# COUNTING DECOMPOSABLE UNIVARIATE POLYNOMIALS 

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#### Abstract

A univariate polynomial $f$ over a field is decomposable if it is the composition $f=g \circ h$ of two polynomials $g$ and $h$ whose degree is at least 2 . We determine an approximation to the number of decomposables over a finite field. The tame case, where the field characteristic $p$ does not divide the degree $n$ of $f$, is reasonably well understood, and we obtain exponentially decreasing relative error bounds. The wild case, where $p$ divides $n$, is more challenging and our error bounds are weaker.


Keywords. computer algebra, polynomial decomposition, multivariate polynomials, finite fields, combinatorics on polynomials

Subject classification. AMS classification: $68 \mathrm{~W} 30,11 \mathrm{~T} 06,12 \mathrm{E} 10$.

## 1. Introduction

It is intuitively clear that the decomposable polynomials form a small minority among all polynomials (univariate over a field). The goal in this work is to give a quantitative version of this intuition.

Our question has two facets: in the geometric view, we want to determine the dimension of the algebraic set of decomposable polynomials, say over an algebraically closed field. The combinatorial task is to approximate the number of decomposables over a finite field, together with a good relative error bound.

The first task is easy. For the second task, one readily obtains an upper bound. The challenge then is to find an essentially matching lower bound. Von zur Gathen (1990a,b) introduced the notion of tame for the case where the field characteristic does not divide the degree of the left component, and wild for the complementary case. (Schinzel (2000), § 1.5, uses tame in a different sense.) Algorithmically, the tame case is well understood since the breakthrough result of Kozen \& Landau (1986); see also von zur Gathen, Kozen \& Landau (1987); Kozen \& Landau (1989); Kozen, Landau \& Zippel (1996); Gutierrez \& Sevilla
(2006), and the survey articles of von zur Gathen (2002) and Gutierrez \& Kozen (2003) with further references. This leads to good estimates of the number of decomposable polynomials, provided that we can also apply a central tool in this area, namely Ritt's Second Theorem. This provision is satisfied if the square of the smallest prime divisor $\ell$ of the degree $n$ does not divide $n$.

In the wild case, the methods from the literature do not yield a satisfactory lower bound. We present in Section 3 a decomposition "algorithm" which fails on some inputs but works on sufficiently many ones. The algorithm is a centerpiece of this paper and yields lower bounds on the number of decomposable polynomials in the wild case.

An important tool for estimating the number of "collisions", where different pairs of components yield the same composition, is Ritt's Second Theorem. Ritt worked with $F=\mathbb{C}$ and used analytic methods. Subsequently, his approach was replaced by algebraic methods, in the work of Levi (1942) and Dorey \& Whaples (1974), and Schinzel (1982) presented an elementary but long and involved argument. Thus Ritt's Second Theorem was also shown to hold in positive characteristic $p$. The original versions of this required $p>\operatorname{deg}(g \circ h)$. Zannier (1993) reduced this to the milder and more natural requirement $g^{\prime}\left(g^{*}\right)^{\prime} \neq 0$. His proof works over an algebraic closed field, and Schinzel's 2000 monograph adapts it to finite fields. In Section 4, we provide a precise quantitative version of this Theorem, by determining exactly the number of such collisions in the tame case, assuming that $p \nmid n / \ell$. This is based on a unique normal form for the polynomials occurring in the Theorem. Furthermore, we give (less precise) substitutes in those cases where the Theorem is not applicable.

A uniqueness property in Ritt's Second Theorem is not obvious, and indeed Beardon \& Ng (2000) are puzzled by its absence. On their page 128, they write, translated to the present notation, "Now these rules are a little less transparent, and a little less independent, than may appear at first sight. First, we note that [the First Case], which is stated in its conventional form, is rather loosely defined, for the $k$ and $w$ are not uniquely determined by the form $x^{k} w\left(x^{\ell}\right)$; for instance, if $w(0)=0$, we can equally well write this expression in the form $x^{k+\ell} \tilde{w}\left(x^{\ell}\right)$, where $\tilde{w}=w / x$. Next, $T_{2}(x, 1)=x^{2}-2$ differs by a linear component from $x^{2}$, so that in some circumstances it is possible to apply [the Second Case] to $T_{2}(x, 1)$, then [a linear composition], and then (on what is essentially the same factor) [the Second Case]. These observations perhaps show why it is difficult to use Ritt's result." These well-motivated concerns are settled by the result of the present paper.

Section 5 presents the resulting estimates in the tame case. Section 6 puts together all our bounds in the general case, resulting in a veritable jungle of
case distinctions. It is not clear whether this is the nature of the problem or an artifact of our approach. The following is proved at the very end of the paper and provides a précis of our results-by necessity less precise than the individual bounds, in particular when $q \leq 4$ or $n$ is (close to) $\ell^{2}$. The basic statement is that $\alpha_{n}$ is an approximation to the number of decomposable polynomials of degree $n$, with relative error bounds of varying quality.

Main Theorem. Let $\mathbb{F}_{q}$ be a finite field with $q$ elements and characteristic $p$, let $\ell$ be the smallest prime divisor of the composite integer $n \geq 2, D_{n}$ the set of decomposable polynomials in $\mathbb{F}_{q}[x]$ of degree $n$, and

$$
\alpha_{n}= \begin{cases}2 q^{\ell+n / \ell}\left(1-q^{-1}\right) & \text { if } n \neq \ell^{2} \\ q^{2 \ell}\left(1-q^{-1}\right) & \text { if } n=\ell^{2}\end{cases}
$$

Then the following hold.
(i) $q^{2 \sqrt{n}} / 2 \leq \alpha_{n}<2 q^{n / 2+2}$.
(ii) $\alpha_{n} / 2 \leq \# D_{n} \leq \alpha_{n}\left(1+q^{-n / 3 \ell^{2}}\right)<2 \alpha_{n}<4 q^{n / 2+2}$.
(iii) If $n \neq p^{2}$ and $q>5$, then $\# D_{n} \geq\left(3-2 q^{-1}\right) \alpha_{n} / 4 \geq q^{2 \sqrt{n}} / 2$.
(iv) Unless $p=\ell$ and $p$ divides $n$ exactly twice, we have $\# D_{n} \geq \alpha_{n}\left(1-2 q^{-1}\right)$.
(v) If $p \nmid n$, then $\left|\# D_{n}-\alpha_{n}\right| \leq \alpha_{n} \cdot q^{-n / 3 \ell^{2}}$.

The upper and lower bounds in (ii) and (v) differ by a factor of $1+\epsilon$, with $\epsilon$ exponentially decreasing in the input size $n \log q$, in the tame case and for growing $n / 3 \ell^{2}$. When the field characteristic is the smallest prime divisor of $n$ and divides $n$ exactly twice, then we have a factor of about 2 , provided that the condition in (iii) is satisfied. In all other cases, the factor is $1+O\left(q^{-1}\right)$ over $\mathbb{F}_{q}$. It remains a challenge whether these gaps can be reduced.

Giesbrecht (1988) was the first to consider our counting problem. He showed that the decomposable polynomials form an exponentially small fraction of all univariate polynomials. My interest, dating back to the supervision of this thesis, was rekindled by a study of similar (but multivariate) counting problems (von zur Gathen 2008b) and during a visit to Pierre Dèbes' group at Lille, where I received a preliminary version of Bodin, Dèbes \& Najib (2009). Multivariate decomposable polynomials are counted in von zur Gathen (2008a).

We use the methods from von zur Gathen (2008b), where the corresponding counting task was solved for reducible, squareful, relatively irreducible,
and singular bivariate polynomials. Von zur Gathen, Viola \& Ziegler (2009) extends those results to multivariate polynomials. Recently, Zieve \& Müller (2008) found interesting characterizations of complete decompositions, where all components are indecomposable.

## 2. Decompositions

A nonzero polynomial $f \in F[x]$ over a field $F$ is monic if its leading coefficient $\operatorname{lc}(f)$ equals 1 . We call $f$ original if its graph contains the origin, that is, $f(0)=0$.

Definition 2.1. For $g, h \in F[x]$,

$$
f=g \circ h=g(h) \in F[x]
$$

is their composition. If $\operatorname{deg} g, \operatorname{deg} h \geq 2$, then $(g, h)$ is a decomposition of $f$. A polynomial $f \in F[x]$ is decomposable if there exist such $g$ and $h$, otherwise $f$ is indecomposable. The decomposition $(g, h)$ is normal if $h$ is monic and original.

Remark 2.2. Multiplication by a unit or addition of a constant does not change decomposability, since

$$
f=g \circ h \Longleftrightarrow a f+b=(a g+b) \circ h
$$

for all $f, g, h$ as above and $a, b \in F$ with $a \neq 0$. In other words, the set of decomposable polynomials is invariant under this action of $F^{\times} \times F$ on $F[x]$.

Furthermore, any decomposition ( $g, h$ ) can be normalized by this action, by taking $a=\operatorname{lc}(h)^{-1} \in F^{\times}, b=-a \cdot h(0) \in F, g^{*}=g\left((x-b) a^{-1}\right) \in F[x]$, and $h^{*}=a h+b$. Then $g \circ h=g^{*} \circ h^{*}$ and $\left(g^{*}, h^{*}\right)$ is normal.

We fix some notation for the remainder of this paper. For $n \geq 0$, we write

$$
P_{n}=\{f \in F[x]: \operatorname{deg} f \leq n\}
$$

for the vector space of polynomials of degree at most $n$, of dimension $n+1$. Furthermore, we consider the subsets

$$
\begin{aligned}
P_{n}^{=} & =\left\{f \in P_{n}: \operatorname{deg} f=n\right\} \\
P_{n}^{0} & =\left\{f \in P_{n}^{=}: f \text { monic and original }\right\} .
\end{aligned}
$$

Over an infinite field, the first of these is the Zariski-open subset $P_{n} \backslash P_{n-1}$ of $P_{n}$, and thus irreducible, taking $P_{-1}=\{0\}$. The second one is obtained by
further imposing one equation and working modulo multiplication by units, so that

$$
\begin{aligned}
\operatorname{dim} P_{n}^{=} & =n+1 \\
\operatorname{dim} P_{n}^{0} & =n-1,
\end{aligned}
$$

with $P_{0}^{0}=\varnothing$. For any divisor $e$ of $n$, we have the normal composition map

$$
\gamma_{n, e}: \begin{aligned}
P_{e}^{=} \times P_{n / e}^{0} & \longrightarrow P_{n}^{=}, \\
(g, h) & \longmapsto g \circ h,
\end{aligned}
$$

corresponding to Definition 2.1, and set

$$
\begin{equation*}
D_{n, e}=\operatorname{im} \gamma_{n, e} . \tag{2.3}
\end{equation*}
$$

The set $D_{n}$ of all decomposable polynomials in $P_{n}^{=}$satisfies

$$
\begin{equation*}
D_{n}=\bigcup_{\substack{e \mid n \\ 1<e<n}} D_{n, e} \tag{2.4}
\end{equation*}
$$

In particular, $D_{n}=\varnothing$ if $n$ is prime. We also let $I_{n}=P_{n}^{=} \backslash D_{n}$ be the set of indecomposable polynomials. Over a finite field $\mathbb{F}_{q}$ with $q$ elements, we have

$$
\begin{aligned}
\# P_{n}^{=} & =q^{n+1}\left(1-q^{-1}\right) \\
\# P_{n}^{0} & =q^{n-1} \\
\# D_{n, e} & \leq q^{e+n / e}\left(1-q^{-1}\right) .
\end{aligned}
$$

Remark 2.5. By Remark 2.2, over an algebraically closed field, the codimension of $D_{n}$ in $P_{n}^{=}$equals that of $D_{n} \cap P_{n}^{0}$ in $P_{n}^{0}$. The same holds for $I_{n}$, and over a finite field for the corresponding fractions:

$$
\frac{\# D_{n}}{\# P_{n}^{=}}=\frac{\#\left(D_{n} \cap P_{n}^{0}\right)}{\# P_{n}^{0}}
$$

Example 2.6. We look at normal decompositions $(g, h)$ of univariate quartic polynomials $f$, so that $n=4$. By Remark 2.2 , we may assume $f \in P_{4}^{0}$, and then also $g$ is monic with constant coefficient 0 . Thus the general case is

$$
\left(x^{2}+a x\right) \circ\left(x^{2}+b x\right)=x^{4}+u x^{3}+v x^{2}+w x \in F[x],
$$

with $a, b, u, v, w \in F$. We find that with $a=2 w / u$ and $b=u / 2$ (assuming $2 u \neq 0$ ), the cubic and linear coefficients match, and the whole decomposition does if and only if

$$
u^{3}-4 u v+8 w=0
$$

This is a defining equation for the hypersurface of decomposable polynomials in $P_{4}^{0}$ (if char $F \neq 2$ ). Translating back to $P_{4}^{=}$, we have

$$
\operatorname{dim} D_{4}=4<5=\operatorname{dim} P_{4}^{=}
$$

This example is also in Barton \& Zippel (1985, 1976).

## 3. Equal-degree collisions

A decomposition $(g, h)$ of $f=g \circ h$ over a field of characteristic $p$ is called tame if $p \nmid \operatorname{deg} g$, and wild otherwise, in analogy with ramification indices. The polynomial $f$ itself is tame if $p \nmid \operatorname{deg} f$, and wild otherwise. The tame case is well understood, both theoretically and algorithmically. The wild case is more difficult and less well understood; there are polynomials with superpolynomially many "inequivalent" decompositions (Giesbrecht 1988).

For $u, v \in F[x]$ and $j \in \mathbb{N}$, we write

$$
u=v+O\left(x^{j}\right)
$$

if $\operatorname{deg}(u-v) \leq j$. We start with two facts from the literature concerning the injectivity of the composition map. When $p \mid n$, a polynomial $f=x^{n}+f_{i} x^{i}+$ $O\left(x^{i-1}\right)$ with $f_{i} \neq 0$ is called simple if $p \nmid i$ or $i<n-p$.

FACT 3.1. Let $F$ be a field of characteristic $p$, and $e$ a divisor of $n \geq 2$.
(i) If $p$ does not divide $e$, then $\gamma_{n, e}$ is injective, and

$$
\# D_{n, e}=q^{e+n / e}\left(1-q^{-1}\right)
$$

(ii) If $p$ divides $n$ exactly $d$ times and $f \in F[x]$ is simple, then $f$ has at most $s<2 p^{d} \leq 2 n$ normal decompositions, where $s=\left(p^{d+1}-1\right) /(p-1)=$ $1+p+\cdots+p^{d}$.

Proof. The uniqueness in (i) is well-known, see e.g., von zur Gathen (1990a) and the references therein. (ii) follows from von zur Gathen (1990b), where the above notion of a simple polynomial is defined, and (the proof of) Corollary 3.6 of that paper shows that there are at most $s$ such decompositions of $f$.

The paper cited for (ii) also gives an algorithm to decide decomposability and, in that case, to compute all such decompositions. This only applies to "simple" polynomials, and no nontrivial general upper bound on the number of decompositions seems to be known.

Algorithm 3.14 below uses a similar approach. On the one hand, it applies to more restricted inputs. On the other hand, it is faster (roughly, $n^{2}$ vs. $n^{4}$ ), more transparent and hence easier to analyze, and yields a lower bound on the number of decomposables at fixed component degrees.

In Section 5 , we find an upper bound $\alpha_{n}$ on $\# D_{n}$, up to some small relative error. When the exact size of the error term is not a concern, then this is quite easy. Furthermore, Fact 3.1 immediately yields a lower bound of $\alpha_{n} / 2$ if $p$ is not the smallest prime divisor $\ell$ of $n$, and of about $\alpha_{n} / 4 n$ in general, since "most" polynomials are simple.

Our goal in this paper is to improve these estimates. For this purpose, we have to address the uniqueness (or lack thereof) of normal compositions

$$
\begin{equation*}
g \circ h=g^{*} \circ h^{*} \tag{3.2}
\end{equation*}
$$

in two situations. We call $\left\{(g, h),\left(g^{*}, h^{*}\right)\right\}$ satisfying (3.2) with $h \neq h^{*}$ an equaldegree collision if $\operatorname{deg} g=\operatorname{deg} g^{*}$ (and hence $\operatorname{deg} h=\operatorname{deg} h^{*}$ ), and a distinctdegree collision if $\operatorname{deg} g=\operatorname{deg} h^{*} \neq \operatorname{deg} h$ (and hence $\operatorname{deg} h=\operatorname{deg} g^{*}$ ). The present section deals with equal-degree collisions, and Section 4 with distinctdegree collisions.

By Fact 3.1(i), there are no equal-degree collisions when $p \nmid \operatorname{deg} g$. In the more interesting case $p \mid \operatorname{deg} g$, collisions are well-known to exist; Example 3.46 exhibits all collisions over $\mathbb{F}_{3}$ at degree 9 . Our goal, then, is to show that there are few of them, so that the decomposable polynomials are still numerous. Algorithm 3.14 provides a constructive proof of this. For many, but not all, $(g, h)$ it reconstructs $(g, h)$ from $g \circ h$. To quantify the benefit provided by the algorithm, we rely on a result by Antonia Bluher (2004).

Distinct-degree collisions are classically taken care of by Ritt's Second Theorem. Some versions put a restriction on $p$ that would make our task difficult, but Umberto Zannier (1993) has cut this restriction down to the bare minimum. The additional common restriction that $\operatorname{gcd}(\operatorname{deg} g, \operatorname{deg} h)=1$ has essentially been removed by Tortrat (1988), but only if $p$ does not divide the degree. If,
in addition, the composition is wild, then a look at derivatives provides a reasonable bound. It is useful to single out a special case of wild compositions.

Definition 3.3. We call Frobenius composition any $f \in F\left[x^{p}\right]$, since then $f=x^{p} \circ h^{*}$ for some $h^{*} \in P_{n / p}^{=}$, and any decomposition $(g, h)$ of $f=g \circ h$ is a Frobenius decomposition. A Frobenius collision is the following example of a collision (3.2). For any integer $j$, we denote by $\varphi_{j}: F \longrightarrow F$ the $j$ th power of the Frobenius automorphism over a field $F$ of characteristic $p$, with $\varphi_{j}(a)=a^{p^{j}}$ for all $a \in F$, and extend it to an $\mathbb{F}_{p}$-linear isomorphism $\varphi_{j}: F[x] \longrightarrow F[x]$ with $\varphi_{j}(x)=x$. Then if $h \in F[x]$, we have

$$
\begin{equation*}
x^{p^{j}} \circ h=\varphi_{j}(h) \circ x^{p^{j}} . \tag{3.4}
\end{equation*}
$$

Thus any Frobenius composition except $x^{p^{2}}$ is the result of a collision. Over $F=\mathbb{F}_{q}$, there are $q^{p^{j}-1}-1$ many $h \in P_{p^{j}}^{0}$ with $h \neq x^{p^{j}}$ and for $m \neq p^{j}$, this produces $q^{m-1}$ collisions with $h \in P_{m}^{0}$. By composing with a linear function, we obtain $q^{p^{j}+1}\left(1-q^{-1}\right)\left(1-q^{-p^{j}+1}\right)$ and $q^{m+1}\left(1-q^{-1}\right)$ Frobenius collisions for $m=p^{j}$ and $m \neq p^{j}$, respectively. This example is noted in Schinzel (1982), Section I.5, page 39 .

The Frobenius compositions from Definition 3.3 are easily described and counted. It is useful to separate them from the others. If $p \mid n$ and $\ell$ is a proper divisor of $n$, we set

$$
\begin{align*}
D_{n}^{\varphi} & =D_{n} \cap F\left[x^{p}\right], \\
D_{n}^{+} & =D_{n} \backslash D_{n}^{\varphi},  \tag{3.5}\\
D_{n, \ell}^{+} & =D_{n, \ell} \cap D_{n}^{+}
\end{align*}
$$

so that $D_{n}^{\varphi}$ comprises exactly the Frobenius compositions of degree $n$.
Von zur Gathen (1990b) presents an algorithm for certain "wild" decompositions $f=g \circ h$ with

$$
\operatorname{deg} f=n=k \cdot m=\operatorname{deg} g \cdot \operatorname{deg} h
$$

and $p \mid k$. It first makes coefficient comparisons to compute $h$, and then a Taylor expansion to find $g$. We now take a simplified version of that method. It does not work for all inputs, but for sufficiently many for our counting purpose. In general, decomposing a polynomial can be done by solving the corresponding system of equations in the coefficients of the unknown components, say, using Gröbner bases.

To fix some notation, we have integers

$$
\begin{equation*}
d \geq 1, r=p^{d}, k=a r, m \geq 2, n=k m, \kappa \text { with } 0 \leq \kappa<k \text { and } p \nmid a \kappa, \tag{3.6}
\end{equation*}
$$

and polynomials

$$
\begin{align*}
& g=x^{k}+\sum_{1 \leq i \leq \kappa} g_{i} x^{i}, \\
& h=\sum_{1 \leq i \leq m} h_{i} x^{i},  \tag{3.7}\\
& f=g \circ h=h^{k}+\sum_{1 \leq i \leq \kappa} g_{i} h^{i},
\end{align*}
$$

with $h_{m}=1, h_{m-1} \neq 0$, and either $g_{\kappa} \neq 0$ or $g=x^{k}$; the latter case corresponds to $\kappa=0$. The idea is to compute $h_{i}$ for $i=m-1, m-2, \ldots, 1$ by comparing the known coefficients of $f$ to the unknown ones of $h^{k}$ and $g_{\kappa} h^{\kappa}$. Special situations arise when the latter two polynomials both contribute to a coefficient. We denote by

$$
h^{(i)}=\sum_{i<b<m} h_{b} x^{b}
$$

the top part of $h$, so that $h^{(m-1)}=0$. Furthermore, we write $\operatorname{coeff}(v, j)$ for the coefficient of $x^{j}$ in a polynomial $v$, and

$$
c_{i, j}(v)=\operatorname{coeff}\left(v \circ\left(h-h^{(i)}\right), j\right) .
$$

Thus $c_{m-1, j}\left(x^{k}\right)=\operatorname{coeff}\left(h^{k}, j\right)$, and in particular, we have $c_{m-1, j}(g)=f_{j}$ for all $j$. To illustrate the usage of these $c_{i j}$, we consider $E_{1}$ below. At some point in the algorithm, we have determined $g_{\kappa}, h_{m}, \ldots, h_{i+1}$. The appropriate $c_{i j}$ exhibits $h_{i}$ in a simple fashion, meaning that we can compute it from $f_{j}$ and $h^{(i)}$. Lastly we define the rational number

$$
\begin{equation*}
i_{0}=m\left(\frac{\kappa-a}{r-1}-a+1\right)=\frac{\kappa m-n}{r-1}+m ; \tag{3.8}
\end{equation*}
$$

thus $i_{0}<m$, and $i_{0}$ is an integer if and only if

$$
r-1 \mid(\kappa-a) m
$$

LEMMA 3.9. For $1 \leq i \leq m$ and $0 \leq j \leq n$, we have the following.
$E_{1}$ : If $i<m$, then

$$
\begin{equation*}
c_{i,(\kappa-1) m+i}\left(g_{\kappa} x^{\kappa}\right)=\kappa g_{\kappa} h_{i}, \tag{3.10}
\end{equation*}
$$

and $c_{m-1, \kappa m}\left(g_{\kappa} x^{\kappa}\right)=g_{\kappa}$.
$E_{2}:$ If $i<m$, then

$$
\begin{equation*}
c_{i, n-r(m-i)}\left(x^{k}\right)=a h_{i}^{r} . \tag{3.11}
\end{equation*}
$$

If $r \nmid j$, then $\operatorname{coeff}\left(h^{k}, j\right)=0$.
$E_{3}:$ If $i_{0} \in \mathbb{N}$, then

$$
\begin{equation*}
c_{i_{0},(\kappa-1) m+i_{0}}\left(x^{k}+g_{\kappa} x^{\kappa}\right)=a h_{i_{0}}^{r}+\kappa g_{\kappa} h_{i_{0}} . \tag{3.12}
\end{equation*}
$$

$E_{4}:$ If $m=r$ and $\kappa=k-1$, then

$$
\begin{align*}
c_{m-1, \kappa m}\left(x^{k}+g_{\kappa} x^{\kappa}\right) & =a h_{m-1}^{r}+g_{\kappa}, \\
c_{m-1, \kappa m-1}\left(x^{k}+g_{\kappa} x^{\kappa}\right) & =-g_{\kappa} h_{m-1} . \tag{3.13}
\end{align*}
$$

Proof. For $E_{1}$, we have to consider

$$
g_{\kappa}\left(x^{m}+h_{i} x^{i}+O\left(x^{i-1}\right)\right)^{\kappa}=g_{\kappa} x^{\kappa a}+\kappa^{\prime} g_{\kappa} h_{i} x^{(\kappa-1) m+i}+O\left(x^{(\kappa-1) m+i-1}\right),
$$

furthermore

$$
\begin{aligned}
c_{i,(\kappa-1) m+i}\left(g_{\kappa} x^{\kappa}\right) & =g_{\kappa} \cdot \kappa h_{i} \\
c_{m, \kappa m}\left(g_{\kappa} x^{\kappa}\right) & =\operatorname{coeff}\left(g_{\kappa} h^{\kappa}, \kappa m\right)=g_{\kappa},
\end{aligned}
$$

and $E_{1}$ follows. For $E_{2}$, we have

$$
h^{a}=x^{a m}+a h_{m-1} x^{a m-1}+O\left(x^{a m-2}\right) .
$$

When $i<m$, then in the coefficient of $x^{(a-1) m+i}$, we have the contribution $a h_{i}$, which comes from taking in the expansion of $h^{a}$ the factor $x^{m}$ exactly $a-1$ times and the factor $h_{i} x^{i}$ exactly once; there are $a$ ways to make these choices. The largest degree to which a summand $h_{j} x^{j}$ contributes in $h^{a}$ is $(a-1) m+j$, so that those with $j<i$ do not appear in the coefficient under consideration, and $c_{i,(a-1) m+i}\left(x^{a}\right)=a h_{i}$. Raising $h^{a}$ to the $r$ th power yields

$$
c_{i,((a-1) m+i) r}\left(x^{k}\right)=c_{i,((a-1) m+i) r}\left(\left(x^{a}\right)^{r}\right)=a^{r} h_{i}^{r}=a h_{i}^{r}
$$

and proves $E_{2}$, since $((a-1) m+i) r=n-r(m-i)$.
For $E_{3}$, we have

$$
\begin{aligned}
(\kappa-1) m+i_{0} & =n-r\left(m-i_{0}\right), \\
c_{i_{0},(\kappa-1) m+i_{0}}\left(x^{\kappa}+g_{\kappa} x^{\kappa}\right) & =c_{i_{0}, n-r\left(m-i_{0}\right)}\left(x^{k}\right)+c_{i_{0},(\kappa-1) m+i_{0}}\left(g_{\kappa} x^{\kappa}\right) \\
& =a h_{i_{0}}^{r}+\kappa g_{\kappa} h_{i_{0}} .
\end{aligned}
$$

For $E_{4}$, we have $\kappa m=n-m$ and from $E_{1}$ and $E_{2}$

$$
\begin{aligned}
c_{m-1, \kappa m}\left(x^{k}+g_{\kappa} x^{\kappa}\right) & =c_{m-1, n-m}\left(x^{k}\right)+c_{m-1, \kappa m}\left(g_{\kappa} x^{\kappa}\right)=a h_{m-1}^{r}+g_{\kappa}, \\
c_{m-1, \kappa m-1}\left(x^{k}+g_{\kappa} x^{\kappa}\right) & =\operatorname{coeff}\left(h^{k}, \kappa m-1\right)+c_{m-1, \kappa m-1}\left(g_{\kappa} x^{\kappa}\right) \\
& =0+\kappa g_{\kappa} h_{m-1}=-g_{\kappa} h_{m-1} .
\end{aligned}
$$

In the following algorithm, the instruction "determine $h_{i}$ (or $g_{\kappa}$ ) by $E_{\mu}$ (at $\left.x^{j}\right)^{\prime \prime}$, for $1 \leq \mu \leq 4$, means that the property $E_{\mu}$ involves some quantity $c_{i j}(\cdot)$ which is a summand in $\operatorname{coeff}(g \circ h, j)=f_{j}$, the other summands are already known, and we can solve for $h_{i}\left(\right.$ or $\left.g_{\kappa}\right)$. When we use $E_{2}$, we first compute $y=h_{i}^{r}$ and then $h_{i}$ by extracting the $r$ th root of $y$. Over a finite field, this always yields a unique answer, since $r$ is a power of $p$. But in general, $y$ might not have an $r$ th root. We say "compute $h_{i}^{r}$ by $E_{2}$, then $h_{i}$ if possible" to mean that first $y$ is determined, then $h_{i}$ as its $r$ th root; if $y$ does not have an $r$ th root, then the empty set is returned.

The main effort in the correctness proof is to show that all data required are available at that point in the algorithm, and that the equation can indeed be solved. The algorithm's basic structure is driven by the relationship between the degrees $\kappa m$ of $g_{\kappa} h^{\kappa}$ and $n-r$ of $h^{k}-x^{n}$.

Algorithm 3.14. Wild decomposition.
Input: $f \in F[x]$ monic and original of degree $n=k m$, where $F$ is a field of characteristic $p \geq 2, d \geq 1, r=p^{d}$, and $k=a r$ with $p \nmid a$.
Output: Either a set of at most $r+1$ pairs $(g, h)$ with $g, h \in F[x]$ monic and original of degrees $k$ and $m$, respectively, and $f=g \circ h$, or "failure".

1. Let $j$ be the largest integer for which $f_{j} \neq 0$ and $p \nmid j$. If no such $j$ exists then if $d \geq 2$ call Algorithm 3.14 recursively and else call a tame decomposition algorithm, in either case with input $f^{*}=f^{1 / p}$ and $k^{*}=k / p$. If a set of $\left(g^{*}, h^{*}\right)$ is output by the call, then return the set of all Frobenius compositions $\left(x^{p} \circ g^{*}, h^{*}\right)$.
2. If $p \nmid m$ then if $m \nmid j$ then return "failure" else set $\kappa=j / m$. If $p \mid m$ then if $m \nmid j+1$ then return "failure" else set $\kappa=(j+1) / m$. If $p \mid \kappa$, then return "failure". Calculate $i_{0}=(\kappa m-n) /(r-1)+m$.
3. If $\kappa m \geq n-r+2$ then do the following.
a. Set $g_{\kappa}=f_{\kappa m}$.
b. Determine $h_{i}$ for $i=m-1, \ldots, 1$ by $E_{1}$.
4. If $\kappa m=n-r+1$ then do the following.
a. Set $g_{\kappa}=f_{\kappa m}$.
b. Determine $h_{m-1}$ by $E_{3}$. If (3.12) does not have a unique solution, then return "failure".
c. Determine $h_{i}$ for $i=m-2, \ldots, 1$ by $E_{1}$.
5. If $\kappa m=n-r$ then do the following.
a. Determine $h_{m-1}$ by $E_{4}$, in the following way. Compute the set $S$ of all nonzero $s \in \mathbb{F}_{q}$ with

$$
\begin{equation*}
a s^{r+1}-f_{\kappa m} s-f_{\kappa m-1}=0 . \tag{3.15}
\end{equation*}
$$

If $S=\varnothing$ then return the empty set, else do steps 5.b and 5.c for all $s \in S$, setting $h_{m-1}=s$.
b. Determine $g_{\kappa}$ by $E_{1}$ and $E_{2}$ at $x^{\kappa m}$, from $f_{\kappa m}=a h_{m-1}^{r}+g_{\kappa}$.
c. For $i=m-2, \ldots, 1$ determine $h_{i}$ by $E_{1}$.
6. If $\kappa m<n-r$ then do the following.
a. Determine $h_{m-1}^{r}$ by $E_{2}$, then $h_{m-1}$ if possible.
b. If $r \nmid m$ then determine $g_{\kappa}$ by $E_{1}$ at $x^{\kappa m}\left(\right.$ as $\left.g_{\kappa}=f_{\kappa m}\right)$, else by $E_{1}$ at $x^{\kappa m-1}\left(\right.$ via $\left.\kappa g_{\kappa} h_{m-1}=f_{\kappa m-1}\right)$.
c. Determine $h_{i}^{r}$ by $E_{2}$, then $h_{i}$ if possible, for decreasing $i$ with $m-2 \geq$ $i>i_{0}$.
d. If $i_{0}$ is a positive integer, then determine $h_{i_{0}}$ by $E_{3}$. If $E_{3}$ does not yield a unique solution, then return "failure".
e. Determine $h_{i}$ for decreasing $i$ with $i_{0}>i \geq 1$ by $E_{1}$.
7. [We now know $h$.] Compute the remaining coefficients $g_{1}, \ldots, g_{\kappa-1}$ as the "Taylor coefficients" of $f$ in base $h$.
8. Return the set of all $(g, h)$ for which $g \circ h=f$. If there are none, then return the empty set.

The Taylor expansion method determines for given $f$ and $h$ the unique $g$ (if one exists) so that $f=g \circ h$; see von zur Gathen (1990a).

We first illustrate the algorithm in some examples.

Example 3.16. We let $p=5, n=50$, and $k=r=5$, so that $a=d=1$ and $m=10$, and start with $\kappa=4=r-1$. We assume $f_{39}=g_{4} h_{9} \neq 0$. Then

$$
\begin{gathered}
h^{5}+g_{4} h^{4}=x^{50}+h_{9}^{5} x^{45}+\left(h_{8}^{5}+g_{4}\right) x^{40}+4 g_{4} h_{9} x^{39}+g_{4}\left(4 h_{8}+h_{9}^{2}\right) x^{38} \\
+x^{36} \cdot O(x)+\left(h_{7}^{5}+g_{4}\left(4 h_{5}+h_{9} h_{6}+h_{8} h_{7}+h_{9}^{2} h_{7}+h_{9} h_{8}^{2}+h_{9}^{3} h_{8}\right)\right) x^{35}+O\left(x^{34}\right) .
\end{gathered}
$$

Step 1 determines $j=39$, and step 2 finds $\kappa=(39+1) / 10$ and $i_{0}=15 / 2 \notin$ $\mathbb{N}$. Since $\kappa m=40<45=n-r$, we go to step 6. Step 6.a computes $h_{9}$ at $x^{45}$, step 6.b yields $g_{4}$ at $x^{39}$, step 6.c determines $h_{8}$ at $x^{40}$ by $E_{2}$, step $6 . d$ is skipped, and then step 6 .e yields $h_{7}, \ldots, h_{1}$ at $x^{37}, \ldots, x^{31}$, respectively, all using $E_{1}$. Step 7 determines $g_{1}, g_{2}, g_{3}$, and step 8 checks whether indeed $f=g \circ h$, and if so, returns $(g, h)$.

With the same values, except that $\kappa=3$, we have

$$
\begin{aligned}
h^{5}+g_{3} h^{3}= & x^{50}+h_{9}^{5} x^{45}+h_{8}^{5} x^{40}+h_{7}^{5} x^{35} \\
& +\left(h_{6}^{5}+g_{3}\right) x^{30}+3 g_{3} h_{9} x^{29}+g_{3}\left(3 h_{9}^{2}+3 h_{8}\right) x^{28}+x^{26} \cdot O(x) \\
& +\left(h_{5}^{5}+g_{3}\left(3 h_{5}+3 h_{9} h_{6}+3 h_{8} h_{7}+3 h_{9}^{2} h_{7}+3 h_{9} h_{8}^{2}\right)\right) x^{25}+O\left(x^{24}\right) .
\end{aligned}
$$

Assuming that $f_{29}=3 g_{3} h_{9} \neq 0$, the algorithm computes $j=29, \kappa=$ $(29+1) / 10, i_{0}=5 \in \mathbb{N}$, goes to step 6 , determines $h_{9}$ at $x^{45}, g_{3}$ at $x^{29}, h_{8}, h_{7}$, $h_{6}$ according to $E_{2}$, then $h_{5}$ at $x^{25}$ via the known value for $h_{5}^{5}+3 g_{3} h_{5}$ in step 6.d with $E_{3}$. Condition (3.18) below requires that $\left(-3 g_{3}\right)^{(q-1) / 4} \neq 1$ and guarantees that $h_{5}$ is uniquely determined, as shown in the proof of Theorem 3.17 below. Finally $h_{4}, \ldots, h_{1}$ and $g_{1}, g_{2}$ are computed.

As a last example, we take $p=5, n=25, k=r=m=5$ and $\kappa=4$, so that $a=1$ and

$$
h^{5}+g_{4} h^{4}=x^{25}+\left(h_{4}^{5}+g_{4}\right) x^{20}+4 g_{4} h_{4} x^{19}+O\left(x^{18}\right) .
$$

Again we assume $f_{19}=4 g_{4} h_{4} \neq 0$. Then steps 1 and 2 determine $j=19$, $\kappa=4$, and $i_{0}=15 / 4 \notin \mathbb{N}$. We have $\kappa m=20=n-r$, so that we go to step 5. In step 5.a, we have to solve (3.15). The number of solutions is discussed starting with Fact 3.25 below. We consider two special cases, namely $q=5$ and $q=125$. For $q=5$, we have 25 pairs $(v, w)=\left(f_{20}, f_{19}\right) \in \mathbb{F}_{5}^{2}$ to consider, with $w \neq 0$. When $v \neq 0$, then the number of solutions is

$$
\begin{cases}2 & \text { if } w v^{-2} \in\{2,0\} \\ 1 & \text { if } w v^{-2}=1 \\ 0 & \text { otherwise }\end{cases}
$$

and when $v=0$ :

$$
\begin{cases}2 & \text { for the squares } w=1,4 \\ 0 & \text { otherwise. }\end{cases}
$$

Over $\mathbb{F}_{125}$, we have the following numbers of nonzero solutions $s$ when $v \neq 0$ :

$$
\begin{cases}6 & \text { for } 1 \cdot 124 \text { values }(v, w) \\ 2 & \text { for } 47 \cdot 124 \text { values }(v, w) \\ 1 & \text { for } 25 \cdot 124 \text { values }(v, w) \\ 0 & \text { for } 52 \cdot 124 \text { values }(v, w)\end{cases}
$$

and when $v=0$ :

$$
\begin{cases}2 & \text { for } 62 \text { values of } w, \text { namely the squares, } \\ 0 & \text { for } 62 \text { values of } w\end{cases}
$$

These numbers are explained below. We run the remaining steps in parallel for each value $h_{4}=s$ with $s \in S$. This yields $g_{4}$ in step 5.b, $h_{3}, h_{2}, h_{1}$ in step 5.c, and $g_{1}, g_{2}, g_{3}$ in step 7 .

We denote by $\mathrm{M}(n)$ a multiplication time, so that polynomials of degree at most $n$ can be multiplied with $\mathrm{M}(n)$ operations in $F$. Then $\mathrm{M}(n)$ is in $O(n \log n \log \log n)$; see von zur Gathen \& Gerhard (2003), Chapter 8, and Fürer (2007) for an improvement.

For an input $f$, we set $\sigma(f)=\# S$ if the precondition of step 5 is satisfied and $S$ computed there, and otherwise $\sigma(f)=1$.

Theorem 3.17. Let $f$ be an input polynomial with parameters $n, p, q=p^{e}$, $d, r, a, k, m$ as specified, $g, h, \kappa, i_{0}$ as in (3.7) and (3.8), so that $f=g \circ h$, set $c=\operatorname{gcd}(d, e)$ and suppose further that

$$
\begin{equation*}
\text { if } i_{0} \in \mathbb{N} \text { and } 1 \leq i_{0}<m, \text { then }\left(-\kappa g_{\kappa} / a\right)^{(q-1) /\left(p^{c}-1\right)} \neq 1 \tag{3.18}
\end{equation*}
$$

On input $f$, Algorithm 3.14 returns either "failure" or a set of at most $\sigma(f)$ normal decompositions $\left(g^{*}, h^{*}\right)$ of $f$, and $(g, h)$ is one of them. Except if returned in step 1, none of them is a Frobenius decomposition. If $F=\mathbb{F}_{q}$ is finite, then the algorithm uses

$$
O(\mathrm{M}(n) \log k(m+\log (k q)))
$$

or $O^{\sim}(n(m+\log q))$ operations in $\mathbb{F}_{q}$.

Proof. Since $r=p^{d} \mid k$, we have coeff $\left(h^{k}, j\right)=0$ unless $r \mid j$. Furthermore $g_{\kappa} h^{\kappa}=g_{\kappa} x^{\kappa m}+\kappa g_{\kappa} h_{m-1} x^{\kappa m-1}+O\left(x^{\kappa m-2}\right)$ and $\kappa g_{\kappa} h_{m-1} \neq 0$, so that $j$ from step 1 equals $\kappa m$ (if $p \nmid m$ ) or $\kappa m-1$ (if $p \mid m$ ). Thus $\kappa$ is correctly determined in step 2. In particular, $f$ is not a Frobenius composition.

We denote by $G$ the set of $(g, h)$ allowed in the theorem. We claim that the equations used in the algorithm involve only coefficients of $f$ and previously computed values, and usually have a unique solution. It follows that most $f \in \gamma_{n, k}(G)$ are correctly and uniquely decomposed by the algorithm. The only exception to the uniqueness occurs in (3.15).

In the remaining steps, we use various coefficients $f_{j}$ for $j=(\kappa-1) m+i$ with $1 \leq i \leq m$ or $j=n-r(m-i)$ with $i_{0} \leq i<m$. The value $i_{0}$ is defined so that $n-r\left(m-i_{0}\right)=(\kappa-1) m+i_{0}$, and thus

$$
\begin{equation*}
n-r(m-i) \geq(\kappa-1) m+i \text { if and only if } i \geq i_{0} \tag{3.19}
\end{equation*}
$$

since the first linear function in $i$ has the slope $r>1$, greater than for the second one. Since $i \geq 1$, it follows that $j>(\kappa-1) m$ for all $j$ under consideration. For the low-degree part of $g$ we have

$$
\operatorname{deg}\left(\left(g-\left(x^{k}+g_{\kappa} x^{\kappa}\right)\right) \circ h\right) \leq(\kappa-1) m<j,
$$

so that

$$
f_{j}=\operatorname{coeff}(g \circ h, j)=\operatorname{coeff}\left(\left(x^{k}+g_{\kappa} x^{\kappa}\right) \circ h, j\right)=\operatorname{coeff}\left(h^{k}+g_{\kappa} h^{\kappa}, j\right)
$$

for all $j$ in the algorithm.
We have to see that the application of $E_{3}$ in steps 4.b (where $i_{0}=m-1$ ) and 6 .d (where $m-2 \geq i_{0} \geq 1$ ) always has a unique solution. The right hand side of (3.12), say $a s^{r}+\kappa g_{\kappa} s$, is an $\mathbb{F}_{p}$-linear function of $s$. The equation has a unique solution if and only if its kernel is $\{0\}$. (Segre 1964, Teil 1 , $\ddot{i}<\frac{1}{2} 3$, and Wan 1990 provide an explicit solution in this case.) But when $s \in \mathbb{F}_{q}$ is nonzero with $a s^{r}+\kappa g_{\kappa} s=0$, then $-\kappa g_{\kappa} / a=s^{r-1}$. Writing $z=p^{c}$, so that $z-1=\operatorname{gcd}(q-1, r-1)$, we have

$$
\left(-\kappa g_{\kappa} / a\right)^{(q-1) /(z-1)}=\left(s^{r-1}\right)^{(q-1) /(z-1)}=\left(s^{(r-1) /(z-1)}\right)^{q-1}=1,
$$

contradicting the condition (3.18).
For the correctness it is sufficient to show that all required quantities are known, in particular $c_{i, j}\left(g_{\kappa} x^{\kappa}\right)$ in $E_{1}$ and $c_{i, j}\left(x^{k}\right)$ in $E_{2}$, and that the equations determine the coefficient to be computed. We have

$$
\begin{equation*}
\operatorname{deg}\left(h^{k}-x^{n}\right)=\operatorname{deg}\left(\left(h^{a}-x^{a m}\right)^{r}\right) \leq(a m-1) r=n-r, \tag{3.20}
\end{equation*}
$$

so that $g_{\kappa}=f_{\kappa m}$ in steps 3.a and 4.a.
The precondition of step 3 implies that for all $i<m$ we have

$$
\begin{gathered}
(\kappa-1) m \geq n-r-m+2>n-m r+(r-1)(m-1) \geq n-r m+(r-1) i, \\
n-r(m-i)<(\kappa-1) m+i .
\end{gathered}
$$

Thus from $E_{1}$ we have with $j=(\kappa-1) m-i$

$$
\begin{aligned}
f_{(\kappa-1) m+i} & =\operatorname{coeff}\left(h^{k}, j\right)+\operatorname{coeff}\left(g_{\kappa} h^{\kappa}, j\right) \\
& =\operatorname{coeff}\left(\left(h^{(i)}\right)^{k}, j\right)+\kappa g_{\kappa} h_{i}
\end{aligned}
$$

with $\kappa g_{\kappa} \neq 0$, so that $h_{i}$ can be computed in step 3.b.
The precondition in step 4 implies that $i_{0}=m-1$, and hence $(r-1) \mid$ $(a-\kappa) m . E_{3}$ says that $f_{\kappa m-1}=c_{m-1, \kappa m-1}\left(x^{k}+g_{\kappa} x^{\kappa}\right)=a h_{m-1}^{r}+\kappa g_{\kappa} h_{m-1}$. We have seen above that under our assumptions the equation $f_{\kappa m-1}=a s^{r}+\kappa g_{\kappa} s$ has exactly one solution $s$. By an argument as for step 3.b, also step 4.c works correctly.

The only usage of $E_{4}$ occurs in step 5.a, where $\kappa=(n-r) / m=k-r / m$. Since $p \mid k, r$ is a power of $p$, and $p \nmid \kappa$, this implies that $r=m$ and $\kappa=k-1$. We have from $E_{4}$

$$
\begin{aligned}
f_{\kappa m} & =a h_{m-1}^{r}+g_{\kappa} \\
f_{\kappa m-1} & =-g_{\kappa} h_{m-1}=-\left(f_{\kappa m}-a h_{m-1}^{r}\right) h_{m-1}=a h_{m-1}^{r+1}-f_{\kappa m} h_{m-1}
\end{aligned}
$$

Thus $h_{m-1} \in S$ as computed in step 5.a and $g_{\kappa}$ is correctly determined in step 5.b. The precondition of step 5 implies that $i_{0}=m-1-1 /(r-1)$, which is an integer only for $r=2$. In that case, $i_{0}=m-2=0$ and no further $h_{i}$ is needed. Otherwise, $m-2<i_{0}<m-1$ and step $5 . c$ works correctly since $i<i_{0}$.

The precondition of step 6 implies that $i_{0}<m-1$. If $r \nmid m$, then $\operatorname{coeff}\left(h^{k}, \kappa m\right)=0$ by $E_{2}$, and otherwise $\operatorname{coeff}\left(h^{k}, \kappa m-1\right)=0$. Thus $g_{\kappa}$ is correctly computed in step 6.b. Correctness of the remaining steps follows as above.

For the cost of the algorithm over $F=\mathbb{F}_{q}$, two contributions are from calculating $\left(h^{(j)}\right)^{\kappa}$ for some $j<m$ and the various $r$ th roots. The first comes to $O(m \cdot \log \kappa \cdot \mathrm{M}(n))$ and the second one to $O\left(m \cdot \log _{p} q\right)$ operations in $\mathbb{F}_{q} . E_{3}$ and $E_{4}$ are applied at most once. We then have to find all roots of a univariate polynomial of degree at most $r+1$. This can be done with $O(\mathrm{M}(r) \log r \log r q)$ operations (see von zur Gathen \& Gerhard (2003), Corollary 14.16). The Taylor coefficients in step 7 can be calculated with $O(\mathrm{M}(n) \log k)$ operations (see
von zur Gathen \& Gerhard (2003), Theorem 9.15). All other costs are dominated by these contributions, and we find the total cost as

$$
O(\mathrm{M}(n) \log k \cdot(m+\log (k q))) .
$$

A more direct way to compute $h$ (say, in step 3 ) is to consider its reversal as the $\kappa$ th root of the reversal of $\left(f-h^{k}\right) / g_{\kappa}$, feeding the contribution of $h^{k}$ into the Newton iteration as in von zur Gathen (1990a). I have not analyzed this procedure.

Our next task is to determine the number $N$ of decomposable $f$ obtained as $g \circ h$ in Theorem 3.17. Since (3.15) is an equation of degree $r+1$, it has at most $r+1$ solutions, and $\sigma(f) \leq r+1 . N$ is at least the number of $(g, h)$ permitted by Theorem 3.17, divided by $r+1$. The following considerations lead to a much better lower bound on $N$.

In the following we write, as usually, $p=\operatorname{char} \mathbb{F}_{q}$, and also

$$
\begin{equation*}
q=p^{e}, r=p^{d}, c=\operatorname{gcd}(d, e), z=p^{c} \tag{3.21}
\end{equation*}
$$

so that $\mathbb{F}_{q} \cap \mathbb{F}_{r}=\mathbb{F}_{z}$ (assuming an embedding of $\mathbb{F}_{q}$ and $\mathbb{F}_{r}$ in a common superfield) and $\operatorname{gcd}(q-1, r-1)=z-1$ (see Lemma 3.29). We have to understand the number of solutions $s$ of (3.15), in other words, the size of

$$
S(v, w)=\left\{s \in \mathbb{F}_{q}^{\times}: s^{r+1}-v s-w=0\right\}
$$

for $v=f_{\kappa m} / a, w=f_{\kappa m-1} / a \in \mathbb{F}_{q}$. (3.15) is only used in step 5 , where $m=r$, as noted above. We have $\kappa=(j+1) / m$ in step 2 and hence $f_{\kappa m-1} \neq 0$ and $w \neq 0$. Furthermore, we define for $u \in \mathbb{F}_{q}$

$$
\begin{equation*}
T(u)=\left\{t \in \mathbb{F}_{q}^{\times}: t^{r+1}-u t+u=0\right\} . \tag{3.22}
\end{equation*}
$$

In (3.15), we have $w \neq 0$, but $v$ might be zero. In order to apply a result from the literature, we first assume that also $v$ is nonzero, make the invertible substitution $s=-v^{-1} w t$, and set $u=v^{r+1}(-w)^{-r}=-v^{r+1} w^{-r} \in \mathbb{F}_{q}$. Then $u \neq 0$ and

$$
\begin{align*}
s^{r+1}-v s-w & =\left(-v^{-1} w\right)^{r+1}\left(t^{r+1}-u t+u\right)  \tag{3.23}\\
\# S(v, w) & =\# T(u)
\end{align*}
$$

This reduces the study of $S(v, w)$, with two parameters, to the one-parameter problem $T(u)$. The polynomial $t^{r+1}-u t+u$ has appeared in other contexts such as the inverse Galois problem, difference sets, and Müller-Cohen-Matthews
polynomials. Bluher (2004) has determined the combinatorial properties that we need here; see her paper also for further references. Bluher allows an infinite ground field $F$, but we only use her results for $F=\mathbb{F}_{q}$.

For $i \geq 0$, let

$$
\begin{align*}
C_{i} & =\#\left\{u \in \mathbb{F}_{q}^{\times}: \# T(u)=i\right\}, \\
c_{i} & =\# C_{i} . \tag{3.24}
\end{align*}
$$

Then $C_{i}=\varnothing$ for $i>r+1$. Bluher (2004) completely determines these $c_{i}$, as follows.

FACT 3.25. With the notations (3.21) and (3.24), let $I=\{0,1,2, z+1\}$. Then

$$
\begin{align*}
c_{1} & =\frac{q}{z}-\gamma, \\
c_{i} & =0 \text { unless } i \in I,  \tag{3.26}\\
c_{z+1} & =\left\lfloor\frac{q}{z^{3}-z}\right\rfloor,
\end{align*}
$$

where

$$
\gamma= \begin{cases}1 & \text { if } q \text { is even and } e / c \text { is odd }  \tag{3.27}\\ 0 & \text { otherwise }\end{cases}
$$

and furthermore

$$
\begin{equation*}
q=1+\sum_{i \in I} c_{i}=2+\sum_{i \in I} i c_{i} . \tag{3.28}
\end{equation*}
$$

Proof. The claims are shown in Bluher (2004), Theorem 5.6. Her statement assumes $t u \neq 0$, which is equivalent to our assumption $t \neq 0$. (3.28) corresponds to the fact that the numbers $c_{i}$ form the preimage statistic of the map from $\mathbb{F}_{q} \backslash\{0,1\}$ to $\mathbb{F}_{q} \backslash\{0\}$ given by the rational function $x^{r+1} /(x-1)$.
(3.26) and (3.28) also determine the remaining two values $c_{0}$ and $c_{2}$, namely $c_{2}=\frac{1}{2}\left(q-2-c_{1}-(z+1) c_{z+1}\right)$ and $c_{0}=1+c_{2}+z c_{z+1}$. For large $z$, we have

$$
c_{2} \approx \frac{q}{2}\left(1-\frac{1}{z}-\frac{z+1}{z^{3}-z}\right)=\frac{q}{2}\left(1-\frac{1}{z-1}\right) \approx \frac{q}{2} .
$$

Thus $x^{r+1} /(x-1)$ behaves a bit like squaring: about half the elements have two preimages, and about half have none.

For the case $v=0$, we have the following facts, which are presumably wellknown. For an integer $m$, we let the integer $\nu(m)$ be the multiplicity of 2 in $m$, so that $m=2^{\nu(m)} m^{*}$ with an odd integer $m^{*}$.

Lemma 3.29. Let $\mathbb{F}_{q}$ have characteristic $p$ with $q=p^{e}, r=p^{d}$ with $d \geq 1$, $b=\operatorname{gcd}(q-1, r+1)$ and $w \in \mathbb{F}_{q}^{\times}$. Then the following hold.
(i)

$$
\# S(0, w)= \begin{cases}b & \text { if } w^{(q-1) / b}=1 \\ 0 & \text { otherwise }\end{cases}
$$

(ii) We let $c=\operatorname{gcd}(d, e), z=p^{c}, \delta=\nu(d), \epsilon=\nu(e), \alpha=\nu\left(r^{2}-1\right), \beta=$ $\nu(q-1)$,

$$
\begin{aligned}
& \lambda= \begin{cases}2 & \text { if } \delta<\epsilon \\
1 & \text { if } \delta \geq \epsilon\end{cases} \\
& \mu= \begin{cases}1 & \text { if } \alpha>\beta \\
0 & \text { if } \alpha \leq \beta\end{cases}
\end{aligned}
$$

Then $\operatorname{gcd}(r-1, q-1)=z-1$ and

$$
b=\frac{\left(z^{\lambda}-1\right) \cdot 2^{\mu}}{z-1}= \begin{cases}2(z+1) & \text { if } \delta<\epsilon \text { and } \alpha>\beta \\ z+1 & \text { if } \delta<\epsilon \text { and } \alpha \leq \beta \\ 2 & \text { if } \delta \geq \epsilon \text { and } \alpha>\beta \\ 1 & \text { if } \delta \geq \epsilon \text { and } \alpha \leq \beta\end{cases}
$$

(iii) If $p$ is odd, then $\alpha>\beta$ if and only if $e / c$ is odd.

Proof. (i) The power function $y \mapsto y^{r+1}$ from $\mathbb{F}_{q}^{\times}$to $\mathbb{F}_{q}^{\times}$maps $b$ elements to one, and its image consists of the $u \in \mathbb{F}_{q}$ with $u^{(q-1) / b}=1$.
(ii) For the first claim that

$$
\begin{equation*}
\operatorname{gcd}(q-1, r-1)=z-1 \tag{3.30}
\end{equation*}
$$

we may assume, by symmetry, that $d>e$ and let $d=i e+j$ be the division with remainder of $d$ by $e$, with $0 \leq j<e$. Then for

$$
a=\frac{x^{j}\left(x^{d-j}-1\right)}{x^{e}-1}=x^{j} \cdot \frac{x^{i e}-1}{x^{e}-1} \in \mathbb{Z}[x],
$$

we have

$$
x^{d}-1=a \cdot\left(x^{e}-1\right)+\left(x^{j}-1\right) .
$$

By induction along the Extended Euclidean Algorithm for $(d, e)$ it follows that all quotients in the Euclidean Algorithm for $\left(x^{d}-1, x^{e}-1\right)$ in $\mathbb{Q}[x]$ are, in fact, in $\mathbb{Z}[x]$, hence also the Bézout coefficients, and that all remainders are of the form $x^{y}-1$, where $y$ is some remainder for $d$ and $e$. For $c=\operatorname{gcd}(d, e)$, there exist $u, v, s, t \in \mathbb{Z}[x]$ so that

$$
\begin{aligned}
u \cdot\left(x^{c}-1\right) & =x^{d}-1, \\
v \cdot\left(x^{c}-1\right) & =x^{e}-1, \\
s \cdot\left(x^{d}-1\right)+t \cdot\left(x^{e}-1\right) & =x^{c}-1 .
\end{aligned}
$$

Substituting any integer $q$ for $x$ into these equations shows the claim (3.30).
We note that $\operatorname{gcd}(2 d, e)=\lambda c$ and

$$
\operatorname{gcd}\left(p^{d}-1, p^{d}+1\right)= \begin{cases}2 & \text { if } p \text { is odd } \\ 1 & \text { if } p \text { is even }\end{cases}
$$

When $p$ is even, then applying (3.30) to $q=p^{e}$ and $r^{2}=p^{2 d}$, we find

$$
\begin{aligned}
p^{\lambda c}-1 & =\operatorname{gcd}\left(\left(p^{d}-1\right)\left(p^{d}+1\right), p^{e}-1\right) \\
& =\operatorname{gcd}\left(p^{d}-1, p^{e}-1\right) \cdot \operatorname{gcd}\left(p^{d}+1, p^{e}-1\right) \\
& =\left(p^{c}-1\right) \cdot b, \\
b & =\frac{p^{\lambda c}-1}{p^{c}-1}= \begin{cases}z+1 & \text { if } \delta<\epsilon, \\
1 & \text { if } \delta \geq \epsilon .\end{cases}
\end{aligned}
$$

For odd $p$, the second equation above is still almost correct, except possibly for factors which are powers of 2 . We note that exactly one of $\nu\left(p^{d}-1\right)$ and $\nu\left(p^{d}+1\right)$ equals 1 , and

$$
\begin{aligned}
p^{\lambda c}-1 & =\operatorname{gcd}\left(\left(p^{d}-1\right)\left(p^{d}+1\right), p^{e}-1\right) \\
& =\operatorname{gcd}\left(p^{d}-1, p^{e}-1\right) \cdot \operatorname{gcd}\left(p^{d}+1, p^{e}-1\right) \cdot 2^{-\mu} \\
& =\left(p^{c}-1\right) \cdot b \cdot 2^{-\mu}, \\
b & =\frac{\left(p^{\lambda c}-1\right) \cdot 2^{\mu}}{p^{c}-1} .
\end{aligned}
$$

(iii) We define the integers $k_{q}$ and $k_{r}$ by

$$
\begin{aligned}
\frac{q-1}{z-1} & =\frac{z^{e / c}-1}{z-1}=z^{e / c-1}+\cdots+1=k_{q} \\
\frac{r^{2}-1}{z-1} & =\frac{(r+1)\left(z^{d / c}-1\right)}{z-1}=(r+1)\left(z^{d / c-1}+\cdots+1\right)=(r+1) k_{r}
\end{aligned}
$$

Now $r+1$ is even and $z$ is odd. If $e / c$ is odd, then $k_{q}$ is odd and hence $\alpha>\beta$. Now assume that $e / c$ is even. Then $d / c$ is odd, and so is $k_{r}$. Hence $\nu(r-1)=\nu(z-1)$, and we denote this integer by $\gamma$. If $\gamma \geq 2$, then $\nu(r+1)=1$ and $\alpha=\nu(r+1)+\gamma \leq \nu\left(k_{q}\right)+\gamma=\beta$.

Now suppose that $\gamma=1$, and let $\tau=\nu(z+1)$ and $m=(z+1) \cdot 2^{-\tau}$. Then $\tau \geq 2, m$ is an odd integer, and

$$
\begin{aligned}
z^{2} & =\left(m 2^{\tau}-1\right)^{2} \equiv-2 \cdot 2^{\tau}+1 \equiv 2^{\tau+1}+1 \bmod 2^{\tau+2} \\
r^{2} & =\left(z^{2}\right)^{d / c}=\left(2^{\tau+1}+1\right)^{d / c} \equiv 2^{\tau+1}+1 \bmod 2^{\tau+2}, \\
q & =\left(z^{2}\right)^{e / 2 c} \equiv\left(2^{\tau+1}+1\right)^{e / 2 c} \bmod 2^{\tau+2} .
\end{aligned}
$$

The last value equals $2^{\tau+1}+1$ or 1 modulo $2^{\tau+2}$ if $e / 2 c$ is odd or even, respectively. In either case, it follows that $\alpha=\nu\left(r^{2}-1\right)=\tau+1 \leq \nu(q-1)=\beta$.

Theorem 3.31. Let $\mathbb{F}_{q}$ have characteristic $p$ with $q=p^{e}$, and take integers $d \geq 1, r=p^{d}, k=a r$ with $p \nmid a, m \geq 2, n=k m, c=\operatorname{gcd}(d, e), z=p^{c}, \mu=$ $\operatorname{gcd}(r-1, m), r^{*}=(r-1) / \mu$, and let $G$ consist of the $(g, h)$ as in Theorem 3.17. Then we have the following lower bounds on the cardinality of $\gamma_{n, k}(G)$.
(i) If $r \neq m$ and $\mu=1$ :

$$
q^{k+m-2}\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)\left(1-q^{-k}\right)
$$

(ii) If $r \neq m$ :

$$
\begin{gathered}
q^{k+m-2}\left(\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)\left(1-q^{-k}\right)\right. \\
\left.-q^{-k-r^{*}-c / e+2} \frac{\left(1-q^{-1}\right)^{2}\left(1-q^{-r^{*}(\mu-1)}\right)}{\left(1-q^{-c / e}\right)\left(1-q^{-r^{*}}\right)}\left(1+q^{-r^{*}(p-2)}\right)\right) .
\end{gathered}
$$

(iii) If $r=m$ :

$$
q^{k+m-2}\left(1-q^{-1}\right)\left(\frac{1}{2}+\frac{1+q^{-1}}{2 z+2}+\frac{q^{-1}}{2}-q^{-k} \frac{1-q^{-p+1}}{1-q^{-p}}-q^{-p+1} \frac{1-q^{-1}}{1-q^{-p}}\right) .
$$

Proof. We have seen at the beginning of the proof of Theorem 3.17 that steps 1 and 2 determine $j$ and $\kappa$. We also know that, given $g_{\kappa}$ and $h_{m-1}$, the remaining coefficients of $g$ and $h$ are uniquely determined by those of $f$.

We count the number of compositions $g \circ h$ according to the four mutually exclusive conditions in steps 3 through 6 , for a fixed $\kappa$. The admissible $\kappa$ are those with $1 \leq \kappa<k$ and $p \nmid \kappa$. $E_{3}$ or $E_{4}$ are used if and only if either $i_{0} \in \mathbb{N}$ or $\kappa m=n-r$, respectively. If neither happens, then the number of $(g, h)$ is

$$
\begin{equation*}
q^{\kappa}\left(1-q^{-1}\right) \cdot q^{m-1}\left(1-q^{-1}\right)=q^{\kappa+m-1}\left(1-q^{-1}\right)^{2} . \tag{3.32}
\end{equation*}
$$

$E_{3}$ is used if and only if $\kappa \in K$, where

$$
K=\left\{\kappa \in \mathbb{N}: 1 \leq \kappa<k, p \nmid \kappa, i_{0} \in \mathbb{N}, 1 \leq i_{0}<m\right\}
$$

which corresponds to steps $4 . \mathrm{b}$ (where $i_{0}=m-1$ ) and 6 .d (where $i_{0} \in \mathbb{N}$ and $1 \leq$ $i_{0} \leq m-2$ ). For $\kappa \in K$, we have the condition (3.18) that $\left(-\kappa g_{\kappa} / a\right)^{(q-1) /(z-1)} \neq$ 1. The exponent is a divisor of $q-1$, and there are exactly $(q-1) /(z-1)$ values of $g_{\kappa}$ that violate (3.18). Thus for $\kappa \in K$ the number of $(g, h)$ equals

$$
\begin{equation*}
\left(q-1-\frac{q-1}{z-1}\right) q^{\kappa-1} \cdot q^{m-1}\left(1-q^{-1}\right)=q^{\kappa+m-1}\left(1-\frac{1}{z-1}\right)\left(1-q^{-1}\right)^{2} . \tag{3.33}
\end{equation*}
$$

The only usage of $E_{4}$ occurs in step $5 . \mathrm{a}$, where $\kappa=(n-r) / m=k-r / m$. We have seen in the proof of Theorem 3.17 that this implies $r=m$ and $\kappa=k-1$. We split $G$ according to whether $\kappa=k-1$ or $\kappa<k-1$, setting

$$
G^{*}=\{(g, h) \in G: \kappa=k-1 \text { in }(3.7)\} .
$$

We define three summands $S_{12}, S_{3}$, and $S_{4}$ according to whether only $E_{1}$ and $E_{2}$, or also $E_{3}$, or $E_{4}$ are used, respectively:

$$
\begin{aligned}
S_{12} & =\sum_{\substack{1 \leq \kappa<k \\
p \not \kappa}} q^{\kappa+m-1}\left(1-q^{-1}\right)^{2} \\
S_{3} & =\sum_{\kappa \in K}\left(q^{\kappa+m-1}\left(1-q^{-1}\right)^{2}-q^{\kappa+m-1}\left(1-q^{-1}\right)^{2}\left(1-\frac{1}{z-1}\right)\right) \\
S_{4} & =q^{k+m-2}\left(1-q^{-1}\right)^{2}-\# \gamma_{n, k}\left(G^{*}\right)
\end{aligned}
$$

We will see below that $K=\varnothing$ if $r=m$. Thus

$$
\# \gamma_{n, k}(G) \geq \begin{cases}S_{12} & \text { if } r \neq m \text { and } K=\varnothing \\ S_{12}-S_{3} & \text { if } r \neq m \\ S_{12}-S_{4} & \text { if } r=m\end{cases}
$$

The subtraction of $S_{3}$ corresponds to replacing the summand (3.32) by (3.33) for $\kappa \in K$. Similarly, $S_{4}$ replaces (3.32) for $\kappa=k-1$ by the correct value if $E_{4}$ is applied.

Since $p \mid k$, the first sum equals

$$
\begin{aligned}
S_{12} & =q^{m-1}\left(1-q^{-1}\right)^{2}\left(\sum_{1 \leq \kappa<k} q^{\kappa}-\sum_{\substack{1 \leq \kappa<k \\
p \mid \kappa}} q^{\kappa}\right) \\
& =q^{m-1}\left(1-q^{-1}\right)^{2}\left(\frac{q^{k}-1}{q-1}-1-\frac{\left(q^{p}\right)^{k / p}-1}{q^{p}-1}+1\right) \\
& =q^{k+m-2}\left(1-q^{-1}\right)\left(1-q^{-k}\right) \frac{1-q^{-p+1}}{1-q^{-p}} \\
& =q^{k+m-2}\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)\left(1-q^{-k}\right) .
\end{aligned}
$$

For $S_{3}$, we describe $K$ more transparently. From (3.8) we find

$$
\begin{align*}
1 \leq i_{0}= & \frac{\kappa m-n}{r-1}+m \leq m-1  \tag{3.34}\\
& \Longleftrightarrow k-(r-1)+\frac{r-1}{m} \leq \kappa \leq k-\frac{r-1}{m} \\
& i_{0} \in \mathbb{Z} \Longleftrightarrow(r-1) \mid(\kappa-a) m \tag{3.35}
\end{align*}
$$

We have $\mu=\operatorname{gcd}(r-1, m)$ and $r^{*}=(r-1) / \mu$, and set $m^{*}=m / \mu$, so that $\operatorname{gcd}\left(r^{*}, m^{*}\right)=1$ and
$(3.34) \Longleftrightarrow k-(r-1)+\frac{r^{*}}{m^{*}} \leq \kappa \leq k-\frac{r^{*}}{m^{*}}$,
$(3.35) \Longleftrightarrow r^{*}\left|(\kappa-a) m^{*} \Longleftrightarrow r^{*}\right|(\kappa-a)$.
Since $r^{*} \mid k-a=a(r-1)$, we have

$$
\begin{align*}
& (3.35) \Longleftrightarrow \exists j \in \mathbb{Z} \quad \kappa=k-(r-1)+j r^{*},  \tag{3.36}\\
& (3.34) \Longleftrightarrow \frac{1}{m^{*}} \leq j \leq \frac{r-1}{r^{*}}-\frac{1}{m^{*}} \Longleftrightarrow 1 \leq j \leq \mu-1 . \tag{3.37}
\end{align*}
$$

Since $\mu \mid(r-1)$ and $r=p^{d}$, we have $p \nmid \mu$. Thus

$$
\begin{align*}
p \mid \kappa & \Longleftrightarrow 1-\frac{j}{\mu} \equiv 1+\frac{j(r-1)}{\mu} \equiv k-(r-1)+j r^{*}=\kappa \equiv 0 \bmod p  \tag{3.38}\\
& \Longleftrightarrow j \equiv \mu \bmod p \Longleftrightarrow \exists i \in \mathbb{Z} \quad j=\mu-i p \\
(3.34) & \Longleftrightarrow 1 \leq j=\mu-i p \leq \mu-1 \Longleftrightarrow 1 \leq i \leq\left\lfloor\frac{\mu-1}{p}\right\rfloor
\end{align*}
$$

Abbreviating $\mu^{*}=\lfloor(\mu-1) / p\rfloor$, it follows that

$$
K=\left\{k-(r-1)+j r^{*}: 1 \leq j \leq \mu-1\right\} \backslash\left\{k-i p r^{*}: 1 \leq i \leq \mu^{*}\right\}
$$

In particular, we have $K=\varnothing$ if $\mu=1$. Assuming $\mu \geq 2$ and using $z=p^{c}=q^{c / e}$, we can evaluate $S_{3}$ as follows.

$$
\begin{aligned}
S_{3}= & \sum_{\kappa \in K} \frac{q^{\kappa+m-1}}{z-1}\left(1-q^{-1}\right)^{2} \\
= & \frac{q^{m-1}\left(1-q^{-1}\right)^{2}}{z-1} \sum_{\kappa \in K} q^{\kappa} \\
= & \frac{q^{m-1}\left(1-q^{-1}\right)^{2}}{z-1}\left(q^{k-(r-1)+r^{*}} \frac{\left(q^{r^{*}}\right)^{\mu-1}-1}{q^{r^{*}}-1}-q^{k-p r^{*}} \frac{\left(q^{-p r^{*}}\right)^{\mu^{*}}-1}{q^{-p r^{*}}-1}\right) \\
= & q^{k+m-1-r^{*}-c / e} \frac{\left(1-q^{-1}\right)^{2}\left(1-q^{-r^{*}(\mu-1)}\right)}{\left(1-q^{-c / e}\right)\left(1-q^{-r^{*}}\right)} \\
& \cdot\left(1-q^{-r^{*}(p-1)} \frac{\left(1-q^{-r^{*}}\right)\left(1-q^{-p r^{*} \mu^{*}}\right)}{\left(1-q^{-r^{*}(\mu-1)}\right)\left(1-q^{-p r^{*}}\right)}\right) \\
\leq & q^{k+m-1-r^{*}-c / e} \frac{\left(1-q^{-1}\right)^{2}\left(1-q^{-r^{*}(\mu-1)}\right)}{\left(1-q^{-c / e}\right)\left(1-q^{-r^{*}}\right)} .
\end{aligned}
$$

In order to evaluate $S_{4}$, we first recall from the above that we have $\kappa m=$ $n-r, \kappa=k-1, m=r$, and any $(g, h) \in G^{*}$ is uniquely determined by $f=g \circ h, g_{k-1}$, and $h_{m-1}$. To any $(g, h) \in G^{*}$, we associate the field elements

$$
\begin{align*}
V(g, h) & =h_{m-1}^{r}+g_{k-1} / a \\
W(g, h) & =-g_{k-1} h_{m-1} / a  \tag{3.39}\\
U(g, h) & =-V(g, h)^{r+1} W(g, h)^{-r}
\end{align*}
$$

Then if $f=g \circ h$, we have $a V(g, h)=f_{n-r}, a W(g, h)=f_{n-r-1} \neq 0$, and for nonzero $s \in \mathbb{F}_{q}$ and $t=-V(g, h) \cdot W(g, h)^{-1} s,(3.23)$ says that

$$
\text { (3.15) holds } \Longleftrightarrow s \in S(V(g, h), W(g, h)) \Longleftrightarrow t \in T(U(g, h))
$$

We recall the sets $C_{i}$ from (3.24) and for $i \in\{1,2, z+1\}$, we set

$$
\begin{aligned}
G_{i} & =\left\{(g, h) \in G: V(g, h) \neq 0, U(g, h) \in C_{i}\right\}, \\
G_{0} & =\{(g, h) \in G: V(g, h)=0\} .
\end{aligned}
$$

Now let $v \in \mathbb{F}_{q}^{\times}, i \in\{1,2, z+1\}, u \in C_{i}$, and $g_{k-2}, \ldots, g_{1}, h_{m-2}, \ldots$, $h_{1} \in \mathbb{F}_{q}$. From these data, we construct $(g, h) \in G_{i}$ with $g=\sum_{1 \leq i \leq k} g_{i} x^{i}$ and
$h=\sum_{1 \leq i \leq m} h_{i} x^{i}$ and $g_{k}=h_{m}=1$, so that only $g_{k-1}$ and $h_{m-1}$ still need to be determined. Furthermore, if $f=g \circ h$, we show that different data lead to different $f$. This will prove that

$$
\begin{equation*}
\gamma_{n, k}\left(G_{i}\right) \geq(q-1) c_{i} \cdot q^{k+m-4} \tag{3.40}
\end{equation*}
$$

By assumption, we have $u \neq 0$ and $\# T(u)=i \geq 1$. We choose some $t \in T(u)$ and define $w, s \in \mathbb{F}_{q}^{\times}$by

$$
\begin{aligned}
w^{r} & =-v^{r+1} u^{-1} \\
s & =-v^{-1} w t
\end{aligned}
$$

Then $s \in S(v, w)$ by (3.23). We set $h_{m-1}=s$ and $g_{k-1}=a v-a s^{r}$. Now $g$ and $h$ are determined, and $E_{1}$ and $E_{2}$ imply that

$$
\begin{aligned}
f_{n-r} & =a h_{m-1}^{r}+g_{\kappa}=a V(g, h)=a v \\
f_{n-r-1} & =-g_{\kappa} h_{m-1}=a W(g, h)=-a\left(v-s^{r}\right) s=a\left(s^{r+1}-v s\right)=a w, \\
U(g, h) & =-v^{r+1} w^{-r}=-v^{r+1}\left(-v^{r+1} u^{-1}\right)^{-1}=u
\end{aligned}
$$

Suppose that $(u, v)$ and $(\tilde{u}, \tilde{v})$ lead to $\left(f_{n-r}, f_{n-r-1}\right)=(a v, a w)$ and $\left(\widetilde{f_{n-r}}, \widetilde{f_{n-r-1}}\right)=$ ( $a \tilde{v}, a \tilde{w}$ ), and that the latter pairs are equal. Then $v=\tilde{v}$ and $u=-v^{r+1} w^{-r}=$ $-\tilde{v}^{r+1} \tilde{w}^{-r}=\tilde{u}$. This concludes the proof of (3.40).

A similar argument works for $G_{0}$. We let $b=\operatorname{gcd}(q-1, r+1)$, take $w \in \mathbb{F}_{q}$ with $w^{(q-1) / b}=1$, and some $s \in \mathbb{F}_{q}$ with $s^{r+1}=w$. There are $(q-1) / b$ such $w$, and according to Lemma 3.29(i), $b$ such values $s$ for each $w$. We set $h_{m-1}=s$ and $g_{k-1}=-a h_{m-1}^{r}$ and, as above, complete them with arbitrary coefficients to $(g, h) \in G_{0}$. When $f=g \circ h$, then $f_{n-r}=0$ and $f_{n-r-1}=-g_{k-1} h_{m-1}=$ $a h_{m-1}^{r+1}=a w=a W(g, h)$, and different $w$ lead to different $f$. It follows that

$$
\begin{equation*}
\gamma_{n, k}\left(G_{0}\right) \geq \frac{q-1}{b} \tag{3.41}
\end{equation*}
$$

The images of $G_{1}, G_{2}, G_{z+1}$, and $G_{0}$ under $\gamma_{n, k}$ are pairwise disjoint, since the map $V \times W \times U: \bigcup_{i=0,1,2, z+1} G_{i} \longrightarrow \mathbb{F}_{q}^{3}$ is injective, and its value together with the lower coefficients of $g$ and $h$ determines $f$, again injectively. It follows that

$$
\begin{align*}
\sum_{i=0,1,2, z+1} \# \gamma_{n, k}\left(G_{i}\right) & \geq \sum_{i=1,2, z+1}(q-1) c_{i} \cdot q^{k+m-4}+\frac{q-1}{b} \cdot q^{k+m-4}  \tag{3.42}\\
& =(q-1) q^{k+m-4}\left(\sum_{i=1,2, z+1} c_{i}+\frac{1}{b}\right)
\end{align*}
$$

We write $q=p^{e}$ and set

$$
z^{*}= \begin{cases}z & \text { if } e / c \text { is odd } \\ z^{2} & \text { if } e / c \text { is even }\end{cases}
$$

Fact 3.25 yields

$$
\begin{gathered}
c_{z+1}=\left\lfloor\frac{q}{z^{3}-z}\right\rfloor=\frac{q-z^{*}}{z^{3}-z} \\
2 \sum_{i=1,2, z+1} c_{i}=2 c_{1}+\left(q-2-c_{1}-(z+1) c_{z+1}\right)+2 c_{z+1} \\
=q-2+\frac{q}{z}-\gamma-(z-1) \frac{q-z^{*}}{z^{3}-z} \\
=q-2+\frac{q}{z}-\gamma-\frac{q-z^{*}}{z^{2}+z} \\
\# \gamma_{n, k}\left(G^{*}\right) \geq q^{k+m-3}\left(1-q^{-1}\right)\left(\frac{1}{2}\left(q-2+\frac{q}{z}-\gamma-\frac{q-z^{*}}{z^{2}+z}\right)+\frac{1}{b}\right)
\end{gathered}
$$

We call the last factor $B$. If $e / c$ is odd, then, in the notation of Lemma 3.29, $\delta=\nu(d) \geq \nu(e)=\epsilon$, so that $b \in\{1,2\}$, and

$$
b= \begin{cases}2 & \text { if } p \text { is odd } \\ 1 & \text { if } p=2\end{cases}
$$

If $p$ is odd, then $\gamma=0$ and $2 / b-\gamma=1$. If $p=2$, then $\gamma=1$ and again $2 / b-\gamma=2-1=1$. It follows that

$$
2 B=q-2+\frac{q}{z}-\frac{q-z}{z^{2}+z}+\frac{2}{b}-\gamma=q\left(1+\frac{1}{z+1}\left(1-\frac{z}{q}\right)\right)
$$

If $e / c$ is even, then $\gamma=0, b=z+1$ and

$$
2 B=q-2+\frac{q}{z}-\frac{q-z^{2}}{z^{2}+z}+\frac{2}{z+1}=q\left(1+\frac{1}{z+1}\left(1-\frac{z}{q}\right)\right) .
$$

It follows that in all cases

$$
\begin{aligned}
\# \gamma_{n, k}\left(G^{*}\right) & \geq \frac{1}{2} q^{k+m-2}\left(1-q^{-1}\right)\left(1+\frac{1}{z+1}\left(1-\frac{z}{q}\right)\right), \\
S_{4} & \leq q^{k+m-2}\left(1-q^{-1}\right)\left(1-q^{-1}-\frac{1}{2}\left(1+\frac{1}{z+1}\left(1-\frac{z}{q}\right)\right)\right) \\
& =q^{k+m-2}\left(1-q^{-1}\right)\left(\frac{1}{2}-q^{-1}-\frac{1}{2 z+2}\left(1-\frac{z}{q}\right)\right) .
\end{aligned}
$$

Together we have found the following lower bounds on $\# \gamma_{n, k}(G)$. If $r \neq m$ and $\mu=1$, then

$$
\# \gamma_{n, k}(G) \geq S_{12}=q^{k+m-2}\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)\left(1-q^{-k}\right)
$$

If $r \neq m$, then

$$
\begin{aligned}
\# \gamma_{n, k}(G) \geq & S_{12}-S_{3} \geq q^{k+m-2}\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)\left(1-q^{-k}\right) \\
& -q^{k+m-k-2 r^{*}-c / e} \frac{\left(1-q^{-1}\right)^{2}\left(1-q^{-r^{*}(\mu-1)}\right)}{\left(1-q^{-c / e}\right)\left(1-q^{-r^{*}}\right)}\left(1+q^{-r^{*}(p-2)}\right) \\
= & q^{k+m-2}\left(\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)\left(1-q^{-k}\right)\right. \\
& \left.-q^{-k-r^{*}-c / e+2} \frac{\left(1-q^{-1}\right)^{2}\left(1-q^{-r^{*}(\mu-1)}\right)}{\left(1-q^{-c / e}\right)\left(1-q^{-r^{*}}\right)}\left(1+q^{-r^{*}(p-2)}\right)\right)
\end{aligned}
$$

If $r=m$, then

$$
\begin{aligned}
\# \gamma_{n, k}(G) \geq & S_{12}-S_{4} \geq q^{k+m-2}\left(1-q^{-1}\right)\left(1-q^{-k}\right) \frac{1-q^{-p+1}}{1-q^{-p}} \\
& -q^{k+m-2}\left(1-q^{-1}\right)\left(\frac{1}{2}-q^{-1}-\frac{1}{2 z+2}\left(1-\frac{z}{q}\right)\right) \\
= & q^{k+m-2}\left(1-q^{-1}\right)\left(\frac{1}{2}+\frac{1+q^{-1}}{2 z+2}+\frac{q^{-1}}{2}\right. \\
& \left.-q^{-k} \frac{1-q^{-p+1}}{1-q^{-p}}-q^{-p+1} \frac{1-q^{-1}}{1-q^{-p}}\right) .
\end{aligned}
$$

Corollary 3.43. With the assumptions and notation of Theorem 3.31, the set $D_{n, k}^{+}$of non-Frobenius compositions has at least the following size.
(i) If $r \neq m$ and $\mu=1$ :

$$
q^{k+m}\left(1-q^{-1}\right)\left(1-q^{-k}\right)\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)
$$

(ii) If $r \neq m$ :

$$
\begin{aligned}
& q^{k+m}\left(1-q^{-1}\right)\left(\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)\left(1-q^{-k}\right)\right. \\
& \left.\quad-q^{-k-r^{*}-c / e+2} \frac{\left(1-q^{-1}\right)^{2}\left(1-q^{-r^{*}(\mu-1)}\right)}{\left(1-q^{-c / e}\right)\left(1-q^{-r^{*}}\right)}\left(1+q^{-r^{*}(p-2)}\right)\right) \\
& \geq q^{k+m}\left(1-q^{-1}\right)\left(\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)\left(1-q^{-k}\right)\right. \\
& \left.\quad-q^{-k-r^{*}+2} \frac{\left(1-q^{-1}\right)^{2}\left(1-q^{-r^{*}(\mu-1)}\right)}{1-q^{-r^{*}}}\left(1+q^{-r^{*}(p-2)}\right)\right)
\end{aligned}
$$

If furthermore $r^{*} \geq 2$ and $p>\mu$, then the latter quantity is at least

$$
q^{k+m}\left(1-q^{-1}\right)\left(\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)\left(1-q^{-k}\right)-\frac{4}{3} q^{-k}\left(1-q^{-1}\right)^{2}\right)
$$

(iii) If $r=m$ :

$$
q^{k+m}\left(1-q^{-1}\right)^{2}\left(\frac{1}{2}+\frac{1+q^{-1}}{2 z+2}+\frac{q^{-1}}{2}-q^{-k} \frac{1-q^{-p+1}}{1-q^{-p}}-q^{-p+1} \frac{1-q^{-1}}{1-q^{-p}}\right) .
$$

Proof. All $g$ and $h$ considered in Theorem 3.31 are monic and original, and so are their compositions $f$. We may replace the left hand component $g$ of any $(g, h) \in G$ by $(a x+b) \circ g$, where $a, b \in \mathbb{F}_{q}$ are arbitrary with $a \neq 0$. Hence

$$
\# D_{n, k}^{\#} \geq q^{2}\left(1-q^{-1}\right) \cdot \# \gamma_{n, k}(G)
$$

and the claims follow from Theorem 3.31. For the first inequality in (ii), we observe that $c \geq 1$ and

$$
\begin{equation*}
\frac{q^{-c / e}}{1-q^{-c / e}}=\frac{p^{-c}}{1-p^{-c}} \leq 1 \tag{3.44}
\end{equation*}
$$

For the last estimate, we have

$$
\begin{gathered}
q^{-r^{*}} \leq 1 / 4 \\
q^{-r^{*}(p-2)} \leq q^{-r^{*}(\mu-1)} \\
\left(1-q^{-r^{*}(\mu-1)}\right)\left(1+q^{-r^{*}(p-2)}\right) \leq \frac{4}{3}\left(1-q^{-r^{*}}\right) .
\end{gathered}
$$

The algorithm works over any field of characteristic $p$ where each element has a $p$ th root; in $\mathbb{F}_{q}$, this is just the $(q / p)$ th power. It even works over an arbitrary extension of $\mathbb{F}_{p}$, rather than just the separable ones, provided we have a subroutine that tests whether a field element is a $p$ th power, and if so, returns a $p$ th root. Then where a $p$ th root is requested in the algorithm (steps $3 \mathrm{a}, 6 \mathrm{a}$, and 6 c ), we either return "no decomposition" or the root, depending on the outcome of the test.

Example 3.45. When $n=p^{2}$, then we have $k=r=m=p$ in Corollary 3.43 (iii), and including the Frobenius compositions (Lemma 4.32(ii)), we obtain

$$
\begin{aligned}
\# D_{n} & \geq \frac{1}{2} q^{2 p}\left(1-q^{-1}\right)^{2}\left(1+\frac{1+q^{-1}}{p+1}+q^{-1}-2 q^{-p+1}\right)+q^{p+1}\left(1-q^{-1}\right) \\
& =\alpha_{n} \cdot\left(\frac{1}{2}\left(1+\frac{1}{p+1}\right)\left(1-q^{-2}\right)+q^{-p}\right)
\end{aligned}
$$

In characteristic 2, the estimate is exact, since we have accounted for all compositions and a monic original polynomial of degree 2 is determined by its linear coefficient. Thus

$$
\begin{aligned}
& \# D_{4}=\alpha_{4} \cdot\left(\frac{2}{3} \cdot\left(1-q^{-2}\right)+q^{-2}\right)=\alpha_{4} \cdot \frac{2+q^{-2}}{3} \\
& \# D_{4}=\frac{3}{4} \alpha_{4} \text { over } \mathbb{F}_{2} \\
& \# D_{4}=\frac{11}{16} \alpha_{4} \text { over } \mathbb{F}_{4}
\end{aligned}
$$

Over an algebraically closed field, a quartic polynomial is decomposable if and only if its cubic coefficient vanishes; compare to Example 2.6. For $p=3$, we find

$$
\begin{aligned}
& \# D_{9} \geq \alpha_{9} \cdot\left(\frac{5}{8}\left(1-q^{-2}\right)+q^{-3}\right)=\alpha_{9} \cdot\left(\frac{5}{8}-q^{-2}\left(\frac{5}{8}-q^{-1}\right)\right), \\
& \# D_{9} \geq \frac{16}{27} \cdot \alpha_{9}>0.59259 \alpha_{9} \text { over } \mathbb{F}_{3} \\
& \# D_{9} \geq \frac{451}{3^{6}} \cdot \alpha_{9}>0.61065 \alpha_{9} \text { over } \mathbb{F}_{9} .
\end{aligned}
$$

Table 6.3 shows that these are serious underestimates of the actual ratios $\approx 0.8518$ and 0.9542 . In the same vein we find, when $p=k$ and $n=a p^{2}>p^{2}$ with $p \nmid a$, that

$$
\# D_{n, n / p} \geq \frac{\alpha_{n}}{2} \cdot\left(\frac{1}{2}\left(1+\frac{1}{p+1}\right)\left(1-q^{-2}\right)+q^{-p}\right)
$$

Example 3.46. In $\mathbb{F}_{3}[x]$, we have, besides the eight Frobenius collisions according to Definition 3.3, four two-way collisions of degree 9:

$$
\begin{gathered}
\left(x^{3}+x\right) \circ\left(x^{3}-x\right)=\left(x^{3}-x\right) \circ\left(x^{3}+x\right)=x^{9}-x, \\
\left(x^{3}+x^{2}\right) \circ\left(x^{3}-x^{2}-x\right)=\left(x^{3}-x^{2}+x\right) \circ\left(x^{3}+x^{2}\right)=x^{9}+x^{5}-x^{4}+x^{3}+x^{2}, \\
\left(x^{3}+x^{2}+x\right) \circ\left(x^{3}-x^{2}\right)=\left(x^{3}-x^{2}\right) \circ\left(x^{3}+x^{2}-x\right)=x^{9}+x^{5}+x^{4}+x^{3}-x^{2}, \\
\left(x^{3}+x^{2}+x\right) \circ\left(x^{3}-x^{2}+x\right)=\left(x^{3}-x^{2}+x\right) \circ\left(x^{3}+x^{2}+x\right)=x^{9}+x^{5}+x .
\end{gathered}
$$

Our general bounds of Theorem 5.2(i), Corollary 3.43, and Example 3.45 say that

$$
18 \cdot 16=288<18 \cdot 17=306<\# D_{9}=414=18 \cdot 23<486=18 \cdot 27=\alpha_{9} . \diamond
$$

## 4. Distinct-degree collisions of compositions

In this section, we turn to the last preparatory task. Namely, for a lower bound on $D_{n}$ we have to understand $D_{n, \ell} \cap D_{n, n / \ell}$, that is, the distinct-degree collisions (3.2) when $\operatorname{deg} g^{*}=\operatorname{deg} h=\ell$. In our application, $\ell$ is the smallest prime divisor of $n$.

The following is an example of a collision:

$$
x^{k} w^{\ell} \circ x^{\ell}=x^{k \ell} w^{\ell}\left(x^{\ell}\right)=x^{\ell} \circ x^{k} w\left(x^{\ell}\right)
$$

for any polynomial $w \in F[x, y]$, where $F$ is a field (or even a ring). We define the (bivariate) Dickson polynomials of the first kind $T_{m} \in F[x, y]$ by $T_{0}=2$, $T_{1}=x$, and

$$
\begin{equation*}
T_{m}=x T_{m-1}-y T_{m-2} \text { for } m \geq 2 \tag{4.1}
\end{equation*}
$$

The monograph of Lidl et al. (1993) provides extensive information about these polynomials. We have $T_{m}(x, 0)=x^{m}$, and $T_{m}(x, 1)$ is closely related to the Chebyshev polynomial $C_{n}=\cos (n \arccos x)$, as $T_{n}(2 x, 1)=2 C_{n}(x) . T_{m}$ is monic (for $m \geq 1$ ) of degree $m$, and

$$
T_{m}=\sum_{0 \leq i \leq m / 2} \frac{m}{m-i}\binom{m-i}{i}(-y)^{i} x^{m-2 i} \in F[x, y] .
$$

Furthermore,

$$
\begin{equation*}
T_{m}\left(x, y^{\ell}\right) \circ T_{\ell}(x, y)=T_{\ell m}(x, y)=T_{\ell}\left(x, y^{m}\right) \circ T_{m}(x, y) \tag{4.2}
\end{equation*}
$$

and if $\ell \neq m$, then substituting any $z \in F$ for $y$ yields a collision.
Ritt's Second Theorem is the central tool for understanding distinct-degree collisions, and the following notions enter the scene. The functional inverse $v^{-1}$ of a linear polynomial $v=a x+b$ with $a, b \in F$ and $a \neq 0$ is defined as $v^{-1}=(x-b) / a$. Then $v^{-1} \circ v=v \circ v^{-1}=x$. Two pairs $(g, h)$ and $\left(g^{*}, h^{*}\right)$ of polynomials are called equivalent if there exists a linear polynomial $v$ such that

$$
g^{*}=g \circ v, h^{*}=v^{-1} \circ h .
$$

Then $g \circ h=g^{*} \circ h^{*}$, and we write $(g, h) \sim\left(g^{*}, h^{*}\right)$. The following result says that, under certain conditions, the examples above are essentially the only distinct-degree collisions. It was first proved by Ritt (1922) for $F=\mathbb{C}$. We use the strong version of Zannier (1993), adapted to finite fileds. The adaption uses Schinzel (2000), Section 1.4, Lemma 2, and leads to his Theorem 8. Further references can be found in this monograph as well.

FACT 4.3. (Ritt's Second Theorem) Let $\ell$ and $m$ be integers, $F$ a field, and $g$, $h, g^{*}, h^{*} \in F[x]$ with

$$
\begin{gather*}
m>\ell \geq 2, \operatorname{gcd}(\ell, m)=1, \operatorname{deg} g=\operatorname{deg} h^{*}=m, \operatorname{deg} h=\operatorname{deg} g^{*}=\ell  \tag{4.4}\\
g^{\prime}\left(g^{*}\right)^{\prime} \neq 0 \tag{4.5}
\end{gather*}
$$

where $g^{\prime}=\partial g / \partial x$ is the derivative of $g$. Then

$$
\begin{equation*}
g \circ h=g^{*} \circ h^{*} \tag{4.6}
\end{equation*}
$$

if and only if

$$
\exists k \in \mathbb{N}, v_{1}, v_{2} \in F[x] \text { linear, } w \in F[x] \text { with } k+\ell \operatorname{deg} w=m, z \in F^{\times}
$$

so that either

First Case

$$
\left\{\begin{aligned}
\left(v_{1} \circ g, h \circ v_{2}\right) & \sim\left(x^{k} w^{\ell}, x^{\ell}\right), \\
\left(v_{1} \circ g^{*}, h^{*} \circ v_{2}\right) & \sim\left(x^{\ell}, x^{k} w\left(x^{\ell}\right)\right),
\end{aligned}\right.
$$

or

Second Case

$$
\left\{\begin{aligned}
\left(v_{1} \circ g, h \circ v_{2}\right) & \sim\left(T_{m}\left(x, z^{\ell}\right), T_{\ell}(x, z)\right) \\
\left(v_{1} \circ g^{*}, h^{*} \circ v_{2}\right) & \sim\left(T_{\ell}\left(x, z^{m}\right), T_{m}(x, z)\right)
\end{aligned}\right.
$$

In principle, one also has to consider the First Case with $(g, h, m)$ and ( $g^{*}, h^{*}, \ell$ ) interchanged; see Zannier (1993), Main Theorem (ii). Then $k+$ $m \operatorname{deg} w=\ell$ and hence $\operatorname{deg} w=0$. But this situation is covered by the First Case in Fact 4.3, with $k=m$. We note that the conclusion of the First Case is asymmetric in $\ell$ and $m$, but in the Second Case it is symmetric, so that there the assumption $m>\ell$ does not intervene.

According to Remark 2.2, we may assume $h$ and $h^{*}$ to be monic and original. If one of $g$ or $g^{*}$ is also monic and original, then so is the other one, and also the composition (4.6). It is convenient to add this condition:

$$
\begin{equation*}
f=g \circ h, \text { and } g, h, g^{*}, h^{*} \text { are monic and original. } \tag{4.7}
\end{equation*}
$$

The transition between the general and this special case is by left composition with a linear polynomial.

The following lemma about Dickson polynomials will be useful for determining the number of collisions exactly. We write $T_{n}^{\prime}(x, y)=\partial T_{n}(x, y) / \partial x$ for the derivative with respect to $x$.

Lemma 4.8. Let $F$ be a field of characteristic $p \geq 0, n \geq 1$, and $z \in F^{\times}$.
(i) If $p=0$, or $p \geq 3$ and $\operatorname{gcd}(n, p)=1$, then the derivative $T_{n}^{\prime}(x, z)$ is squarefree in $F[x]$.
(ii) If $p=0$ or $\operatorname{gcd}(n, p)=1$, and $n$ is odd, then there exists some monic squarefree $u \in F[x]$ of degree $(n-1) / 2$ so that $T_{n}\left(x, z^{2}\right)=(x-2 z) \cdot u^{2}+$ $2 z^{n}$.
(iii) Let $\gamma=(-y)^{\lfloor n / 2\rfloor}$. $T_{n}$ is an odd or even polynomial in $x$ if $n$ is odd or even, respectively. It has the form

$$
T_{n}= \begin{cases}x^{n}-n y x^{n-2}+-\cdots+\gamma x & \text { if } n \text { is odd } \\ x^{n}-n y x^{n-2}+-\cdots+2 \gamma & \text { if } n \text { is even }\end{cases}
$$

(iv) If $p \geq 2$, then $T_{p^{j}}=x^{p^{j}}$ for $j \geq 0$.
(v) If $p \geq 2$ and $p \mid n$, then $T_{n}^{\prime}=0$.
(vi) For a new indeterminate $t$, we have $t^{n} T_{n}(x, y)=T_{n}\left(t x, t^{2} y\right)$.
(vii) $T_{n}\left(2 z, z^{2}\right)=2 z^{n}$.

Proof. (i) Williams (1971) and Corollary 3.14 of Lidl et al. (1993) show that if $F$ contains a primitive $n$th root of unity $\rho$, then $T_{n}^{\prime}(x, z) / n c$ factors over $F$ completely into a product of quadratic polynomials $\left(x^{2}-\alpha_{k}^{2} z\right)$, where $1 \leq k<n / 2$, the $\alpha_{k}=\rho^{k}+\rho^{-k}$ are Gauß periods derived from $\rho$, and the $\alpha_{k}^{2}$ are pairwise distinct, with $c=1$ if $n$ is odd and $c=x$ otherwise. We note that $\alpha_{k}=\alpha_{n-k}$. We take an extension $E$ of $F$ that contains a primitive $n$th root of unity and a square root $z_{0}$ of $z$. This is possible since $p=0$ or $\operatorname{gcd}(n, p)=1$. Thus $x^{2}-\alpha_{k}^{2} z=\left(x-\alpha_{k} z_{0}\right)\left(x+\alpha_{k} z_{0}\right)$, and the $\pm \alpha_{k} z_{0}$ for $1 \leq k<n / 2$ are pairwise distinct, using that $p \neq 2$. It follows that $T_{n}^{\prime}(x, z)$ is squarefree over $E$. Since squarefreeness is a rational condition, equivalent to the nonvanishing of the discriminant, $T_{n}^{\prime}(x, z)$ is also squarefree over $F$.

For (ii), we take a Galois extension field $E$ of $F$ that contains a primitive $n$th root of unity $\rho$, and set $\alpha_{k}=\rho^{k}+\rho^{-k}$ and $\beta_{k}=\rho^{k}-\rho^{-k}$ for all $k \in \mathbb{Z}$. We have $T_{n}\left(2 z, z^{2}\right)=2 z^{n}$ by (vii), proven below, and Theorem 3.12(i) of Lidl \& Mullen (1993) states that

$$
T_{n}\left(x, z^{2}\right)-2 z^{n}=(x-2 z) \prod_{1 \leq k<n / 2}\left(x^{2}-2 \alpha_{k} z x+4 z^{2}+\beta_{k}^{2} z^{2}\right)
$$

see also Turnwald (1995), Proposition 1.7. Now $-\alpha_{k}^{2}+4+\beta_{k}^{2}=-\left(\rho^{k}+\rho^{-k}\right)^{2}+$ $\left(\rho^{k}-\rho^{-k}\right)^{2}+4=0$, so that $x^{2}-2 \alpha_{k} z x+4 z^{2}+\beta_{k}^{2} z^{2}=\left(x-\alpha_{k} z\right)^{2}$. We set $u=\prod_{1 \leq k<n / 2}\left(x-\alpha_{k} z\right) \in E[x]$. Then $T_{n}\left(x, z^{2}\right)-2 z^{n}=(x-2 z) u^{2}$, and $u$ is squarefree. It remains to show that $u \in F[x]$. We take some $\sigma \in \operatorname{Gal}(E: F)$. Then $\sigma(\rho)$ is also a primitive $n$th root of unity, say $\sigma(\rho)=\rho^{i}$ with $1 \leq i<n$ and $\operatorname{gcd}(i, n)=1$. We take some $k$ with $1 \leq k<n / 2$, and $j$ with $i k \equiv j \bmod n$ and $0<|j|<n / 2$. Then $\sigma\left(\alpha_{k}\right)=\alpha_{|j|}$. Hence, $\sigma$ induces a permutation on $\left\{\alpha_{1}, \ldots, \alpha_{(n-1) / 2}\right\}$. It follows that

$$
u=\prod_{1 \leq k<n / 2}\left(x-\alpha_{k} z\right)=\prod_{1 \leq k<n / 2}\left(x-\sigma\left(\alpha_{k} z\right)\right)=\sigma u
$$

Since this holds for all $\sigma$, we have $u \in F[x]$.
(iii) follows from the recursion (4.1), and (iv) from Lidl et al. (1993), Lemma 2.6(iii). (v) follows from (4.2) and (iv). The claim in (vi) is Lemma 2.6(ii) of Lidl et al. (1993). It also follows inductively from (4.1), as does (vii).

In the following, we present several pairs of results. In each pair, the first item is a theorem, valid over fairly general fields, that describes the structure of distinct-degree collisions. The second one is a corollary, valid over finite fields, giving bounds on the number of such collisions. We start with the following
normal form for the decompositions in Ritt's Second Theorem. The uniqueness result is not obvious, as witnessed by the quotes in the Introduction.

Theorem 4.9. Let $F$ be a field of characteristic $p$, let $m>\ell \geq 2$ be integers, and $n=\ell m$. Furthermore, we have monic original $f, g, h, g^{*}, h^{*} \in F[x]$ satisfying (4.4) through (4.7). Then either (i) or (ii) hold, and (iii) is also valid.
(i) (First Case) There exists a monic polynomial $w \in F[x]$ of degree $s$ and $c \in F$ so that

$$
\begin{equation*}
f=\left(x-a^{k \ell} w^{\ell}\left(a^{\ell}\right)\right) \circ x^{k \ell} w^{\ell}\left(x^{\ell}\right) \circ(x+a), \tag{4.10}
\end{equation*}
$$

where $m=s \ell+k$ is the division with remainder of $m$ by $\ell$, with $1 \leq k<\ell$. Furthermore

$$
\begin{align*}
& \quad k w+\ell x w^{\prime} \neq 0 \text { and } p \nmid \ell,  \tag{4.11}\\
& g=\left(x-a^{k \ell} w^{\ell}\left(a^{\ell}\right)\right) \circ x^{k} w^{\ell} \circ\left(x+a^{\ell}\right), \\
& h=\left(x-a^{\ell}\right) \circ x^{\ell} \circ(x+a), \\
& g^{*}=\left(x-a^{k \ell} w^{\ell}\left(a^{\ell}\right)\right) \circ x^{\ell} \circ\left(x+a^{k} w\left(a^{\ell}\right)\right), \\
& h^{*}=\left(x-a^{k} w\left(a^{\ell}\right)\right) \circ x^{k} w\left(x^{\ell}\right) \circ(x+a) .
\end{align*}
$$

Conversely, any ( $w, a$ ) as above for which (4.11) holds yields a collision satisfying (4.4) through (4.7), via the above formulas. If $p \nmid m$, then $(w, a)$ is uniquely determined by $f$ and $\ell$.
(ii) (Second Case) There exist $z, a \in F$ with $z \neq 0$ so that

$$
\begin{equation*}
f=\left(x-T_{n}(a, z)\right) \circ T_{n}(x, z) \circ(x+a) \tag{4.12}
\end{equation*}
$$

Now $(z, a)$ is uniquely determined by $f$. Furthermore we have

$$
\begin{align*}
& p \nmid n,  \tag{4.13}\\
g & =\left(x-T_{n}(a, z)\right) \circ T_{m}\left(x, z^{\ell}\right) \circ\left(x+T_{\ell}(a, z)\right), \\
h & =\left(x-T_{\ell}(a, z)\right) \circ T_{\ell}(x, z) \circ(x+a), \\
g^{*} & =\left(x-T_{n}(a, z)\right) \circ T_{\ell}\left(x, z^{m}\right) \circ\left(x+T_{m}(a, z)\right), \\
h^{*} & =\left(x-T_{m}(a, z)\right) \circ T_{m}(x, z) \circ(x+a) .
\end{align*}
$$

Conversely, if (4.13) holds, then any $(z, a)$ as above yields a collision satisfying (4.4) through (4.7), via the above formulas.
(iii) When $\ell \geq 3$, the First and Second Cases are mutually exclusive. For $\ell=2$, the Second Case is included in the First Case.

Proof. By assumption, either the First or the Second Case of Ritt's Second Theorem (Fact 4.3) applies.
(i) From the First Case in Fact 4.3, we have a positive integer $K$, linear polynomials $v_{1}, v_{2}, v_{3}, v_{4}$ and a nonzero polynomial $W$ with $d=\operatorname{deg} W=$ $(m-K) / \ell$ and (renaming $v_{2}$ as $v_{2}^{-1}$ )

$$
\begin{aligned}
x^{K} W^{\ell} & =v_{1} \circ g \circ v_{3}, \\
x^{\ell} & =v_{3}^{-1} \circ h \circ v_{2}^{-1}, \\
x^{\ell} & =v_{1} \circ g^{*} \circ v_{4}, \\
x^{K} W\left(x^{\ell}\right) & =v_{4}^{-1} \circ h^{*} \circ v_{2}^{-1} .
\end{aligned}
$$

We abbreviate $r=\operatorname{lc}(W)$, so that $r \neq 0$, and write $v_{i}=a_{i} x+b_{i}$ for $1 \leq i \leq 4$ with all $a_{i}, b_{i} \in F$ and $a_{i} \neq 0$, and first express $v_{3}, v_{4}$, and $v_{1}$ in terms of $v_{2}$. We have

$$
\begin{aligned}
h & =v_{3} \circ x^{\ell} \circ v_{2}=a_{3}\left(a_{2} x+b_{2}\right)^{\ell}+b_{3}, \\
h^{*} & =v_{4} \circ x^{K} W\left(x^{\ell}\right) \circ v_{2}=a_{4}\left(a_{2} x+b_{2}\right)^{K} \cdot W\left(\left(a_{2} x+b_{2}\right)^{\ell}\right)+b_{4} .
\end{aligned}
$$

Since $h$ and $h^{*}$ are monic and original and $K+\ell d=m$, it follows that

$$
a_{3}=a_{2}^{-\ell}, b_{3}=-a_{2}^{-\ell} b_{2}^{\ell}, a_{4}=a_{2}^{-m} r^{-1}, b_{4}=-a_{2}^{-m} b_{2}^{K} r^{-1} W\left(b_{2}^{\ell}\right)
$$

Playing the same game with $g$, we find

$$
\begin{aligned}
g=v_{1}^{-1} \circ x^{K} W^{\ell} \circ v_{3}^{-1} & =a_{1}^{-1}\left(\left(\frac{x-b_{3}}{a_{3}}\right)^{K} W^{\ell}\left(\frac{x-b_{3}}{a_{3}}\right)-b_{1}\right), \\
a_{1} & =a_{2}^{n} r^{\ell}, \\
b_{1} & =b_{2}^{K \ell} W^{\ell}\left(b_{2}^{\ell}\right) .
\end{aligned}
$$

We note that then

$$
g^{*}=v_{1}^{-1} \circ x^{\ell} \circ v_{4}^{-1}=a_{1}^{-1}\left(\left(\frac{x-b_{4}}{a_{4}}\right)^{\ell}-b_{1}\right)
$$

is automatically monic and original. Furthermore, we have $d=(m-K) / \ell \leq$ $\lfloor m / \ell\rfloor=s$ and

$$
\begin{equation*}
f=v_{1}^{-1} \circ\left(v_{1} \circ g \circ v_{3}\right) \circ\left(v_{3}^{-1} \circ h \circ v_{2}^{-1}\right) \circ v_{2}=v_{1}^{-1} \circ x^{K \ell} \cdot W^{\ell}\left(x^{\ell}\right) \circ v_{2} . \tag{4.14}
\end{equation*}
$$

We set

$$
\begin{aligned}
a & =\frac{b_{2}}{a_{2}} \in F, \quad u_{1}=x+\frac{b_{1}}{a_{1}}=\frac{v_{1}}{a_{1}}, \quad u_{2}=x+a=\frac{v_{2}}{a_{2}}, \\
w & =r^{-1} a_{2}^{-\ell d} x^{s-d} \cdot W\left(a_{2}^{\ell} x\right) \in F[x] .
\end{aligned}
$$

Then $b_{1} / a_{1}=a^{k \ell} w^{\ell}\left(a^{\ell}\right), w$ is monic of degree $s, u_{1}^{-1}=x-b_{1} / a_{1}=x-a^{k \ell} w^{\ell}\left(a^{\ell}\right)$, and

$$
\begin{equation*}
W(x)=\operatorname{lc}(W) a_{2}^{\ell s} x^{-(s-d)} w\left(a_{2}^{-\ell} x\right) \tag{4.15}
\end{equation*}
$$

Noting that $m=\ell d+K=\ell s+k$, the equation analogous to (4.14) reads

$$
\begin{align*}
u_{1}^{-1} \circ x^{k \ell} w^{\ell}\left(x^{\ell}\right) \circ u_{2} & =a_{1} \cdot v_{1}^{-1} \circ x^{k \ell} \cdot \frac{x^{\ell^{2}(s-d)} W^{\ell}\left(a_{2}^{\ell} x^{\ell}\right)}{a_{2}^{d \ell^{2}} r^{\ell}} \circ \frac{v_{2}}{a_{2}} \\
& =v_{1}^{-1} \circ a_{2}^{n} r^{\ell} \cdot\left(\frac{v_{2}}{a_{2}}\right)^{k \ell} \cdot\left(\frac{v_{2}}{a_{2}}\right)^{\ell^{2}(s-d)} \cdot \frac{W^{\ell}\left(v_{2}^{\ell}\right)}{a_{2}^{d \ell^{2}} r^{\ell}} \\
& =v_{1}^{-1} \circ x^{K \ell} \cdot W^{\ell}\left(x^{\ell}\right) \circ v_{2}=f . \tag{4.16}
\end{align*}
$$

This proves the existence of $w$ and $a$, as claimed in (4.10).
In order to express the four components in the new parameters, we note
that $K=k+\ell(s-d)$. Thus

$$
\begin{aligned}
g= & v_{1}^{-1} \circ x^{K} W^{\ell} \circ v_{3}^{-1} \\
= & \left(r^{-\ell} a_{2}^{-n} x-a^{k \ell} w^{\ell}\left(a^{\ell}\right)\right) \circ\left(a_{2}^{\ell}\left(x+a^{\ell}\right)\right)^{K} \cdot W^{\ell}\left(a_{2}^{\ell}\left(x+a^{\ell}\right)\right) \\
= & r^{-\ell} a_{2}^{-n}\left(a_{2}^{K \ell}\left(x+a^{\ell}\right)^{K} \cdot r^{\ell} a_{2}^{\ell^{2} s} a_{2}^{-\ell^{2}(s-d)}\left(x+a^{\ell}\right)^{-\ell(s-d)} w^{\ell}\left(x+a^{\ell}\right)\right) \\
& -a^{k \ell} w^{\ell}\left(a^{\ell}\right) \\
= & a_{2}^{-n+K \ell+\ell^{2} s-\ell^{2} s+\ell^{2} d}\left(x+a^{\ell}\right)^{K-\ell s+\ell d} w^{\ell}\left(x+a^{\ell}\right)-a^{k \ell} w^{\ell}\left(c a^{\ell}\right) \\
= & \left(x+a^{\ell}\right)^{k} w^{\ell}\left(x+a^{\ell}\right)-a^{k \ell} w^{\ell}\left(a^{\ell}\right) \\
= & \left(x-a^{k \ell} w^{\ell}\left(a^{\ell}\right)\right) \circ x^{k} w^{\ell} \circ\left(x+a^{\ell}\right), \\
h= & v_{3} \circ x^{\ell} \circ v_{2}=a_{2}^{-\ell}\left(a_{2} x+b_{2}\right)^{\ell}-a_{2}^{-\ell} b_{2}^{\ell} \\
= & \left(x-a^{\ell}\right) \circ x^{\ell} \circ(x+a), \\
g^{*}= & v_{1}^{-1} \circ x^{\ell} \circ v_{4}^{-1} \\
= & \left(r^{-\ell} a_{2}^{-n} x-a^{k \ell} w^{\ell}\left(a^{\ell}\right)\right) \circ\left(r a_{2}^{m}\left(x+r^{-1} a_{2}^{-m} b_{2}^{K} \cdot W\left(b_{2}^{\ell}\right)\right)\right)^{\ell} \\
= & \left(x+r^{-1} a_{2}^{-m} b_{2}^{K} \cdot r a_{2}^{\ell s} b_{2}^{-\ell(s-d)} w\left(a^{\ell}\right)\right)^{\ell}-a^{k \ell} w^{\ell}\left(a^{\ell}\right) \\
= & \left(x+a_{2}^{-k} b_{2}^{k} w\left(a^{\ell}\right)\right)^{\ell}-a^{k \ell} w^{\ell}\left(a^{\ell}\right) \\
= & \left(x-a^{k \ell} w^{\ell}\left(a^{\ell}\right)\right) \circ x^{\ell} \circ\left(x+a^{k} w\left(a^{\ell}\right)\right), \\
h^{*}= & v_{4} \circ x^{K} W\left(x^{\ell}\right) \circ v_{2} \\
= & \left(r^{-1} a_{2}^{-m}\left(x-b_{2}^{K} W\left(b_{2}^{\ell}\right)\right)\right) \circ\left(a_{2}(x+a)\right)^{K} W\left(a_{2}^{\ell}(x+a)^{\ell}\right) \\
= & r^{-1} a_{2}^{-m} \cdot r a_{2}^{\ell s} \cdot\left(\left(a_{2}^{K}(x+a)^{K}\left(a_{2}^{\ell}(x+a)^{\ell}\right)\right)^{-(s-d)} w\left((x+a)^{\ell}\right)\right. \\
& \left.-b_{2}^{K} b_{2}^{-\ell(s-d)} w\left(a^{\ell}\right)\right) \\
= & a_{2}^{-k}\left(a_{2}^{K-\ell(s-d)}(x+a)^{K-\ell(s-d)} w\left((x+a)^{\ell}\right)-b_{2}^{K-\ell(s-d)} w\left(a^{\ell}\right)\right) \\
= & (x+a)^{k} w\left((x+a)^{\ell}\right)-a^{k} w\left(a^{\ell}\right) \\
= & \left(x-a^{k} w\left(a^{\ell}\right)\right) \circ x^{k} w\left(x^{\ell}\right) \circ(x+a) .
\end{aligned}
$$

(4.10) has been shown above. We note that in the right hand component $x+a$, the constant $a$ is arbitrary. All other linear components follow automatically from the required form of $g, h, g^{*}, h^{*}$, namely, being monic and original, and from the condition that $g$ and $h$ (and $g^{*}$ and $h^{*}$ ) have to match up with their "middle" components. Furthermore, we have

$$
\begin{gather*}
0=g^{\prime}=\left(x^{k-1} w^{\ell-1}\left(k w+\ell x w^{\prime}\right)\right) \circ\left(x+a^{\ell}\right) \Longleftrightarrow k w+\ell x w^{\prime}=0, \\
0=\left(g^{*}\right)^{\prime}=\ell x^{\ell-1} \circ\left(x+a^{k} w\left(a^{\ell}\right)\right) \Longleftrightarrow p \mid \ell . \tag{4.17}
\end{gather*}
$$

Thus (4.11) follows from (4.5).

In order to prove the uniqueness if $p \nmid n$, we take monic $w, \tilde{w} \in F[x]$ of degree $s$, and $a, \tilde{a} \in F$ and the unique monic linear polynomials $v$ and $\tilde{v}$ for which

$$
\begin{equation*}
f=v \circ x^{k \ell} w^{\ell}\left(x^{\ell}\right) \circ(x+a)=\tilde{v} \circ x^{k \ell} \tilde{w}^{\ell}\left(x^{\ell}\right) \circ(x+\tilde{a}) \tag{4.18}
\end{equation*}
$$

By composing on the left and right with $\tilde{v}^{-1}$ and $(x+\tilde{a})^{-1}$, respectively, and abbreviating $u=\tilde{v}^{-1} \circ v$, we find

$$
\begin{aligned}
x^{k \ell} \tilde{w}^{\ell}\left(x^{\ell}\right) & =\tilde{v}^{-1} \circ v \circ x^{k \ell} w^{\ell}\left(x^{\ell}\right) \circ(x+a) \circ(x-\tilde{a}) \\
& =u \circ x^{k \ell} w^{\ell}\left(x^{\ell}\right) \circ(x+a-\tilde{a}) .
\end{aligned}
$$

Since $\ell \geq 2$ and the left hand side is a polynomial in $x^{\ell}$, its second highest coefficient (of $x^{n-1}$ ) vanishes. Equating this with the same coefficient on the right, and abbreviating $a^{*}=a-\tilde{a}$, we find

$$
0=k \ell a^{*}+s \ell^{2} a^{*}=n a^{*},
$$

so that $a^{*}=0$, since $p \nmid n$. Thus $a=\tilde{a}$ and

$$
\begin{gathered}
x^{k} \tilde{w}^{\ell} \circ x^{\ell}=x^{k \ell} \tilde{w}^{\ell}\left(x^{\ell}\right)=u \circ x^{k \ell} w^{\ell}\left(x^{\ell}\right)=u \circ x^{k} w^{\ell} \circ x^{\ell}, \\
x^{k} \tilde{w}^{\ell}=u \circ x^{k} w^{\ell} .
\end{gathered}
$$

Now $x^{k} \tilde{w}^{\ell}$ and $x^{k} w^{\ell}$ are monic and original, since $k \geq 1$. It follows that $u=x$ and $w^{\ell}=\tilde{w}^{\ell}$. Both polynomials are monic, so that $w=\tilde{w}$, as claimed. (The equation for $h$ in Theorem 4.9(i) determines $a$ uniquely provided that $p \nmid \ell$, even if $p \mid m$. However, the value of $h$ is not unique in this case.)

Conversely, we take some ( $w, a$ ) satisfying (4.11) and define $f, g, h, g^{*}, h^{*}$ via the formulas in (i). Then (4.4), (4.6), and (4.7) hold. As to (4.5), we have $p \nmid \ell$ from (4.11), and hence $\left(g^{*}\right)^{\prime} \neq 0$. Furthermore,

$$
\left(x^{k} w^{\ell}\right)^{\prime}=x^{k-1} w^{\ell-1} \cdot\left(k w+\ell x w^{\prime}\right) \neq 0,
$$

so that also $g^{\prime} \neq 0$.
(ii) In the Second Case, again renaming $v_{2}$ as $v_{2}^{-1}$, and also $z$ as $z_{2}$, we have from Fact 4.3

$$
\begin{aligned}
T_{m}\left(x, z_{2}^{\ell}\right) & =v_{1} \circ g \circ v_{3}, \\
T_{\ell}\left(x, z_{2}\right) & =v_{3}^{-1} \circ h \circ v_{2}^{-1}, \\
T_{\ell}\left(x, z_{2}^{m}\right) & =v_{1} \circ g^{*} \circ v_{4}, \\
T_{m}\left(x, z_{2}\right) & =v_{4}^{-1} \circ h^{*} \circ v_{2}^{-1}, \\
h & =v_{3} \circ T_{\ell}\left(x, z_{2}\right) \circ v_{2}=a_{3} T_{\ell}\left(a_{2} x+b_{2}, z_{2}\right)+b_{3}, \\
h^{*} & =v_{4} \circ T_{m}\left(x, z_{2}\right) \circ v_{2}=a_{4} T_{m}\left(a_{2} x+b_{2}, z_{2}\right)+b_{4} .
\end{aligned}
$$

As before, it follows that

$$
a_{3}=a_{2}^{-\ell}, \quad b_{3}=-a_{2}^{-\ell} T_{\ell}\left(b_{2}, z_{2}\right), \quad a_{4}=a_{2}^{-m}, \quad b_{4}=-a_{2}^{-m} T_{m}\left(b_{2}, z_{2}\right)
$$

Furthermore, we have

$$
\begin{aligned}
g & =v_{1}^{-1} \circ T_{m}\left(x, z_{2}^{\ell}\right) \circ v_{3}^{-1}=a_{1}^{-1}\left(T_{m}\left(a_{3}^{-1}\left(x-b_{3}\right), z_{2}^{\ell}\right)-b_{1}\right), \\
a_{1} & =a_{2}^{n}, \\
b_{1} & =T_{m}\left(T_{\ell}\left(b_{2}, z_{2}\right), z_{2}^{\ell}\right)=T_{n}\left(b_{2}, z_{2}\right), \\
f & =\left(a_{2}^{-n}\left(x-T_{n}\left(b_{2}, z_{2}\right)\right)\right) \circ T_{n}\left(x, z_{2}\right) \circ\left(a_{2} x+b_{2}\right) .
\end{aligned}
$$

We now set $a=b_{2} / a_{2}$ and $z=z_{2} / a_{2}^{2}$ and show that the preceding equation holds with $(1, a, z)$ for $\left(a_{2}, b_{2}, z_{2}\right)$. Lemma $4.8(v i)$ with $t=a_{2}^{-1}$ says that

$$
\begin{aligned}
a_{2}^{-n} T_{n}\left(a_{2} x+b_{2}, z_{2}\right) & =T_{n}(x+a, z), \\
a_{2}^{-n} T_{n}\left(b_{2}, z_{2}\right) & =T_{n}(a, z), \\
f & =\left(x-T_{n}(a, z)\right) \circ T_{n}(x, z) \circ(x+a) .
\end{aligned}
$$

Thus the first claim in (ii) holds with these values. In the same vein, applying Lemma 4.8(vi) with $t$ equal to $a_{2}^{-1}, a_{2}^{-\ell}, a_{2}^{-m}, a_{2}^{-1}$, respectively, yields

$$
\begin{aligned}
a_{2}^{-\ell} T_{\ell}\left(a_{2} x+b_{2}, z_{2}\right) & =T_{\ell}(x+a, z), \\
a_{2}^{-n} T_{m}\left(a_{2}^{\ell} x+T_{\ell}\left(b_{2}, z_{2}\right), z_{2}^{\ell}\right) & =T_{m}\left(x+a_{2}^{-\ell} T_{\ell}\left(b_{2}, z_{2}\right), z^{\ell}\right) \\
& =T_{m}\left(x+T_{\ell}(a, z), z^{\ell}\right), \\
a_{2}^{-n} T_{\ell}\left(a_{2}^{m} x+T_{m}\left(b_{2}, z_{2}\right), z_{2}^{m}\right) & =T_{\ell}\left(x+a_{2}^{-m} T_{m}\left(b_{2}, z_{2}\right), z^{m}\right) \\
& =T_{\ell}\left(x+T_{m}(a, z), z^{m}\right), \\
a_{2}^{-m} T_{m}\left(a_{2} x+b_{2}, z_{2}\right) & =T_{m}(x+a, z) .
\end{aligned}
$$

For the four components, we have

$$
\begin{aligned}
g & =v_{1}^{-1} \circ T_{m}\left(x, z_{2}^{\ell}\right) \circ v_{3}^{-1} \\
& =a_{2}^{-n}\left(x-T_{n}\left(b_{2}, z_{2}\right)\right) \circ T_{m}\left(x, z_{2}^{\ell}\right) \circ\left(a_{2}^{\ell} x+T_{\ell}\left(b_{2}, z_{2}\right)\right) \\
& =a_{2}^{-n} T_{m}\left(a_{2}^{\ell} x+T_{\ell}\left(b_{2}, z_{2}\right), z_{2}^{\ell}\right)-a_{2}^{-n} T_{m}\left(T_{\ell}\left(b_{2}, z_{2}\right), z_{2}^{\ell}\right) \\
& =T_{m}\left(x+T_{\ell}(a, z), z^{\ell}\right)-T_{n}(a, z) \\
& =\left(x-T_{n}(a, z)\right) \circ T_{m}\left(x, z^{\ell}\right) \circ\left(x+T_{\ell}(a, z)\right), \\
h & =v_{3} \circ T_{\ell}\left(x, z_{2}\right) \circ v_{2}=a_{2}^{-\ell} T_{\ell}\left(a_{2} x+b_{2}, z_{2}\right)-a_{2}^{-\ell} T_{\ell}\left(b_{2}, z_{2}\right) \\
& =a_{2}^{-\ell}\left(x-T_{\ell}\left(b_{2}, z_{2}\right)\right) \circ T_{\ell}\left(x, z_{2}\right) \circ\left(a_{2} x+b_{2}\right) \\
& =T_{\ell}(x+a, z)-T_{\ell}(a, z) \\
& =\left(x-T_{\ell}(a, z)\right) \circ T_{\ell}(x, z) \circ(x+a), \\
g^{*} & =v_{1}^{-1} \circ T_{\ell}\left(x, z_{2}^{m}\right) \circ v_{4}^{-1} \\
& =a_{2}^{-n}\left(x-T_{n}\left(b_{2}, z_{2}\right)\right) \circ T_{\ell}\left(x, z_{2}^{m}\right) \circ\left(a_{2}^{m} x+T_{m}\left(b_{2}, z_{2}\right)\right) \\
& =a_{2}^{-n} T_{\ell}\left(a_{2}^{m} x+T_{m}\left(b_{2}, z_{2}\right), z_{2}^{m}\right)-a_{2}^{--n} T_{n}\left(b_{2}, z_{2}\right) \\
& =T_{\ell}\left(x+T_{m}(a, z), z^{m}\right)-T_{n}(a, z) \\
& =\left(x-T_{n}(a, z)\right) \circ T_{\ell}\left(a, z^{m}\right) \circ\left(x+T_{m}(a, z)\right), \\
h^{*} & =v_{4} \circ T_{m}\left(x, z_{2}\right) \circ v_{2} \\
& =a_{2}^{-m}\left(x-T_{m}\left(b_{2}, z_{2}\right)\right) \circ T_{m}\left(x, z_{2}\right) \circ\left(a_{2} x+b_{2}\right) \\
& =a_{2}^{-m} T_{m}\left(a_{2} x+b_{2}, z_{2}\right)-a_{2}^{-m} T_{m}\left(b_{2}, z_{2}\right) \\
& =T_{m}(x+a, z)-T_{m}(a, z) \\
& =\left(x-T_{m}(a, z)\right) \circ T_{m}(x, z) \circ(x+a) .
\end{aligned}
$$

Since

$$
0 \neq g^{\prime}=T_{m}^{\prime}\left(x, z^{\ell}\right) \circ\left(x+T_{\ell}(a, z)\right)
$$

Lemma 4.8(v) implies that $p \nmid m$. Similarly, the non-vanishing of $\left(g^{*}\right)^{\prime}$ implies that $p \nmid \ell$, and (4.13) follows.

Next we claim that the representation of $f$ is unique. So we take some $(z, a),\left(z^{*}, a^{*}\right) \in F^{2}$ with $z z^{*} \neq 0$ and

$$
\begin{equation*}
\left(x-T_{n}(a, z)\right) \circ T_{n}(x, z) \circ(x+a)=\left(x-T_{n}\left(a^{*}, z^{*}\right)\right) \circ T_{n}\left(x, z^{*}\right) \circ\left(x+a^{*}\right) \tag{4.19}
\end{equation*}
$$

Comparing the coefficients of $x^{n-1}$ in (4.19) and using Lemma 4.8(iii) yields $n a=n a^{*}$, hence $a=a^{*}$, since $p \nmid n$. We now compose (4.19) with $x-a$ on the right and find

$$
\left(x-T_{n}(a, z)\right) \circ T_{n}(x, z)=\left(x-T_{n}\left(a, z^{*}\right)\right) \circ T_{n}\left(x, z^{*}\right) .
$$

Now the coefficients of $x^{n-2}$ yield $-n z=-n z^{*}$, so that $z=z^{*}$.
The converse claim that any $(z, a)$ with $z \neq 0$ and (4.13) yields a collision as prescribed follows since (4.13) and Lemma 4.8(v) imply that $T_{m}^{\prime}\left(x, z^{\ell}\right) T_{\ell}^{\prime}\left(x, z^{m}\right) \neq$ 0.
(iii) We first assume $\ell \geq 3$ and show that the First and Second Cases are mutually exclusive. Assume, to the contrary, that in our usual notation we have

$$
\begin{equation*}
f=v_{1} \circ x^{k \ell} w^{\ell}\left(x^{\ell}\right) \circ(x+a)=v_{2} \circ T_{n}(x, z) \circ\left(x+a^{*}\right), \tag{4.20}
\end{equation*}
$$

where $v_{1}$ and $v_{2}$ are the unique linear polynomials that make the composition monic and original, as specified in (i) and (ii). Then

$$
\begin{aligned}
f & =\left(v_{1} \circ x^{k} w^{\ell} \circ\left(x+a^{\ell}\right)\right) \circ\left((x+a)^{\ell}-a^{\ell}\right) \\
& =\left(v_{2} \circ T_{m}\left(x+T_{\ell}\left(a^{*}, z\right), z^{\ell}\right)\right) \circ\left(T_{\ell}\left(x+a^{*}, z\right)-T_{\ell}\left(a^{*}, z\right)\right) .
\end{aligned}
$$

These are two normal decompositions of $f$, and since $p \nmid m$ by (4.13), the uniqueness of Fact 3.1(i) implies that

$$
\begin{align*}
h & =(x+a)^{\ell}-a^{\ell}=T_{\ell}\left(x+a^{*}, z\right)-T_{\ell}\left(a^{*}, z\right),  \tag{4.21}\\
h^{\prime} & =\ell(x+a)^{\ell-1}=T_{\ell}^{\prime}\left(x+a^{*}, z\right) .
\end{align*}
$$

If $p=0$ or $p \geq 3$, then according to Lemma 4.8(i), $T_{\ell}^{\prime}(x, z)$ is squarefree, while $(x+a)^{\ell-1}$ is not, since $\ell \geq 3$. This contradiction refutes the assumption (4.20).

If $p=2$, then $\ell$ is odd by (4.13). After adjoining a square root $z_{0}$ of $z$ to $F$ (if necessary), Lemma 4.8(ii) implies that $T_{\ell}^{\prime}(x, z)=\left(\left(x-2 z_{0}\right) u^{2}+2 z_{0}^{n}\right)^{\prime}=u^{2}$ has $(\ell-1) / 2$ distinct roots in an algebraic closure of $F$, while $(x+a)^{\ell-1}$ has only one. This contradiction is sufficient for $\ell \geq 5$. For $\ell=3$, we have $T_{3}=x^{3}-3 y x$ and there are no $a, a^{*}, z \in F$ with $z \neq 0$ so that

$$
\begin{aligned}
x^{3}+a x^{2}+a^{2} x & =(x+a)^{3}-a^{3}=\left(x+a^{*}\right)^{3}-3 z\left(x+a^{*}\right)-\left(\left(a^{*}\right)^{3}-3 z a^{*}\right) \\
& =x^{3}+a^{*} x^{2}+\left(\left(a^{*}\right)^{2}+z\right) x
\end{aligned}
$$

Again, (4.20) is refuted.
For $\ell=2$, we claim that any composition

$$
f=v_{1} \circ T_{m}\left(x, z^{2}\right) \circ T_{2}(x, z) \circ v_{2}
$$

of the Second Case already occurs in the First Case. We have $T_{2}=x^{2}-2 y$. Since $m$ is odd by (4.4) and $p \nmid m$ by (4.13), Lemma $4.8(i i)$ guarantees a
monic $u \in F[x]$ of degree $d=(m-1) / 2$ with $T_{m}\left(x, z^{2}\right)=T_{m}\left(x,(-z)^{2}\right)=$ $(x+2 z) u^{2}-2 z^{m}$. Then for $\tilde{u}=u \circ(x-2 z)$ we have

$$
f=v_{1} \circ\left((x+2 z) u^{2}-2 z^{m}\right) \circ\left(x^{2}-2 z\right) \circ v_{2}=\left(v_{1}-2 z^{m}\right) \circ x^{2} \tilde{u}^{2}\left(x^{2}\right) \circ v_{2},
$$

which is of the form (4.10), with $k=m-2 d=1$.

Remark 4.22. Other parametrizations are possible. As an example, in the Second Case, for odd $q=p$, one can choose a nonsquare $z_{0} \in F=\mathbb{F}_{q}$ and $B=\{1, \ldots,(q-1) / 2\}$. Then all $f$ in (4.12) can also be written as

$$
f=b^{-n}\left(x-T_{n}(a, z)\right) \circ T_{n}(x, z) \circ(b x+a)
$$

with unique $(z, a, b) \in\left\{1, z_{0}\right\} \times F \times B=Z$. To wit, let $z, a \in F$ with $z \neq 0$. Take the unique $\left(z^{*}, a^{*}, b\right) \in Z$, so that $z^{*}=b^{2} z$ and $a^{*}=a b$. Then $z^{*}$ is determined by the quadratic character of $z$, and $b$ by the fact that every square in $F^{\times}$has a unique square root in $A$; the other one is $-b \in F^{\times} \backslash A$. Lemma 4.8(vi) says that

$$
\begin{gathered}
b^{n} T_{n}(x, z)=T_{n}\left(b x, z^{*}\right) \\
\left(x-T_{n}(a, z)\right) \circ T_{n}(x, z) \circ(x+a) \\
=b^{-n}\left(x-T_{n}\left(a^{*}, z^{*}\right)\right) \circ T_{n}\left(b x, z^{*}\right) \circ(x+a) \\
\\
=b^{-n}\left(x-T_{n}\left(a^{*}, z^{*}\right)\right) \circ T_{n}\left(x, z^{*}\right) \circ\left(b x+a^{*}\right),
\end{gathered}
$$

as claimed. If $F$ is algebraically closed, as in Zannier (1993), we can take $z=1$. The reduction from finite fields to this case is provided by Schinzel (2000), Section 1.4, Lemma 2.

Remark 4.23. Given just $f \in F[x]$, how can we determine whether Ritt's Second Theorem applies to it, and if so, compute $(w, a)$ or ( $z, a$ ), as appropriate? We may assume $f$ to be monic and original of degree $n$. The divisor $\ell$ of $n$ might be given as a further input, or we perform the following for all divisors $\ell$ of $n$ with $2 \leq \ell \leq \sqrt{n}$ and $\operatorname{gcd}(\ell, n / \ell)=1$. If $p \nmid n$, the task is easy. We compute decompositions

$$
f=g \circ h=g^{*} \circ h^{*}
$$

with $\operatorname{deg} h=\operatorname{deg} g^{*}=\ell$ and all components monic and original. If one of these decompositions does not exist, Ritt's Second Theorem does not apply;
otherwise the components are uniquely determined. If $h_{\ell-1}$ is the coefficient of $x^{\ell-1}$ in $h$, then $a=h_{\ell-1} / \ell$ in (4.10). Furthermore,

$$
\begin{aligned}
g\left(-a^{\ell}\right) & =-a^{k \ell} w^{\ell}\left(a^{\ell}\right), \\
g \circ\left(x-a^{\ell}\right)-g\left(-a^{\ell}\right) & =x^{k} w^{\ell},
\end{aligned}
$$

from which $w$ is easily determined via an $x$-adic Newton iteration for extracting an $\ell$ th root of the reversal of the left hand side, divided by $x^{k}$. Actually only a single Newton step is required to compute the root modulo $x^{2}$.

If the Second Case applies, then by Lemma 4.8(iii) the three highest coefficients in $f$ are

$$
\begin{aligned}
f & =x^{n}+f_{n-1} x^{n-1}+f_{n-2} x^{n-2}+O\left(x^{n-3}\right) \\
& =(x+a)^{n}-n z(x+a)^{n-2}+O\left(x^{n-4}\right) \\
& =x^{n}+n a x^{n-1}+\left(\frac{n(n-1)}{2} a^{2}-n z\right) x^{n-2}+O\left(x^{n-3}\right) ;
\end{aligned}
$$

this determines $a$ and $z$.

REmARK 4.24. If $p \nmid n$, then we can get rid of the right hand component $x+a$ by a further normalization. Namely, when $f=x^{n}+\sum_{0 \leq i<n} f_{i} x^{i}$, then $f \circ(x+a)=x^{n}+\left(n a+f_{n-1}\right) x^{n-1}+O\left(x^{n-2}\right)$. We call $f$ second-normalized if $f_{n-1}=0$. (This has been used at least since the times of Cardano and Tartaglia.) For any $f$, the composition $f \circ\left(x-f_{n-1} / n\right)$ is second-normalized, and if

$$
\begin{equation*}
\operatorname{deg} g=m \text { and } f=g \circ h=x^{n}+m h_{n / m-1} x^{n-1}+O\left(x^{n-2}\right) \tag{4.25}
\end{equation*}
$$

is second-normalized, then so is $h$ (but not necessarily $g$ ).

Corollary 4.26. In Theorem 4.9, if $p \nmid n$ and $f$ is second-normalized, then all claims hold with $a=0$.

Example 4.27. We note two instances of misreading Ritt's Second Theorem. Bodin et al. (2009) claim in the proof of their Lemma 5.8 that $t \leq q^{5}$ in the situation of Corollary $4.30(\mathrm{i})$. This contradicts the fact that the exponent $s+3$ of $q$ is unbounded. A second instance is in Corrales-Rodrigáñez (1990). The author claims that his following example contradicts the Theorem. He takes (in our language) positive integers $b, c, d$, $t$, sets $m=b p^{c}+d$, and
$\ell=p^{c}+1$, elements $h_{0}, \ldots, h_{t} \in F$, where $c<p$ and $t \ell \leq m$ and $F$ is a field of characteristic $p>0$, and

$$
\begin{aligned}
h & =\sum_{0 \leq i \leq t} h_{i} x^{m-i \ell} \\
g^{*} & =\sum_{o \leq i, j \leq t} h_{i} h_{j} x^{m-i p^{c}-j}
\end{aligned}
$$

Then

$$
x^{\ell} \circ h=g^{*} \circ x^{\ell}
$$

provided that all $h_{i}$ are in $\mathbb{F}_{p^{n}}$. If $d>b$, we have $m=b \ell+(d-b)$, so that $s=b$ and $k=d-b$.

Applying Theorem 4.9, we find $w=\sum_{0 \leq i \leq t} h_{i} x^{b-i}$ and $a=0$. Then

$$
\begin{aligned}
h & =x^{4} w\left(x^{\ell}\right) \\
g^{*} & =x^{k} w^{\ell}
\end{aligned}
$$

Thus the example falls well within Ritt's Second Theorem. Zannier (1993) points out that this was also remarked by A. Kondracki, a student of Andrzej Schinzel.

For the arguments below, it is convenient to assume $F$ to be perfect. Then each element of $F$ has a $p$ th root, where $p \geq 2$ is the characteristic. Any finite field is perfect.

For the next result, we have to make the first condition in (4.11) more explicit.

Lemma 4.28. Let $F$ be a perfect field, $\ell$ and $m$ positive integers with $\operatorname{gcd}(\ell, m)=$ $1, m=\ell s+k$ and $s=t p+r$ divisions with remainder, so that $1 \leq k<\ell$ and $0 \leq r<p$, and $w \in F[x]$ monic of degree $s$. Then

$$
\begin{equation*}
p \nmid \ell \text { and } k w+\ell x w^{\prime}=0 \Longleftrightarrow p \mid m \text { and } \exists u \in F[x] \quad w=x^{r} u^{p}, u \text { monic. } \tag{4.29}
\end{equation*}
$$

If the conditions in (4.29) are satisfied, then $u$ is uniquely determined.
Proof. For " $\Longrightarrow$ ", we denote by $w^{(i)}$ the $i$ th derivative of $w$. By induction on $i \geq 0$, we find that

$$
\begin{aligned}
(k+i \ell) w^{(i)}+\ell x w^{(i+1)} & =0 \\
(k+i \ell) w^{(i)}(0) & =0 .
\end{aligned}
$$

Now $p \nmid s-i$ for $0 \leq i<r, p \mid m=k+\ell s=\operatorname{lc}\left(k w+\ell x w^{\prime}\right)$, and $p \nmid \ell$. Thus

$$
p \nmid m-(s-i) \ell=k+\ell s-\ell s+i \ell=k+i \ell
$$

for $0 \leq i<r$, and hence $w^{(i)}(0)=0$ for these $i$. Since $r<p$, this implies that the lowest $r$ coefficients of $w$ vanish, so that $x^{r} \mid w$ and $v=x^{-r} w \in F[x]$. Then

$$
\begin{aligned}
\ell v^{\prime} & =\ell\left(-r x^{-r-1} w+x^{-r} w^{\prime}\right)=x^{-r-1}(-\ell r w-k w) \\
& =-x^{-r-1} w \cdot(\ell r+k)=-x^{-r-1} w \cdot(m-\ell(s-r))=0 .
\end{aligned}
$$

This implies that $v^{\prime}=0$ and $v=u^{p}$ for some $u \in F[x]$, since $F$ is perfect.
For " $\Longleftarrow ", p \nmid \ell$ follows from $\operatorname{gcd}(\ell, m)=1$, and we verify

$$
\begin{aligned}
k w+\ell x w^{\prime} & =k x^{r} u^{p}+\ell x \cdot r x^{r-1} u^{p}=x^{r} u^{p}(k+\ell r) \\
& =w \cdot(m-\ell(s-r))=0 .
\end{aligned}
$$

The uniqueness of $u$ is immediate, since $x^{r} u^{p}=x^{r} \tilde{u}^{p}$ implies $u=\tilde{u}$.
We can now estimate the number of distinct-degree collisions. If $p \nmid m$, the bound is exact. We use Kronecker's $\delta$ in the statement.

Corollary 4.30. Let $\mathbb{F}_{q}$ be a finite field of characteristic $p$, let $\ell$ and $m$ be integers with $m>\ell \geq 2$ and $\operatorname{gcd}(\ell, m)=1, n=\ell m, s=\lfloor m / \ell\rfloor$, and $t=\#\left(D_{n, \ell} \cap D_{n, m} \cap D_{n}^{+}\right)$. Then the following hold.
(i) If $p \nmid n$, then

$$
\begin{gathered}
t=\left(q^{s+3}+\left(1-\delta_{\ell, 2}\right)\left(q^{4}-q^{3}\right)\right)\left(1-q^{-1}\right) \\
q^{s+3}\left(1-q^{-1}\right) \leq t \leq\left(q^{s+3}+q^{4}\right)\left(1-q^{-1}\right)
\end{gathered}
$$

(ii) If $p \mid \ell$, then $t=0$.
(iii) If $p \mid m$, then

$$
t \leq\left(q^{s+3}-q^{\lfloor s / p\rfloor+3}\right)\left(1-q^{-1}\right) .
$$

Proof. (i) The monic original polynomials $f \in D_{n, \ell} \cap D_{n, m} \cap D_{n}^{+}=T$ fall either into the First or the Second Case of Ritt's Second Theorem. In the First Case, such $f$ are injectively parametrized by $(w, a)$ in Theorem 4.9(i). Condition (4.11) is satisfied, since $p \nmid m=k+\ell s=\operatorname{lc}\left(k w+\ell x w^{\prime}\right)$. Thus there are $q^{s+1}$ such pairs. Allowing composition by an arbitrary linear polynomial on the left, we get $q^{s+3}\left(1-q^{-1}\right)$ elements of $T$. In the Second Case, we have the
parameters $(z, a), q^{2}\left(1-q^{-1}\right)$ in number, from Theorem 4.9(ii). Composing with a linear polynomial yields a total of $q^{4}\left(1-q^{-1}\right)^{2}$. Furthermore, Theorem 4.9(iii) says that $t$ equals the sum of the two contributions if $\ell \geq 3$, and it equals the first summand for $\ell=2$; in the letter case, we have $p \neq 2$. Both claims in (i) follow.
(ii) (4.11) and (4.13) are never satisfied, so that $t=0$.
(iii) We have essentially the same situation as in (i), with $p \nmid \ell$ and $(w, a)$ parametrizing our $f$ in the First Case, albeit not injectively. Thus we only obtain an upper bound. The first condition in (4.11) holds if and only if $w$ is not of the form $x^{r} u^{p}$ as in (4.29). We note that $\operatorname{deg} u=(s-r) / p=\lfloor s / p\rfloor$ in (4.29), so that the number of $(w, a)$ satisfying (4.11) equals $q^{s+1}-q^{\lfloor s / p\rfloor+1}$. Since $p|m| n$, (4.13) does not hold, and there is no non-Frobenius decomposition in the Second Case.

Example 4.31. We note two instances of misreading Ritt's Second Theorem. Bodin et al. (2009) claim in the proof of their Lemma 5.8 that $t \leq q^{5}$ in the situation of Corollary 4.30(i). This contradicts the correct statement, where the exponent $s+3$ of $q$ is unbounded. A second instance is in Corrales-Rodrigáñez (1990). The author claims that his following example contradicts the Theorem. He takes (in our language) positive integers $b, c, d, t$ and elements $h_{0}, \ldots, h_{t} \in F$ and sets $m=b p^{c}+d$ and $\ell=p^{c}+1$, where $c<p, t \ell \leq m$, and $F$ is a field of characteristic $p>0$. Then for

$$
\begin{aligned}
h & =\sum_{0 \leq i \leq t} h_{i} x^{m-i \ell}, \\
g^{*} & =\sum_{o \leq i, j \leq t} h_{i} h_{j} x^{m-i p^{c}-j} .
\end{aligned}
$$

we have

$$
x^{\ell} \circ h=g^{*} \circ x^{\ell},
$$

provided that all $h_{i}$ are in $\mathbb{F}_{p^{n}}$. If $d>b$, we have $m=b \ell+(d-b)$, so that $s=b$ and $k=d-b$. Applying Theorem 4.9, we find $w=\sum_{0 \leq i \leq t} h_{i} x^{b-i}$ and $a=0$. Then

$$
\begin{aligned}
h & =x^{k} w\left(x^{\ell}\right) \\
g^{*} & =x^{k} w^{\ell}
\end{aligned}
$$

Thus the example falls well within Ritt's Second Theorem. Zannier (1993) points out that this was also remarked by A. Kondracki, a student of Andrzej Schinzel.

Lemma 4.32. Let $F$ be a perfect field, let $\ell$, $m \geq 2$ be integers for which $p$ divides $n=\ell m$, and let $g$ and $h$ in $F[x]$ have degrees $\ell$ and $m$, respectively. Then the following hold.
(i) $g \circ h \in D_{n}^{\varphi} \Longleftrightarrow g^{\prime} h^{\prime}=0 \Longleftrightarrow g \in D_{\ell}^{\varphi}$ or $h \in D_{m}^{\varphi}$,
(ii) $\# D_{n}^{\varphi}=q^{n / p+1}\left(1-q^{-1}\right)$,
(iii)

$$
\# D_{n, \ell}^{\varphi} \begin{cases}=\# D_{n / p, \ell} & \text { if } p \nmid \ell, \\ =\# D_{n / p, \ell / p} & \text { if } p \nmid m \\ \leq \# D_{n / p, \ell}+\# D_{n / p, \ell / p} & \text { always. }\end{cases}
$$

Proof. i is clear. For (ii), all Frobenius compositions are of the form $g^{*} \circ x^{p}$ with $g^{*} \in P_{n / p}^{=}$, and $g^{*}$ is uniquely determined by the composition. In (iii), if $p \nmid \ell$, then $p \mid m$, and according to (3.4), any $g \circ h \in D_{n, \ell}^{\varphi}$ can be uniquely rewritten as $g \circ h^{*} \circ x^{p}$, with $h^{*} \in P_{m / p}^{0}$. If $p \nmid m$, then the corresponding argument works. For the third line, we may assume that $p$ divides $\ell$ and $m$, and then have both possibilities above for Frobenius compositions.

A particular strength of Zannier's and Schinzel's result in Fact 4.3 is that, contrary to earlier versions, the characteristic of $F$ appears only very mildly, namely in (4.5). We now elucidate the case excluded by (4.5), namely $g^{\prime}\left(g^{*}\right)^{\prime}=$ 0 , which is mentioned in Zannier (1993), page 178 . This case can only occur when $p \geq 2$. We recall the Frobenius power $\varphi_{j}: F[x] \rightarrow F[x]$ from Definition 3.3.

Lemma 4.33. In the above notation, assume that $\left(\ell, m, g, h, g^{*}, h^{*}\right)$ and $f$ satisfy (4.4), (4.6), and (4.7), and that $F$ is perfect.
(i) The following are equivalent:
(a) $f$ is a Frobenius composition,
(b) $f^{\prime}=0$,
(c) $g^{\prime}\left(g^{*}\right)^{\prime}=0$.
(ii) If $g^{\prime}=0$, then $p \nmid \ell$ and $\left(g^{*}\right)^{\prime} \neq 0$, and there exist positive integers $j$ and $M$, and monic original $G, G^{*}, H^{*} \in F[x]$ so that

$$
\begin{align*}
& m=p^{j} M, \operatorname{deg} G=\operatorname{deg} H^{*}=M, \operatorname{deg} G^{*}=\ell, \\
& g=x^{p^{j}} \circ G, g^{*} \circ x^{p^{j}}=x^{p^{j}} \circ G^{*}, h^{*}=x^{p^{j}} \circ H^{*}, \\
& G^{\prime}\left(G^{*}\right)^{\prime} \neq 0, G \circ h=G^{*} \circ H^{*},  \tag{4.34}\\
& f=x^{p^{j}} \circ G \circ h=x^{p^{j}} \circ\left(G^{*} \circ H^{*}\right) .
\end{align*}
$$

In particular, $\left(\ell, M, G, h, G^{*}, H^{*}\right)$ satisfies (4.4) through (4.6) if $M>\ell$, and $\left(M, \ell, G^{*}, H^{*}, G, h\right)$ does if $2 \leq M<\ell$. If $M=1$, then $G$ and $H^{*}$ are linear.
(iii) If $\left(g^{*}\right)^{\prime}=0$, then $p \nmid m$ and $g^{\prime} \neq 0$, and there exist positive integers $d$ and $L$, and monic original $G, H, G^{*} \in F[x]$ with

$$
\begin{align*}
\ell & =p^{d} L, p \nmid L, g=\varphi_{d}(G), h=x^{p^{d}} \circ H, g^{*}=x^{p^{d}} \circ G^{*}, \\
G^{\prime}\left(G^{*}\right)^{\prime} & \neq 0,  \tag{4.35}\\
G \circ H & =G^{*} \circ h^{*}, f=x^{p^{d}} \circ G \circ H .
\end{align*}
$$

with $\varphi_{d}$ from Definition 3.3. In particular, $\left(L, m, G, H, G^{*}, h^{*}\right)$ satisfies (4.4) through (4.6) if $L \geq 2$.
(iv) The data derived in (ii) and (iii) are uniquely determined. Conversely, given such data, the stated formulas yield $\left(\ell, m, g, h, g^{*}, h^{*}\right)$ and $f$ that satisfy (4.4), (4.6), and (4.7).

Proof. (i) If $f=x^{p} \circ G$ is a Frobenius composition, then $f^{\prime}=0$. We have

$$
\begin{equation*}
f^{\prime}=\left(g^{\prime} \circ h\right) \cdot h^{\prime}=\left(\left(g^{*}\right)^{\prime} \circ h^{*}\right) \cdot\left(h^{*}\right)^{\prime} . \tag{4.36}
\end{equation*}
$$

If (b) holds, then $p \mid \operatorname{deg} f=n=\ell m$, hence $p \mid \ell$ or $p \mid m$. In the case $p \mid \ell$, (4.4) implies that $p \nmid m$ and $g^{\prime}\left(h^{*}\right)^{\prime} \neq 0$, hence $h^{\prime}=\left(g^{*}\right)^{\prime}=0$ by (4.36). Symmetrically, $p \mid m$ implies that $g^{\prime}=\left(h^{*}\right)^{\prime}=0$, so that (c) follows in both cases.

If (c) holds, say $g^{\prime}=0$, then the coefficient of $x^{i}$ in $g$ is zero unless $p \mid i$. Since $F$ is perfect, every element has a $p$ th root, and it follows that $g=x^{p} \circ G$ for some $G \in F[x]$. Thus $g$ is a Frobenius composition, and so is $f=g \circ h$.
(ii) Let $j \geq 1$ be the largest integer for which there exists some $G \in F[x]$ with $g=x^{p^{j}} \circ G$. Then $j$ and $G$ are uniquely determined, $G$ is monic and
original, $G^{\prime} \neq 0, p^{j} \mid m, \operatorname{deg} G=m p^{-j}=M$, and $p \nmid \ell$ by (4.4). Furthermore, we have

$$
\begin{equation*}
g^{*} \circ h^{*}=g \circ h=x^{p^{j}} \circ G \circ h . \tag{4.37}
\end{equation*}
$$

Writing $h^{*}=\sum_{1 \leq i \leq m} h_{i}^{*} x^{i}$ with $h_{m}^{*}=1$, we let $I=\left\{i \leq m: h_{i}^{*} \neq 0\right\}$ be the support of $h^{*}$. Assume that there is some $i \in I$ with $p^{j} \nmid i$, and let $k$ be the largest such $i$. Then $k<m, m(\ell-1)+k$ is not divisible by $p^{j}$, the coefficient of $x^{m(\ell-1)+k}$ in $\left(h^{*}\right)^{\ell}$ is $\ell h_{k}^{*}$, and in $g^{*} \circ h^{*}$ it is $\operatorname{lc}\left(g^{*}\right) \cdot \ell h_{k}^{*} \neq 0$; see $E_{1}$ in Lemma 3.9. This contradicts (4.37), so that the assumption is false and $h^{*}=\left(H^{*}\right)^{p^{j}}$ for a unique monic original $H^{*} \in F[x]$, of degree $M=m p^{-j}$.

Setting $G^{*}=\varphi_{j}^{-1}\left(g^{*}\right)$, we have $\operatorname{deg} G^{*}=\operatorname{deg} g^{*}=\ell$ and hence $\left(G^{*}\right)^{\prime} \neq 0$, $x^{p^{j}} \circ G^{*}=\varphi_{j}\left(G^{*}\right) \circ x^{p^{j}}$, and

$$
\begin{aligned}
& x^{p^{j}} \circ G \circ h=g \circ h=f=g^{*} \circ h^{*}=\varphi_{j}\left(G^{*}\right) \circ x^{p^{j}} \circ H^{*}=x^{p^{j}} \circ G^{*} \circ H^{*} \\
& G \circ h=G^{*} \circ H^{*} .
\end{aligned}
$$

(iii) Since $p \mid \ell=\operatorname{deg} g^{*}$, (4.4) implies that $p \nmid m, g^{\prime} \neq 0$, and $g^{\prime} \circ h \neq 0$. In (4.36), we have $f^{\prime}=0$ and hence $h^{\prime}=0$. There exist monic original $G_{1}$, $H_{1} \in F[x]$ with $g^{*}=x^{p} \circ G_{1}, h=x^{p} \circ H_{1}$, and

$$
\begin{aligned}
x^{p} \circ G_{1} \circ h^{*} & =f=g \circ x^{p} \circ H_{1}=x^{p} \circ \varphi_{1}^{-1}(g) \circ H_{1}, \\
G_{1} \circ h^{*} & =\varphi_{1}^{-1}(g) \circ H_{1} .
\end{aligned}
$$

If $G_{1}^{\prime}=0$, then $H_{1}^{\prime}=0$ and we can continue this transformation. Eventually we find an integer $j \geq 1$ and monic original $G_{j}, H_{j} \in F[x]$ with $p^{j} \mid \ell, g^{*}=x^{p^{j}} \circ G_{j}$, $h=x^{p^{j}} \circ H_{j}$, and $G_{j}^{\prime} \neq 0$. We set $G=\varphi_{j}^{-1}(g), G^{*}=G_{j}$, and $H=H_{j}$. Then $G^{\prime}\left(G^{*}\right)^{\prime} \neq 0, \operatorname{deg} G^{*}=\operatorname{deg} H=L, \operatorname{deg} G=m$. As above, we have

$$
\begin{gathered}
G^{*} \circ h^{*}=G_{j} \circ h^{*}=\varphi_{j}^{-1}(g) \circ H_{j}=G \circ H, \\
f=\left(x^{p^{d}} \circ G^{*}\right) \circ h^{*}=g \circ\left(x^{p^{d}} \circ H\right)=x^{p^{d}} \circ G \circ H .
\end{gathered}
$$

According to (iii), $d$ is the multiplicity of $p$ in $\ell$. We now show that $j=d$. We set $\ell^{*}=\ell p^{-j}$. If $\ell^{*} \geq 2$, then the above collision satisfies the assumptions (4.4) through (4.6), with $\ell^{*}<m$ instead of $\ell$. Thus Theorem 4.9 applies.

In the First Case, (4.11) shows that $p \nmid \ell^{*}$. It follows that $j=d$ and $\ell^{*}=L$. In the Second Case, we have $p \nmid \ell^{*} m=\ell p^{-j} m$ by (4.10), so that again $j=d$ and $\ell^{*}=L$. In the remaining case $\ell^{*}=1$, we have $L=1$ and $G^{*}=H=x$.
(iv) The uniqueness of all quantities is clear.

We need some simple properties of the Frobenius map $\varphi_{j}$ from (3.4).

Lemma 4.38. Let $F$ be a field of characteristic $p \geq 2, f, g \in F[x], a \in F$, let $i, j \geq 1$, and denote by $f^{\prime}$ the derivative of $f$. Then
(i) $\varphi_{j}(f g)=\varphi_{j}(f) \varphi_{j}(g)$,
(ii) $\varphi_{j}\left(f^{i}\right)=\varphi_{j}(f)^{i}$,
(iii) $\varphi_{j}(f \circ g)=\varphi_{j}(f) \circ \varphi_{j}(g)$,
(iv) $\varphi_{j}(f(a))=\varphi_{j}(f)\left(a^{p^{j}}\right)$,
(v) $\varphi_{j}\left(f^{\prime}\right)=\varphi_{j}(f)^{\prime}$.

Proof. (i) is immediate, and (ii) follows. For (iii), we write $f=\sum f_{i} x^{i}$ with all $f_{i} \in F$. Then

$$
\varphi_{j}(f \circ g)=\varphi_{j}\left(\sum f_{i} g^{i}\right)=\sum f_{i}^{p^{j}} \varphi_{j}\left(g^{i}\right)=\varphi_{j}(f) \circ \varphi_{j}(g)
$$

(iv) is a special case of (iii). For (v), we have

$$
\varphi_{j}\left(f^{\prime}\right)=\varphi_{j}\left(\sum i f_{i} x^{i-1}\right)=\sum i^{p^{j}} f_{i}^{p^{j}} x^{i-1}=\sum i f_{j}^{p^{j}} x^{i-1}=\varphi_{j}(f)^{\prime}
$$

Our next goal is to get rid of the assumption (4.5), namely that $g^{\prime}\left(g^{*}\right)^{\prime} \neq 0$, in Theorem 4.9. This is achieved by the following result. Its statement is lengthy, and the simple version is: if (4.5) is violated, remove the component $x^{p}$ from the culprit as long as you can. Then Theorem 4.9 applies.

Theorem 4.39. Let $F$ be a perfect field of characteristic $p \geq 0$. Let $m>\ell \geq 2$ be integers with $\operatorname{gcd}(\ell, m)=1$, set $n=\ell m$ and let $f, g, h, g^{*}, h^{*} \in F[x]$ be monic original of degrees $n, m, \ell, \ell, m$, respectively, with $f=g \circ h=g^{*} \circ h^{*}$. Then the following hold.
(i) If $g^{\prime}=0$, then there exists a uniquely determined positive integer $j$ so that $p^{j}$ divides $m$ and either (a) or (b) hold; furthermore, (c) is true. We set $M=p^{-j} m$.
(a) (First Case)
(1) If $M>\ell$, then there exist a monic $W \in F[x]$ of degree $S=$ $\lfloor M / \ell\rfloor$ and $a \in F$ so that

$$
K W+\ell x W^{\prime} \neq 0
$$

for $K=M-\ell\lfloor M / \ell\rfloor$, and all conclusions of Theorem 4.9(i), except (4.11) and $k<\ell$, hold for $k=p^{j} K, s=p^{j} S$, and $w=W^{p^{j}}$. Conversely, any $W$ and $a$ as above yield via these formulas a collision satisfying (4.4), (4.6) and (4.7), with $g^{\prime}=0$. If $p \nmid M$, then $W$ and $a$ are uniquely determined by $f$ and $\ell$.
(2) If $M<\ell$, then there exist a monic $W \in F[x]$ of degree $S=$ $\lfloor\ell / M\rfloor$ and $a \in F$ so that

$$
\begin{gathered}
f=\left(x-a^{k M} w^{M}\left(a^{M}\right)\right) \circ x^{k M} w^{M}\left(x^{M}\right) \circ(x+a), \\
K W+\ell x W^{\prime} \neq 0
\end{gathered}
$$

for $K=\ell-M\lfloor\ell / M\rfloor$, and all conclusions of Theorem 4.9(i), with $\ell$ replaced by $M$ and excepting (4.11) and the division with remainder, hold for $k=p^{j} K, s=p^{j} S$, and $w=W^{p^{j}}$. Conversely, any $W$ and $a$ as above yield via these formulas a collision satisfying (4.4), (4.6) and (4.7), with $g^{\prime}=0$. Furthermore, $W$ and $a$ are uniquely determined by $f$ and $\ell$.
(3) If $m=p^{j}$, then $g=h^{*}=x^{p^{j}}$ and $g^{*}=\varphi_{j}(h)$.
(b) (Second Case) $p \nmid M$, and all conclusions of Theorem 4.9(ii) hold, except (4.13).
(c) Assume that $M \geq 2$, and let $f$ be a collision of the Second Case. Then $f$ belongs to the First Case if and only if $\min (\ell, M)=2$.
(ii) If $\left(g^{*}\right)^{\prime}=0$, then there exists a unique positive integer $d$ such that $p^{d} \mid \ell$, $p \nmid p^{-d} \ell=L$, and either (a) or (b) holds; furthermore, (c) is true.
(a) (First Case) There exist a monic $w \in F[x]$ of degree $\lfloor m / L\rfloor$ and $a \in F$ so that

$$
\begin{aligned}
f & =\left(x-a^{k \ell} w^{L}\left(a^{\ell}\right)\right) \circ x^{k \ell} w^{L}\left(x^{\ell}\right) \circ(x+a), \\
g & =\left(x-a^{k \ell} w^{L}\left(a^{\ell}\right)\right) \circ x^{k} w^{L} \circ\left(x+a^{\ell}\right), \\
h & =\left(x-a^{\ell}\right) \circ x^{\ell} \circ(x+a), \\
g^{*} & =\left(x-a^{k \ell} w^{L}\left(a^{\ell}\right)\right) \circ x^{\ell} \circ\left(x+a^{k} \varphi_{d}^{-1}(w)\left(a^{L}\right)\right), \\
h^{*} & =\left(x-a^{k} \varphi_{d}^{-1}(w)\left(a^{L}\right)\right) \circ x^{k} \varphi_{d}^{-1}(w)\left(x^{L}\right) \circ(x+a),
\end{aligned}
$$

where $m=L\lfloor m / L\rfloor+k$. The quantities $w$ and $a$ are uniquely determined by $f$ and $\ell$. Conversely, any $w$ and $a$ as above yield via these formulas a collision satisfying (4.4), (4.6), and (4.7). Furthermore, $k w+\ell x w^{\prime} \neq 0$.
(b) (Second Case) There exist $z, a \in F$ with $z \neq 0$ for which all conclusions of Theorem 4.9(ii) hold, except (4.13). Conversely, any ( $z, a$ ) as above yields a collision satisfying (4.4), (4.6) and (4.7).
(c) When $L \geq 3$, then (a) and (b) are mutually exclusive. For $L \leq 2$, (b) is included in (a).

Proof. (i) We take the quantities $j, M, G, G^{*}, H^{*}$ from Lemma 4.33(ii) and apply Theorem 4.9 to the collision $G \circ h=G^{*} \circ H^{*}$ in (4.34). We start with the First Case (Theorem 4.9(i)). If $M>\ell$, it yields a monic $W \in F[x]$ of degree $\lfloor M / \ell\rfloor$ and $a \in F$ with

$$
\begin{align*}
& G \circ h=G^{*} \circ h^{*}=\left(x-a^{*}\right) \circ x^{K \ell} W^{\ell}\left(x^{\ell}\right) \circ(x+a), \\
& K W+\ell x W^{\prime} \neq 0, \tag{4.40}
\end{align*}
$$

where $K=M-\ell\lfloor M / \ell\rfloor$ and $a^{*}=a^{K \ell} W^{\ell}\left(a^{\ell}\right)$. We set $k=p^{j} K$ and $w=W^{p^{j}}$. Then

$$
\begin{aligned}
f & =g \circ h=G^{p^{j}} \circ h=x^{p^{j}} \circ G \circ h \\
& =x^{p^{j}} \circ\left(x-a^{*}\right) \circ x^{K \ell} W^{\ell}\left(x^{\ell}\right) \circ(x+a) \\
& =\left(x-\left(a^{*}\right)^{p^{j}}\right) \circ x^{p^{j} K \ell}\left(W^{p^{j}}\right)^{\ell}\left(x^{\ell}\right) \circ(x+a) \\
& =\left(x-a^{k \ell} w^{\ell}\left(a^{\ell}\right)\right) \circ x^{k \ell} w^{\ell}\left(a^{\ell}\right) \circ(x+a) .
\end{aligned}
$$

Furthermore, we have

$$
\ell s+k=\ell p^{j}\lfloor M / \ell\rfloor+p^{j}(M-\ell\lfloor M / \ell\rfloor)=m .
$$

If $2 \leq M<\ell$, we have to reverse the roles of $M$ and $\ell$ in the application of Theorem 4.9(i). Thus we now find a monic $W \in F[x]$ of degree $\lfloor\ell / M\rfloor$ and $a \in F$ with

$$
G \circ h=\left(x-a^{*}\right) \circ x^{K M} W^{M}\left(x^{M}\right) \circ(x+a),
$$

with $K=\ell-M\lfloor\ell / M\rfloor, a^{*}=a^{K M} W^{M}\left(a^{M}\right)$, and $K W+M x W^{\prime} \neq 0$. We set $k=p^{j} K$ and $w=W^{p^{j}}$. Then

$$
\begin{aligned}
f & =x^{p^{j}} \circ G \circ h=\varphi_{j}\left(x-a^{*}\right) \circ x^{p^{j}} \circ x^{K M} W^{M}\left(x^{M}\right) \circ(x+a) \\
& =\left(x-a^{k M} w^{M}\left(a^{M}\right)\right) \circ x^{k M} w^{M}\left(x^{M}\right) \circ(x+a) .
\end{aligned}
$$

Furthermore we have

$$
M s+k=M p^{j}\lfloor\ell / M\rfloor+p^{j}(\ell-M\lfloor\ell / M\rfloor)=p^{j} \ell .
$$

Since $p \nmid \ell, W$ and $a$ are uniquely determined.
If $M=1$, then $g=x^{p^{j}}, f=x^{p^{j}} \circ h=\varphi_{j}(h) \circ x^{p^{j}}$, and $g^{*}=\varphi_{j}(h)$ by Fact 3.1(i).

In the Second Case of Theorem 4.9, we use $T_{p^{j}}=x^{p^{j}}$ from Lemma 4.8(iv). Now Theorem 4.9(ii) provides $z, a \in F$ with $z \neq 0$ and

$$
\begin{aligned}
G \circ h & =G^{*} \circ H^{*}=\left(x-T_{\ell M}(a, z)\right) \circ T_{\ell M}(x, z) \circ(x+a), \\
G & =\left(x-T_{\ell M}(a, z)\right) \circ T_{M}\left(x, z^{\ell}\right) \circ\left(x+T_{\ell}(a, z)\right) .
\end{aligned}
$$

Since $G^{\prime} \neq 0$, we have $p \nmid M$, and hence $p \nmid \ell M$. Thus $z$ and $a$ are uniquely determined. Furthermore

$$
\begin{aligned}
f & =g \circ h=x^{p^{j}} \circ G \circ h \\
& =\left(x^{p^{j}}-\left(T_{\ell M}(a, z)\right)^{p^{j}}\right) \circ T_{\ell M}(x, z) \circ(x+a) \\
& =\left(x-T_{n}(a, z)\right) \circ x^{p^{j}} \circ T_{\ell M}(x, z) \circ(x+a) \\
& =\left(x-T_{n}(a, z)\right) \circ T_{n}(x, z) \circ(x+a) .
\end{aligned}
$$

In (i.c), we have $p \nmid \ell M=p^{-j} n$. By Theorem 4.9(iii), $G \circ h$ belongs to the First Case if and only if $\min \{\ell, M\}=2$.
(ii) We take $d, L, G, H, G^{*}$ from Lemma 4.33(iii), and apply Theorem 4.9 to the collision $G \circ H=G^{*} \circ h^{*}$. In the First Case, this yields a monic $W \in F[x]$ of degree $\lfloor m / L\rfloor$ and $a \in F$ so that the conclusions of Theorem 4.9(i) hold for these values, with $k=m-L \cdot\lfloor m / L\rfloor$. We set $w=\varphi_{d}(W)$. Then

$$
\begin{aligned}
\operatorname{deg} G & =\operatorname{deg}\left(x^{k} W^{L}\right)=(m-L \cdot\lfloor m / L\rfloor)+L \cdot\lfloor m / L\rfloor=m, \\
g & =\varphi_{d}(G)=\varphi_{d}\left(\left(x-a^{k L} W^{L}\left(a^{L}\right)\right) \circ x^{k} W^{L} \circ\left(x+a^{L}\right)\right) \\
& =\varphi_{d}\left(x-a^{k L} W^{L}\left(a^{L}\right)\right) \circ \varphi_{d}\left(x^{k} W^{L}\right) \circ \varphi_{d}\left(x+a^{L}\right) \\
& =\left(x-a^{k \ell} w^{L}\left(a^{\ell}\right)\right) \circ x^{k} w^{L} \circ\left(x+a^{\ell}\right) . \\
h & =x^{p^{d}} \circ H=x^{p^{d}} \circ\left(x-a^{L}\right) \circ x^{L} \circ(x+a) \\
& =\left(x-a^{\ell}\right) \circ x^{\ell} \circ(x+a), \\
g^{*} & =x^{p^{d}} \circ G^{*}=x^{p^{d}} \circ\left(x-a^{k L} W^{L}\left(a^{L}\right)\right) \circ x^{L} \circ\left(x+a^{k} W\left(a^{L}\right)\right) \\
& =\left(x-a^{k \ell} W^{p^{d} L}\left(a^{L}\right)\right) \circ x^{\ell} \circ\left(x+a^{k} W\left(a^{L}\right)\right) \\
& =\left(x-a^{k \ell} w^{L}\left(a^{\ell}\right)\right) \circ x^{\ell} \circ\left(x+a^{k} \varphi_{d}^{-1}(w)\left(a^{L}\right)\right), \\
h^{*} & =\left(x-a^{k} W\left(a^{L}\right)\right) \circ x^{k} W\left(x^{L}\right) \circ(x+a) \\
& =\left(x-a^{k} \varphi_{d}^{-1}(w)\left(a^{L}\right)\right) \circ x^{k} \varphi_{d}^{-1}(w)\left(x^{L}\right) \circ(x+a), \\
f & =\left(x-a^{k \ell} w^{L}\left(a^{\ell}\right)\right) \circ x^{k \ell} w^{L}\left(x^{\ell}\right) \circ(x+a) .
\end{aligned}
$$

Furthermore, Lemma 4.38 implies that

$$
k w+\ell x w^{\prime}=k \varphi_{d}(W)+\ell x \varphi_{d}(W)^{\prime}=\varphi_{d}\left(k W+\ell x W^{\prime}\right) \neq 0
$$

In the Second Case, Theorem 4.9(ii) provides $z, a \in F$ with $z \neq 0$ and

$$
\begin{aligned}
g & =\varphi_{d}(G)=\varphi_{d}\left(\left(x-T_{m L}(a, z)\right) \circ T_{m}\left(x, z^{L}\right) \circ\left(x+T_{L}(a, z)\right)\right) \\
& =\left(x-\varphi_{d}\left(T_{m L}(a, z)\right)\right) \circ \varphi_{d}\left(T_{m}\left(x, z^{L}\right)\right) \circ\left(x+\varphi_{d}\left(T_{L}(a, z)\right)\right) \\
& =\left(x-T_{m L}(a, z)^{p^{d}}\right) \circ T_{m}\left(x,\left(z^{L}\right)^{p^{d}}\right) \circ\left(x+T_{L}(a, z)^{p^{d}}\right) \\
& =\left(x-T_{n}(a, z)\right) \circ T_{m}\left(x, z^{\ell}\right) \circ\left(x+T_{\ell}(a, z)\right), \\
h & =x^{p^{d}} \circ H=x^{p^{d}} \circ\left(x-T_{L}(a, z)\right) \circ T_{L}(x, z) \circ(x+a) \\
& =\left(x-T_{\ell}(a, z)\right) \circ T_{\ell}(x, z) \circ(x+a), \\
g^{*} & =x^{p^{d}} \circ G^{*}=x^{p^{d}} \circ\left(x-T_{L m}(a, z)\right) \circ T_{L}\left(x, z^{m}\right) \circ\left(x+T_{m}(a, z)\right) \\
& =\left(x-T_{n}(a, z)\right) \circ x^{p^{d}} \circ T_{L}\left(x, z^{m}\right) \circ\left(x+T_{m}(a, z)\right) \\
& =\left(x-T_{n}(a, z)\right) \circ T_{\ell}\left(x, z^{m}\right) \circ\left(x+T_{m}(a, z)\right), \\
h^{*} & =\left(x-T_{m}(a, z)\right) \circ T_{m}(x, z) \circ(x+a), \\
f & =\left(x-T_{n}(a, z)\right) \circ T_{n}(x, z) \circ(x+a) .
\end{aligned}
$$

(ii.c) follows from Theorem 4.9(iii) for $L \geq 2$. If $L=1$, then $\ell=p^{d}$ and $k=0$ in (ii.a). For any

$$
f=\left(x-T_{n}(a, z)\right) \circ T_{n}(x, z) \circ(x+a)
$$

in (ii.b), we take $w=T_{m}\left(x, z^{p^{d}}\right)$. Then

$$
\begin{aligned}
T_{n}(x, z) & =T_{m}\left(x, z^{p^{d}}\right) \circ T_{p^{d}}(x, z)=w \circ x^{p^{d}}, \\
f & =\left(x-w\left(a^{\ell}\right)\right) \circ w\left(x^{\ell}\right) \circ(x+a),
\end{aligned}
$$

which is an instance of (ii.a).
If $p \nmid n$, then the case where $\operatorname{gcd}(\ell, m) \neq 1$ is reduced to the previous one by the following result of Tortrat (1988). We will only use the special case where $\ell=\ell^{*}$ and $m=m^{*}$.

Fact 4.41. Suppose we have a field $F$ of characteristic $p \geq 0$, integers $\ell, \ell^{*}, m, m^{*}$ $\geq 2$ with $p \nmid \ell m$, monic original polynomials $g, h, g^{*}, h^{*} \in F[x]$ of degrees $m, \ell, \ell^{*}, m^{*}$, respectively, with $g \circ h=g^{*} \circ h^{*}$. Furthermore, let $i=\operatorname{gcd}\left(m, \ell^{*}\right)$ and $j=\operatorname{gcd}\left(\ell, m^{*}\right)$. Then the following hold.
(i) There exist monic original polynomials $u, v, \tilde{g}, \tilde{h}, \tilde{g}^{*}, \tilde{h}^{*} \in F[x]$ of degrees $i, j, m / i, \ell / j, \ell^{*} / i, m^{*} / j$, respectively, so that

$$
\begin{align*}
g & =u \circ \tilde{g}, \\
h & =\tilde{h} \circ v,  \tag{4.42}\\
g^{*} & =u \circ \tilde{g}^{*}, \\
h^{*} & =\tilde{h}^{*} \circ v .
\end{align*}
$$

(ii) Assume that $\ell=\ell^{*}<m=m^{*}$. Then $i=j$ and $m / i, \ell / i, \tilde{f}=\tilde{g} \circ$ $\tilde{h}, \tilde{g}, \tilde{h}, \tilde{g}^{*}, \tilde{h}^{*}$ satisfy the assumptions of Theorem 4.9.

Proof. (i) Tortrat (1988) proves the claim if $F$ is algebraically closed, but without the condition of being monic original. Thus we have four decompositions (4.42) over an algebraic closure of $F$. We may choose all six components in (4.42) to be monic original. They are then uniquely determined. Since $p \nmid n$, decomposition is rational; see Schinzel (2000), I.3, Theorem 6, and Kozen \& Landau (1989) or von zur Gathen (1990a) for an algorithmic proof. It follows that the six components are in $F[x]$.
(ii) We have $\operatorname{gcd}(\ell / i, m / i)=1$, and

$$
f=(u \circ \tilde{g}) \circ(\tilde{h} \circ v)=\left(u \circ \tilde{g}^{*}\right) \circ\left(\tilde{h}^{*} \circ v\right) .
$$

The uniqueness of tame decompositions (Fact 3.1) implies that $\tilde{g} \circ \tilde{h}=\tilde{g}^{*} \circ \tilde{h}^{*}$. The other requirements are immediate.

Tortrat's result, together with the preceding material, determines $D_{n, \ell} \cap$ $D_{n, m}$ completely, if $p \nmid n=\ell m$.

Corollary 4.43. Let $\mathbb{F}_{q}$ be a finite field of characteristic $p$, and let $m>$ $\ell \geq 2$ be integers with $p \nmid n=\ell m, i=\operatorname{gcd}(\ell, m)$ and $s=\lfloor m / \ell\rfloor$. Let $t=\#\left(D_{n, \ell} \cap D_{n, m}\right)$. Then the following hold.
(i)

$$
t= \begin{cases}q^{2 \ell+s-1}\left(1-q^{-1}\right) & \text { if } \ell \mid m \\ q^{2 i}\left(q^{s+1}+\left(1-\delta_{\ell, 2}\right)\left(q^{2}-q\right)\right)\left(1-q^{-1}\right) & \text { otherwise }\end{cases}
$$

(ii)

$$
t \leq 2 q^{2 \ell+s-1}\left(1-q^{-1}\right)
$$

Proof. (i) Let $T=D_{n, \ell} \cap D_{n, m} \cap D_{n}^{0}$ consist of the monic original polynomials in the intersection, and similarly $U=D_{n / i^{2}, \ell / i} \cap D_{n / i^{2}, m / i} \cap D_{n / i^{2}}^{0}$. Then Fact 4.41(ii) implies that $T=P_{i}^{0} \circ U \circ P_{i}^{0}$, using $G \circ H=\{g \circ h: g \in G, h \in H\}$ for sets $G, H \subseteq F[x]$. Furthermore, the composition maps involved are injective. Thus

$$
\begin{aligned}
\# T & =\left(\# P_{i}^{0}\right)^{2} \cdot \# U=q^{2 i-2} \cdot \# U, \\
\#\left(D_{n, \ell} \cap D_{n, m}\right) & =q^{2}\left(1-q^{-1}\right) \cdot q^{2 i-2} \cdot \# U .
\end{aligned}
$$

If $\ell \nmid m$, then $\ell / i \geq 2$ and from Corollary 4.30 (i) we have

$$
\# U=\frac{q^{-2}}{1-q^{-1}} \cdot\left(q^{s+3}+\left(1-\delta_{\ell, 2}\right)\left(q^{4}-q^{3}\right)\right)\left(1-q^{-1}\right)
$$

which implies the claim in this case. If $\ell \mid m$, then $\ell / i=1$ and Corollary 4.30 is inapplicable. Now

$$
\begin{aligned}
U & =D_{m / \ell, 1} \cap D_{m / \ell, m / \ell} \cap P_{m / \ell}^{0}=P_{m / \ell}^{0} \\
\# U & =\# P_{m / \ell}^{0}=q^{m / \ell-1}=q^{s-1}
\end{aligned}
$$

which again shows the claim.
(ii) We have $q^{2} \leq q^{s+1}$, and if $\ell \nmid m$, then $2 i \leq \ell \leq 2 \ell-2$.

This result shows that there are more polynomials in the intersection when $\ell^{2} \mid n$ than otherwise.

We now have determined the size of the intersection if either $p \nmid n$ or $\operatorname{gcd}(\ell, m)=1$. It remains a challenge to do this with the same precision when both conditions are violated. The following approach yields a rougher estimate.

THEOREM 4.44. Let $F$ be a field of characteristic $p \geq 2$, let $\ell, m, n \geq 2$ be integers with $p \mid n=\ell m$, and set $T=D_{n, \ell} \cap D_{n, m} \cap D_{n}^{+}$. Then the following hold.
(i) If $p \nmid \ell$, then for any monic original $f \in T$ there exist monic original $g^{*}$ and $h^{*}$ in $F[x]$ of degrees $\ell$ and $m$, respectively, with $f=g^{*} \circ h^{*}$, $\left(g^{*}\right)^{\prime}\left(h^{*}\right)^{\prime} \neq 0$, and $0 \leq \operatorname{deg}\left(h^{*}\right)^{\prime}<m-\ell$.
(ii) If $p \mid \ell$, then for any monic original $f \in T$ there exist monic original $g$ and $h \in F[x]$ of degrees $m$ and $\ell$, respectively, with $f=g \circ h$ and $\operatorname{deg} g^{\prime} \leq m-(m+1) / \ell$.

Proof. We take a collision (4.7) and its derivative (4.36). Since $f \in D_{n}^{+}$, we have $f^{\prime} \neq 0$.
(i) Since $p \mid m$, we have $\operatorname{deg} g^{\prime} \leq m-2,\left(h^{*}\right)^{\prime} \neq 0$, and $\operatorname{deg}\left(h^{*}\right)^{\prime} \geq 0$, so that

$$
\begin{aligned}
n-m+\operatorname{deg}\left(h^{*}\right)^{\prime} & =(\ell-1) \cdot m+\operatorname{deg}\left(h^{*}\right)^{\prime}=\operatorname{deg} f^{\prime} \\
& \leq(m-2) \cdot \ell+\ell-1=n-\ell-1 \\
0 & \leq \operatorname{deg}\left(h^{*}\right)^{\prime}<m-\ell
\end{aligned}
$$

(ii) We have $g^{\prime} h^{\prime} \neq 0, \operatorname{deg}\left(g^{*}\right)^{\prime} \leq \ell-2, \operatorname{deg} h^{\prime} \geq 0$, and

$$
\begin{aligned}
\ell \cdot \operatorname{deg} g^{\prime} & \leq \ell \cdot \operatorname{deg} g^{\prime}+\operatorname{deg} h^{\prime}=\operatorname{deg} f^{\prime} \\
& \leq(\ell-2) \cdot m+m-1=\ell m-m-1 \\
\operatorname{deg} g^{\prime} & \leq m-\frac{m+1}{\ell}
\end{aligned}
$$

We deduce the following upper bounds on $\# T$.
Corollary 4.45. Let $\mathbb{F}_{q}$ be a finite field of characteristic $p, \ell$ a prime number dividing $m>\ell$, assume that $p \mid n=\ell m$, and set $t=\#\left(D_{n, \ell} \cap D_{n, m} \cap D_{n}^{+}\right)$. Then the following hold.
(i) If $p \nmid \ell$, then

$$
t \leq q^{m+\lceil\ell / p\rceil}\left(1-q^{-1}\right)
$$

(ii) If $p \mid \ell$, we set $c=\lceil(m-\ell+1) / \ell\rceil$. Then

$$
t \leq q^{m+\ell-c+\lceil c / p\rceil}\left(1-q^{-1}\right)
$$

If $\ell \mid m$, then $c=m / \ell$.
Proof. (i) Any $h^{*}$ permitted in Theorem 4.44(i) has nonzero coefficients only at $x^{i}$ with $p \mid i$ or $i \leq m-\ell$. Since $p \mid m$, the number of such $i$ is $m-\ell+\lceil\ell / p\rceil$. Taking into account that $h^{*}$ is monic, the number of $g^{*} \circ h^{*}$, composed on the left with a linear polynomial, is at most

$$
q^{2}\left(1-q^{-1}\right) \cdot q^{\ell-1} \cdot q^{m-\ell+\lceil\ell / p\rceil-1}=q^{m+\lceil\ell / p\rceil}\left(1-q^{-1}\right) .
$$

(ii) The polynomials $g$ permitted in Theorem 4.44(ii) are monic of degree $m$ and satisfy

$$
\begin{aligned}
& \operatorname{deg} g^{\prime} \leq m-\frac{m+1}{\ell} \\
& \operatorname{deg} g^{\prime} \leq m-2
\end{aligned}
$$

Thus $p \mid m$, and $g$ has nonzero coefficients only at $x^{i}$ with $i \leq m$ and $p \mid i$ or $1 \leq i \leq m-c$. The number of such $i$ is $m-c+\lceil c / p\rceil$. By composing with a linear polynomial on the left and by $h$ on the right and using that $g$ is monic, we find

$$
t \leq q^{2}\left(1-q^{-1}\right) \cdot q^{m-c+\lceil c / p\rceil-1} \cdot q^{\ell-1}=q^{m+\ell-c+\lceil c / p\rceil}\left(1-q^{-1}\right)
$$

If $\ell \mid m$, then $c=m / \ell-1+\lceil 1 / \ell\rceil=m / \ell$.
For perspective, we also note the following lower bounds on $\# T$. Unlike the results up to Corollary 4.43, there is a substantial gap between the upper and lower bounds.

Corollary 4.46. Let $\mathbb{F}_{q}$ be a finite field of characteristic $p$, $\ell$ a prime number dividing $m>\ell$, assume that $p \mid n=\ell m$, and set $t=\#\left(D_{n, \ell} \cap D_{n, m} \cap D_{n}^{+}\right)$. Then the following hold.
(i) If $p \neq \ell$ divides $m$ exactly $d \geq 1$ times, then

$$
q^{2 \ell+m / \ell-1}\left(1-q^{-1}\right)\left(1-q^{-m / \ell}\right)\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right) \leq t
$$

if $\ell \nmid p^{d}-1$. Otherwise we set $\mu=\operatorname{gcd}\left(p^{d}-1, \ell\right), r^{*}=\left(p^{d}-1\right) / \mu$ and have

$$
\begin{aligned}
& q^{2 \ell+m / \ell-1}\left(1-q^{-1}\right)\left(\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)\left(1-q^{-m / \ell}\right)\right. \\
& \left.-q^{-m / \ell-r^{*}+2} \frac{\left(1-q^{-1}\right)^{2}\left(1-q^{-r^{*}(\mu-1)}\right)}{1-q^{-r^{*}}}\left(1+q^{-r^{*}(p-2)}\right)\right) \leq t
\end{aligned}
$$

(ii) If $p=\ell, p \nmid m / p$, and $m$ has no prime divisor smaller than $p$, then

$$
q^{2 p+m / p-1}\left(1-q^{-1}\right)^{2}\left(1-q^{-p+1}\right) \leq t .
$$

Proof. (i) For any monic original $g, w, h \in \mathbb{F}_{q}[x]$ of degrees $\ell, m / \ell, \ell$, respectively, we have $g \circ w \circ h \in D_{n, \ell} \cap D_{n, m} \cap D_{n}^{0}$. We now estimate the number of such compositions.

Since $p \nmid \ell=\operatorname{deg} g$, Fact 3.1(i) implies that the composition map $(g, w \circ h) \mapsto$ $g \circ w \circ h$ is injective. To estimate from below the number $N$ of $w \circ h$, we use Theorem 3.31 with $r=p^{d}, a=m / \ell p^{d}, k=m / \ell, \tilde{m}=\ell \neq r, \mu=\operatorname{gcd}(r-1, \ell)$, and $r^{*}=(r-1) / \mu$. (Here $\tilde{m}$ is the value called $m$ in Theorem 3.31, whose name conflicts with the present value of $m$.)

If $\mu=1$, we obtain from Theorem 3.31(i)

$$
N \geq q^{\ell+m / \ell-2}\left(1-q^{-m / \ell}\right)\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)
$$

If $\mu \neq 1$, Theorem 3.31(ii) says that

$$
\begin{aligned}
N \geq & q^{\ell+m / \ell-2}\left(\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)\left(1-q^{-m / \ell}\right)\right. \\
& \left.-q^{-m / \ell-r^{*}+2} \frac{\left(1-q^{-1}\right)^{2}\left(1-q^{-r^{*}(\mu-1)}\right)}{1-q^{-r^{*}}}\left(1+q^{-r^{*}(p-2)}\right)\right),
\end{aligned}
$$

where we have used the simplification of (3.44). (We note that Corollary 3.43 provides a simplified bound if $r^{*} \geq 2$ and $p>\mu$; when $p>\ell$, then these two inequalities hold unless $\ell=2$ and $r=3$.)

We compose these $w \circ h$ with $v \circ g$ on the left, where $v$ is linear and $g$ monic original of degree $\ell$. This gives the lower bound

$$
q^{2}\left(1-q^{-1}\right) \cdot q^{\ell-1} \cdot N=q^{\ell+1}\left(1-q^{-1}\right) N
$$

on $t$, as claimed.
Thus $g$ has nonzero coefficients only at $x^{i}$ with $p \mid i$ or $i \leq a p-a$. It follows that

$$
t \leq q^{a-1+a p-(a-\lfloor a / p\rfloor)} \cdot q^{p-1}=q^{a p+p-a+\lfloor a / p\rfloor-2} .
$$

(ii) Clearly, $t$ is at least the number of $v \circ g \circ w \circ h$ with $v$ linear and $g, w, h \in F[x]$ monic original of degrees $p, m / p, p$, respectively.

We first bound the number $t^{*}$ of $h^{*}=w \circ h$ with $h_{m-1}^{*} \neq 0$. We denote as $h_{p-1}$ the second highest coefficient of $h$. Then $h_{m-1}^{*}=m / p \cdot h_{p-1}$, and $h_{m-1}^{*}$ vanishes if and only if $h_{p-1}$ does. By Fact 3.1(i), $\gamma_{m, m / p}$ is injective, so that

$$
t^{*}=q^{m / p-1} \cdot q^{p-1}\left(1-q^{-1}\right)=q^{m / p+p-2}\left(1-q^{-1}\right) .
$$

We now consider $g \circ h^{*}$ as input to Algorithm 3.14.
We have $r=p \neq m$ and $\mu=\operatorname{gcd}(p-1, m)=1$. In the proofs of Theorem 3.31(i) and Corollary 3.43(i), no special properties of $h$ are used, except (3.18). In the notation used there, we have $i_{0} \in \mathbb{N}$ if and only if $p-1 \mid(\kappa-1) m$. Now $\kappa<p$ and $m$ has no divisors less than $p$, so that $i_{0} \notin \mathbb{N}$ and (3.18) holds vacuously for all $h$. Thus the lower bound also applies when we replace the number $q^{m-1}\left(1-q^{-1}\right)$ of all possible second components by $t^{*}$. Thus
$t \geq q^{p+m}\left(1-q^{-1}\right)\left(1-q^{-p}\right)\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right) \cdot \frac{q^{m / p+p-2}\left(1-q^{-1}\right)}{q^{m-1}\left(1-q^{-1}\right)}$

$$
\begin{gathered}
=q^{2 p+m / p-1}\left(1-q^{-1}\right)\left(1-q^{-p}\right)\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right) \\
=q^{2 p+m / p-1}\left(1-q^{-1}\right)^{2}\left(1-q^{-p+1}\right)
\end{gathered}
$$

Example 4.47. We study the particular example $p=\ell=2$ and $m=6$, so that $n=12$. Let $t_{1}=t \cdot q^{-2}\left(1-q^{-1}\right)^{-1}$ denote the number of monic original polynomials in $D_{12,2} \cap D_{12,6} \cap D_{12}^{+}$. Then Corollary 4.45 (ii) says that $t_{1} \leq q^{5}$. By coefficient comparison, we now find a better bound. Namely, we are looking for $g \circ h=g^{*} \circ h^{*}$ with $g, h, g^{*}, h^{*} \in \mathbb{F}_{q}[x]$ monic original of degrees $2,6,6,2$, respectively. (We have reversed the usual degrees of $g, h$ and $g^{*}, h^{*}$ for notational convenience.) We write $h=\sum_{i} h_{i} x^{i}$, and similarly for the other polynomials. Then we choose any $h_{2}, h_{4}, h_{5} \in \mathbb{F}_{q}$, and either $g_{1}$ arbitrary and $h_{1}=u h_{5}$, or $h_{1}$ arbitrary and $g_{1}=h_{5}\left(h_{1}+u h s\right)$, where $u=h_{5}^{4}+h_{5}^{2} h_{4}+h_{2}$. Furthermore, we set $h_{3}=h_{5}^{3}$ and $h_{1}^{*}=h_{5}$. Then the coefficients of $g^{*}$ are determined. If $g^{\prime}\left(g^{*}\right)^{\prime} \neq 0$, then the above constitute a collision, and by comparing coefficients, one finds that these are all. Their number is at most $2 q^{4}$, so that $t_{1} \leq 2 q^{4}$ and $t \leq 2 q^{6}\left(1-q^{-1}\right)$.

For an explicit description of $g$, we set $u_{2}=h_{4}+h_{5}^{2}$. In the first case, where $h_{1}=u h_{5}$, we have

$$
g^{*}=x^{6}+u_{2}^{2} x^{4}+g_{1} x^{3}+\left(u^{2}+u_{2} g_{1}\right) x^{2}+g_{1} u x .
$$

In the second case, we have

$$
g^{*}=x^{6}+u_{2}^{2} x^{4}+h_{5}\left(h_{1}+u h_{5}\right) x^{3}+\left(u_{2} h_{1} h_{5}+u h_{2}\right) x^{2}+h_{1}\left(h_{1}+u h_{5}\right) x .
$$

In both cases, $g_{1}=g^{\prime} \neq 0$ implies that $\left(g^{*}\right)^{\prime} \neq 0$.
Giesbrecht (1988), Theorem 3.8, shows that there exist polynomials of degree $n$ over a field of characteristic $p$ with super-polynomially many decompositions, namely at least $n^{\lambda \log n}$ many, where $\lambda=(6 \log p)^{-1}$.

## 5. Counting tame decomposable polynomials

This section estimates the dimension and number of decomposable univariate polynomials. We start with the dimension of decomposables over an algebraically closed field. Over a finite field, Theorem 5.2 below provides a general upper bound on the number in (i), and an almost matching lower bound. The latter applies only to the tame case, where $p \nmid n$, and both bounds carry a relative error term. Lower bounds in the more difficult wild case are the subject of Section 6.

Giesbrecht (1988) was the first work on our counting problem. He proves (in his Section 1.G and translated to our notation) an upper bound of $d(n) q^{2+n / 2}$ $\left(1-q^{-1}\right)$ on the number of decomposable polynomials, where $d(n)$ is the number of divisors of $n$. This is mildly larger than our bound of about $2 q^{\ell+n / \ell}\left(1-q^{-1}\right)$, in Theorem 5.2(i), with its dependence on $\ell$ replaced by the "worst case" $\ell=2$, as in the Main Theorem (i). With the same replacement, Giesbrecht's thesis contains the upper bound in the following result, which is the geometric bound for our current problem.

Theorem 5.1. Let $F$ be an algebraically closed field, $n \geq 2$, and $\ell$ the smallest prime divisor of $n$. Then $D_{n}=\varnothing$ if $n$ is prime, and otherwise

$$
\operatorname{dim} D_{n}=\ell+n / \ell
$$

Proof. We may assume that $n$ is composite. By Fact 3.1 , the fibers of $\gamma_{n, \ell}$ are finite, and hence

$$
\operatorname{dim} D_{n} \geq \operatorname{dim} D_{n, \ell}=\operatorname{dim}\left(P_{\ell}^{=} \times P_{n / \ell}^{0}\right)=\ell+n / \ell
$$

Now $D_{n, n / \ell}$ has the same dimension, and $D_{n, e}$ has smaller dimension for all other divisors $e$ of $n$.

The argument for Corollary 4.30(i) shows that if $n$ is composite, $p \nmid n$, and $\ell^{2} \nmid n$, then $\operatorname{dim}\left(D_{n, \ell} \cap D_{n, n / \ell}\right) \leq\left\lfloor n / \ell^{2}\right\rfloor+3<\ell+n / \ell$. Thus $\gamma_{n, \ell}$ and $\gamma_{n, n / \ell}$ describe two different irreducible components of $D_{n}$, both of dimension $\ell+n / \ell$.

Zannier (2008) studies a different but related question, namely compositions $f=g \circ h$ in $\mathbb{C}[x]$ with a sparse polynomial $f$, having $t$ terms. The degree is not bounded. He gives bounds, depending only on $t$, on the degree of $g$ and the number of terms in $h$. Furthermore, he gives a parametrization of all such $f$, $g, h$ in terms of varieties (for the coefficients) and lattices (for the exponents).

We now present a generally valid upper bound on the number of decomposables and a lower bound in the tame case $p \nmid n$.

Theorem 5.2. Let $\mathbb{F}_{q}$ be a field of characteristic $p$ and with $q$ elements, and $n \geq 2$. Let $\ell$ and $\ell_{2}$ be the smallest and second smallest nontrivial divisors of
$n$, respectively (with $\ell_{2}=1$ if $n=\ell$ or $n=\ell^{2}$ ), $s=\left\lfloor n / \ell^{2}\right\rfloor$, and

$$
\begin{align*}
\alpha_{n} & = \begin{cases}0 & \text { if } n=\ell, \\
q^{2 \ell}\left(1-q^{-1}\right) & \text { if } n=\ell^{2}, \\
2 q^{\ell+n / \ell}\left(1-q^{-1}\right) & \text { otherwise },\end{cases}  \tag{5.3}\\
c & =\frac{\left(n-\ell \ell_{2}\right)\left(\ell_{2}-\ell\right)}{\ell \ell_{2}}, \\
\beta_{n} & = \begin{cases}0 & \text { if } n \in\left\{\ell, \ell^{2}, \ell^{3}, \ell \ell_{2}\right\}, \\
\frac{q^{-c}}{1-q^{-1}} & \text { otherwise },\end{cases} \\
\beta_{n}^{*} & =q^{-\ell-n / \ell+s+3},  \tag{5.4}\\
t & = \begin{cases}0 & \text { if } n \in\left\{\ell, \ell^{2}\right\}, \\
\#\left(D_{n, \ell} \cap D_{n, n / \ell}\right) & \text { otherwise } .\end{cases} \tag{5.5}
\end{align*}
$$

Then the following hold.
(i) $\# D_{n} \leq \alpha_{n}\left(1+\beta_{n}\right)$. If $n \notin\left\{\ell^{2}, \ell^{3}\right\}$, then $\# D_{n} \leq \alpha_{n}\left(1-\alpha_{n}^{-1} t+\beta_{n}\right)$.
(ii) $\# I_{n} \geq \# P_{n}^{=}-2 \alpha_{n}$.
(iii) If $p \nmid n$ and $\ell^{2} \nmid n$, then

$$
\alpha_{n}\left(1-q^{-n / \ell+\ell+s-1}\right) \leq \alpha_{n}\left(1-\beta_{n}^{*}\right) \leq \# D_{n} \leq \alpha_{n}\left(1-\frac{\beta_{n}^{*}}{2}+\beta_{n}\right)
$$

(iv) If $p \nmid n$, then

$$
\alpha_{n}\left(1-q^{-n / \ell+\ell+s-1}\right) \leq \# D_{n} \leq \alpha_{n}\left(1-\frac{\beta_{n}^{*}}{2}+\beta_{n}\right)
$$

(v) If $p \neq \ell$, then $\# D_{\ell^{2}}=\alpha_{\ell^{2}}$ and $\# D_{\ell^{3}}=\alpha_{\ell^{3}}\left(1-q^{-(\ell-1)^{2}} / 2\right)$.
(vi) If $p \nmid n \neq \ell^{2}$ and $n / \ell$ is prime, then

$$
\# D_{n}=\alpha_{n}\left(1-\frac{1}{2} q^{-n / \ell-\ell+3}\left(q^{s}+\left(1-\delta_{\ell, 2}\right)(q-1)\right)\right)
$$

Proof. When $n=\ell$ is prime, then $D_{n}=\varnothing$ and all claims are clear (reading $\alpha_{n}^{-1} t$ as 0 ). We may now assume that $n$ is composite.
(i) The claim for $n \in\left\{\ell^{2}, \ell^{3}\right\}$ follows from (v), and we now exclude these cases. We write $u(e)=e+n / e$ for the exponent in Fact 3.1(i). We have the two largest subsets $D_{n, \ell}$ and $D_{n, n / \ell}$ of $D_{n}$, both of size at most
$\frac{\alpha_{n}}{2}=q^{u(\ell)}\left(1-q^{-1}\right)=q^{\ell+n / \ell}\left(1-q^{-1}\right)=\#\left(P_{\ell}^{=} \times P_{n / \ell}^{0}\right)=\#\left(P_{n / \ell}^{=} \times P_{\ell}^{0}\right)$.
Their joint contribution to $\# D_{n}$ is at most

$$
\begin{equation*}
\alpha_{n}-t \tag{5.7}
\end{equation*}
$$

Since $n$ is not $\ell$ or $\ell^{2}$, we have $\ell<\ell_{2} \leq n / \ell$, and $\ell_{2}$ is either $\ell^{2}$ or a prime number larger than $\ell$. The index set $E$ in (2.4) consists of all proper divisors of $n$. If $n=\ell \ell_{2}$, then $E=\left\{\ell, \ell_{2}\right\}$, and from (5.7) we have

$$
\# D_{n} \leq \alpha_{n}-t
$$

We may now assume that $n \neq \ell \ell_{2}$. For any $e \in E$, we have $u(e)=e+n / e=$ $u(n / e)$. Furthermore

$$
\begin{equation*}
u(e)-u\left(e^{\prime}\right)=\frac{\left(n-e e^{\prime}\right)\left(e^{\prime}-e\right)}{e e^{\prime}} \tag{5.8}
\end{equation*}
$$

holds for $e, e^{\prime} \in E$, and in particular

$$
\begin{equation*}
u(\ell)-u\left(\ell_{2}\right)=\left(n-\ell \ell_{2}\right)\left(\ell_{2}-\ell\right) / \ell \ell_{2}=c \tag{5.9}
\end{equation*}
$$

Considered as a function of a real variable $e, u$ is convex on the interval [1..n], since $\partial^{2} u / \partial e^{2}=2 n / e^{3}>0$. Thus $u(\ell)-u(e) \geq c$ for all $e \in E_{2}=E \backslash\{\ell, n / \ell\}$. Then

$$
\begin{aligned}
\sum_{e \in E_{2}} q^{u(e)-u(\ell)} & =q^{-c} \sum_{e \in E_{2}} q^{u(e)-u(\ell)+c} \\
& <q^{-c} \cdot 2 \sum_{i \geq 0} q^{-i}=\frac{2 q^{-c}}{1-q^{-1}}
\end{aligned}
$$

since each value $u(e)$ is assumed at most twice, namely for $e$ and $n / e$, according to (5.8). Using (5.7), it follows for $n \neq \ell^{2}$ that

$$
\begin{align*}
\# D_{n}+t & \leq \sum_{e \in E} \# D_{n, e} \leq \sum_{e \in E} q^{u(e)}\left(1-q^{-1}\right) \\
& \leq q^{\ell+n / \ell}\left(1-q^{-1}\right)\left(2+\sum_{e \in E_{2}} q^{u(e)-u(\ell)}\right)  \tag{5.10}\\
& \leq q^{\ell+n / \ell}\left(1-q^{-1}\right)\left(2+\frac{2 q^{-c}}{1-q^{-1}}\right)=\alpha_{n}\left(1+\beta_{n}\right) .
\end{align*}
$$

This implies the claim in (i).
(ii) follows from $\beta_{n} \leq 1$.

For (iii), we have $D_{n, \ell} \cup D_{n, n / \ell} \subseteq D_{n}$. Since $p \nmid n$, both $\gamma_{n, \ell}$ and $\gamma_{n, n / \ell}$ are injective, by Fact 3.1(i). From Corollary 4.30(i), we find

$$
\begin{aligned}
\# D_{n} & \geq \# D_{n, \ell}+\# D_{n, n / \ell}-\#\left(D_{n, \ell} \cap D_{n, n / \ell}\right) \\
& \geq 2 q^{\ell+n / \ell}\left(1-q^{-1}\right)-\left(q^{s+3}+q^{4}\right)\left(1-q^{-1}\right) \\
& =\alpha_{n}\left(1-\frac{q^{s+3}+q^{4}}{2 q^{\ell+n / \ell}}\right) \geq \alpha_{n}\left(1-\frac{q^{s+3}}{q^{\ell+n / \ell}}\right)=\alpha_{n}\left(1-\beta_{n}^{*}\right), \\
\# D_{n} & \leq \alpha_{n}\left(1-\frac{q^{s+3}\left(1-q^{-1}\right)}{\alpha_{n}}+\beta_{n}\right)=\alpha_{n}\left(1-\frac{\beta_{n}^{*}}{2}+\beta_{n}\right) .
\end{aligned}
$$

Furthermore, we have $1 \leq s \leq n / \ell^{2}$ (since $n$ is composite), $s+3 \geq 4, \ell \geq 2$, and hence

$$
-\ell-\frac{n}{\ell}+s+3 \leq-\frac{n}{\ell}+\ell+s-1 .
$$

It follows that

$$
\beta^{*} \leq q^{-n / \ell+\ell+s-1} .
$$

(iv) For the lower bound if $\ell^{2} \mid n$, we replace the upper bound from Corollary $4.30(\mathrm{i})$ by the one from Corollary 4.43 (ii).

In (v), for $n=\ell^{2}$, we have $D_{n}=D_{n, \ell}$ and

$$
\# D_{n}=q^{\ell+n / \ell}\left(1-q^{-1}\right)=\alpha_{n}
$$

using the injectivity of $\gamma_{\ell^{2}, \ell}$ (Fact 3.1(i)). When $n=\ell^{3}$, then Corollary 4.43 says that

$$
\begin{aligned}
t & =q^{3 \ell-1}\left(1-q^{-1}\right) \\
\# D_{\ell^{3}} & =\alpha_{\ell^{3}}\left(1-\frac{t}{\alpha_{\ell^{3}}}\right)=\alpha_{n}\left(1-\frac{q^{-(\ell-1)^{2}}}{2}\right) .
\end{aligned}
$$

This shows (v). For (vi), we replace the bound on $\#\left(D_{n, \ell} \cap D_{n, n / \ell}\right)$ by its exact value from Corollary 4.30(i).

Bodin et al. (2009) state an upper bound as in Theorem 5.2(i), with an error term which is only $O(n)$ worse than $\beta_{n}$.

Remark 5.11. How often does it happen that the smallest prime factor $\ell$ of $n$ actually divides $n$ at least twice? The answer: almost a third of the time.

For a prime $\ell$, let

$$
S_{\ell}=\left\{n \in \mathbb{N}: \ell^{2} \mid n, \forall \text { primes } r<\ell \quad r \nmid n\right\},
$$

so that $\bigcup_{\ell} S_{\ell}$ is the set in question. The union is disjoint, and its density is

$$
\sigma=\sum_{\ell} \frac{1}{\ell^{2}} \prod_{r<\ell}\left(1-\frac{1}{r}\right) \approx 0.330098
$$

If we take a prime $p$ and further ask that $p \nmid n$, then we have the density

$$
\sigma_{p}=\sigma-\frac{1}{p^{2}} \prod_{r<p}\left(1-\frac{1}{r}\right)-\frac{1}{p} \sum_{\ell<p} \frac{1}{\ell^{2}} \prod_{r<\ell}\left(1-\frac{1}{r}\right) .
$$

The correction terms $\sigma-\sigma_{p}$ are $\approx 0.25,0.13889,0.07444$ for $p=2,3,5$, respectively.

The upper and lower bounds in Theorem 5.2(i) and (iii) have distinct relative error estimates. We now compare the two.

Proposition 5.12. In the notation of Theorem 5.2, assume that $n \neq \ell, \ell^{2}, \ell \ell_{2}$.
(i) If $\ell_{2} \leq \ell^{2}$, then $\beta_{n}>\beta_{n}^{*}$. If furthermore $\ell^{2} \nmid n$ and $p \nmid n$, then

$$
\left|\# D_{n}-\alpha_{n}\right| \leq \alpha_{n} \beta_{n}
$$

(ii) If $\ell_{2} \geq \ell^{2}+\ell$, then $\beta_{n} \leq \beta_{n}^{*}$. If furthermore $\ell^{2} \nmid n$ and $p \nmid n$, then

$$
\left|\# D_{n}-\alpha_{n}\right| \leq \alpha_{n} \beta_{n}^{*}
$$

Proof. We let $\mu=-\log _{q}\left(1-q^{-1}\right)$ and $\sigma=n / \ell^{2}-s$, so that $0<\mu \leq 1$, $0 \leq \sigma \leq 1-1 / \ell<1$, and

$$
\begin{aligned}
& \beta_{n}=q^{-c+\mu} \\
& \beta_{n}^{*}=q^{-\ell-n / \ell+n / \ell^{2}-\sigma+3} .
\end{aligned}
$$

Furthermore,

$$
\begin{align*}
\beta_{n} \leq \beta_{n}^{*} & \Longleftrightarrow \ell \ell_{2}\left(\ell+\frac{n}{\ell}-\frac{n}{\ell^{2}}+\sigma+\mu-3\right) \leq\left(n-\ell \ell_{2}\right)\left(\ell_{2}-\ell\right) \\
& \Longleftrightarrow \ell \ell_{2}\left(\ell_{2}+\sigma+\mu-3\right) \leq \frac{n}{\ell}\left(\ell_{2}-\ell^{2}\right) . \tag{5.13}
\end{align*}
$$

We note that $\ell_{2}>\ell_{2}+\sigma+\mu-3>0$. If $\ell_{2} \leq \ell^{2}$, it follows that $\beta_{n}>\beta_{n}^{*}$. If $\ell_{2} \geq \ell^{2}+\ell$, then $a=n / \ell \ell_{2}$ is a proper divisor of $n$, since $n \neq \ell \ell_{2}$. It follows that $a \geq \ell_{2}$, since $a=\ell$ would mean that $\ell^{2}$ is a divisor of $n$ with $\ell<\ell^{2}<\ell_{2}$, contradicting the minimality of $\ell_{2}$. Then

$$
\frac{n}{\ell}\left(\ell_{2}-\ell^{2}\right) \geq \ell_{2}^{2} \cdot \ell>\ell \ell_{2}\left(\ell_{2}+\sigma+\mu-3\right)
$$

and $\beta_{n} \leq \beta_{n}^{*}$.
The claims about $\# D_{n}$ follow from Theorem 5.2.
There remains the "gray area" of $\ell^{2}<\ell_{2}<\ell^{2}+\ell$, where (5.13) has to be evaluated. The three equivalent properties in (5.13) hold when $n$ has at least four prime factors, and do not hold when $n=\ell \ell_{2}$.

We can simplify the bounds of Theorem 5.2, at the price of a slightly larger relative error.

Corollary 5.14. We assume the notation of Theorem 5.2.
(i) If $n$ is prime, then $D_{n}=\varnothing$.
(ii) For all $n$, we have

$$
\begin{equation*}
\# D_{n} \leq \alpha_{n}\left(1+q^{-n / 3 \ell^{2}}\right) \tag{5.15}
\end{equation*}
$$

(iii) If $p \nmid n$, then

$$
\left|\# D_{n}-\alpha_{n}\right| \leq \alpha_{n} \cdot q^{-n / 3 \ell^{2}}
$$

Proof. (i) follows from Theorem 5.2(i), since $\alpha_{n}=0$. For (ii), we claim that $\beta_{n} \leq q^{-n / 3 \ell^{2}}$. The cases where $n \in\left\{\ell, \ell^{2}, \ell \ell_{2}\right\}$ are trivial, and we may now assume that $a=n / \ell \ell_{2} \geq 2$. We set $\mu=-\log _{q}\left(1-q^{-1}\right)$, so that $0<\mu \leq 1$ and $\beta_{n}=q^{-c+\mu}$.

We have

$$
\frac{3 \ell^{3}+3 \ell}{3 \ell-2} \geq \frac{3 \ell^{2}}{3 \ell-1}
$$

If

$$
\begin{equation*}
\ell_{2} \geq \frac{3 \ell^{2}+3 \ell}{3 \ell-2}=\ell+\frac{5}{3}+\frac{10}{9 \ell-6} \tag{5.16}
\end{equation*}
$$

then $\ell_{2}-\ell-\ell_{2} / 3 \ell \geq 0$ and

$$
\begin{align*}
& a\left(\ell_{2}-\ell-\frac{\ell_{2}}{3 \ell}\right) \geq 2\left(\ell_{2}-\ell-\frac{\ell_{2}}{3 \ell}\right) \geq \ell_{2}-\ell+1, \\
&(a-1)\left(\ell_{2}-\ell\right)-1 \geq \frac{a \ell_{2}}{3 \ell}=\frac{n}{3 \ell^{2}}, \tag{5.17}
\end{align*}
$$

from which the claim follows. (5.16) is satisfied except when $\left(\ell, \ell_{2}\right)$ is $(2,3)$, $(2,4)$ or $(3,5)$.

In the first case, (5.17) is satisfied for $a \geq 4$, and in the other two for $a \geq 3$. The latter always holds in the case $(3,5)$, and we are left with $n \in\{12,16,18\}$. For these values of $n$, we use a direct bound on the sum in (5.10), namely

$$
\sum_{e \in E_{2}} q^{u(e)-u(\ell)} \leq \# E_{2} \cdot q^{-c}=2 \epsilon q^{-c}
$$

where $\epsilon=\# E_{2} / 2$, so that

$$
\# D_{n} \leq \alpha_{n}\left(1+\epsilon q^{-c}\right)-t
$$

The required values are given in Table 5.1. In all cases, we conclude from Theorem 5.2(i) that $\# D_{n} \leq \alpha_{n}\left(1+q^{-n / 3 \ell^{2}}\right)$.

| $n$ | 12 | 16 | 18 |
| :--- | :---: | :---: | :---: |
| $\epsilon$ | 1 | $1 / 2$ | 1 |
| $c$ | 1 | 2 | 2 |
| $n / 3 \ell^{2}$ | 1 | $4 / 3$ | $3 / 2$ |

Table 5.1: Parameters for three values of $n$.
(iii) Our claim is that $q^{-n / \ell+\ell+s-1} \leq q^{-n / 3 \ell^{2}}$. Since $n \geq \ell^{2}$, we have

$$
\begin{aligned}
\ell^{2}(3 \ell-3) & \leq \ell^{2}(3 \ell-2) \leq n(3 \ell-2) \\
2 n+3 \ell^{3} & \leq 3 \ell n+3 \ell^{2} \\
\frac{n}{3 \ell^{2}}+\ell+s & \leq \frac{n}{3 \ell^{2}}+\ell+\frac{n}{\ell^{2}}=\frac{2 n}{3 \ell^{2}}+\ell \leq \frac{n}{\ell}+1
\end{aligned}
$$

This proves the claim, and (iii) follows from (ii) and Theorem 5.2.

## 6. Counting general decomposable polynomials

Theorem 5.2 provides a satisfactory result in the tame case, where $p \nmid n$. Most of the preparatory work in Sections 3 and 4 is geared towards the wild case. The upper bound of Theorem 5.2(i) still holds. We now present the resulting lower bounds.

We have to deal with an annoyingly large jungle of case distinctions. To keep an overview, we reduce it to the single tree of Figure 6.1. Its branches correspond to the various bounds on equal-degree collisions (Corollary 3.43)


Figure 6.1: The tree of case distinctions for estimating $\# D_{n}$.
and on distinct-degree collisions (Corollaries 4.30, 4.43, and 4.45). Since at each internal vertex, the two branches are complementary, the leaves cover all possibilities. We use a top down numbering of the vertices according to the branches; as an example, II.B.ii.b. $\beta$ is the rightmost leaf at the lowest level. Furthermore, if a branching is left out, as in II.B, then a bound at that vertex holds for all descendants, which comprise three internal vertices and five leaves in this example.

Theorem 6.1. Let $\mathbb{F}_{q}$ be a finite field of characteristic $p$ with $q$ elements, and $\ell$ the smallest prime divisor of the composite integer $n \geq 2$. Then we have the following bounds on $\# D_{n}$ over $\mathbb{F}_{q}$.
(i) If the "upper" column in Table 6.1 contains a 1, then

$$
\# D_{n} \leq \alpha_{n}
$$

| leaf in |  | up- |
| :--- | :--- | :--- |
| Figure 6.1 | lower bound on $\# D_{n} / \alpha_{n}$ | per |
| I.A | 1 | 1 |
| I.B | $\frac{1}{2}\left(1+\frac{1}{p+1}\right)\left(1-q^{-2}\right)+q^{-p}>1 / 2$ | 1 |
| II.A.i | $1-\beta_{n}^{*} \geq 1-q^{-n / \ell-\ell+n / \ell^{2}+3}$ |  |
| II.A.ii | $1-q^{-n / \ell+\ell+n / \ell^{2}-1} / 2$ |  |
| II.B.i.a | $1-\left(q^{-1}+q^{-p+1}+q^{-n / \ell-\ell+n / \ell^{2}+3}\right) / 2$ | 1 |
| II.B.i.b | $1-\left(q^{-1}-q^{-p}\right) / 2$ |  |
| II.B.ii.a | $1-\left(q^{-1}+q^{-p+1}-q^{-p}+q^{-\ell+1}\right) / 2$ |  |
| II.B.ii.b. $\alpha$ | $\frac{1}{2}\left(\frac{3}{2}+\frac{1}{2 p+2}-q^{-1}-\frac{q^{-2}}{2}\left(1+\frac{1}{p+1}\right)-\frac{q^{-p+1}}{1-q^{-p}}\right)$ | 1 |
| II.B.ii.b. $\beta$ | $1-q^{-1}-q^{-p+1}$ | 1 |

Table 6.1: The bounds at the leaves of Figure 6.1.
(ii) The lower bounds in Table 6.1 hold.

Proof. We recall $D_{n, e}$ from (2.3) and $\alpha_{n}$ from (5.3), the superscript + for non-Frobenius from (3.5), and set at each vertex

$$
\nu=\frac{\# D_{n}}{\alpha_{n}}, \nu_{0}=\frac{\# D_{n, \ell}^{+}}{\alpha_{n}}, \nu_{1}=\frac{\# D_{n, n / \ell}^{+}}{\alpha_{n}}, \nu_{2}=\frac{\#\left(D_{n, \ell}^{+} \cap D_{n, n / \ell}^{+}\right)}{\alpha_{n}}, \nu_{3}=\frac{\# D_{n}^{\varphi}}{\alpha_{n}} .
$$

Then $\nu=\nu_{0}+\nu_{3}$ if $n=\ell^{2}$, and otherwise

$$
\begin{equation*}
\nu_{0}+\nu_{1}-\nu_{2}+\nu_{3} \leq \nu \leq 1+\beta_{n}-\nu_{2}-\nu_{3} . \tag{6.2}
\end{equation*}
$$

In the lower bound, $\nu_{0}+\nu_{1}-\nu_{2}$ counts the non-Frobenius compositions of the dominant contributions $D_{n, \ell}$ and $D_{n, n / \ell}$, and $\nu_{3}$ adds the Frobenius compositions. In the upper bound, $1-\nu_{2}$ bounds the two dominant contributions from above, $\beta_{n}$ accounts for the non-dominant contributions. We may subtract $\nu_{3}$ since the Frobenius compositions have been counted twice, in $D_{n, p}$ and $D_{n, n / p}$; of course, $\nu_{3}$ is nonzero only if $p \mid n$.

The proof proceeds in two stages. In the first one, we indicate for some vertices $V$ bounds $\lambda_{i}(V)$ with the following properties:

$$
\nu_{0} \geq \lambda_{0}, \nu_{1} \geq \lambda_{1}, \lambda_{2} \geq \nu_{2} \geq \lambda_{4}
$$

Such a bound at $V$ applies to all descendants of $V$. The value $\lambda_{4}$ only intervenes in the upper bound on $\nu$, and we sometimes forego its detailed calculation and
simply use $\lambda_{4}=0$. In the second stage, we assemble those bounds for each leaf, according to (6.2).

Throughout the proof, $d \geq 0$ denotes the multiplicity of $p$ in $n$, and $s=$ $\left\lfloor n / \ell^{2}\right\rfloor$. In the first stage, we use Theorem $5.2(\mathrm{v})$ at I.A:

$$
\nu(\mathrm{I} . \mathrm{A})=1 .
$$

At I.B, we have from Example 3.45

$$
\lambda_{0}(\mathrm{I} . \mathrm{B}) \geq \frac{1}{2}\left(1+\frac{1}{p+1}\right)\left(1-q^{-2}\right)+q^{-p} .
$$

Furthermore,

$$
\left(1+\frac{1}{p+1}\right)\left(1-q^{-2}\right) \geq\left(1+\frac{1}{p+1}\right)\left(1-p^{-2}\right)=1+\frac{p-2}{p^{2}} \geq 1
$$

so that $\lambda_{0}($ I.B $)>1 / 2$. Lemma 4.32 (ii) says that

$$
\lambda_{3}(\mathrm{I} . \mathrm{B})=q^{-p+1}
$$

From Fact 3.1(i), we have

$$
\lambda_{0}(\mathrm{II} . \mathrm{A})=\lambda_{1}(\mathrm{II} . \mathrm{A})=\frac{1}{2}
$$

and since $p \nmid n$,

$$
\nu_{3}(\text { II.A })=0 .
$$

Vertex II.A.i has been dealt with in Corollary 4.30(i):

$$
\begin{aligned}
& \lambda_{2}(\text { II.A.i })=\beta_{n}^{*} \geq \frac{1}{2} q^{-n / \ell-\ell}\left(q^{s+3}+q^{4}\right) \\
& \lambda_{4}(\text { II.A.i })=\frac{1}{2} q^{-n / \ell-\ell+s+3}
\end{aligned}
$$

Since $\ell \mid n / \ell$, Corollary 4.43 yields

$$
\lambda_{2}(\text { II.A.ii })=\lambda_{4}(\text { II.A.ii })=\frac{1}{2} q^{-n / \ell+\ell+s-1} .
$$

Since $p \mid n$ at II.B, Lemma 4.32(ii) implies that

$$
\nu_{3}(\mathrm{II} . \mathrm{B})=\frac{1}{2} q^{-\ell-n / \ell+n / p+1} .
$$

We now let $V$ be one of II.B.i.a or II.B.ii.a. Then we have

$$
\lambda_{0}(V)=\frac{1}{2},
$$

by Fact 3.1(i). Applying Corollary 3.43 to $D_{n, n / \ell}$ at $V$, we have $d \geq 1, r=p^{d} \neq$ $\ell=m, k=n / \ell$, and

$$
\begin{equation*}
\mu=\operatorname{gcd}\left(p^{d}-1, \ell\right) \text { is either } 1 \text { or } \ell . \tag{6.3}
\end{equation*}
$$

In the first case, where $\mu=1$, we have

$$
\nu_{1}(V) \geq \frac{1}{2}\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)\left(1-q^{-n / \ell}\right)
$$

from Corollary 3.43(i). In the second case, where $\mu=\ell$, we have $p>\ell=\mu \geq 2$. We first assume that $r \neq 3$. Then $r-1=p^{d}-1$ is not a prime number, and $r^{*}=(r-1) / \ell \geq 2$, so that the last bound in Corollary 3.43(ii) applies and

$$
\nu_{1}(V) \geq \frac{1}{2}\left(\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)\left(1-q^{-n / \ell}\right)-\frac{2}{3} q^{-n / \ell}\left(1-q^{-1}\right)^{2} .\right.
$$

If $r=3$, then $p=3, \mu=\ell=2, r^{*}=1$, and according to the second bound in Corollary 3.43(ii), we have to replace the last summand above by

$$
-\frac{1}{2} q^{-n / \ell+1}\left(1-q^{-1}\right)^{2}\left(1+q^{-1}\right) .
$$

Since $2 / 3 \leq q\left(1+q^{-1}\right) / 2$, the latter term dominates in absolute value the one for $r \neq 3$. Its value is at least $q^{-n / \ell+1} / 2$, and we find for $\mu=\ell$ that

$$
\begin{aligned}
\nu_{1}(V) \geq & \frac{1}{2}-\frac{q^{-1}}{2}\left(1+q^{-p+2}\left(1-q^{-1}\right)\right) \\
& -\frac{q^{-n / \ell}}{2}\left(1-q^{-1}-q^{-p+1} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}+q\right) \\
\geq & \frac{1}{2}-\frac{q^{-1}}{2}\left(1+q^{-p+2}\right)+\frac{q^{-p}}{2}-\frac{q^{-n / \ell}(q+1)}{2} .
\end{aligned}
$$

Thus we may take the last value as $\lambda_{1}$ (II.B.i.a) and $\lambda_{1}$ (II.B.ii.a). Furthermore, Corollary 4.30(iii) yields

$$
\lambda_{2}(\text { II.B.i.a })=\frac{1}{2} q^{-n / \ell-\ell}\left(q^{s+3}-q^{\lfloor s / p\rfloor+3}\right) .
$$

When $V$ is II.B.i.b or II.B.ii.b, we have for $\lambda_{0}$ in the notation of Corollary 3.43 that $k=r=p \neq n / p=m$ and $\mu=\operatorname{gcd}(p-1, n / p)=1$, since all proper divisors of $n / p$ are at least $\ell=p$. Thus we may apply Corollary 3.43(i) to find

$$
\begin{aligned}
\lambda_{0}(V) & =\frac{1}{2}\left(1-q^{-p}\right)\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right) \\
& =\frac{1}{2}\left(1-q^{-1}-q^{-p+1}+q^{-p}\right) .
\end{aligned}
$$

At II.B.i.b, we have $p \nmid n / p$, so that Fact 3.1(i) for $D_{n, n / p}$ implies

$$
\lambda_{1}(\text { II.B.i.b })=\frac{1}{2},
$$

and Corollary 4.30 (ii) yields

$$
\lambda_{2}(\text { II.B.i.b })=\lambda_{4}(\text { II.B.i.b })=0
$$

At II.B.ii.a, we have $\ell<p$, and Corollary 4.45 (i) says that

$$
\lambda_{2}(\text { II.B.ii.a })=\frac{1}{2} q^{-\ell+\lceil\ell / p\rceil}=\frac{1}{2} q^{-\ell+1} .
$$

At II.B.ii.b. $\alpha$, we have $k=n / p$ and $r=p=z=m$ in Corollary 3.43(iii) for $D_{n, n / p}$, so that

$$
\begin{aligned}
\lambda_{1}(\text { II.B.ii.b. } \alpha)= & \frac{1}{2}\left(1-q^{-1}\right)\left(\frac{1}{2}+\frac{1+q^{-1}}{2 p+2}+\frac{q^{-1}}{2}\right. \\
& \left.-q^{-n / p} \frac{1-q^{-p+1}}{1-q^{-p}}-q^{-p+1} \frac{1-q^{-1}}{1-q^{-p}}\right)
\end{aligned}
$$

Furthermore, from Corollary 4.45(ii) we have

$$
\lambda_{2}(\text { II.B.ii.b })=\frac{1}{2} q^{-n / p^{2}+\left\lceil n / p^{3}\right\rceil} .
$$

At II.B.ii.b. $\beta$, we have for $D_{n, n / p}$ that $k=n / p, r=p^{d-1} \neq p=m$, since $d \geq 3$, and $\mu=\operatorname{gcd}(r-1, m)=\operatorname{gcd}\left(p^{d-1}-1, p\right)=1$, so that Corollary 3.43(i) yields

$$
\begin{aligned}
\lambda_{1}(\text { II.B.ii.b. } \beta) & =\frac{1}{2}\left(1-q^{-1}\left(1+q^{-p+2} \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}\right)\right)\left(1-q^{-n / p}\right) \\
& =\frac{\left(1-q^{-1}\right)\left(1-q^{-p+1}\right)\left(1-q^{-n / p}\right)}{2\left(1-q^{-p}\right)}
\end{aligned}
$$

Corollary 4.46 (ii) says that

$$
\lambda_{4}(\text { II.B.ii.b. } \alpha)=\frac{1}{2} q^{-n / p+p+n / p^{2}-1}\left(1-q^{-1}\right)\left(1-q^{-p+1}\right) .
$$

We find the following bounds on $\nu$ at the leaves.
I.A:

$$
\nu=\lambda_{0}(\mathrm{I} . \mathrm{A})=1,
$$

I.B: We have $\lambda_{3}($ I.B $)=q^{-p+1}$, and all Frobenius compositions except $x^{p} \circ x^{p}$ are collisions. Thus

$$
1-q^{-p+1}\left(1-q^{-p+1}\right) \geq \nu \geq \frac{1}{2}\left(1+\frac{1}{p+1}\right)\left(1-q^{-2}\right)+q^{-p}>1 / 2
$$

II.A.i:

$$
\begin{aligned}
& \nu \leq 1+\beta_{n}-\lambda_{4}(\text { II.A.i })=1+\beta_{n}-\frac{1}{2} q^{-n / \ell-\ell+s+3} \leq 1+\beta_{n} \\
& \nu \geq \lambda_{0}(\text { II.A })+\lambda_{1}(\text { II.A })-\lambda_{2}(\text { II.A.i })=1-\beta_{n}^{*}
\end{aligned}
$$

II.A.ii:

$$
\begin{aligned}
\nu & \leq 1+\beta_{n}-\lambda_{4}(\text { II.A.ii })=1+\beta_{n}-\frac{1}{2} q^{-n / \ell+\ell+n / \ell^{2}-1} \leq 1+\beta_{n} \\
\nu & \geq \lambda_{0}(\text { II.A })+\lambda_{1} \text { (II.A) }-\lambda_{2} \text { (II.A.ii) } \\
& =\frac{1}{2}+\frac{1}{2}-\frac{1}{2} q^{-n / \ell+\ell+s-1}=1-\frac{1}{2} q^{-n / \ell+\ell+s-1} .
\end{aligned}
$$

II.B.i.a:

For the lower bound, we find

$$
\begin{align*}
\nu \geq & \lambda_{0}(\text { II.B.i.a })+\lambda_{1}(\text { II.B.i.a })-\lambda_{2}(\text { II.B.i.a })+\nu_{3} \text { (II.B) } \\
= & \frac{1}{2}+\frac{1}{2}\left(1-q^{-1}\left(1+q^{-p+2}\right)+q^{-p}-q^{-n / \ell}(q+1)\right) \\
& -\frac{1}{2} q^{-n / \ell-\ell}\left(q^{s+3}-q^{\lfloor s / p\rfloor+3}\right)+\frac{1}{2} q^{-\ell-n / \ell+n / p+1} \\
\geq & 1-\frac{1}{2}\left(q^{-1}+q^{-p+1}\right)+\frac{q^{-p}}{2}-\frac{q^{-n / \ell}}{2}\left(q+1+q^{s-\ell+3}-q^{n / p-\ell+1}\right) . \tag{6.4}
\end{align*}
$$

At the present leaf, we have $n=a \ell p$ with $p>\ell \geq 2$ and $a \geq 1$. Thus $n / \ell \geq p$ and

$$
q^{-p} \geq q^{-n / \ell}
$$

Furthermore, $n / p \geq \ell$ and

$$
q^{n / p-\ell+1} \geq q
$$

It follows that

$$
\begin{equation*}
\nu \geq 1-\frac{1}{2}\left(q^{-1}+q^{-p+1}+q^{-n / \ell-\ell+s+3}\right) \tag{6.5}
\end{equation*}
$$

II.B.i.b:

$$
\nu \leq 1+\beta_{n}-\lambda_{4}(\text { II.B.i.b })-\nu_{3}(\text { II.B })=1+\beta_{n}-0-\frac{1}{2} q^{-p+1}
$$

We claim that $\beta_{n} \leq \frac{1}{2} q^{-p+1}$, so that $\nu \leq 1$. We may assume that $n \notin$ $\left\{\ell^{2}, \ell \ell_{2}\right\}$, since otherwise $\beta_{n}=0$. Setting $\mu=\log _{q}\left(2 /\left(1-q^{-1}\right)\right)$, we have $0<\mu \leq 2$ and $2 \beta_{n}=q^{-c+\mu} \leq q^{-c+2}$, so that it suffices to show

$$
\ell-1=p-1 \leq c-2=\frac{\left(n-\ell \ell_{2}\right)\left(\ell_{2}-\ell\right)}{\ell \ell_{2}}-2 .
$$

Abbreviating $a=n / \ell \ell_{2}$, this is equivalent to

$$
\begin{equation*}
\frac{\ell+1}{\ell_{2}-\ell}+1 \leq a \tag{6.6}
\end{equation*}
$$

Since $p=\ell$ and $p^{2} \nmid n$, we have $\ell \nmid a$ and $a \geq \ell_{2}>\ell$, by the minimality conditions on $\ell$ and $\ell_{2}$. If $\ell_{2} \geq \ell+2$, (6.6) holds. If $\ell_{2}=\ell+1$, then $\ell=2$ and $a \geq 4$ is required for (6.6). Since $2 \nmid a$, it remains the case $a=3$, corresponding to $n=18$ and $p=2$. One checks that $\beta_{18} \leq \frac{1}{2} q^{-1}$ for $q \geq 4$. For $q=2$, we have to go back to (5.10) and check that $\nu_{3}=q^{10}\left(1-q^{-1}\right)$ and

$$
\# D_{18} \leq \alpha_{18}-\nu_{3}+2 q^{9}\left(1-q^{-1}\right)=\alpha_{18}
$$

For the lower bound, we have

$$
\begin{aligned}
\nu & \geq \lambda_{0}(\text { II.B.i.b })+\lambda_{1}(\text { II.B.i.b })-\lambda_{2}(\text { II.B.i.b })+\nu_{3}(\text { II.B }) \\
& =\frac{1}{2}\left(1-q^{-1}-q^{-p+1}+q^{-p}\right)+\frac{1}{2}-0+\frac{1}{2} q^{-p+1} \\
& =1-\frac{1}{2}\left(q^{-1}-q^{-p}\right) .
\end{aligned}
$$

At II.B.ii.a, we have

$$
\begin{aligned}
\nu \geq & \lambda_{0}(\text { II.B.ii.a })+\lambda_{1} \text { (II.B.ii.a) }-\lambda_{2} \text { (II.B.ii.a) }+\nu_{3} \text { (II.B) } \\
= & \frac{1}{2}+\frac{1}{2}-\frac{q^{-1}}{2}\left(1+q^{-p+2}\right)+\frac{q^{-p}}{2}-\frac{q^{-n / \ell}(q+1)}{2} \\
& -\frac{q^{-\ell+1}}{2}+\frac{q^{-\ell-n / \ell+n / p+1}}{2} \\
= & 1-\frac{1}{2}\left(q^{-1}+q^{-p+1}\right)+\frac{q^{-p}}{2}-\frac{q^{-\ell+1}}{2}+\frac{q^{-n / \ell}}{2}\left(q^{n / p-\ell+1}-q-1\right) .
\end{aligned}
$$

Since $n=a \ell^{2} p$ with $a \geq 1$, we have $n / p \geq \ell^{2}>\ell+1$, and

$$
\begin{aligned}
q^{n / p-\ell+1} & >q^{2}>q+1 \\
\nu & >1-\frac{1}{2}\left(q^{-1}+q^{-p+1}-q^{-p}+q^{-\ell+1}\right) .
\end{aligned}
$$

II.B.ii.b. $\alpha$ :

$$
\begin{aligned}
\nu \geq & \lambda_{0}(\text { II.B.ii.b })+\lambda_{1}(\text { II.B.ii.b. } \alpha)-\lambda_{2} \text { (II.B.ii.b) }+\nu_{3}(\text { II.B) }) \\
= & \frac{1}{2}\left(1-q^{-1}-q^{-p+1}+q^{-p}\right)+\frac{1}{2}\left(1-q^{-1}\right)\left(\frac{1}{2}+\frac{1+q^{-1}}{2 p+2}+\frac{q^{-1}}{2}\right. \\
& \left.\quad-q^{-n / p} \frac{1-q^{-p+1}}{1-q^{-p}}-q^{-p+1} \frac{1-q^{-1}}{1-q^{-p}}\right)-\frac{1}{2} q^{-n / p^{2}+\left\lceil n / p^{3}\right\rceil}+\frac{1}{2} q^{-p+1} \\
= & \frac{1}{2}\left(\frac{3}{2}+\frac{1}{2 p+2}-q^{-1}-\frac{q^{-2}}{2}\left(1+\frac{1}{p+1}\right)+\frac{q^{-p}\left(2-q^{-1}-q^{-p}\right)}{1-q^{-p}}\right. \\
& \left.\quad-q^{-n / p^{2}+\left\lceil n / p^{3}\right\rceil}-q^{-n / p} \frac{\left(1-q^{-1}\right)\left(1-q^{-p+1}\right)}{1-q^{-p}}\right) .
\end{aligned}
$$

We have $n=a p^{2}$ with $a>p$ and all prime divisors of $a$ larger than $p$. If $p \geq 3$, then $a \geq p+2$ and

$$
\begin{align*}
a & \geq p+2>p+1+\frac{1}{p-1}=\frac{p^{2}}{p-1}, \\
a & \geq p+\frac{a}{p}, \\
a & \geq p+\left\lceil\frac{a}{p}\right\rceil \\
q^{-p} & \geq q^{-n / p^{2}+\left\lceil n / p^{3}\right\rceil} . \tag{6.8}
\end{align*}
$$

We may now assume that $p=2$. If $a \geq 5$, then

$$
a-\frac{a}{2}=\frac{a}{2} \geq 2=p
$$

and (6.8) again holds. In the remaining case $p=2$ and $a=3$, we have $n=12$ and (6.8) is false. Furthermore, we have $p<n / p$ and

$$
\begin{aligned}
\frac{q^{-p}}{1-q^{-p}} & >q^{-n / p} \frac{\left(1-q^{-1}\right)\left(1-q^{-p+1}\right)}{1-q^{-p}} \\
q^{-p+1}\left(1-q^{-1}\right)^{2} & =q^{-p+1}-\left(2-q^{-1}\right) q^{-p} \geq q^{-p+1}-2 q^{-p}
\end{aligned}
$$

so that for $n \neq 12$ the following holds:

$$
\nu \geq \frac{1}{2}\left(\frac{3}{2}+\frac{1}{2 p+2}-q^{-1}-\frac{q^{-2}}{2}\left(1+\frac{1}{p+1}\right)-\frac{q^{-p+1}}{1-q^{-p}}\right) .
$$

For $n=12$, we have calculated in Example 4.47 that $\lambda_{2}$ (II.B.ii.b) $=t / \alpha_{12} \leq$ $q^{-2}=q^{-p}$, and we may use this to the same cancellation effect as (6.8), so that the last inequality also holds for $n=12$.
II.B.ii.b. $\beta$ :

$$
\begin{align*}
\nu \geq & \lambda_{0}\left(\text { II.B.ii.b) }+\lambda_{1}(\text { II.B.ii.b. } \beta)-\lambda_{2} \text { (II.B.ii.b) }+\nu_{3}(\text { II.B })\right. \\
= & \frac{1}{2}\left(1-q^{-1}-q^{-p+1}+q^{-p}\right)+\frac{1}{2} \frac{\left(1-q^{-1}\right)\left(1-q^{-p+1}\right)\left(1-q^{-n / p}\right)}{1-q^{-p}} \\
& -\frac{1}{2} q^{-n / p^{2}+\left\lceil n / p^{3}\right\rceil}+\frac{1}{2} q^{-p+1} \\
= & 1-q^{-1}-\frac{q^{-p+1}}{2} \cdot \frac{\left(1-q^{-1}\right)^{2}}{1-q^{-p}}+\frac{q^{-p}}{2}  \tag{6.9}\\
& -\frac{q^{-n / p}\left(1-q^{-1}-q^{-p+1}+q^{-p}\right)}{2\left(1-q^{-p}\right)}-\frac{1}{2} q^{-n / p^{2}+n / p^{3}} .
\end{align*}
$$

Since $n \geq p^{3}$, we have

$$
\begin{aligned}
n / p & \geq p^{2}>p, \\
q^{-p} & >q^{-n / p} \\
n(p-1) & \geq p^{3}(p-1), \\
-p+1 & \geq-\frac{n}{p^{2}}+\frac{n}{p^{3}}, \\
\nu & \geq 1-q^{-1}-\frac{q^{-p+1}}{2}-\frac{1}{2} q^{-n / p^{2}+n / p^{3}} \geq 1-q^{-1}-q^{-p+1} .
\end{aligned}
$$

Except at I.B and II.B.ii.b. $\alpha$, the lower bounds are of the satisfactory form $1-O\left(q^{-1}\right)$. The leaf I.B is discussed in Example 3.45. For small values of $q$, the entry in Table 6.1 at II.B.ii.b. $\alpha$ provides the lower bounds in Table 6.2.

| $q$ | $\# D_{n} / \alpha_{n} \geq$ |
| :--- | :--- |
| 2 | $1 / 6>0.1666$ |
| 3 | $259 / 468>0.5534$ |
| 4 | $133 / 240>0.5541$ |
| 5 | $106091 / 156200>0.6791$ |
| 7 | $56824055 / 80707116>0.7040$ |
| 8 | $2831 / 4032>0.7021$ |
| 9 | $88087 / 117936>0.7469$ |

Table 6.2: Lower bounds at the leaf II.B.ii.b. $\alpha$, where $\ell^{2}=p^{2} \| n \neq p^{2}$.
The multitude of bounds, driven by the estimates of Section 3 and 4, is quite confusing. The Main Theorem in the introduction provides simple and universally applicable estimates. Before we come to its proof, we note that for special values, in particular for small ones, of our parameters one may find better bounds in other parts of this paper.

Proof (Main Theorem). (i) follows from $2 \leq \ell \leq \sqrt{n}$. The first upper bound on $\# D_{n}$ in (ii) follows from Corollary 5.14(ii). It remains to deduce the lower bounds. Starting with the last claim, we note that (v) is Corollary 5.14(iii). In the assumption of (iv), the leaves I.B and II.B.ii.b. $\alpha$ are disallowed. We claim that Theorem 6.1 implies

$$
\begin{equation*}
\nu \geq 1-2 q^{-1} \tag{6.11}
\end{equation*}
$$

at all leaves but these two. Leaf I.A is clear. At II.A.i, we have $n=a \ell$, where $a>\ell$ and all prime factors of $a$ are larger than $\ell$. When $a \geq \ell+2$, then

$$
\begin{aligned}
& \frac{n}{\ell}-\frac{n}{\ell^{2}}=a\left(1-\frac{1}{\ell}\right) \geq(\ell+2)\left(1-\frac{1}{\ell}\right)=\ell+1-\frac{2}{\ell} \geq \ell \\
& \beta_{n}^{*} \leq q^{-n / \ell-\ell+n / \ell^{2}+3} \leq q^{3-2 \ell} \leq q^{-1} \\
& \nu \geq 1-\beta_{n}^{*} \geq 1-q^{-1}
\end{aligned}
$$

When $a=\ell+1$, then $\ell=2, a=3, n=6$, and by Theorem 5.2(iii) we have again

$$
\frac{\# D_{6}}{\alpha_{6}} \geq 1-\beta_{6}^{*}=1-q^{-1}
$$

At II.A.ii, we have $n=a \ell^{2}$ with $a \geq \ell$ and

$$
\begin{aligned}
\frac{n}{\ell}-\frac{n}{\ell^{2}} & =a(\ell-1) \geq \ell(\ell-1) \geq \ell \\
q^{-n / \ell+\ell+n / \ell^{2}-1} & \leq q^{-1} \\
\nu & \geq 1-q^{-1} / 2
\end{aligned}
$$

At II.B.i.a, we consider the inequality

$$
\begin{equation*}
-\frac{n}{\ell}-\ell+s+3 \leq-1 \tag{6.12}
\end{equation*}
$$

with $s=\left\lfloor n / \ell^{2}\right\rfloor \leq n / \ell^{2}$. It holds for $\ell \geq 3$. When $\ell=2$, it holds for $n \geq 8$, and one checks it for $n=6$. Now $n=4$ is case II.B and excepted here. Thus (6.12) holds in all cases at II.B.i.a, and (6.5) implies that $\nu \geq 1-3 q^{-1} / 2>1-2 q^{-1}$.
(6.11) is clear for II.B.i.b and II.B.ii.b. $\beta$. At II.B.ii.a, we have $p>\ell \geq 2$, and (6.11) follows from Table 6.1. This concludes the proof of (iv).

In (iii), the second inequality follows from $\left(3-2 q^{-1}\right) \cdot\left(1-q^{-1}\right) / 4>1 / 2$ when $q \geq 5$. For the first inequality, we have $1-2 q^{-1} \geq\left(3-2 q^{-1}\right) / 4$ when $q>5$. Thus it remains to prove (iii) at II.B.ii.b. $\alpha$. It is convenient to show (ii) and (iii) together at this leaf.

We have for $p \geq 3$ and $q \geq 5$ that

$$
\begin{aligned}
1-q^{-3} & \geq q^{-2}(3 q+4)>q^{-2}(3 p+4)-q^{-5}(p+2) \\
& =q^{-2}(p+2)\left(1-q^{-3}\right)+q^{-2}(2 p+2) \\
\frac{1}{2 p+2} & >\frac{q^{-2}(p+2)}{2 p+2}+\frac{q^{-2}}{1-q^{-3}} \geq \frac{q^{-2}}{2}\left(1+\frac{1}{p+1}\right)+\frac{q^{-p+1}}{1-q^{-p}}
\end{aligned}
$$

and from Table 6.1

$$
\begin{align*}
\nu & \geq \frac{1}{2}\left(\frac{3}{2}+\frac{1}{2 p+2}-q^{-1}-\frac{q^{-2}}{2}\left(1+\frac{1}{p+1}\right)-\frac{q^{-p+1}}{1-q^{-p}}\right)  \tag{6.13}\\
& >\frac{3}{4}-\frac{q^{-1}}{2}=\frac{3-2 q^{-1}}{4} .
\end{align*}
$$

For the remaining cases $q=3$ or $p=2$, we use (6.7). At the current leaf, we can write $n=a p^{2}>p^{2}$ with all prime divisors of $a$ greater than $p$, and split the lower bound into two summands:

$$
\begin{aligned}
\nu_{q} & =\frac{1}{2}\left(\frac{3}{2}+\frac{1}{2 p+2}-q^{-1}-\frac{q^{-2}}{2}\left(1+\frac{1}{p+1}\right)+\frac{q^{-p}\left(2-q^{-1}-q^{-p}\right)}{1-q^{-p}}\right), \\
\epsilon_{q, n} & =\frac{1}{2}\left(q^{-a+\lceil a / p\rceil}+q^{-a p} \frac{\left(1-q^{-1}\right)\left(1-q^{-p+1}\right)}{1-q^{-p}}\right)
\end{aligned}
$$

so that $\nu \geq \nu_{q}-\epsilon_{q, n}$, and $\epsilon_{q, n}$ is monotonically decreasing in $a$.
For $q=3$, we have $a \geq 5$,

$$
\begin{aligned}
\nu_{3} & =\frac{203}{27 \cdot 13}>0.5783 \\
\epsilon_{3, n} & \leq \frac{1}{2}\left(3^{-a+\lceil a / 3\rceil}+\frac{8}{13} \cdot 3^{-3 a}\right) \leq \epsilon_{3,45}=\frac{1}{2}\left(3^{-5+2}+\frac{8}{13} \cdot 3^{-15}\right) \\
& =\frac{1}{54}+\frac{4}{13} \cdot 3^{-15}<0.0186 \\
\nu & \geq \nu_{3}-\epsilon_{3, n}>0.5598>1 / 2 .
\end{aligned}
$$

For $p=2$, we find

$$
\begin{aligned}
\nu_{q} & =\frac{5}{6}-q^{-1}+\frac{q^{-2}}{6} \\
\epsilon_{q, n} & =\frac{1}{2}\left(q^{-(a-1) / 2}+q^{-2 a} \cdot \frac{1-q^{-1}}{1+q^{-1}}\right) .
\end{aligned}
$$

When $q \geq 8$ and $n \geq 28$, so that $a \geq 7$, we have

$$
\begin{aligned}
\frac{q^{-2}}{6} & \geq \frac{1}{2}\left(q^{-3}+q^{-14} \cdot \frac{1-q^{-1}}{1+q^{-1}}\right)=\epsilon_{q, 28} \geq \epsilon_{q, n} \\
\nu & \geq \nu_{q}-\epsilon_{q, n} \geq \frac{5}{6}-q^{-1} \geq \frac{3-2 q^{-1}}{4}
\end{aligned}
$$

For the remaining values $q \in\{2,4\}$ or $n \in\{12,20\}$, we note the values

$$
\begin{aligned}
\nu_{2} & =\frac{3}{8}, \\
\nu_{4} & =\frac{19}{32}, \\
\epsilon_{q, 12} & =\frac{1}{2}\left(q^{-1}+q^{-6} \cdot \frac{1-q^{-1}}{1+q^{-1}}\right), \\
\epsilon_{q, 20} & =\frac{1}{2}\left(q^{-2}+q^{-10} \cdot \frac{1-q^{-1}}{1+q^{-1}}\right) .
\end{aligned}
$$

We find that $\nu \geq\left(3-2 q^{-1}\right) / 4$ for $q \geq 8$ and $n=20$, and for $q \geq 16$ and $n=12$. Table 6.3 shows that this also holds for $(q, n)=(8,12)$. When $q=4$, we have $\nu \geq 1 / 2$ for $n \geq 20$ by the above, and according to Table 6.3 also for $n=12$.

When $q=2$, the values above only show that $\nu \geq 1 / 4$ for $n \geq 28$. However, a different and simple approach gives a better bound for $n=4 a$ with an odd

| $q, n$ | $\# D_{n}$ | $\alpha_{n}$ | $\# D_{n} / \alpha_{n} \geq$ |
| ---: | ---: | ---: | ---: |
| 2,4 | 6 | 8 | 0.7500 |
| 2,8 | 36 | 64 | 0.5625 |
| 2,12 | 236 | 256 | 0.9218 |
| 2,16 | 762 | 1024 | 0.7441 |
| 2,20 | 3264 | 4096 | 0.7968 |
| 2,24 | 14264 | 16384 | 0.8706 |
| 2,28 | 49920 | 65536 | 0.7617 |
| 2,36 | 821600 | 1048576 | 0.7835 |
| 4,4 | 132 | 192 | 0.6875 |
| 4,12 | 100848 | 98304 | 1.0258 |
| 8,4 | 2408 | 3584 | 0.6718 |
| 8,12 | 30382016 | 29360128 | 1.0348 |
| 16,4 | 41040 | 61440 | 0.6679 |
| 32,4 | 677536 | 1015808 | 0.6669 |
| 64,4 | 11011392 | 16515072 | 0.6667 |
| 128,4 | 177564288 | 266338304 | 0.6666 |
| 256,4 | 2852148480 | 4278190080 | 0.6666 |
| 3,9 | 414 | 486 | 0.8518 |
| 9,9 | 450792 | 472392 | 0.9542 |
| 5,5 | 7798100 | 7812500 | 0.9981 |

Table 6.3: Decomposable polynomials of degree $n$ over $\mathbb{F}_{q}$.
$a \geq 3$ over $\mathbb{F}_{2}$. We exploit the special fact that $x^{2}+x \in \mathbb{F}_{2}[x]$ is the only quadratic original polynomial that is not a square.

Any $g \in \mathbb{F}_{2}[x]$ is uniquely determined by $f=g \circ\left(x^{2}+x\right)$, due to the uniqueness of the Taylor expansion. The number of original $g$ of degree $2 a$ and that are not a square is $2^{2 a-1}-2^{a-1}$, and by composing with a linear polynomial on the left, we have $\# D_{n, n / 2}^{+}=2^{2 a}-2^{a}=2^{n / 2}-2^{n / 4}$. Similary, $\left(x^{2}+x\right) \circ h=\left(x^{2}+x\right) \circ h^{*}$ with $h \neq h^{*}$ implies that $-1=h^{*}-h$, so that one of the two polynomials is not original. Thus $\gamma_{n, 2}$ is also injective on the original polynomials, and $\# D_{n, 2}^{+}=2^{n / 2}-2^{n / 4}$. Furthermore, Corollary 4.45(ii) says that

$$
t=\#\left(D_{n, 2}^{+} \cap D_{n, n / 2}^{+}\right) \leq 2^{n / 4+\lceil n / 8\rceil+1}=2^{3 n / 8+3 / 2}
$$

The number of Frobenius compositions (that is, squares) of degree $n$ equals
$\# D_{n}^{\varphi}=2^{2 a}$, and $\alpha_{n}=2^{n / 2+2}$. It follows that

$$
\begin{align*}
\# D_{n} & \geq \# D_{n, 2}^{+}+\# D_{n, n / 2}^{+}-t+\# D_{n}^{\varphi} \\
& \geq 2 \cdot 2^{n / 2}\left(1-2^{-n / 4}\right)-2^{3 n / 8+3 / 2}+2^{n / 2} \\
& =\left(\frac{3}{4}-2^{-n / 8-1 / 2}-2^{-n / 4-1}\right) \alpha_{n}  \tag{6.14}\\
\nu & \geq \frac{3}{4}-2^{-5 / 2-1 / 2}-2^{-5-1}=\frac{39}{64}>0.6093>1 / 2
\end{align*}
$$

for $n \geq 20$. Using Table 6.3 for $n=12$, we find $\nu>1 / 2$ also for $q=2$, and hence for all values at leaf II.B.ii.b. $\alpha$. Now it only remains to prove $\nu \geq 1 / 2$ in (ii). The leaf II.B.ii.b. $\alpha$ has just been dealt with. Since $1-q^{-1} \geq 1 / 2$ for all $q$, the claim follows from the previous bounds at the leaves I.A, II.A.i, II.A.ii, and II.B.i.b. At II.B.i.a, we have shown $\nu \geq 1-3 q^{-1} / 2 \geq 1 / 2$ for $q \geq 3$; since $p \neq \ell$ and hence $p \geq 3$ at this leaf, the claim follows. Similarly, we have at II.B.ii.a that $q \geq p \geq 3$ and $\nu \geq 1-\frac{1}{2}\left(q^{-1}+q^{-\ell+1}+q^{-p+1}-q^{-p}\right) \geq 1-q^{-1}-q^{-2} \geq 1 / 2$. Now remain the two leaves I.B and II.b.ii.b. $\beta$.

At leaf I.B, we have $n=p^{2}$ and

$$
\begin{gathered}
q^{2} \geq q+2 \geq p+2 \\
\frac{1}{p+1}+2 q^{-p}>q^{-2}\left(1+\frac{1}{p+1}\right)
\end{gathered}
$$

From Example 3.45 we find

$$
\nu \geq \frac{1}{2}\left(1+\frac{1}{p+1}\right)\left(1-q^{-2}\right)+q^{-p} \geq \frac{1}{2} .
$$

Table 6.3 gives the exact values of $\nu$ for $p=2$ and $q \leq 256$.
At the final leaf II.B.ii.b. $\beta$, we have $\ell=p$ and $p^{3} \mid n$. The lower bound in Table 6.1 implies $\nu \geq 1 / 2$ for $q \geq 4$. When $q=3$, (6.10) yields

$$
\nu \geq 1-\frac{1}{3}-\frac{1}{9}=\frac{5}{9}>\frac{1}{2} .
$$

For $q=2$, we have from (6.9)

$$
\nu \geq \frac{1}{2}+\frac{1}{24}-\frac{2^{-n / 2-1}}{3}-2^{-n / 8-1} .
$$

When $n \geq 32$, this shows $\nu \geq 1 / 2$. For the smaller values 8,16 , and 24 of $n$, the data in Table 6.3 are sufficient.

Two features are worth noting. Firstly, our lower bounds are rather pessimistic when $q=2$, yielding for $n=12$ that $\nu \geq 47 / 384>0.1223$ by $(6.7), \nu \geq 3 / 16=$ 0.1875 from the special argument, compared to $\nu=59 / 64>0.9218$ from our experiments. Secondly, our lower bounds are strictly increasing in $n$, while the experiments show a decrease in $\nu$ from $n=12$ to $n=20$. Both features show that more work is needed to understand the case $p=\ell$ and $p^{2} \| n$, where the latter means that $p^{2} \mid n$ and $p^{3} \nmid n$.

Much effort has been spent here in arriving at precise bounds, without asymptotics or unspecified constants. We now derive some conclusions about the asymptotic behavior. There are two parameters: the field size $q$ and the degree $n$. When $n$ is prime, then $\# D_{n}=\alpha_{n}=0$, and prime values of $n$ are excepted in the following. We consider the asymptotics in one parameter, where the other one is fixed, and also the special situations where $\operatorname{gcd}(q, n)=1$. Furthermore, we denote as " $q, n \longrightarrow \infty$ " the set of all infinite sequences of pairwise distinct $(q, n)$. The cases $p^{2} \| n$ are the only ones where Table 6.1 does not show that $\nu \longrightarrow 1$.

THEOREM 6.15. Let $\nu_{q, n}=\# D_{n} / \alpha_{n}$ over $\mathbb{F}_{q}$. We only consider composite $n$.
(i) For any $q$, we have

$$
\begin{gathered}
\limsup _{n \rightarrow \infty} \nu_{q, n}=1, \\
\lim _{\substack{n \rightarrow \infty \\
\operatorname{gcd}(q, n)=1}} \nu_{q, n}=1, \\
\frac{1}{2} \leq \nu_{q, n} \text { for any } n, \\
\frac{3-2 q^{-1}}{4} \leq \nu_{q, n} \text { for any } n, \text { if } q \geq 5 .
\end{gathered}
$$

(ii) Let $n$ be a composite integer and $\ell$ its smallest prime divisor. Then

$$
\begin{gathered}
\limsup _{q \rightarrow \infty} \nu_{q, n}=1, \\
\liminf _{q \rightarrow \infty} \nu_{q, n} \begin{cases}\geq \frac{1}{2}\left(1+\frac{1}{\ell+1}\right) \geq \frac{2}{3} & \text { if } n=\ell^{2}, \\
\geq \frac{1}{4}\left(3+\frac{1}{\ell+1}\right) \geq \frac{5}{6} & \text { if } \ell^{2} \| n \text { and } n \neq \ell^{2}, \\
=1 & \text { otherwise, } \\
\lim _{\substack{q \rightarrow \infty \\
\operatorname{gcd}(q, n)=1}} \nu_{q, n}=1 .\end{cases}
\end{gathered}
$$

(iii) For any sequence $q, n \rightarrow \infty$, we have

$$
\begin{gathered}
\frac{1}{2} \leq \liminf _{q, n \rightarrow \infty} \nu_{q, n} \leq \limsup _{q, n \rightarrow \infty} \nu_{q, n}=1 \\
\lim _{\substack{q, n \rightarrow \infty \\
\operatorname{gcd}(q, n)=1}} \nu_{q, n}=1
\end{gathered}
$$

Proof. (i) We start with an upper bound. The conclusions of the Main Theorem are too weak for our current purpose, and we have to resort to Theorem 5.2. For the special $n$ which are a square or a cube of primes, or a product of two distinct primes, Theorem 5.2(i) says that $\nu_{q, n} \leq 1$. For the other values, we set $d=n / \ell \ell_{2}$, and the upper bound on the limsup follows if we show that $c=(d-1)\left(\ell_{2}-\ell\right)$ is unbounded as $n$ grows, since then $\beta_{n}=q^{-c} /\left(1-q^{-1}\right)$ tends to zero, and $\nu_{q, n} \leq 1+\beta_{n}$. Since $\ell_{2}-\ell \geq 1$, it is sufficient to show the unboundedness of $d$. When $n=\ell^{e}$ is a power of a prime, we may assume by the above that $e \geq 4$. Then $\ell_{2}=\ell^{2}, \ell \leq n^{1 / 4}$ and $d=\ell^{e-3} \geq \ell^{e / 4}=n^{1 / 4}$ is unbounded.

If $n=\ell^{e} \ell_{+}^{e_{+}}$has exactly two prime factors $\ell<\ell_{+}$, we may assume that $e+e_{+} \geq 3$. If $e=1$, then $\ell_{2}=\ell_{+}, e_{+} \geq 2$, and $d=\ell_{+}^{e_{+}-1} \geq \ell_{+}^{\left(e_{+}+1\right) / 3}>n^{1 / 3}$. We now assume that $e \geq 2$. Then

$$
\begin{aligned}
& \ell_{2}= \begin{cases}\ell^{2} & \text { if } \ell^{2}<\ell_{+}, \\
\ell_{+} & \text {otherwise },\end{cases} \\
& d= \begin{cases}\frac{n}{\ell^{3}} & \text { if } \ell^{2}<\ell_{+}, \\
\frac{n}{\ell_{+}} & \text {otherwise. } .\end{cases}
\end{aligned}
$$

We first treat the case where $\ell^{2}<\ell_{+}$. If $e=2$, then

$$
d=\ell_{+}^{e_{+}} / \ell>\ell_{+}^{e_{+}-1 / 2} \geq \ell_{+}^{\left(1+e_{+}\right) / 4}>n^{1 / 4} .
$$

If $e=3$, then

$$
d=\ell_{+}^{e_{+}}>\ell_{+}^{\left(e_{+}+3 / 2\right) / 3}>\left(\ell^{2}\right)^{1 / 2} \ell_{+}^{e_{+} / 3}=n^{1 / 3} .
$$

If $e \geq 4$, then $d=\ell^{e-3} \ell_{+}^{e_{+}} \geq \ell^{e / 4} \ell_{+}^{e_{+}}>n^{1 / 4}$. Next we deal with $\ell_{+}>\ell^{2}$. If $e=1$, we have $e_{+} \geq 2$, and then

$$
d=\ell_{+}^{e_{+}-1} \geq \ell_{+}^{\left(e_{+}+1\right) / 3}>n^{1 / 3} .
$$

If $e_{+}=1$ we have $e \geq 2$, and then

$$
d=\ell^{e-1} \geq \ell^{(e+2) / 4}>\ell^{e / 4} \ell_{+}^{1 / 4}=n^{1 / 4}
$$

In the remaining case, where $e, e_{+} \geq 2$, we have

$$
d=\ell^{e-1} \ell_{+}^{e_{+}-1} \geq \ell^{e / 2} \ell_{+}^{e / 2}=n^{1 / 2}
$$

In the last case, $n=\ell^{e} \ell_{+}^{e_{+}} \ell_{++}^{e_{++}} \cdots$ has at least three distinct prime factors $\ell<\ell_{+}<\ell_{++}<\cdots$, and

$$
d= \begin{cases}\frac{n}{\ell^{3}} & \text { if } e \geq 2 \text { and } \ell^{2}<\ell_{+}, \\ \frac{n}{\ell \ell_{+}} & \text {otherwise } .\end{cases}
$$

If $e=e_{+}=1$, then $\ell \ell_{+}<n^{2 / 3}$ and $d \geq n^{1 / 3}$. Otherwise, we apply the previous argument to $n^{*}=\ell^{e} \ell_{e}^{e_{+}}=n / m$ and $d^{*}=d / m$, where $m=\ell_{++}^{e_{++}} \cdots=$ $n \ell^{-e} \ell_{+}^{-e_{+}}$. Then $d^{*}$ equals the value $d$ defined above for $n^{*}$, and

$$
d=d^{*} m \geq\left(n^{*}\right)^{1 / 4} m>n^{1 / 4} .
$$

In all cases, $d$ is unbounded if $n$ is. Thus $\lim \sup _{n \rightarrow \infty} \nu_{q, n} \leq 1$, and Theorem $5.2(\mathrm{v})$ for $n=\ell^{2}$ implies that $\limsup _{n \rightarrow \infty} \geq 1$.

If we only consider $n$ with $\operatorname{gcd}(q, n)=1$, then Theorem $5.2(\mathrm{vi})$ says that

$$
\nu_{q, n} \geq 1-2 q^{-n / \ell+\ell+n / \ell^{2}-1} \geq 1-q^{-n / \ell+\ell+n / \ell^{2}}
$$

When $n$ is the product of two prime numbers, then $\nu_{q, n}$ tends to 1 for these special $n$. We may now assume that $n$ has at least three prime factors. Then $n \geq \ell^{3}$, and

$$
\begin{aligned}
-\frac{n}{\ell}+\ell+\frac{n}{\ell^{2}} & =-\frac{n}{\ell}\left(1-\frac{1}{\ell}\right)+\ell \leq-\frac{n}{2 \ell}+\ell \leq-\frac{n}{2 n^{1 / 3}}+n^{1 / 3} \\
& =-\frac{n^{2 / 3}}{2}+n^{1 / 3} \leq-n^{1 / 2}
\end{aligned}
$$

for $n \geq 512$, say. The second claim in (i) follows. The other two inequalities are in the Main Theorem.
(ii) The first claim follows from Corollary 5.14(ii), since $n \geq \ell^{2}$ and hence $\nu_{q, n} \leq 1+q^{-1 / 3}$. For the other claims, we consider two subsequences of $q: q=\ell^{e}$ with $e \rightarrow \infty$, and $q$ with $\operatorname{gcd}(q, \ell)=1$; we denote the latter as $q^{\prime}$. For $n=\ell^{2}$, the lower bound follows from the entry at I.B in Table 6.1, and for $\ell^{2} \| n \neq \ell^{2}$
from the entry at II.B.ii.b. $\alpha$. In all other cases, the Main Theorem guarantees that $\nu_{\ell^{e}, n}$ and $\nu_{q^{\prime}, n}$ tend to 1 ; see also (6.11).
(iii) We take some infinite sequence of $(q, n)$ for which $\nu_{q, n}$ tends to $s=$ limsup. If all $q$ occurring in the sequence are bounded, then (i) implies that $s \leq 1$. Otherwise, $\nu_{q, n} \leq 1+q^{-1 / 3}$ is sufficient. The same case distinction yields the lower bound on the limit, using the Main Theorem (vi). The lower bound on liminf follows from (i).

Example 6.17. Let $p^{2} \| n$ and $n \neq p^{2}$. We study $D_{n}$ over $\mathbb{F}_{q}$, using the notation of (the proof of) Theorem 5.2. We have $\ell=p<\ell_{2} \leq p^{2}$,

$$
c=\frac{\left(n-\ell \ell_{2}\right)\left(\ell_{2}-\ell\right)}{\ell \ell_{2}} \geq \frac{n-\ell(\ell+1)}{\ell(\ell+1)} \geq \frac{n}{2 \ell^{2}} .
$$

With

$$
E_{2}=\left\{e \in \mathbb{N}: e \mid n, \ell_{2} \leq e \leq n / \ell_{2}\right\}
$$

we have

$$
\begin{aligned}
\sum_{e \in E_{2}} \# D_{n, e} & \leq \sum_{e \in E_{2}} q^{u(e)}\left(1-q^{-1}\right) \leq q^{u(\ell)}\left(1-q^{-1}\right) \frac{2 q^{-c}}{1-q^{-1}} \\
& =\frac{q^{-c}}{1-q^{-1}} \cdot \alpha_{n} \leq 2 q^{-n / 2 \ell^{2}} \cdot \alpha_{n}
\end{aligned}
$$

We let

$$
\begin{aligned}
\lambda_{q, n} & =\frac{\# D_{n, p}^{+}+\# D_{n, n / p}^{+}}{\alpha_{n}}, \\
t & =\#\left(D_{n, p}^{+} \cap D_{n, n / p}^{+}\right) .
\end{aligned}
$$

Then

$$
\begin{aligned}
\nu_{q, n} & =\frac{\# D_{n}}{\alpha_{n}} \leq \lambda_{q, n}-\frac{t}{\alpha_{n}}+\frac{\# D_{n}^{\varphi}}{\alpha_{n}}+\frac{\sum_{e \in E_{2}} \# D_{n, e}}{\alpha_{n}} \\
& \leq \lambda_{q, n}+\frac{q^{n / p+1}\left(1-q^{-1}\right)}{\alpha_{n}}+2 q^{-n / 2 \ell^{2}}=\lambda_{q, n}+\frac{q^{-p+1}}{2}+2 q^{-n / 2 p^{2}} .
\end{aligned}
$$

On the other hand, Corollary 4.45 (ii) says that

$$
\begin{aligned}
& t \leqq n / p+p-n / p^{2}+\left\lfloor n / p^{3}\right\rfloor+1 \\
&\left(1-q^{-1}\right) \\
& \nu_{q, n} \geq \lambda_{q, n}-\frac{t}{\alpha_{n}}+\frac{\# D_{n}^{\varphi}}{\alpha_{n}} \geq \lambda_{q, n}-\frac{1}{2} q^{-n / p^{2}+\left\lfloor n / p^{3}\right\rfloor+1}+\frac{q^{-p+1}}{2} .
\end{aligned}
$$

For $p \geq 3$ we have

$$
\begin{gathered}
-\frac{n}{p^{2}}+\frac{n}{p^{3}}+1 \leq-\frac{n}{2 p^{2}}, \\
\left|\nu_{q, n}-\left(\lambda_{q, n}+q^{-p+1}\right)\right| \leq 2 q^{-n / 2 p^{2}} .
\end{gathered}
$$

We have presented some bounds on $\lambda_{q, n}$, but they are not sufficient to determine its value in general, not even asymptotically. However, for $q=2$ we have from (6.14)

$$
\begin{array}{r}
\lambda_{q, n}=\frac{2^{n / 2+1}\left(1-2^{-n / 4}\right)}{2 \cdot 2^{n / 2+2}}=\frac{1-2^{-n / 4}}{2} \\
\frac{3}{4}-2^{-n / 8-1 / 2}-2^{n / 4-1} \leq \nu_{2, n} \leq \frac{3}{4}+2^{-n / 8+1}-2^{-n / 4-1} \tag{6.18}
\end{array}
$$

We have seen that $\nu_{q, n}$ tends to 1 unless $p^{2} \| n$. Example 6.17 suggests to use a correction factor $\gamma$ so that $\nu_{q, n} / \gamma$ tends to 1 also in those cases.

Conjecture 6.19. For any prime $p$ and power $q$ of $p$ there exist $\gamma_{p}, \delta_{q} \in \mathbb{R}$ so that

$$
\begin{aligned}
\lim _{e \longrightarrow \infty} \nu_{p^{e}, p^{2}} & =\gamma_{p}, \\
\lim _{n \longrightarrow \infty p^{2} \| n} \nu_{q, n} & =\delta_{q} .
\end{aligned}
$$

If true, this would imply that $\# D_{p^{2}} \sim \gamma_{p} \alpha_{p^{2}}$ over extensions $\mathbb{F}_{q}$ of $\mathbb{F}_{p}$, and $\# D_{n} \sim \delta_{q} \alpha_{n}$ for growing $n$ with $p^{2} \| n$. Example 3.45 shows that the first part is true for $p=2$ and $\gamma_{2}=2 / 3$, and (6.18) that the second part holds for $q=2$ and $\delta_{2}=3 / 4$.

Bodin et al. (2009) state without proof that $\# D_{n} \approx \frac{3}{4} \alpha_{n}$ over $\mathbb{F}_{2}$ for even $n \geq 6$. Assuming a standard meaning of the $\approx$ symbol, this is false unless $4 \| n$, in which case it is proven by (6.18).

Example 6.20. Theorem 6.1(i) exhibits several situations where $\# D_{n} \leq \alpha_{n}$. One might wonder whether this always happens. We show that this is not the case. Table 6.3 gives an example. More generally, we take three primes $2<\ell_{1}<\ell_{2}<\ell_{3}, n=\ell_{1} \ell_{2} \ell_{3}$, and an odd $q$ with $\operatorname{gcd}(n, q)=1$. For $i \leq 3$, we
set

$$
\begin{aligned}
B_{i} & =D_{n, \ell_{i}} \cup D_{n, n / \ell_{i}} \\
S_{i} & =\left\lfloor\frac{n}{\ell_{i}^{2}}\right\rfloor \\
t_{i} & =\frac{1}{2}\left(2 q^{s_{i}+3}+q^{4}-q^{3}\right)\left(1-q^{-1}\right) .
\end{aligned}
$$

Then

$$
\begin{aligned}
D_{n} & =B_{1} \cup B_{2} \cup B_{3} \\
\# B_{i} & =2 q^{n / \ell_{i}+\ell_{i}}\left(1-q^{-1}\right)-t_{i} .
\end{aligned}
$$

For a permutation $\pi \in S_{3}$, we set

$$
\begin{aligned}
C_{\pi} & =\gamma_{\pi}\left(P_{\ell_{\pi 1}}^{=} \times P_{\ell_{\pi 2}}^{0} \times P_{\ell_{\pi 3}}^{0}\right) \\
C & =\bigcup_{\pi \in S_{3}} C_{\pi}
\end{aligned}
$$

where $\gamma_{\pi}$ is the composition map for three components. Then for any $\pi \in S_{3}$

$$
\# C_{\pi}=q^{\ell_{1}+\ell_{2}+\ell_{3}-1}\left(1-q^{-1}\right)
$$

Now let $i \neq j$ and $f=g \circ h=g^{*} \circ h^{*} \in B_{i} \cap B_{j}$, with $\{\operatorname{deg} g, \operatorname{deg} h\}=\left\{\ell_{i}, n / \ell_{i}\right\}$ and $\left\{\operatorname{deg} g^{*}, \operatorname{deg} h^{*}\right\}=\left\{\ell_{j}, n / \ell_{j}\right\}$. To simplify notation, suppose that $i=1$ and $j=2$. We refine both decompositions into complete ones. Then for $g \circ h$, the set of degrees is either $\left\{\ell_{1}, \ell_{2} \ell_{3}\right\}$ or $\left\{\ell_{1}, \ell_{2}, \ell_{3}\right\}$, and for $g^{*} \circ h^{*}$ it is either $\left\{\ell_{2}, \ell_{1} \ell_{3}\right\}$ or $\left\{\ell_{1}, \ell_{2}, \ell_{3}\right\}$. This set of degrees is unique, so that it equals $\left\{\ell_{1}, \ell_{2}, \ell_{3}\right\}$. It follows that $f \in C$ and $B_{i} \cap B_{j} \subseteq C$. Thus

$$
\begin{align*}
\# D_{n} & \geq \sum_{1 \leq i \leq 3} \# B_{i}-\# C \\
& \geq\left(1-q^{-1}\right) \sum_{1 \leq i \leq 3}\left(2 q^{n / \ell_{i}+\ell_{i}}-\frac{1}{2}\left(2 q^{s_{i}+3}+q^{4}\right)\right)-6 q^{\ell_{1}+\ell_{2}+\ell_{3}-1} \\
1) & =\left(1-q^{-1}\right)\left(2 \sum_{1 \leq i \leq 3} q^{n / \ell_{i}+\ell_{i}}-\sum_{1 \leq i \leq 3} q^{s_{i}+3}-\frac{3}{2} q^{4}-6 q^{\ell_{1}+\ell_{2}+\ell_{3}-1}\right) . \tag{6.21}
\end{align*}
$$

Now suppose further that

$$
\ell_{3} \leq 2+\left(\ell_{1}-1\right)\left(\ell_{2}-1\right), \quad 5 \leq \ell_{2} \leq \ell_{1}^{2}, \quad q \geq 7
$$

Then

$$
\begin{aligned}
& \ell_{1}+\ell_{2}+\ell_{3}-1 \leq \ell_{1}+\ell_{2}+1+\left(\ell_{1}-1\right)\left(\ell_{2}-1\right) \\
&=\ell_{1} \ell_{2}+2, \\
& 6 q^{\ell_{1}+\ell_{2}+\ell_{3}-1} \leq 6 q^{\ell_{1} \ell_{2}+2} \leq q^{\ell_{1} \ell_{2}+3} \\
& 4 \ell_{3} \leq 10\left(\ell_{3}-1\right) \leq \ell_{1} \ell_{2}\left(\ell_{3}-1\right), \\
& \frac{\ell_{1} \ell_{2}}{\ell_{3}}+4 \leq \ell_{1} \ell_{2}<\ell_{1} \ell_{2}+\ell_{3}, \\
& q^{\ell_{1} \ell_{2} / \ell_{3}+3}+\frac{3}{2} q^{4}+6 q^{\ell_{1}+\ell_{2}+\ell_{3}-1}<q^{\ell_{1} \ell_{2}+\ell_{3}}\left(q^{-1}+\frac{3}{2} q^{4-\ell_{3}}+q^{3-\ell_{3}}\right) \\
&<2 q^{\ell_{1} \ell_{2}+\ell_{3}}
\end{aligned},
$$

Finally, (6.21) implies that

$$
\begin{aligned}
\frac{\# D_{n}}{1-q^{-1}} & \geq \frac{\alpha_{n}}{1-q^{-1}}+2 q^{\ell_{1} \ell_{3}+\ell_{2}}+2 q^{\ell_{1} \ell_{2}+\ell_{3}}-\sum_{1 \leq i \leq 3} q^{\left\lfloor n / \ell_{i}^{2}\right\rfloor+3}-\frac{3}{2} q^{4}-6 q^{\ell_{1}+\ell_{2}+\ell_{3}-1} \\
& >\frac{\alpha_{n}}{1-q^{-1}} .
\end{aligned}
$$

As a small example, we take $\ell_{1}=3, \ell_{2}=5, \ell_{3}=7, q=11$, so that $n=105$ and $\alpha_{105}=2 q^{38}\left(1-q^{-1}\right)$. The lower bound in (6.21) evaluates to

$$
\begin{aligned}
\# D_{105} & \geq \alpha_{105}+\left(1-q^{-1}\right)\left(2\left(q^{26}+q^{22}\right)-\left(q^{14}+q^{7}+q^{5}+\frac{3}{2} q^{4}+6 q^{15}\right)\right) \\
& >\alpha_{105}+2 q^{26}\left(1-q^{-1}\right)
\end{aligned}
$$

The general bounds of Theorem 5.2(i) and Corollary 4.30(i) specialize to

$$
\# D_{105} \leq \alpha_{105}\left(1+\frac{q^{-12}}{1-q^{-1}}\right)=\alpha_{105}+2 q^{26}
$$

The closeness of these two estimates indicates a certain precision in our bounds.

REmARK 6.22. We claim that if $p \nmid n$, then

$$
\# D_{n} \geq \alpha_{n}\left(1-q^{-1}\right)
$$

By Corollary 5.14(iii), this is satisfied if $n \geq 3 \ell^{2}$. So we now assume that $n<3 \ell^{2}$. Then $n / \ell<3 \ell$, and all prime factors of $n / \ell$ are at least $\ell$. It follows that either $n=8$ or $n / \ell=\ell_{2}$ is prime. If $\ell_{2}=\ell$, then $\# D_{n}=\alpha_{n}$, by Theorem 5.2(v). Otherwise we have $s=\left\lfloor n / \ell^{2}\right\rfloor=\left\lfloor\ell_{2} / \ell\right\rfloor \leq\lfloor(3 \ell-1) / \ell\rfloor \leq 2$ and from Theorem 5.2(iii) that

$$
\# D_{n} \geq \alpha_{n}\left(1-\beta_{n}^{*}\right) \geq \alpha_{n}\left(1-q^{-\ell-\ell_{2}+5}\right)
$$

It is now sufficient to show

$$
\ell+\ell_{2} \geq 6 .
$$

This holds unless $n \in\{4,6,9\}$, so that only $n=6$ needs to be further considered. We have $\beta_{6}^{*}=q^{-2-3}\left(q^{1+3}+q^{4}-q^{3}\right) / 2 \leq q^{-1}$, and the claim follows from Theorem 5.2(iii).

Open Question 6.23. ○ Some polynomials have more than a polynomial number of decompositions. Can we find them in time polynomial in the output size? Or even a "description" of them in time polynomial in the input size? If not: prove (by a reduction) that this is hard?

- In the case where $p=\ell$ and $p^{2} \| n$, can one tighten the gap between upper and lower bounds in the Main Theorem (ii), maybe to within a factor $1+O\left(q^{-1}\right)$ ?
- Can one simplify the arguments and reduce the number of cases, yet obtain results of a quality as in the Main Theorem? The bounds in Theorem 3.31 are based on "low level" coefficient comparisons. Can these results be proved (or improved) by "higher level" methods?


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