Sperner's problem for G-independent families

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Abstract

Given a graph G, let Q(G) denote the collection of all independent (edge-free) sets of vertices in G. We consider the problem of determining the size of a largest antichain in Q(G). When G is the edge-less graph, this problem is resolved by Sperner's Theorem. In this paper, we focus on the case where G is the path of length n-1, proving the size of a maximal antichain is of the same order as the size of a largest layer of Q(G).

1 Introduction

1.1 The G-independent hypercube: definition and motivation

Let $n \in \mathbb{N}$ and let G = (V, E) be a graph on $V(G) = [n] = \{1, 2 \dots n\}$.

Definition 1. A subset $A \subseteq [n]$ is G-independent if A is an edge-free set of vertices in G. The G-independent hypercube Q(G) is the collection of all G-independent subsets of [n].

G-independent hypercubes are our main object of study in this paper. By definition, the G-independent hypercube is a subset of the n-dimensional hypercube Q_n . Indeed, if G is the graph with no edges then Q(G) is exactly Q_n , the collection of all subsets of [n].

We will be particularly interested in Q(G) when G is the path of length n-1, P_n , or the cycle of length n, C_n . These can be thought of as the collection of zero-one strings of length n with no consecutive ones (with winding round in the case of C_n). These are natural combinatorial spaces, which

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have already appeared in a variety of contexts. Considered as graphs, the G-independent hypercubes $Q(P_n)$ and $Q(C_n)$ have been studied as an efficient network topology in parallel computing [8, 9, 16]. In this setting, they are known as the $Fibonacci\ cube$ and the $Lucas\ cube$ respectively. Cohen, Fachini and Körner [2] gave bounds for the size of large antichains in $Q(P_n)$ in connection with skewincidence, a new class of problems lying halfway between intersection problems and capacity problems for graphs. Talbot [18] proved a direct analogue of the Erdős–Ko–Rado theorem [5] for the Lucas cube $Q(C_n)$. To state his result, we need to make a standard definition.

Definition 2. Let r be an integer with $0 \le r \le n$. The r^{th} layer of the G-independent hypercube, denoted by $Q^{(r)}(G)$, is the collection of all G-independent subsets of [n] of size r.

We can now state Talbot's theorem.

Theorem 3 (Talbot). Let $A \subseteq Q^{(r)}(C_n)$ be a family of pairwise intersecting sets, and let A^* be the collection of all C_n -independent r-sets containing 1. Then $|A| \leq |A^*|$.

Talbot's proof used an ingenious cyclic compression argument and easily adapts to the $Q(P_n)$ setting as well. In this case, the study of $Q(C_n)$ was motivated by a conjecture of Holroyd and Johnson [7] on the independence number of a vertex-critical subset of the Kneser graph first identified by Schrijver [14].

1.2 Antichains and G-independent families

Our efforts in this paper are directed towards finding G-independent analogues of another classical combinatorial result in the hypercube, namely Sperner's theorem.

Definition 4. A subset of the hypercube $A \subseteq Q_n$ is an *antichain* if for all $A, B \in A$ with $A \neq B$, A is not a subset of B and B is not a subset of A.

How large an antichain can we find? Clearly for all integers r with $0 \le r \le n$, the r^{th} layer of Q_n is an antichain. So certainly we can find an antichain at least as large as the largest layer of Q_n , and a celebrated theorem of Sperner [15] asserts this is in fact the best we can do.

Theorem 5 (Sperner's Theorem). Let $n \in \mathbb{N}$, and $A \subseteq Q_n$ be an antichain. Then

$$|\mathcal{A}| \le \max_{r} |Q_n^{(r)}| = \binom{n}{\lfloor n/2 \rfloor}.$$

We consider the following generalisation of Sperner's problem.

Problem 6. Let $n \in \mathbb{N}$, and let G be a graph on [n]. What is the maximum size of an antichain in Q(G)?

Write s(G) for the maximum size of an antichain in Q(G). We call s(G) the width of Q(G). As in Sperner's theorem the size of a largest layer in Q(G) gives us a lower bound on the width s(G). This is not sharp in general: if G is the star on [n] with edges $\{1i: 2 \le i \le n\}$, then it is easy to see that s(G) is larger than the largest layer of Q(G) by 1. The width s(G) can in fact be much larger than a largest layer of Q(G), as the following example shows.

Example 7. Let $m \in \mathbb{N}$. Let G be a complete multipartite graph having for each integer $i \in [m]$ exactly $|2^{2^m-2^i+\frac{i}{2}}|$ parts of size 2^i .

The graph G in the example above is T-partite, where $T = \sum_{i=1}^{m} \lfloor 2^{2^m-2^i+\frac{i}{2}} \rfloor$, and has $n = \sum_{i=1}^{m} \lfloor 2^{2^m-2^i+\frac{i}{2}} \rfloor 2^i$ vertices, which is of order 2^{2^m} . A set of vertices in G is independent if and only if it meets at most one of the parts of G. An antichain in Q(G) is therefore the disjoint union of a collection of antichains, each lying inside a distinct part of G. It then follows from Sperner's theorem that the size of a maximal antichain in Q(G) is

$$s(G) = \sum_{i=1}^{m} \lfloor 2^{2^m - 2^i + \frac{i}{2}} \rfloor \binom{2^i}{2^{i-1}} = \sum_{i=1}^{m} 2^{2^m - 2^i + \frac{i}{2}} \frac{2^{2^i}}{\sqrt{2^i}} \sqrt{\frac{2}{\pi}} \left(1 + O\left(\frac{1}{i}\right) \right)$$
$$= \sqrt{\frac{2}{\pi}} m 2^{2^m} (1 + o(1)).$$

(Here in the first line we have used Stirling's approximation for the factorial.)

On the other hand, the layers of Q(G) are much smaller: the size of the r^{th} layer oscillates between peaks which have order 2^{2^m} , one for each i with $1 \leq i \leq m$. These peaks occur when r is close to 2^{i-1} , and correspond to the largest layer for the parts of size 2^i . Close to the peak corresponding to i, the sum of the contribution from the parts of size 2^j for $j \neq i$ has order dominated by the contribution from the parts of size 2^i . It follows that

$$\max_{0 \le r \le n} |Q^{(r)}(G)| = O\left(2^{2^m}\right) = o\left(s(G)\right).$$

In general s(G) and $\max\{|Q^{(r)}(G)|: 0 \le r \le n\}$ need thus not even be of the same order.

Question 8. When is

$$s(G) = \max_{0 \le r \le n} |Q^{(r)}(G)|$$
?

A natural guess is that it is sufficient for most vertices in G to look more or less the same. Let G be a graph. Recall that an automorphism of G is a bijection $\phi: V(G) \to V(G)$ such that ϕ maps edges to edges and non-edges to non-edges. A graph is $vertex\ transitive$ if for every $x,y\in V(G)$ there exists an automorphism of G mapping x to y.

Conjecture 9. Let G be a vertex-transitive graph. Then

$$s(G) = \max_{0 \le r \le n} |Q^{(r)}(G)|.$$

Of course, vertex-transitivity is not a necessary condition for the width of Q(G) to coincide with the size of the largest layer. Indeed, consider the complete graph on n vertices with one edge removed. This is not vertex-transitive, but the largest antichain is exactly the largest layer, i.e. the collection of all singletons. Similarly the path P_n , while not vertex-transitive, is close to the vertex-transitive cycle C_n , and we believe the conclusion of Conjecture 9 holds for $G = P_n$ also.

Conjecture 10.

$$s(P_n) = \max_{0 \le r \le n} |Q^{(r)}(P_n)|.$$

1.3 Results and structure of the paper

In their study of skewincident families, Cohen, Fachini and Körner [2] found themselves needing to give a bound on $s(P_n)$. They showed

$$s(P_n) \le |Q(P_{n-1})| = \left(\frac{2}{1+\sqrt{5}} + o(1)\right)|Q(P_n)|,$$

a bound which was sufficient for their purposes, but which, as they observed, is fairly weak. They asked for the value of $s(P_n)$, and remarked that none of the classical proofs of Sperner's theorem seemed to adapt to this setting. The main purpose of this paper is to try and answer their question. We shall focus on $Q(P_n)$ and Conjecture 10, though our techniques also apply in a more general setting (see Theorem 28 in Section 5). We show the following.

Theorem 11. There exists a constant C > 1 such that

$$s(P_n) \le C \max_{0 \le r \le n} |Q^{(r)}(P_n)|.$$

This improves the earlier bound of Cohen, Fachini and Körner [2] by a multiplicative factor of $O(n^{-1/2})$. It is however a far cry from Conjecture 10, and in addition has a rather calculation-intensive proof.

Our paper is structured as follows. In Section 2, we run through some preliminaries. In Section 3, we prove Theorem 11. We then prove small cases of Conjecture 10 in Section 4, and briefly discuss why some classical proofs of Sperner's theorem do not adapt well to the Fibonacci cube setting. In Section 5 we explain how the proof of Theorem 11 can be made to work in a more general setting. We end in Section 6 with some questions on isoperimetric problems in $Q(P_n)$.

2 Preliminaries

2.1 Counting in the Fibonacci cube

The Fibonacci sequence $(F_n)_{n\in\mathbb{Z}_{\geq 0}}$ is the sequence defined by the initial values $F_0=0$, $F_1=1$ and the recurrence relation $F_{n+2}=F_{n+1}+F_n$ for $n\geq 0$. It is a well-known fact (and an easy exercise) that the sizes of Fibonacci cubes are given by terms of the Fibonacci sequence: $|Q(P_n)|=F_{n+2}$. We now compute the size $q_n^r=|Q^{(r)}(P_n)|$ of a layer in $Q(P_n)$.

Lemma 12.
$$q_n^r = \binom{n-r+1}{r}$$
.

(We follow the standard convention that a binomial coefficient $\binom{a}{b}$ with b > a or b < 0 evaluates to zero.)

Proof. This is again an easy exercise in enumeration, but as we use the same counting technique later on in the paper, we write out the proof in full here.

Note that $Q^{(r)}(P_n)$ is empty for $r > \lceil n/2 \rceil$, so we may assume $r \le \lceil n/2 \rceil$. We build all zero-one sequences of length n containing exactly r ones and such that all ones are separated by at least one zero as follows. We begin with the separated sequence $1010101\dots01$ of length 2r-1 and containing r ones and (r-1) zeroes. Then we insert zeroes in the (r+1) 'bins' defined by the gaps between successive 1s, the gap to the left of the leftmost 1 and the gap to the right of the rightmost 1. We have n-2r+1 zeroes to insert into these bins. The number of ways of partitioning n-2r+1 objects into r+1 labelled lots is just $\binom{n-r+1}{r}$, proving our claim.

Next, let us identify the largest layers of $Q(P_n)$.

Lemma 13. Let r_{\star} be an integer maximising the layer size $|Q^{(r)}(P_n)|$. Then,

$$r_{\star} = \left\lceil \frac{5n + 2 - \sqrt{5n^2 + 20n + 24}}{10} \right\rceil$$

or

$$r_{\star} = \frac{5n + 2 - \sqrt{5n^2 + 20n + 24}}{10} + 1.$$

[theorem]Remark 1. The maximal layer thus satisfies $r_{\star} = \frac{5-\sqrt{5}}{10}n + O(1)$, and is unique unless $5n + 2 - \sqrt{5n^2 + 20n + 24}$ is an integer multiple of 10.

Proof. We consider the ratio between the sizes of two consecutive layers of $Q(P_n)$.

$$\frac{|Q^{(r+1)}(P_n)|}{|Q^{(r)}(P_n)|} = \binom{n-r}{r+1} / \binom{n-r+1}{r}$$

This is greater or equal to 1 if and only if r satisfies

$$5r^2 - r(5n+2) + (n^2 - 1) \ge 0,$$

which in the range $0 \le r \le \lceil n/2 \rceil$ happens if and only if

$$r \le \frac{1}{10} \left\{ 5n + 2 - \sqrt{5n^2 + 20n + 24} \right\}.$$

The lemma follows.

Now let us consider $Q(P_n)$ as a directed graph $D(P_n)$ by setting a directed edge from A to B if $B = A \cup \{b\}$ for some $b \notin A$, i.e. if B covers A in the partial order induced by \subseteq .

Definition 14. The *in-degree* $d^-(A)$ of a set $A \in Q(P_n)$ is the number of edges of $D(P_n)$ directed into A, while the *out-degree* $d^+(A)$ is the number of edges of $D(P_n)$ directed out of A.

Given a set $A \in Q^{(r)}(P_n)$, its in-degree $d^-(A)$ is always exactly r; however, as we shall see next, its out-degree could take any integer value between n-3r and n-2r.

Write $Q^{(r,d)}(P_n)$ for the collection of elements of $Q^{(r)}(P_n)$ with out-degree equal to d, and let $q_n^{r,d} = |Q^{(r,d)}(P_n)|$.

Lemma 15.
$$q_n^{r,d} = \binom{r+1}{d-n+3r} \binom{n-2r}{n-2r-d}$$
.

Proof. We can characterise the out-degree in terms of 'empty bins'. Recall that in Lemma 12 we built $Q^{(r)}(P_n)$ from the zero-one sequence of length 2r-1, 1010...101 by placing the n-2r+1 remaining zeroes into the r+1 'bins' defined by the gaps between consecutive 1s. Suppose i zeroes have been placed in bin j. Then the corresponding interval of zeroes will contribute i-1 to the out-degree. Thus the out-degree associated with a zero-one sequence \mathbf{s} is

$$d = n - 2r + 1 - (r + 1 - z(\mathbf{s})) = n - 3r + z(\mathbf{s}),$$

where $z(\mathbf{s})$ is the number of bins which have not received any zero.

Now, how many of our zero-one strings have z empty bins? There are $\binom{r+1}{z}$ ways of choosing the bins which will be empty, whereupon we need to put at least one zero into the remaining r+1-z bins. We then have to allocate the remaining n-2r+1-(r+1-z)=n-3r+z zeroes to the r+1-z non-empty bins; there are, as we observed in the proof of Lemma 12, $\binom{n-2r}{r-z}$ ways of doing this. Setting z=d-n+3r concludes the proof of the lemma.

Note Lemma 15 implies that $q_n^{r,d} \neq 0$ if and only if $n-3r \leq d \leq n-2r$. These bounds are attained by, for example, the zero-one sequence consisting of r 010-blocks followed by a single block consisting of n-3r zeroes (outdegree n-3r), and the zero-one sequence consisting of r 10-blocks followed by a single block consisting of n-2r zeroes (out-degree n-2r). These two examples are the extremes we have to contend with inside a layer of the Fibonacci cube.

Lemma 15 has the following corollary.

Corollary 16. Let r, n be fixed, and let $d_{\star} = d_{\star}(r, n)$ be an integer maximising $q_n^{r,d}$. Then

$$d_{\star} = \left\lceil \frac{(n-2r)^2 + 2n - 5r - 1}{n - r + 3} \right\rceil$$

or

$$d_{\star} = \frac{(n-2r)^2 + 2n - 5r - 1}{n - r + 3} + 1.$$

Thus if $r = \alpha n$ for some $\alpha > 0$, then the most common out-degree in $Q^{(r)}(P_n)$ is $d_{\star}(r,n) = \frac{(1-2\alpha)^2}{1-\alpha}n + O(1)$. Before we give a proof of Corollary 16, let us give a heuristic justification of why we expect d_{\star} to be about this. In the proof of Lemma 15 we established a correspondence between out-degree and (roughly speaking) the number of occurences of gaps of length one between successive 1s (ie occurences of 101). Now what is the probability that the gap between the first two 1s has length 1? Contracting a gap of length 1 between the first two 1s gives us a member of $Q^{(r-1)}(P_{n-2})$. Thus the likelihood of this occuring is roughly

$$|Q^{(r-1)}(P_{n-2})|/|Q^{(r)}(P_n)| = r/(n-r+1) \approx \alpha/(1-\alpha)$$

when $r = \alpha n$. Since there are $r+1 \approx \alpha n$ gaps, the expected number of short gaps is $z \approx n\alpha^2/(1-\alpha)$, which implies in turn that the expected out-degree is $d = n - 3r + z \approx n(1-2\alpha)^2/(1-\alpha)$. Unsurprisingly the maximum of $q_n^{r,d}$ is attained when d is close to the expected out-degree. Having said this, we turn to a formal argument.

Proof of Corollary 16. Consider the ratio $q_n^{r,d+1}/q_n^{r,d}$. By Lemma 15, this is equal to

$$\frac{q_n^{r,d+1}}{q_n^{r,d}} = \binom{r+1}{d+1-n+3r} \binom{n-2r}{n-2r-d-1} / \binom{r+1}{d-n+3r} \binom{n-2r}{n-2r-d}.$$

Solving the associated linear inequality, we see that $q_n^{r,d+1}/q_n^{r,d} \leq 1$ if and only if

$$d \le \frac{(n-2r)^2 + 2n - 5r - 1}{n - r + 3},\tag{1}$$

with equality if and only if we have equality in (1). The Corollary follows.

[theorem]Remark 2. Note that the proof of Corollary 16 establishes in fact that $q_n^{r,d}$ is strictly increasing in d until it hits its (at most two) maxima, and then becomes strictly decreasing in d. We shall use this monotonicity later on.

Corollary 17. Let r_{\star} be an integer maximising q_n^r , and let $r = r_{\star} + c\sqrt{n}$ for some $c \in [-\sqrt{\log n}, +\sqrt{\log n}]$. Then for $d_{\star}(r, n)$ an integer maximising $q^{r,d}$, we have

$$d_{\star}(r,n) = \left(\frac{5-\sqrt{5}}{10}\right)n - \left(\frac{5\sqrt{5}-7}{2}\right)c\sqrt{n} + (20-8\sqrt{5})c^2 + O(1).$$

Proof. This is a straightforward calculation from Corollary 16, from the fact $r_{\star} = \frac{5-\sqrt{5}}{10}n + O(1)$ (Lemma 13), and from the hypothesis on r:

$$d_{\star}(r,n) = \frac{(n-2r)^2}{n-r} + O(1) = \frac{\left(\frac{2\sqrt{5}}{10}n - 2c\sqrt{n}\right)^2}{\frac{5+\sqrt{5}}{10}n - c\sqrt{n}} + O(1),$$

which, expanded to second order, yields the desired result. \Box

2.2 Concentration

With the combinatorial preliminaries out of the way, let us obtain some concentration results for $q_n^r = |Q^{(r)}(P_n)|$ and $q_n^{r,d} = |Q^{(r,d)}(P_n)|$. Given the binomial coefficients appearing in Lemmas 12 and 15, we expect Chernoff-type concentration of both the weight in $Q(P_n)$ around the heaviest layer(s) $Q^{(r_*)}(P_n)$ and of the out-degrees in $Q^{(r)}(P_n)$ around the likeliest out-degree(s) $d_* = d_*(r,n)$. By double counting, we also expect, analogously to Q_n , that the largest layer in $Q(P_n)$ will occur when the in-degree and the average

out-degree are the same – that is, by the observation after Corollary 16, when $r \approx (n-2r)^2/(n-r)$. Solving this yields $r \approx \frac{(5-\sqrt{5})}{10}n$, matching the estimate we made after Lemma 13 and giving perhaps better intuition as to why the maximum occurs at this point.

These heuristic observations we have made regarding concentration are indeed correct, and can be proved formally using Stirling's approximation,

$$m! = \left(1 + O\left(\frac{1}{m}\right)\right)\sqrt{2\pi m}\left(\frac{m}{e}\right)^m,$$

and some simple calculus.

Let F be the function

$$F: x \mapsto (1-x)\log(1-x) - x\log x - (1-2x)\log(1-2x).$$

Lemma 18. Let $\alpha = \alpha(n)$ be a sequence of real numbers with $10^{-9} < \alpha(n) < \frac{1}{2} - 10^{-9}$ and $\alpha n \in \mathbb{N}$ for $n \geq 4$. Then

$$q_n^{\alpha n} = \left(\frac{(1-\alpha)\sqrt{1-\alpha}}{\sqrt{2\pi\alpha(1-2\alpha)}(1-2\alpha)} + O\left(\frac{1}{n}\right)\right)n^{-1/2}\exp\left(nF(\alpha)\right).$$

Proof. This is a straightforward calculation from Lemma 12 and Stirling's formula:

$$q_n^{\alpha n} = \binom{n - \alpha n + 1}{\alpha n} = \frac{((1 - \alpha)n)!}{(\alpha n)!((1 - 2\alpha)n)!} \left(\frac{(1 - \alpha)n + 1}{(1 - 2\alpha)n + 1}\right).$$

Substituting Stirling's approximation in the above (which we can do since α and $1-2\alpha$ are both bounded away from 0) then yields the claimed equality.

As expected given that the maximum of $q_n^{\alpha n}$ occurs when $\alpha = \frac{5-\sqrt{5}}{10} + O(n^{-1})$, we find that F attains a global maximum at $\frac{5-\sqrt{5}}{10}$:

$$F'(x) = \log\left(\frac{(1-2x)^2}{x(1-x)}\right),$$

which is strictly positive for $x < \frac{5-\sqrt{5}}{10}$, vanishes at $\frac{5-\sqrt{5}}{10}$ and becomes strictly negative for $x > \frac{5-\sqrt{5}}{10}$. Computing the second derivative, we find $F''(\frac{5-\sqrt{5}}{10}) = -5\sqrt{5}$.

Corollary 19. Let $Q^{(r_{\star})}(P_n)$ be a largest layer of $Q(P_n)$. Then the following hold:

(i) if $r = r_{\star} + c\sqrt{n}$ for some $c \in [-\sqrt{\log n}, \sqrt{\log n}]$, then

$$q_n^r = \exp\left(-\frac{5\sqrt{5}c^2}{2} + o(1)\right)q_n^{r_*};$$

(ii) there are

$$O\left(n\exp\left(-\frac{5\sqrt{5}}{2}\log n\right)q_n^{r_\star}\right) = o(q_n^{r_\star})$$

sets in $Q(P_n)$ with size differing from r_{\star} by more than $\sqrt{n \log n}$.

Proof. Immediate from Lemma 18 and the calculation above.

We now turn to out-degree concentration. Define

$$G(x,y) = x \log x + (1-2x) \log(1-2x) - y \log y - 2(x-y) \log(x-y) - (1-3x+y) \log(1-3x+y).$$

Lemma 20. Let $\alpha = \alpha(n)$, and $\beta = \beta(n)$ be sequences of real numbers satisfying $10^{-9} < \beta(n) < \alpha(n) - 10^{-9}$, and $\alpha(n) < (1 + \beta - 10^{-9})/3$ and $n\alpha, n\beta \in \mathbb{N}$ for $n \geq 9$. Then,

$$q_n^{\alpha n, (1-3\alpha+\beta)n} = \left(\frac{\alpha\sqrt{\alpha(1-2\alpha)}}{2\pi(\alpha-\beta)^2\sqrt{\beta(1-3\alpha+\beta)}} + O\left(\frac{1}{n}\right)\right)n^{-1}\exp\left(nG(\alpha,\beta)\right).$$

Proof. This is a straightforward calculation from Lemma 15 and Stirling's formula:

$$q_n^{\alpha n,(1-3\alpha+\beta)n} = \binom{\alpha n+1}{\beta n} \binom{(1-2\alpha)n}{(\alpha-\beta)n}$$

$$= \left(\frac{\alpha n+1}{(\alpha-\beta)n+1}\right) \frac{(\alpha n)!}{((\alpha-\beta)n)!(\beta n)!} \frac{((1-2\alpha)n)!}{((\alpha-\beta)n)!((1-3\alpha+\beta)n!)}$$

Substituting Stirling's approximation in the above then yields the claimed equality. (We can do this since α , β , $(\alpha - \beta)$, $(1 - 2\alpha)$ and $(1 - 3\alpha + \beta)$ are all bounded away from 0. Note that for $n \geq 9$ there exist at least two distinct integers m_1 and m_2 with $\frac{n}{4} \leq m_1 < m_2 \leq \frac{n}{3}$, and hence legal choices of $\alpha(n)$ and $\beta(n)$, so that our claim is not vacuous.)

Again it is no surprise that for a fixed α , the function $G_{\alpha}: y \mapsto G(\alpha, y)$ attains a global maximum at $\beta = \frac{\alpha^2}{1-\alpha}$:

$$G'_{\alpha}(y) = \log\left(\frac{(\alpha - y)^2}{y(1 - 3\alpha + y)}\right),$$

which is strictly positive for $y < \frac{\alpha^2}{1-\alpha}$, vanishes at $\frac{\alpha^2}{1-\alpha}$ and becomes strictly negative for $y > \frac{\alpha^2}{1-\alpha}$. Computing the second derivative, we find

$$G_{\alpha}''\left(\frac{\alpha^2}{1-\alpha}\right) = -\frac{(1-\alpha)^3}{\alpha^2(1-2\alpha)^2}.$$

In particular for $\alpha = \frac{5-\sqrt{5}}{10} + O\left(\sqrt{\frac{\log n}{n}}\right)$, we have $G''_{\alpha}(\frac{\alpha^2}{1-\alpha}) = \frac{25+11\sqrt{5}}{2} + O\left(\sqrt{\frac{\log n}{n}}\right)$.

Corollary 21. Let r_{\star} be an integer maximising q_n^r , and let $r = r_{\star} + O(\sqrt{n \log n})$. Let $d_{\star} = d_{\star}(r, n)$ be an integer maximising $q_n^{r,d}$. Then

(i) if $d = d_{\star} + c\sqrt{n}$ for some $c \in [-2\sqrt{\log n}, +2\sqrt{\log n}]$, then

$$q_n^{r,d} = \exp\left(-\frac{(25+11\sqrt{5})c^2}{4} + o(1)\right)q_n^{r,d_{\star}}$$

(ii) there are

$$O\left(n\exp\left(-\frac{(25+11\sqrt{5})}{4}\log n\right)q_n^{r,d_\star}\right) = o\left(n^{-2}q_n^{r,d_\star}\right)$$

sets in $Q^{(r)}(P_n)$ with out-degree differing from d_{\star} by more than $\sqrt{n \log n}$.

2.3 Summation bounds

We shall also need the following simple bounds on a sum of exponentials.

Lemma 22. Let $p(x) = a_0 + a_1x + a_2x^2$ be a quadratic polynomial with $a_2 > 0$. Then

$$C_1(p) \le \sum_{i \in \mathbb{Z}} e^{-p(i)} \le C_2(p),$$

where $C_1(p), C_2(p)$ are strictly positive constants depending only on a_0 , a_1 and a_2 .

Proof. This is an easy exercise — just use comparison with integrals to bound the sum, and then elementary calculus to evaluate $\int e^{-p(x)} dx$.

Corollary 23. There are constants $C_1 > 0$ and $C_2 > 0$ such that if r is an integer with $|r - r_{\star}| \leq \sqrt{n \log n}$ and $d_{\star} = d_{\star}(r, n)$ is an integer maximising $q_n^{r,d}$, then

$$C_1 \frac{q_n^r}{\sqrt{n}} \le q_n^{r,d_\star} \le C_2 \frac{q_n^r}{\sqrt{n}}$$

(We could also have proved this directly by calculating the ratio $q_n^{r,d_{\star}}/q_n^r$ using Lemmas 12, 15 and Corollary 16.)

Proof. By Corollary 21 part (ii) we may discard sets in $Q^{(r)}(P_n)$ with outdegree differing from d_{\star} by more than $\sqrt{n \log n}$. Divide the remaining sets in $Q^{(r)}(P_n)$ into out-degree intervals of width \sqrt{n} :

$$I_i = \{ A \in Q^{(r)}(P_n) : d_{\star} + i\sqrt{n} \le d^+(A) < d_{\star} + (i+1)\sqrt{n} \},$$

for $i \in \mathbb{Z} \cap [-\sqrt{\log n}, \sqrt{\log n}]$. Then we have

$$q_n^r = \sum_{i} |I_i| + o(q_n^r)$$
by Corollary 21 part (ii)
$$\leq \sum_{i\geq 0} q_n^{r,\lceil d_\star + i\sqrt{n} \rceil} \sqrt{n} + \sum_{i< 0} q_n^{r,\lfloor d_\star + (i+1)\sqrt{n} \rfloor} \sqrt{n}$$

since $q_n^{r,d}$ monotonically decreases as d moves away from d_\star

$$= 2\sqrt{n} \sum_{i \ge 0} \exp\left(-\left(\frac{25 + 11\sqrt{5}}{4}\right) i^2 + o(1)\right) q_n^{r, d_{\star}}$$

by Corollary 21 part (i),

which by Lemma 22 is at most $\frac{\sqrt{n}}{C_1}q_n^{r,d_{\star}}$ for some absolute constant $C_1 > 0$. The inequality in the other direction follows in much the same way.

3 Proof of Theorem 11

We can now proceed to the proof of Theorem 11 proper. Let $Q^{(r_{\star})}(P_n)$ be a largest layer of $Q(P_n)$, and for every r let $d_{\star}(r,n)$ be an integer maximising $q_n^{r,d_{\star}}$. By Corollary 19, we can restrict our attention in a proof of Theorem 11 to layers r with $|r-r_{\star}| \leq \sqrt{n \log n}$. We denote by $Q'(P_n)$ the corresponding subset of $Q(P_n)$. Note that for n sufficiently large (say n > 100) every element of $Q'(P_n)$ has nonzero out-degree in the directed graph $D(P_n)$.

3.1 Dissection into blocks and overlapping trapeziums

Let $c_1 = 1/100$. We divide $Q'(P_n)$ into (overlapping) blocks of layers

$$B_t = \bigcup \left\{ Q^{(r)}(P_n) : r_{\star} + c_1 t \sqrt{n} \le r \le r_{\star} + c_1 (t+1) \sqrt{n} \right\},$$

each of which is roughly $c_1\sqrt{n}$ layers wide. (Here t takes integer values in $[-\frac{\sqrt{\log n}}{c_1}, \frac{\sqrt{\log n}}{c_1}]$.)

If $t \geq 0$, we divide the top layer $Q^{(r_+)}(P_n)$ of B_t into out-degree intervals

$$I_{s,t} = \bigcup \left\{ Q^{(r_+,d)}(P_n): d_{\star}(r_+,n) + s\sqrt{n} \le d \le d_{\star}(r_+,n) + (s+1)\sqrt{n} \right\},$$

each of which ranges over roughly \sqrt{n} different out-degrees.

Each such interval $I_{s,t}$ defines a trapezium

$$T_{s,t} = \{ A \in B_t : \exists A' \in I_{s,t} \text{ with } A \subseteq A' \}.$$

For n sufficiently large, the union of these (overlapping) trapeziums covers all of B_t (since all sets in B_t have positive out-degree).

If on the other hand t < 0, we divide the bottom layer $Q^{(r_{-})}(P_n)$ of B_t into out-degree intervals

$$I_{s,t} = \bigcup \left\{ Q^{(r_-,d)}(P_n) : d_{\star}(r_-,n) + s\sqrt{n} \le d \le d_{\star}(r_-,n) + (s+1)\sqrt{n} \right\},$$

with again each interval defining a trapezium

$$T_{s,t} = \{A \in B_t : \exists A' \in I_{s,t} \text{ with } A' \subseteq A\}.$$

Taken together, the overlapping trapeziums $T_{s,t}$ cover all of B_t in this case also.

3.2 Strategy

The heart of our proof of Theorem 11 is the following lemma.

Lemma 24. There is an absolute constant $C_3 > 1$ such that for every antichain $A \subseteq Q(P_n)$ and every integer $t \in [-\frac{\sqrt{\log n}}{c_1}, \frac{\sqrt{\log n}}{c_1}]$ we have

$$|\mathcal{A} \cap B_t| \le C_3 \max\{q_n^r: Q^{(r)}(P_n) \subseteq B_t\}.$$

Provided we are able to prove Lemma 24, Theorem 11 is straightforward from our concentration result on the layer size, Corollary 19:

Proof of Theorem 11 from Lemma 24. Let \mathcal{A} be an antichain. Then,

$$\begin{aligned} |\mathcal{A}| &= \sum_{t} |\mathcal{A} \cap B_{t}| + o(q_{n}^{r_{\star}}) \\ & \text{by Corollary 19 part (ii)} \\ &\leq \sum_{t} C_{3} \max\{q_{n}^{r}: \ Q^{(r)}(P_{n}) \subseteq B_{t}\} + o(q_{n}^{r_{\star}}) \\ & \text{by Lemma 24} \\ &= C_{3} \left(\sum_{t \geq 0} q_{n}^{r_{\star} + \lceil c_{1}t\sqrt{n} \rceil} + \sum_{t < 0} q_{n}^{r_{\star} + \lfloor c_{1}(t+1)\sqrt{n} \rfloor} \right) + o(q_{n}^{r_{\star}}) \\ &\leq 2C_{3} \left(\sum_{t \geq 0} \exp\left(-\frac{5\sqrt{5}c_{1}^{2}}{2}t^{2} + o(1)\right) q_{n}^{r_{\star}}\right) + o(q_{n}^{r_{\star}}) \\ & \text{by Corollary 19 part (ii)} \\ &\leq Cq_{n}^{r_{\star}} \end{aligned}$$

for some absolute constant C > 0, by Lemma 22.

Let us therefore turn to the proof of Lemma 24. This will be a *shadow* argument.

Definition 25. Let $\mathcal{B} \subseteq Q(P_n)$ be a subset of the Fibonacci cube. The lower shadow of \mathcal{B} is the family

$$\partial^-(\mathcal{B}) = \{ B \in Q(P_n) : \exists b \notin B \text{ such that } B \cup \{b\} \in \mathcal{B} \}.$$

The *upper shadow* of \mathcal{B} is the family

$$\partial^+(\mathcal{B}) = \{ B \in Q(P_n) : \exists b \in B \text{ such that } B \setminus \{b\} \in \mathcal{B} \}.$$

Recalling the directed graph $D(P_n)$ we associated with $Q(P_n)$, the lower shadow is the *in-neighbourhood* of \mathcal{B} in $D(P_n)$ while the upper shadow is the *out-neighbourhood* of \mathcal{B} .

Let $t \geq 0$, and let $\mathcal{A} \subseteq B_t$ be an antichain contained in the block B_t . Write $\mathcal{A}^{(r)}$ for the r^{th} layer of \mathcal{A} , $\mathcal{A}^{(r)} = \mathcal{A} \cap Q^{(r)}(P_n)$.

Let $\mathcal{A}^{(r_+)}$ be the topmost non-empty layer of $\mathcal{A} \subseteq B_t$. Since $\mathcal{A} = \mathcal{A}_0$ is an antichain, the family

$$\mathcal{A}_1 = \left(\mathcal{A} \setminus \mathcal{A}^{(r_+)}\right) \cup \partial^-(\mathcal{A}^{(r_+)})$$

is also an antichain. Repeating this procedure with A_1 , then A_2 , etc, we can 'push down' our family into the bottom layer of B_t . We will thus be done in the proof of Lemma 24 if we can show we have not shrunk the size of our family by more than a constant factor in the process. (The t < 0 case proceeds identically with upper shadows instead of lower shadows.)

To do this, we perform some careful accounting, and this is where our trapeziums (and, unfortunately, some tedious calculations) come in. Roughly speaking, the further away the out-degree lies from the layer's average out-degree, the more we could be shrinking our family when taking lower shadows. This effect is balanced out by the fact that the further we are from the average out-degree the fewer sets we have at our disposal.

3.3 Shadows in the trapeziums

In this subsection, we prove the case $t \ge 0$ of Lemma 24 by taking shadows in trapeziums. We first introduce some notation.

Let $t \geq 0$. Let $r_- = r_\star + \lceil c_1 t \sqrt{n} \rceil$ and $r_+ = r_\star + \lfloor c_1 (t+1) \sqrt{n} \rfloor$ be the size of sets in the bottom-most and top-most layers of B_t respectively. Given a family $\mathcal{C} \subseteq \mathcal{B}$, we let

$$\phi(\mathcal{C}) = \left\{ A \in Q^{(r_{-})}(P_n) : \exists A' \in \mathcal{C} \text{ such that } A \subseteq A' \right\}$$

denote the collection of sets in the bottom-most layer of B_t which are contained in an element of \mathcal{C} . In other words, $\phi(\mathcal{C})$ is obtained from \mathcal{C} by repeatedly replacing the highest non-empty layer of \mathcal{C} by its lower shadow until the entire family lies inside $Q^{(r_-)}(P_n)$.

Proof of case $t \geq 0$ of Lemma 24. Let \mathcal{A} be an antichain. Without loss of generality, we may assume $\mathcal{A} \subseteq B_t$. We shall show that

$$|\mathcal{A}| - |\phi(\mathcal{A})| \le (1 + C_4)q_n^{r_-}$$

for some absolute constant $C_4 > 0$, from which Lemma 24 follows with $C_3 = C_4 + 2$.

Let $A_s = A \cap T_{s,t}$ be the intersection of A with the trapezium $T_{s,t}$. By Corollary 21 and the monotonicity of q_n^r , we have that

$$\left| \bigcup \{ \mathcal{A}_s : \ s \in \mathbb{Z} \setminus [-\sqrt{\log n}, \sqrt{\log n}] \} \right| \le c_1 \sqrt{n} \cdot o\left(n^{-2} q_n^{r_-}\right) = o(q_n^{r_-}).$$

Thus for the purpose of proving Lemma 24, it is enough to consider only the sets A_s with $s \in [-\sqrt{\log n}, \sqrt{\log n}]$.

Observe that deleting an element from a set in $Q(P_n)$ can increase its out-degree by at most 3. It follows that sets in $\phi(A_s)$ have out-degree d satisfying

$$d_{\star}(r_{+}, n) + s\sqrt{n} \le d \le d_{\star}(r_{+}, n) + (s+1)\sqrt{n} + 3c_{1}\sqrt{n}.$$

As $c_1 = 1/100$ it follows that $\phi(\mathcal{A}_s)$ is disjoint from $\phi(\mathcal{A}_{s+2})$ for all s (since $3c_1\sqrt{n} < \sqrt{n}$). In particular, sets in $Q^{(r-)}(P_n)$ are contained in at most two distinct $\phi(\mathcal{A}_s)$, whence

$$\left(\sum_{s} |\phi(\mathcal{A}_s)|\right) - \left|\bigcup_{s} \phi(\mathcal{A}_s)\right| \le |Q^{(r_-)}(P_n)| = q_n^{r_-}.$$
 (2)

Now we shall show $|\phi(\mathcal{A}_s)|$ is not much smaller than $|\mathcal{A}_s|$. To obtain $\phi(\mathcal{A}_s)$ from \mathcal{A}_s , we repeatedly replace the highest non-empty layer by its lower shadow. Since \mathcal{A} (and hence \mathcal{A}_s) is an antichain, we know that the shadow of the family's highest layer is disjoint from the rest of the family. Thus our only concern is that the family could be shrinking every time we take a lower shadow.

Observe that if $\mathcal{B} \subset Q^{(r)}(P_n)$ and the maximum out-degree in the lower shadow of \mathcal{B} is Δ^+ , then, by counting edges from $\partial^-\mathcal{B}$ to $Q^{(r)}(P_n)$ we have:

$$|\partial^- \mathcal{B}| \ge \frac{r}{\Delta^+} |\mathcal{B}|.$$

Going from \mathcal{A}_s to $\phi(\mathcal{A}_s)$, the worst ratio we would have to contend with at any stage of the process is thus when $r = r_- = r_\star + c_1 t \sqrt{n} + O(1)$ and $\Delta^+ = d_\star(r_+, n) + (s+1)\sqrt{n} + 3c_1\sqrt{n} + O(1)$. Now by Lemma 13,

$$r_{\star} = \frac{5 - \sqrt{5}}{10}n + O(1)$$

and by Corollary 17

$$d_{\star}(r_{+},n) = \frac{5-\sqrt{5}}{10}n - \left(\frac{5\sqrt{5}-7}{2}\right)c_{1}(t+1)\sqrt{n} + (20-8\sqrt{5})c_{1}^{2}(t+1)^{2} + O(1).$$

A quick calculation then shows that the worst-case ratio is

$$\begin{split} \frac{r_{-}}{\Delta^{+}} &= \frac{\frac{5-\sqrt{5}}{10}n + c_{1}t\sqrt{n}}{\frac{5-\sqrt{5}}{10}n - \left(\frac{5\sqrt{5}-7}{2}c_{1}(t+1) - (s+1) - 3c_{1}\right)\sqrt{n} + (20 - 8\sqrt{5})c_{1}^{2}(t+1)^{2}} + O\left(\frac{1}{n}\right) \\ &= \frac{1 + \frac{10}{5-\sqrt{5}}c_{1}tn^{-1/2}}{1 + \frac{10}{5-\sqrt{5}}\left(s+1 - \frac{5\sqrt{5}-7}{2}c_{1}t + \frac{13-5\sqrt{5}}{2}c_{1}\right)n^{-1/2}} + O\left(\frac{\log n}{n}\right) \\ &= 1 - \frac{10}{5-\sqrt{5}}\left(s+1 - \frac{5\sqrt{5}-5}{2}c_{1}t + \frac{13-5\sqrt{5}}{2}c_{1}\right)n^{-1/2} + O\left(\frac{\log n}{n}\right). \end{split}$$

(Note we used in the second line the fact that $t = O(\sqrt{\log n})$.) Write $f_t(s)$ for the expression

$$f_t(s) = \frac{10}{5 - \sqrt{5}} \left(s + 1 - \frac{5\sqrt{5} - 5}{2} c_1 t + \frac{13 - 5\sqrt{5}}{2} c_1 \right).$$

If $f_t(s) < 0$, then we have nothing to worry about: our family does not shrink as we take successive shadows. On the other hand if $f_t(s) \ge 0$, then we have

$$|\phi(\mathcal{A}_s)| \ge \left(1 - f_t(s)n^{-1/2} + O\left(\frac{\log n}{n}\right)\right)^{c_1\sqrt{n}} |\mathcal{A}_s|$$

$$= \exp\left(-c_1 f_t(s) + O\left(\frac{\log n}{\sqrt{n}}\right)\right) |\mathcal{A}_s|. \tag{3}$$

We now give an upper bound on the size of $\phi(A_s)$ (and hence, by (3), on $|A_s|$) when $f_t(s) \geq 0$ using our concentration results. Write s_0 for the unique real solution to $f_t(s) = 0$,

$$s_0 = -1 + \frac{5\sqrt{5} - 5}{2}c_1t - \frac{13 - 5\sqrt{5}}{2}c_1,$$

Since $c_1 = 1/100$ and $t \ge 0$, we certainly have $s_0 > -2$. By Corollary 17,

$$d_{\star}(r_{-},n) - d_{\star}(r_{+},n) = \left(\frac{5\sqrt{5}-7}{2}\right)c_{1}\sqrt{n} + O\left(\sqrt{\log n}\right).$$

(Since $t = O(\sqrt{\log n})$). The out-degrees found in $\phi(A_s) \subseteq Q^{(r_-)}(P_n)$ are thus at least

$$\begin{split} \delta_s &= d_\star(r_+, n) + s\sqrt{n} \\ &= d_\star(r_-, n) + s\sqrt{n} - \frac{5\sqrt{5} - 7}{2}c_1\sqrt{n} + O\left(\sqrt{\log n}\right) \\ &= d_\star(r_-, n) + g(s)\sqrt{n} + O\left(\sqrt{\log n}\right) \end{split}$$

where g denotes the linear function $s \mapsto s - \left(\frac{5\sqrt{5}-7}{2}\right)c_1$. As $s_0 > -2$, as $c_1 = 1/100$ and as s is an integer, it follows from the above that apart from at most two values of $s \ge s_0$ (namely s = -1 and s = 0), the minimum out-degree in $\phi(\mathcal{A}_s)$ is greater than $d_{\star}(r_-, n)$ by a term of order \sqrt{n} . We can then use our concentration result and the monotonicity of $q_n^{r_-, d}$ away

from $d_{\star}(r_{-}, n)$ to bound $|\phi(\mathcal{A}_{s})|$ for $s \geq 1$:

$$|\phi(\mathcal{A}_s)| \le (\sqrt{n} + 3c_1\sqrt{n})q_n^{r_-,\delta_s}$$

$$\le (3c_1 + 1)\sqrt{n}q_n^{r_-,d_\star(r_-,n)} \exp\left(-\left(\frac{25 + 11\sqrt{5}}{4}\right)g(s)^2 + o(1)\right)$$
(4)
$$\le (3c_1 + 1)C_2q_n^{r_-} \exp\left(-\left(\frac{25 + 11\sqrt{5}}{4}\right)g(s)^2 + o(1)\right),$$
(5)

by applying Corollary 21 in (4) and Corollary 23 in (5).

Now $f_t(s) \leq f_0(s)$ for all $t \geq 0$, so that we have

$$\sum_{s \geq s_0} |\mathcal{A}_s| \leq \sum_{s \geq s_0} |\phi(\mathcal{A}_s)| \exp\left(c_1 f_t(s) + O\left(\frac{\log n}{\sqrt{n}}\right)\right)
\text{(by (3))}$$

$$\leq (3c_1 + 1)C_2 q_n^{r_-} \left(e^{c_1 f_0(-1) + o(1)} + e^{c_1 f_0(0) + o(1)} + \sum_{s \geq 1} \exp\left(c_1 f_0(s) - \left(\frac{25 + 11\sqrt{5}}{4}\right)g(s)^2 + o(1)\right)\right)$$
(by (5))
$$\leq C_4 q_n^{r_-} \tag{6}$$

for some absolute constant $C_4 > 0$, by observing that $g(s)^2$ is quadratic in s while $f_0(s)$ is only linear and applying Lemma 22.

We are then essentially done:

$$|\mathcal{A}| - |\phi(\mathcal{A})| \le \left(\sum_{s} |\mathcal{A}_{s}| - |\phi(\mathcal{A}_{s})|\right) + \left(\sum_{s} |\phi(\mathcal{A}_{s})| - \left|\bigcup_{s} \phi(\mathcal{A}_{s})\right|\right)$$

$$\le \left(\sum_{s} |\mathcal{A}_{s}| - |\phi(\mathcal{A}_{s})|\right) + q_{n}^{r_{-}} \qquad \text{(by (2))}$$

$$\le \left(\sum_{s \ge s_{0}} |\mathcal{A}_{s}|\right) + q_{n}^{r_{-}} \le (C_{4} + 1)q_{n}^{r_{-}} \qquad \text{(by (6))}$$

from which it follows that

$$|\mathcal{A}| \le |\phi(\mathcal{A})| + (C_4 + 1)q_n^{r-1}$$

$$\le (C_4 + 2)q_n^{r-1},$$

with $C_4 + 2$ a constant independent of t and n as required.

The proof of the case t < 0 of Lemma 24 is essentially the same as the above, except that we use upper shadows instead of lower shadows (so as to push the family towards the largest layer rather than away from it). We conclude here the proof of Lemma 24 and with it the proof of Theorem 11.

4 Small cases of Conjecture 10

We have not tried to optimise the constant C we get in our proof of Theorem 11, as our methods will give a constant strictly greater than 1 when we believe the correct answer should be exactly 1. We have however established Conjecture 10 for some small values of n. Details follow below.

4.1 Partition into chains

A classical proof of Sperner's Theorem consists in partitioning Q_n into symmetric *chains*, each of which intersects the largest layer(s) of Q_n .

Definition 26. An *l*-chain in Q(G) is a family of l distinct elements of Q(G), $\{A_1, \ldots A_l\}$, with $A_1 \subset A_2 \subset \ldots \subset A_l$.

If Conjecture 11 is true, then it follows from a theorem of Dilworth [3] that $Q(P_n)$ can also be partitioned into disjoint chains each of which intersects the largest layer(s) of $Q(P_n)$. Finding an explicit construction of such a partition appears difficult however: $Q(P_n)$ is asymmetric, and which layer is largest changes in an awkward and aperiodic way with n. It is fairly straightforward however to find such a partition for small n.

We begin with a partition of $Q(P_1)$ into a single chain $(\emptyset, \{1\})$, then build a partition for $Q(P_n)$ iteratively for $2 \le n \le 9$.

Our chains shall come in three types: $type\ A$ chains are chains in $Q(P_n)$ every member of which contains n; $type\ B$ chains are chains in $Q(P_n)$ no member of which contains n; and $type\ C$ chains are chains in $Q(P_n)$ of length at least two where only the last member contains n. Our initial partition of $Q(P_1)$ thus consisted of a single C-chain.

Given such a partition of $Q(P_n)$, we build a partition of $Q(P_{n+1})$ into chains in the following way.

- An A-chain $(C_1 \cup \{n\}, C_2 \cup \{n\}, \dots C_l \cup \{n\})$ in $Q(P_n)$ gives rise to a B-chain in $Q(P_{n+1})$, namely $(C_1 \cup \{n\}, C_2 \cup \{n\}, \dots C_l \cup \{n\})$.
- A B-chain $(C_1, C_2, \dots C_l)$ in $Q(P_n)$ gives rise to (potentially) two chains in $Q(P_{n+1})$: a C-chain $(C_1, C_2, \dots C_l, C_l \cup \{n+1\})$, and (if l > 1), to an A-chain $(C_1 \cup \{n+1\}, C_2 \cup \{n+1\}, \dots C_{l-1} \cup \{n+1\})$.
- A C-chain $(C_1, C_2, \dots C_{l-1}, C_{l-1} \cup \{n\})$ in $Q(P_n)$ gives rise to two chains in $Q(P_{n+1})$: a B-chain $(C_1, C_2, \dots C_{l-1}, C_{l-1} \cup \{n\})$ and an A-chain $(C_1 \cup \{n+1\}, C_2 \cup \{n+1\}, \dots C_{l-2} \cup \{n+1\}, C_{l-1} \cup \{n+1\})$. (Note that by construction all C-chains have length at least 2, so that each of them does indeed produce an A-chain.)

It is easy to check that this iterative construction yields a partition of $Q(P_n)$ into chains through the largest layer for $n=1,2,\ldots 7$ and n=9. For n=8, we obtain a partition of $Q(P_8)$ containing one chain not intersecting the largest layer, $Q^{(3)}(P_8)$. However we can fix this by replacing the three chains ($\{258\}$), ($\{25\}, \{257\}$) and ($\{57\}, \{257\}$). This establishes Conjecture 10 for all $n \leq 9$. The argument in the next subsection gives a simpler proof for $n=2,3\ldots 7,9$, and proves the additional case n=10.

4.2 Shadows

Another standard proof of Sperner's theorem (indeed Sperner's original proof) is to 'push' an antichain towards the largest layer of Q_n by repeatedly replacing the antichain's top-most layer by its lower shadow and the antichain's bottom-most layer by its upper shadow. Our proof of Theorem 11 is essentially a variant of this. Unfortunately, the out-degrees in $Q(P_n)$ are not sufficiently concentrated for this technique to give us even an approximate form of Conjecture 10. We can however use shadow arguments to establish some small cases of Conjecture 10.

For $n \geq 2$, set

$$Q'(P_n) = \bigcup_{\frac{n-1}{4} < r < \frac{n+2}{2}} Q^{(r)}(P_n).$$

Lemma 27. Let $n \geq 2$ and let A be an antichain in $Q(P_n)$. Then there exists an antichain A' in $Q'(P_n)$ with $|A| \leq |A'|$.

Proof. Let \mathcal{A} be an antichain, and assume \mathcal{A} is nonempty (for otherwise we have nothing to prove). Write $\mathcal{A}^{(r)}$ for the r^{th} layer of \mathcal{A} ,

$$\mathcal{A}^{(r)} = \mathcal{A} \cap Q^{(r)}(P_n).$$

Let $r_+(\mathcal{A}) = \max\{r : \mathcal{A}^{(r)} \neq \emptyset\}$ and $r_-(\mathcal{A}) = \min\{r : \mathcal{A}^{(r)} \neq \emptyset\}$. Suppose $r_+(\mathcal{A}) \geq \frac{n+2}{3}$. As $\mathcal{A}_0 = \mathcal{A}$ is an antichain, we have that the family

$$\mathcal{A}_1 = \left(\mathcal{A} \setminus \mathcal{A}^{(r_+)}\right) \cup \partial^- \mathcal{A}^{(r_+)}$$

is also an antichain. Now by counting edges between $\partial^- \mathcal{A}^{(r_+)}$ and $\mathcal{A}^{(r_+)}$ in the directed graph $D(P_n)$ we see that

$$|\partial^{-} \mathcal{A}^{(r_{+})}| \ge \frac{r_{+}}{n - 2r_{+} + 2} |\mathcal{A}^{(r_{+})}| \ge |\mathcal{A}^{(r_{+})}| \qquad \text{(since } r_{+} \ge \frac{n + 2}{3}\text{)}.$$

In particular $|\mathcal{A}_1| \geq |\mathcal{A}_0|$. Repeating this procedure as many times as necessary, we can produce an antichain at least as large as \mathcal{A} with no set of size greater than or equal to $\frac{n+2}{3}$.

In the other direction, suppose $r_{-}(\mathcal{A}) \leq \frac{n-1}{4}$. As $\mathcal{A}_{0} = \mathcal{A}$ is an antichain, we have that the family

$$\mathcal{A}_1 = \left(\mathcal{A} \setminus \mathcal{A}^{(r_-)}\right) \cup \partial^+ \mathcal{A}^{(r_-)}$$

is also an antichain. Counting edges between $\mathcal{A}^{(r_-)}$ and $\partial^+ \mathcal{A}^{(r_-)}$ we have

$$|\partial^+ \mathcal{A}^{(r_-)}| \ge \frac{n - 3r_-}{r_- + 1} |\mathcal{A}^{(r_-)}| \ge |\mathcal{A}^{(r_-)}|$$
 (since $r_- \le \frac{n - 1}{4}$).

In particular $|\mathcal{A}_1| \geq |\mathcal{A}_0|$. Repeating this procedure as many times as necessary, we can produce an antichain at least as large as \mathcal{A} with no set of size less than or equal to $\frac{n-1}{4}$.

Now $\frac{n+2}{3} - \frac{n-1}{4} = \frac{n+11}{12}$, thus for $n \geq 2$ there always exists an integer $r: \frac{n-1}{4} < r < \frac{n+2}{3}$, so that the upper and lower shifting processes described above don't interfere with each other. So we can obtain from any antichain \mathcal{A} an antichain \mathcal{A}' which is at least as large and which lies in $Q'(P_n)$, as claimed

Observe now that for n=2,3,4,5,6,7,9 and 10 there is a unique integer r satisfying $\frac{n-1}{4} < r < \frac{n+2}{3}$. Thus Conjecture 10 holds for these n. As we gave a partition of $Q(P_8)$ into chains meeting the largest layer in the previous subsection (and as the case n=1 is trivial), this means Conjecture 10 holds for all n<11.

By Lemma 27, there is an antichain of maximum size in $Q(P_{11})$ which lies entirely inside $Q^{(3)}(P_{11}) \cup Q^{(4)}(P_{11})$. The union of these two layers has size 154, and the largest layer of $Q(P_{11})$ is $Q^{(3)}(P_{11})$ which has size 84. Thus the first open case of our conjecture asks whether we can find an antichain in $Q^{(3)}(P_{11}) \cup Q^{(4)}(P_{11})$ with 85 or more elements. This already does not look amenable to a pure brute force search.

5 Theorem 11 for other graphs

Our proof of Theorem 11 needed very little structural information about $Q(P_n)$. What we actually used was:

- (i) the layer size $|Q^{(r)}(P_n)|$ increases monotonically with r until it hits a maximum (or two consecutive maxima) and then decreases monotonically, and this maximum (or maxima) occurs (occur) when $r = r_{\star} = \alpha_{\star} n + O(1)$, where $\alpha_{\star} = \frac{5 5\sqrt{5}}{10}$;
- (ii) for $\alpha = \alpha_{\star} + cn^{-\frac{1}{2}}$ and $c = o(\sqrt{n})$, we have $|Q^{(r)}(P_n)| \leq e^{-\gamma_1 c^2 + o(1)} |Q^{(r_{\star})}(P_n)|$, where $\gamma_1 > 0$ is a constant, and there are $o(|Q^{(r_{\star})}(P_n)|)$ sets in $Q(P_n)$ with size differing from r_{\star} by more than o(n);

- (iii) within a layer, the number of sets with a given out-degree $|Q^{(r,d)}(P_n)|$ increases monotonically with d until it hits a maximum (or two consecutive maxima) and then decreases monotonically. For $r = \alpha n$ and $\alpha = \alpha_{\star} + o(1)$, this maximum (or maxima) occurs (occur) when $d = d_{\star}(r, n) = \beta_{\star}(\alpha)n + O(1)$, where β_{\star} is a continuous function of α ;
- (iv) for $\alpha = \alpha_{\star} + o(1)$, $\beta = \beta_{\star}(\alpha) + cn^{-\frac{1}{2}}$ and $c = o(\sqrt{n})$, we have $|Q^{(r,d)}(P_n)| \leq e^{-\gamma_2 c^2 + o(1)} |Q^{(r,d_{\star}(r))}(P_n)|$, where $\gamma_2 > 0$ is a constant, and there are $o\left(\frac{|Q^{(r)}(P_n)|}{\sqrt{n}}\right)$ sets in $Q^{(r)}(P_n)$ with out-degree differing from $d_{\star}(r,n)$ by more than o(n);
- (v) for $r = r_{\star} + o(n)$ and $A \in Q^{(r)}(G_n)$, removing a vertex from A increases its out-degree by at most 3, and adding a vertex to A decreases its out-degree by at most 3.

In fact, we could weaken (v): considering the case $t \geq 0$ only (the case $t \leq 0$ is similar) and re-using the notation from Section 3.3, it is sufficient for our argument that in each block B_t there at most $O(q_n^{r-})$ 'bad' sets A from which we can remove a vertex and thereby increase the out-degree by more than γ_3 , where $\gamma_3 > 0$ is a constant. Thus in turn it is enough if for each layer $Q^{(r)}(P_n)$ with $r = r_\star + o(n)$ there are at most $O\left(\frac{|Q^{(r)}(P_n)|}{\sqrt{n}}\right)$ 'bad' sets A.

In particular, our proof of Theorem 11 actually gives the following more general result:

Theorem 28. Let $\gamma_1, \gamma_2, \gamma_3 > 0$. Suppose $(G_n)_{n \in \mathbb{N}}$ is a sequence of n-vertex graphs satisfying the following properties:

- (i) the layer size $|Q^{(r)}(G_n)|$ increases monotonically with r until it hits a maximum (or two consecutive maxima) and then decreases monotonically, and this maximum (or maxima) occurs (occur) when $r = r_{\star} = \alpha_{\star} n + O(1)$, where $\alpha_{\star} \in (0,1)$ is a constant;
- (ii) for $\alpha = \alpha_{\star} + cn^{-\frac{1}{2}}$ and $c = o(\sqrt{n})$, we have $|Q^{(r)}(G_n)| \leq e^{-\gamma_1 c^2 + o(1)} |Q^{(r_{\star})}(G_n)|$, and there are $o(|Q^{(r_{\star})}(G_n)|)$ sets in $Q(G_n)$ with size differing from r_{\star} by more than o(n);
- (iii) within a layer, the number of sets with a given out-degree $|Q^{(r,d)}(G_n)|$ increases monotonically with d until it hits a maximum (or two consecutive maxima) and then decreases monotonically. For $r = \alpha n$ and $\alpha = \alpha_{\star} + o(1)$, this maximum (or maxima) occurs (occur) when $d = d_{\star}(r, n) = \beta_{\star}(\alpha)n + O(1)$, where β_{\star} is a continuous function of α ;

- (iv) for $\alpha = \alpha_{\star} + o(1)$, $\beta = \beta_{\star}(\alpha) + cn^{-\frac{1}{2}}$ and $c = o(\sqrt{n})$, we have $|Q^{(r,d)}(G_n)| \leq e^{-\gamma_2 c^2 + o(1)} |Q^{(r,d_{\star}(r))}(G_n)|$, and there are $o\left(\frac{|Q^{(r)}(G_n)|}{\sqrt{n}}\right)$ sets in $Q^{(r)}(G_n)$ with out-degree differing from $d_{\star}(r,n)$ by more than o(n);
- (v) for $r = r_{\star} + o(n)$ there are at most $O\left(\frac{|Q^{(r)}(G_n)|}{\sqrt{n}}\right)$ sets $A \in Q^{(r)}(G_n)$ such that we can remove a vertex from A and thereby increase its outdegree by more than γ_3 , or add vertex to A and thereby decrease its outdegree by more than γ_3 .

Then there exists a constant $\gamma_4 > 1$ such that

$$s(G_n) \le \gamma_4 \max_{0 \le r \le n} |Q^{(r)}(G_n)|.$$

Theorem 28 covers for example the case when G_n is the cycle C_n , or some finite power of P_n or C_n . The calculations required to check that all the conditions above are satisfied in these cases are very similar to those we performed in Section 2. For other graph families where the theorem might apply, the checks could however become more involved.

We remark that the monotonicity condition in (i) is rather natural. Indeed, our example in Section 1.2 of a graph sequence G_n for which the width was of larger order than the size of a largest layer exploited precisely the non-monotonicity of the layer sizes.

Similarly, (iii) and (iv) feel like reasonable conditions if we want to rule out antichain constructions spread over two consecutive layers and having size larger than the largest of the two layers by a factor of $1 + \varepsilon$ for some $\varepsilon > 0$ (e.g. by taking the union of the low out-degree sets in the bottom layer and the complement of their upper shadow).

The requirement that $\alpha_{\star} \in (0,1)$ in condition (i) forces G_n to have linearsized independent sets. Given this and the monotonicity part of condition (i), the Chernoff-type concentration we require in condition (ii) is in fact what we would expect to see.

Finally, (v) is a kind of homogeneity condition, chiming in with our intuition that a graph G where 'most' vertices look 'more or less the same' should have width s(G) 'more or less the same' as the size of the largest layer in Q(G).

Question 29. Suppose $(G_n)_{n\in\mathbb{N}}$ is a sequence of graphs satisfying all the conditions in Theorem 28. Is it the case that

$$s(G_n) = (1 + o(1)) \max_{r} |Q^{(r)}(G_n)|$$
?

6 Concluding remarks

6.1 The LYM inequality

Sperner's theorem has over time given rise to an entire field, called Sperner Theory. We refer the reader to the monograph of Engel [4] for more details on the subject. We have already briefly discussed two different proofs of Sperner's theorem in the previous section (via a partition into disjoint chains and via shadow arguments) and the reasons why they do not adapt well to the $Q(P_n)$ setting. Let us make a remark here about a third classical approach to Sperner's theorem, via the elegant LYM inequality of Bollobás, Lubell, Meshalkin and Yamamoto [1, 12, 13, 19].

Theorem 30 (LYM inequality). Let $n \in \mathbb{N}$ and $A \subseteq Q_n$ be an antichain. Then

$$\sum_{r=0}^{n} \frac{|\mathcal{A} \cap Q_n^{(r)}|}{|Q_n^{(r)}|} \le 1.$$

Note that Sperner's theorem is instant from LYM. Unfortunately we have been unable to find a good analogue of the LYM inequality for $Q(P_n)$. Not all maximal chains in $Q(P_n)$ have the same length, nor are elements in a given layer of $Q(P_n)$ contained in the same number of chains. Indeed, even restricting to 'typical' layers and 'typical' elements of those layers does not help us. As for shadows, the out-degrees are insufficiently concentrated for a uniform random chain to prove even an approximate form of Conjecture 10: a divergence in the out-degree by an additive factor of $O(\sqrt{n})$ blows up to a divergence by a constant multiplicative factor in the number of chainextensions of order $O(\sqrt{n})$. So to adapt the LYM strategy to our $Q(P_n)$ setting, we would need to construct a biased random chain which samples layers in a uniform manner. We could for example associate an 'energy' to sets, which would be high on high out-degree sets, and then give our random chain a slight bias toward lower energy configurations. Though we have been unable to do this, it is probably one of the more promising approaches left open by our investigations.

6.2 Isoperimetric questions

One way we might try to construct a partition of $Q(P_n)$ into chains is to find for any pair of consecutive layers a matching in (the undirected version of) $D(P_n)$ from the smaller layer to the larger one. By Hall's marriage theorem [6], such matchings exist if and only if Hall's condition is satisfied in the bipartite subgraphs of $D(P_n)$ corresponding to consecutive layers of

 $Q(P_n)$ — i.e. if and only if for every $r > r_{\star}$ and every $\mathcal{A} \subseteq Q^{(r)}(P_n)$ we have $|\mathcal{A}| \leq |\partial^{-}\mathcal{A}|$, and for every $r < r_{\star}$ and every $\mathcal{A} \subseteq Q^{(r)}(P_n)$ we have $|\mathcal{A}| \leq |\partial^{+}\mathcal{A}|$.

This makes us interested more generally in the following isoperimetric problems.

Problem 31. Let $0 \le r \le \lceil \frac{n}{2} \rceil$ and let $0 \le s \le q_n^r$. Identify the families $\mathcal{A} \subseteq Q^{(r)}(P_n)$ of size s that minimise the size of the lower shadow.

Problem 32. Let $0 \le r \le \lceil \frac{n}{2} \rceil$ and let $0 \le s \le q_n^r$. Identify the families $\mathcal{A} \subseteq Q^{(r)}(P_n)$ of size s that minimise the size of the upper shadow.

[theorem]Remark 3. Since $Q(P_n)$ is not closed under complements, these two problems are not equivalent.

In the usual hypercube Q_n , these problems were solved by Kruskal and Katona [10, 11] using shifting techniques that cannot be adapted to $Q(P_n)$ without additional ideas. Talbot [17] has moreover exhibited examples which show that the families minimising the size of the lower shadow in $Q(P_n)$ are not nested, suggesting the problem may be quite difficult.

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