thus it suffices to establish the upper bound. Let w be a weight function such that $\delta_w(K_n) \geq \frac{1+t}{2}n$. Now for any vertex $v \in G_w(t)$, we have $\deg_w(v) < (n - 1 - \deg(v)) \cdot t + \deg(v) \cdot 1$, where $\deg(v)$ is the degree of v in $G_w(t)$. But then the minimum weighted degree condition implies that $\deg(v) \geq \frac{n}{2}$, and so by the Hajnal-Szemerédi theorem there is a K_2 -factor in $G_w(t)$. By the definition of $G_w(t)$, this establishes the bound $\delta(2, t) \leq \frac{1+t}{2}$. \square

Even for $r \geq 3$, if t is small enough, then we can determine the correct value of the function $\delta(r, t)$.

Theorem 3.1. For every $r \geq 3$, there exists a positive real t_r such that for every $t \in (0, t_r)$ we have

$$\delta(r, t) = \frac{1}{r} + \left(1 - \frac{1}{r}\right)t.$$

Proof. We have $\delta(r, t) \geq \frac{1}{r} + (1 - \frac{1}{r})t$ by Proposition 2.1. It remains for us to establish the upper bound. Let ε be an arbitrary positive real. For n sufficiently large, let w be a weight function such that $\delta_w(K_n) \geq (\frac{1}{r} + (1 - \frac{1}{r})t + \varepsilon)n$. We will say a copy of a K_r is *heavy* if it has weight at least $\binom{r}{2}t$. A collection of vertex-disjoint copies of K_r is *heavy* if each K_r in the collection is heavy. An edge is *overweight* if it has weight at least $\binom{r}{2}t$. Let t_r be a sufficiently small positive real depending on r to be determined later. We will find a heavy K_r -factor given that $t < t_r$ and n is a large enough integer divisible by r .

Take a maximum heavy collection of vertex-disjoint copies of K_r , that maximizes the number of overweight edges. Call this collection \mathcal{R} , and suppose that $|\mathcal{R}| = \rho$. Denote by V_R be the vertices covered by \mathcal{R} , thus $|V_R| = r\rho$. We may assume that $\rho < \frac{n}{r}$, as otherwise we have a heavy K_r -factor. Then there exist r distinct vertices $v_1, v_2, \dots, v_r \notin V_R$. Let $L = \{v_1, v_2, \dots, v_r\}$. If there is an overweight edge whose endpoints are both in $V(K_n) \setminus V_R$, then we can find a larger collection than \mathcal{R} by taking the union of this edge with $r - 2$ vertices of L . Thus all the edges within $V(K_n) \setminus V_R$ have weight at most $\binom{r}{2}t$.

Fact 1. For every $R \in \mathcal{R}$, if there exists an overweight edge between $V(R)$ and L , then there exists a unique vertex in R which intersects every overweight edge between $V(R)$ and L .

Proof. Fix a copy of K_r in \mathcal{R} and denote it by R . If there are two vertex-disjoint overweight edges between $V(R)$ and L , then we can find two heavy vertex-disjoint copies of K_r over $V(R) \cup L$. Therefore all the overweight edges between $V(R)$ and L share a common endpoint. In particular, there are at most r overweight edges between $V(R)$ and L .

Now suppose that there are at least two overweight edges between $V(R)$ and L , and that the common endpoint is in L . Without loss of generality, let $x \in V(R)$ and $v_1 \in L$ be vertices such that there are at least two overweight edges of the form $\{y, v_1\}$ for $y \in V(R) \setminus \{x\}$. Then by the assumption that we maximized the number of overweight edges, there are at least two overweight edges among the edges $\{y, x\}$ for $y \in V(R) \setminus \{x\}$ (otherwise we can replace R by $R \setminus \{x\} \cup \{v_1\}$). However, if this is the case, then we can find two independent overweight edges in $V(R) \cup L$, and this contradicts the maximality of \mathcal{R} . Thus if there are at least two overweight edges between $V(R)$ and L , then they share a common endpoint in $V(R)$. \square

Let \mathcal{R}' be the subset of copies of K_r of \mathcal{R} which have at least $r - 1$ overweight edges incident to it whose other endpoint is in L . Let $\rho' = |\mathcal{R}'|$.

Fact 2. There exists a real t'_r such that if $t < t'_r$, then $\rho' \geq 1$.

Proof. By Fact 1 we have

$$\begin{aligned} (1 + (r - 1)t)n &\leq \sum_{i=1}^r \deg_w(v_i) \leq \left(r + (r^2 - r) \binom{r}{2} t \right) \rho' \\ &\quad + \left(r - 2 + (r^2 - r + 2) \binom{r}{2} t \right) (\rho - \rho') + r \binom{r}{2} t (n - r\rho). \end{aligned}$$

By way of contradiction, let $\rho' = 0$ and thus,

$$(1 + (r-1)t)n \leq (r-2) \left(1 - \binom{r}{2}t\right) \rho + r \binom{r}{2}tn.$$

Now for sufficiently small t , the coefficient of ρ is positive so we may substitute $\rho \leq \frac{n}{r}$ and obtain,

$$0 \leq \left(\frac{r-2}{r} \left(1 - \binom{r}{2}t\right) + r \binom{r}{2}t - 1 - (r-1)t\right)n = \left((r-1) \binom{r}{2}t - \frac{2}{r}\right)n$$

If t is sufficiently small this is a contradiction, and thus $\rho' \geq 1$. \square

Fact 3. For $R \in \mathcal{R}'$, there exists a unique vertex $x_R \in V(R)$ incident to all the overweight edges within $V(R) \cup L$. Moreover, all the edges incident to x_R within R are overweight.

Proof. For a fixed copy $R \in \mathcal{R}'$, let $x_R \in V(R)$ be the vertex guaranteed by Fact 1. If there is an overweight edge in R which does not intersect x_R , then we can find two heavy vertex-disjoint copies of K_r over the set of vertices $V(R) \cup L$, and this violates the maximality of \mathcal{R} . Therefore, all the overweight edges within R are incident to x_R . Moreover, if there are less than $r-1$ such edges, then we can find a copy of K_R over the vertex set $\{x_R\} \cup L$ which contains at least $r-1$ overweight edges. This contradicts the maximality of overweight edges of \mathcal{R} . Therefore, all the edges incident to x_R within R are overweight. \square

Let X be the subset of vertices which are covered by copies of K_r in \mathcal{R}' that are incident to an overweight edge (guaranteed by Fact 3), and let Y be the vertices which are covered by copies of K_r in \mathcal{R}' that are not in X .

Fact 4. For every $y \in Y$ and $R \in \mathcal{R} \setminus \mathcal{R}'$, y is incident to R by at most one overweight edge.

Proof. Suppose that we are given vertices $x \in X$ and $y \in Y$ covered by $R \in \mathcal{R}$. Without loss of generality, suppose that x is adjacent to $v_1, \dots, v_{r-1} \in L$ by overweight edges. By way of contradiction, suppose that there exists $R' \in \mathcal{R} \setminus \mathcal{R}'$ with $V(R') = \{z_1, z_2, \dots, z_r\}$ such that $\{y, z_1\}$ and $\{y, z_2\}$ are both overweight. If R' contains an edge e other than $\{z_1, z_2\}$ that is overweight, then among the edges $\{x, v_1\}, \{y, z_1\}, \{y, z_2\}, e$ (which are all overweight), we can find at least three vertex-disjoint edges. Therefore we can find three vertex-disjoint copies of K_r over the vertex set $V(R) \cup V(R') \cup L$. However, this contradicts the maximality of \mathcal{R} . If there are no overweight edges within R' other than (possibly) $\{z_1, z_2\}$, then the two copies of K_r over the vertex sets $\{x, v_1, \dots, v_{r-1}\}, \{y, z_1, z_2, \dots, z_{r-1}\}$ contain at least $r+1$ overweight edges, while R and R' combined contain at most r overweight edges (see Fact 3). Therefore we conclude that there exists at most one overweight edge of the form $\{y, z_i\}$. \square

Fact 5. There does not exist a heavy K_r over a vertex set of the form $\{v_i, y_1, y_2, \dots, y_{r-1}\}$ for $v_i \in L$ and $y_1, \dots, y_{r-1} \in Y$.

Proof. Suppose that for some $v_i \in L$ and $y_1, \dots, y_{r-1} \in Y$ the vertices $\{v_1, y_1, \dots, y_{r-1}\}$ induce a heavy K_r . Suppose that $\{y_1, \dots, y_{r-1}\}$ are contained in s disjoint copies R_1, \dots, R_s of K_r belonging to \mathcal{R} , and let x_1, \dots, x_s be the ‘dominating’ vertices of these K_r guaranteed by Fact 3 (note that $s \leq r-1$). If $s \leq r-2$, then since each x_i are incident to L by at least $r-1$ overweight edges, we can find $s+1$ vertex-disjoint copies of a heavy K_r over the vertices $L \cup V(R_1) \cup \dots \cup V(R_s)$. On the other hand, if $s = r-1$, then there exists an index j such that there exists $z \in V(R_j) \setminus \{x_j, y_1, \dots, y_{r-1}\}$. Then by using the overweight edge $\{x_j, z\}$ (see Fact 3) and the overweight edges between $\{x_1, \dots, x_s\}$ and L , we can find at least $s+1 = r$ vertex-disjoint copies of a heavy K_r over the vertices $L \cup V(R_1) \cup \dots \cup V(R_s)$. This contradicts the maximality of \mathcal{R} . \square

For a set of vertices T , let $w(T) = \sum_{v_1, v_2 \in T} w(v_1, v_2)$. We will show

$$\sum_{i=1}^r \sum_{T \subset \binom{Y}{r-1}} w(\{v_i\} \cup T) > \binom{r}{2} tr \binom{|Y|}{r-1}.$$

This inequality contradicts Fact 5, showing that our original assumption $\rho < \frac{n}{r}$ must be false. Note that

$$\begin{aligned} \sum_{i=1}^r \sum_{T \subset \binom{Y}{r-1}} w(\{v_i\} \cup T) &= \binom{|Y|-1}{r-2} \sum_{i=1}^r \sum_{y \in Y} w(v_i, y) + \frac{1}{2} r \binom{|Y|-2}{r-3} \sum_{y_1 \in Y} \sum_{y_2 \in Y \setminus \{y_1\}} w(y_1, y_2) \\ &= \binom{|Y|}{r-1} \left(\frac{r-1}{|Y|} \sum_{i=1}^r \deg_w(v_i, Y) + \frac{r(r-1)(r-2)}{2|Y|(|Y|-1)} \sum_{y \in Y} \deg_w(y, Y) \right) \\ &\geq \binom{|Y|}{r-1} \left(\frac{r-1}{|Y|} \sum_{i=1}^r \deg_w(v_i, Y) + \frac{r(r-1)(r-2)}{2|Y|^2} \sum_{y \in Y} \deg_w(y, Y) \right) \end{aligned} \quad (1)$$

where $\deg_w(v, Y)$ is the weighted degree of v to vertices in Y .

For the first term on the right hand side of (1), we have

$$\begin{aligned} &\sum_{i=1}^r \deg_w(v_i, Y) \\ &= \sum_{i=1}^r \left(\deg_w(v_i) - \deg_w(v_i, X) - \deg_w(v_i, V_R \setminus (X \cup Y)) - \deg_w(v_i, V \setminus V_R) \right) \\ &\geq (1 + (r-1)t + r\varepsilon)n - r\rho' - \left((r-2) + (r^2 - r + 2) \binom{r}{2} t \right) (\rho - \rho') - r \binom{r}{2} t(n - r\rho). \end{aligned}$$

Since the coefficient of n is positive for small enough t , we can substitute $n > r\rho$ to get

$$\sum_{i=1}^r \deg_w(v_i, Y) > r(r-1)t\rho + \left(2 - (r^2 - r + 2) \binom{r}{2} t \right) (\rho - \rho') + r\varepsilon n. \quad (2)$$

For the second term on the right hand side of (1), by Fact 4 and the fact $|Y| = (r-1)\rho'$, for a vertex $y \in Y$, we have

$$\begin{aligned}\deg_w(y, Y) &\geq \left(\frac{1}{r} + \frac{r-1}{r}t + \varepsilon\right)n - \rho - \binom{r}{2}t(n - \rho - |Y|) \\ &= \left(\frac{1}{r} + \frac{r-1}{r}t - \binom{r}{2}t + \varepsilon\right)n + \binom{r}{2}t(r-1)\rho' - \left(1 - \binom{r}{2}t\right)\rho.\end{aligned}$$

Since the coefficient of n is positive for small enough t , we can substitute $n > r\rho$ to get

$$\deg_w(y, Y) > (r-1)t\rho - (r-1)\binom{r}{2}t(\rho - \rho') + \varepsilon n. \quad (3)$$

Using (2) and (3) in (1),

$$\begin{aligned}&\frac{1}{\binom{|Y|}{r-1}} \sum_{i=1}^r \sum_{T \subset \binom{Y}{r-1}} w(\{v_i\} \cup T) \\ &\geq \frac{r-1}{|Y|} \left(r(r-1)t\rho + \left(2 - (r^2 - r + 2)\binom{r}{2}t\right)(\rho - \rho') + r\varepsilon n \right) \\ &\quad + \frac{r(r-1)(r-2)}{2|Y|} \left((r-1)t\rho - (r-1)\binom{r}{2}t(\rho - \rho') + \varepsilon n \right)\end{aligned}$$

If t is small enough, then the coefficient of ρ in the right hand side is positive. Hence we can substitute $\rho \geq \rho'$ to get,

$$\begin{aligned}\frac{1}{\binom{|Y|}{r-1}} \sum_{i=1}^r \sum_{T \subset \binom{Y}{r-1}} w(\{v_i\} \cup T) &\geq \left(\frac{r(r-1)^2}{|Y|} + \frac{r(r-1)^2(r-2)}{2|Y|} \right) \left(t\rho' + \frac{\varepsilon n}{r-1} \right) \\ &= \frac{r^2(r-1)^2 t\rho'}{2|Y|} + \frac{r^2(r-1)}{2|Y|} \varepsilon n \\ &> \frac{r^2(r-1)^2 t\rho'}{2|Y|} \\ &= \frac{r(r-1)t}{|Y|} \binom{r}{2} \rho' \\ &= rt \binom{r}{2}.\end{aligned}$$

Since $|Y| = (r-1)\rho'$ and $\rho' \neq 0$ by Fact 2, we have

$$\frac{1}{\binom{|Y|}{r-1}} \sum_{i=1}^r \sum_{T \subset \binom{Y}{r-1}} w(\{v_i\} \cup T) > r \binom{r}{2} t.$$

contradicting Fact 5. Thus our initial assumption that $\rho < \frac{n}{r}$ must be false. Hence $\delta(r, t) \leq \frac{1}{r} + \frac{r-1}{r}t + \varepsilon$ for every positive ε , and our claimed upper bound follows. \square

It is worth noting that this proof gives a value of t_r of $\frac{4}{\binom{r}{2}(r^3-r^2-2r+4)}$. Thus for $r = 3$, we have $t_3 \geq \frac{1}{12}$.

For general values of t , we suggest two approaches which establish some upper bound (that unfortunately does not match the lower bound given in Proposition 2.1).

3.1 First approach: hypergraphs

In our first approach, we reduce our problem into the problem of finding a perfect matching in hypergraphs as in Observation 1. The following lemma establishes the minimum number of heavy K_r 's that each vertex must belong to in a given edge-weighted graph.

Lemma 3.2. *If w is a weight function with minimum weighted degree at least δn , then every vertex is in at least $\left(1 - \frac{1-\delta}{1-t}\right) \binom{n-1}{r-1}$ cliques of size $r \geq 3$ with weight at least $t \binom{r}{2}$.*

Proof. Let w be an arbitrary weight function, v be an arbitrary vertex, and let α_v be the number of K_r 's of weight at least $t \binom{r}{2}$ containing v . Now letting S_v be the sum of the weights of all $\binom{n-1}{r-1}$ K_r 's containing v , we have that

$$S_v \leq \alpha_v \binom{r}{2} + \left(\binom{n-1}{r-1} - \alpha_v \right) t \binom{r}{2}.$$

Let W be the total weight of w . Since edges incident with v occur in $\binom{n-2}{r-2}$ K_r 's containing v and the edges not incident to v occur in $\binom{n-3}{r-3}$ such K_r 's, we have

$$S_v \geq \binom{n-3}{r-3} W + \left(\binom{n-2}{r-2} - \binom{n-3}{r-3} \right) \deg_w(v).$$

Combining these inequalities we have

$$\begin{aligned} \alpha_v &\geq \frac{1}{1-t} \binom{n-1}{r-1} \left(\frac{2(n-r)}{r(n-1)(n-2)} \deg_w(v) + \frac{2(r-2)}{r(n-1)(n-2)} W - t \right) \\ &\geq \frac{1}{1-t} \binom{n-1}{r-1} \left(\delta \frac{n}{n-1} - t \right) \\ &\geq \frac{\delta - t}{1-t} \binom{n-1}{r-1}. \end{aligned}$$

□

We now apply Daykin and Häggkvist's theorem [2] which asserts that an r -uniform hypergraph has a perfect matching if every vertex of it lies in at least $\left(1 - \frac{1}{r}\right) \left(\binom{n-1}{r-1} - 1\right)$ hyperedges. This gives the following bound:

Proposition 3.3. *For every $t \in (0, 1]$ and $r \geq 3$ we have $\delta(r, t) \leq 1 - \frac{1-t}{r}$.*

Hàn, Person and Schacht [4] have conjectured that Daykin and Häggkvist's theorem can be improved, and an r -uniform hypergraph has a perfect matching if every vertex lies in at least

$(1 - (\frac{r-1}{r})^{r-1} - o(1))\binom{n}{r-1}$ hyperedges. If this conjecture were proved, then we would have $\delta(r, t) \leq 1 - (1 - t)(\frac{r-1}{r})^{r-1}$.

Since in [4] the conjecture was proved for $r = 3$, we have that $\delta(3, t) \leq \frac{5}{9} + \frac{4}{9}t$. It is worth noting that this technique cannot be applied to obtain an upper bound matching Proposition 2.1. Consider the case when $r = 3$ and $t = \frac{2}{3}$. The lower bound from Proposition 2.1 reads as $\delta(3, \frac{2}{3}) \geq \frac{7}{9}$. To obtain a matching upper bound using this method, we would need to improve the conclusion of Lemma 3.2 so that in every edge-weighted graph of minimum degree at least $\frac{7}{9}n$, every vertex is contained in at least $(\frac{5}{9} + o(1))\binom{n}{2}$ copies of K_3 . However, the following graph has minimum degree $\frac{29}{36}n$, and there are vertices which are contained in at most $\frac{319}{648}n^2$ copies of K_3 . Let $A \cup B$ be a vertex partition such that A has size $\frac{29}{36}n$ and B has size $\frac{7}{36}n$. First, assign weight 1 to all the edges connecting A and B and give weight 1 to an $\frac{11}{18}n$ -regular graph on A . Give weight 0 to each of the remaining edges. The minimum weighted degree of this graph is $\frac{29}{36}n > \frac{7}{9}n > \frac{2}{3}n$, so this graph has a triangle factor by the Hajnal-Szemerédi theorem. However, every vertex in B is only in $\frac{29}{36}n \cdot \frac{11}{18}n \cdot \frac{1}{2} = (\frac{319}{648} + o(1))\binom{n}{2} < (\frac{5}{9} + o(1))\binom{n}{2}$ triangles. Similar constructions can be made for other values of r and t as well.

3.2 Second approach: induction

We improve the upper bound by using two reductive schemes to build a K_r -factor out of a $K_{r'}$ -factor of the graph (or a large portion of the graph).

Scheme 1. Suppose $r = pq$ with $p, q > 1$, and let w be an arbitrary weight function with minimum weighted degree δn . Let \mathcal{K} be an arbitrary K_p -factor of K_n with minimum average weight t_p and consider the weight function $w_{\mathcal{K}}$ on $K_{n/p}$ defined as follows. Associate to each vertex in $K_{n/p}$ a distinct clique in \mathcal{K} ; the weight of an edge is the average weight in w of the edges between the corresponding cliques. Now the minimum weighted degree under $w_{\mathcal{K}}$ is at least $\frac{p\delta n - p(p-1)}{p^2} = (\delta - \frac{p-1}{n})\frac{n}{p}$. Letting K' be an arbitrary K_q -factor of this graph with minimum average weight t_q , the factors \mathcal{K} and K' induce a K_{pq} -factor in K_n with minimum weight at least $t_q(\frac{q}{2})p^2 + t_p(\frac{p}{2})q$. Thus $\delta(pq, t, n) \leq \max\left\{\delta(p, t, n), \delta(q, t, \frac{n}{p}) + \frac{p-1}{p}\right\}$. Consequently, we have $\delta(pq, t) \leq \max\{\delta(p, t), \delta(q, t)\}$.

Scheme 2. Let $\delta' = \max\left\{\delta(r-1, t), \frac{1}{2} + \frac{t}{2}\right\}$. We prove that $\delta(r, t) \leq \delta'$. Let ε be an arbitrary fixed positive real, and assume that n_0 is large enough so that $\delta(r-1, t, n) \leq (\delta(r-1, t) + \frac{\varepsilon}{2})n$ for all $n \geq n_0$. Assume that we are given an edge-weighted graph G on $n \geq 2n_0$ vertices with minimum degree at least $(\delta' + \varepsilon)n$. We partition randomly the vertices of G into a set A of size $\frac{r-1}{r}n$ and a set B of size $\frac{1}{r}n = k$. By the Chernoff-Hoeffding inequalities, for large enough n , there is such a partition which additionally satisfies that for every vertex the weighted degree into A is at least $(\delta' + \frac{\varepsilon}{2})\frac{r-1}{r}n$ and into B is at least $\delta'\frac{1}{r}n$. By the assumption on δ' , we can find a K_{r-1} -factor \mathcal{K}_A on A with minimum average weight t .

Using \mathcal{K}_A we construct a complete weighted bipartite graph H , where the vertices on one side are associated with cliques in \mathcal{K}_A and the vertices on the other side are associated with vertices in B . For a clique $K \in \mathcal{K}_A$ and a vertex $v \in B$, we assign as weight of the edge (K, v) , the average of the weights of the edges between v and the vertices in K . Notice that the minimum

weighted degree of H is at least $\delta'k \geq (\frac{1}{2} + \frac{t}{2})k$. Recall that $H(t)$ is the unweighted subgraph of H consisting of edges with weight at least t . By a similar argument as in Observation 1, the minimum degree in $H(t)$ is at least $\frac{k}{2}$. Thus by Hall's theorem, there is a perfect matching \mathcal{M} in $H(t)$.

Now notice that \mathcal{K}_A and \mathcal{M} lift to a K_r -factor of G with minimum weight $t\binom{r-1}{2} + t(r-1) = t\binom{r}{2}$. Consequently, $\delta(r, t, n) \leq (\delta' + \varepsilon)n$. Since ε can be arbitrarily small, we have $\delta(r, t) \leq \delta' = \max\{\delta(r-1, t), \frac{1}{2} + \frac{t}{2}\}$.

By Proposition 2.1, Observation 1, and Scheme 2, we obtain the following theorem.

Theorem 3.4. *For every $r \geq 3$ and $t \in (0, 1]$,*

$$\frac{1}{r} + \left(1 - \frac{1}{r}\right)t \leq \delta(r, t) \leq \frac{1}{2} + \frac{t}{2}.$$

For the special case related to triangle factors that we discussed in the beginning, we have $\frac{7}{9} \leq \delta(3, \frac{2}{3}) \leq \frac{5}{6}$. We note that Theorem 3.4 has been proved without using Scheme 1, however, Scheme 1 implies that if there is an improvement on the upper bound for any r , then there is an improvement in the upper bound for an infinite class of r' . For example, for any fixed k , $\delta(r^k, t) \leq \delta(r, t)$. Because of the dependence on the bipartite matching result (which cannot be improved) a similar statement does not hold using just Scheme 2.

3.3 Open Question

In this article, we proposed the study of the function $\delta(r, t)$. Based on the evidence given by Proposition 2.1 and Theorem 3.1, we make the following conjecture.

Conjecture 1. For every $r \geq 2$ and $t \in (0, 1]$,

$$\delta(r, t) = \frac{1}{r} + \left(1 - \frac{1}{r}\right)t.$$

The function $\delta(r, t)$ shows different behavior from its non-weighted counterpart (which is related to the Hajnal-Szemerédi theorem). As one can see from the discussion of Subsection 3.2, the approach of examining the function by fixing t and varying r , opens up new possibilities which have no counterpart in the Hajnal-Szemerédi theorem. We note that our results suggest, but do not quite establish, the fact that for fixed t , $\delta(r, t)$ is a *decreasing* function of r . Further note that the weighted case has an extra power coming from the ability to include any edge in a K_r -factor, even if that edge has weight 0. This suggests that there could be a relation to results of Kuhn and Osthus [5] on the existence of H -factors in graphs.

Acknowledgement. We thank Wojciech Samotij for his valuable comments.

References

- [1] K. CORRÁDI AND A. HAJNAL, *On the maximal number of independent circuits in a graph*, Acta Math. Acad. Sci. Hungar., 14 (1963), pp. 423–439.

- [2] D. E. DAYKIN AND R. HÄGGKVIST, *Degrees giving independent edges in a hypergraph*, Bull. Austral. Math. Soc., 23 (1981), pp. 103–109.
- [3] A. HAJNAL AND E. SZEMERÉDI, *Proof of a conjecture of P. Erdős*, in Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969), North-Holland, Amsterdam, 1970, pp. 601–623.
- [4] H. HÀN, Y. PERSON, AND M. SCHACHT, *On perfect matchings in uniform hypergraphs with large minimum vertex degree*, SIAM J. Discrete Math., 23 (2009), pp. 732–748.
- [5] D. KÜHN AND D. OSTHUS, *The minimum degree threshold for perfect graph packings*, Combinatorica, 29 (2009), pp. 65–107.