# Generalised Majority Colourings of Digraphs 

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#### Abstract

We almost completely solve a number of problems related to a concept called majority colouring recently studied by Kreutzer, Oum, Seymour, van der Zypen and Wood. They raised the problem of determining, for a natural number $k$, the smallest number $m=m(k)$ such that every digraph can be coloured with $m$ colours where each vertex has the same colour as at most a $1 / k$ proportion of its out-neighbours. We show that $m(k) \in\{2 k-$ $1,2 k\}$. We also prove a result supporting the conjecture that $m(2)=3$. Moreover, we prove similar results for a more general concept called majority choosability.


For a natural number $k \geq 2$, a $\frac{1}{k}$-majority colouring of a digraph is a colouring of the vertices such that each vertex receives the same colour as at most a $1 / k$ proportion of its out-neighbours. We say that a digraph $D$ is $\frac{1}{k}$-majority $m$-colourable if there exists a $\frac{1}{k}$ majority colouring of $D$ using $m$ colours. The following natural question was recently raised by Kreutzer, Oum, Seymour, van der Zypen and Wood [6].

Question 1. Given $k \geq 2$, determine the smallest number $m=m(k)$ such that every digraph is $\frac{1}{k}$-majority $m$-colourable.

In particular, they asked whether $m(k)=O(k)$. Let us first observe that $m(k) \geq 2 k-1$. Consider a tournament on $2 k-1$ vertices where every vertex has out-degree $k-1$. Any $\frac{1}{k}-$ majority colouring of this tournament must be a proper vertex-colouring, and hence it needs at least $2 k-1$ colours. Conversely, we prove that $m(k) \leq 2 k$.
Theorem 2. Every digraph is $\frac{1}{k}$-majority $2 k$-colourable for all $k \geq 2$.
This is an immediate consequence of a result of Keith Ball (see [3]) about partitions of matrices. We shall use a slightly more general version proved by Alon [1].
Lemma 3. Let $A=\left(a_{i j}\right)$ be an $n \times n$ real matrix where $a_{i i}=0$ for all $i, a_{i j} \geq 0$ for all $i \neq j$, and $\sum_{j} a_{i j} \leq 1$ for all $i$. Then, for every $t$ and all positive reals $c_{1}, \ldots, c_{t}$ whose sum is 1 , there is a partition of $\{1,2, \ldots, n\}$ into pairwise disjoint sets $S_{1}, S_{2}, \ldots, S_{t}$, such that for every $r$ and every $i \in S_{r}$, we have $\sum_{j \in S_{r}} a_{i j} \leq 2 c_{r}$.

[^0]Proof of Theorem [2. Let $D$ be a digraph on $n$ vertices with vertex set $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ and write $d^{+}\left(v_{i}\right)$ for the out-degree of $v_{i}$. Let $A=\left(a_{i j}\right)$ be an $n \times n$ matrix where $a_{i j}=\frac{1}{d^{+}\left(v_{i}\right)}$ if there is a directed edge from $v_{i}$ to $v_{j}$ and $a_{i j}=0$ otherwise. We apply Lemma 3 with $t=2 k$ and $c_{i}=\frac{1}{2 k}$ for $1 \leq i \leq 2 k$ obtaining a partition of $\{1,2, \ldots, n\}$ into sets $S_{1}, S_{2}, \ldots, S_{2 k}$, such that for every $r$ and every $i \in S_{r}, \sum_{j \in S_{r}} a_{i j} \leq \frac{1}{k}$. Equivalently, the number of out-neighbours of $v_{i}$ that have the same colour as $v_{i}$ is at most $\frac{d^{+}\left(v_{i}\right)}{k}$ where the colouring of $D$ is defined by the partition $S_{1} \cup S_{2} \cup \cdots \cup S_{2 k}$.

Question 1 has now been reduced to whether $m(k)$ is $2 k-1$ or $2 k$.
Question 4. Is every digraph $\frac{1}{k}$-majority $(2 k-1)$-colourable?
Surprisingly, this is open even for $k=2$. Kreutzer, Oum, Seymour, van der Zypen and Wood [6] gave an elegant argument showing that every digraph is $\frac{1}{2}$-majority 4-colourable and they conjectured that $m(2)=3$.

Conjecture 5. Every digraph is $\frac{1}{2}$-majority 3-colourable.
We provide evidence for this conjecture by proving that tournaments are almost $\frac{1}{2}$-majority 3-colourable.

Theorem 6. Every tournament can be 3-coloured in such a way that all but at most 205 vertices receive the same colour as at most half of their out-neighbours.

Proof. The proof relies on an observation that in a tournament $T$, the set $S_{i}=\{x \in V(T)$ : $\left.2^{i-1} \leq d^{+}(x)<2^{i}\right\}$ has size at most $2^{i+1}$. Indeed, the sum of the out-degrees of the vertices of $S_{i}$ is at least $\binom{\left|S_{i}\right|}{2}$, the number of edges inside $S_{i}$. On the other hand, this sum is at most $\left(2^{i}-1\right)\left|S_{i}\right|$ by the definition of $S_{i}$. Therefore, $\binom{\left|S_{i}\right|}{2} \leq\left(2^{i}-1\right)\left|S_{i}\right|$ and hence, $\left|S_{i}\right| \leq 2^{i+1}-1$. We proceed by randomly assigning one of three colours to each vertex independently with probability $1 / 3$. Given a vertex $x$, let $B_{x}$ be the number of out-neighbours of $x$ which receive the same colour as $x$. We say that $x$ is bad if $B_{x}>d^{+}(x) / 2$. Trivially $\mathbb{E}\left(B_{x}\right)=d^{+}(x) / 3$, and hence by a Chernoff-type bound, it follows that, for $x \in S_{i}$,

$$
\begin{aligned}
\mathbb{P}(x \text { is bad }) & =\mathbb{P}\left(B_{x}>d^{+}(x) / 2\right)=\mathbb{P}\left(B_{x}>(1+1 / 2) \mathbb{E}(B(x))\right) \\
& \leq \exp \left(-\frac{(1 / 2)^{2}}{3} \mathbb{E}\left(B_{x}\right)\right)=\exp \left(-d^{+}(x) / 36\right) \leq \exp \left(-2^{i-1} / 36\right)
\end{aligned}
$$

Notice that if $i \geq 11$ then $\mathbb{P}(x$ is bad $) \leq 2^{-(2 i-7)}$. Let $X$ denote the total number of bad vertices. Since the vertices of out-degree 0 cannot be bad,

$$
\begin{aligned}
\mathbb{E}(X) & =\sum_{i \geq 1} \sum_{x \in S_{i}} \mathbb{P}(x \text { is bad }) \leq \sum_{i=1}^{10} 2^{i+1} \exp \left(-2^{i-1} / 36\right)+\sum_{i \geq 11} 2^{i+1} 2^{-(2 i-7)} \\
& \leq 205+\sum_{i \geq 11} 2^{-i+8}=205+\frac{1}{4}<206
\end{aligned}
$$

Hence, there is a 3-colouring such that all but at most 205 vertices receive the same colour as at most half of their out-neighbours.

Observe also that the same argument proves a special case of Conjecture 5 .
Theorem 7. Every tournament with minimum out-degree at least $2^{10}$ is $\frac{1}{2}$-majority 3 -colourable.

We remark that Theorem 6 can be strengthened (205 can be replaced by 7) by solving a linear programming problem. Recall that the expected number of bad vertices of out-degree at least 1024 is at most $1 / 4$. We shall use linear programming to show that the expected number of bad vertices of out-degree less than 1024 is less than 7.75 . Let $V_{i}$ be the set of vertices of out-degree $i$ for $i \in\{1,2, \ldots, 1023\}$ and note that the expected number of bad vertices of out-degree at most 1023 is $f\left(v_{1}, v_{2}, \ldots, v_{1023}\right)=\sum_{i=1}^{1023} v_{i} p_{i}$ where $v_{i}=\left|V_{i}\right|$ and $p_{i}=\sum_{j=\left\lceil\frac{i+1}{2}\right\rceil}^{i}\binom{i}{j}(1 / 3)^{j}(2 / 3)^{i-j}$. As before, observe that the number of vertices of degree at most $i$ is at most $2 i+1$, and therefore, $\sum_{j=1}^{i} v_{i} \leq 2 i+1$, leading to the following linear program.

$$
\begin{aligned}
& \text { Maximize: } f\left(v_{1}, v_{2}, \ldots, v_{1023}\right) \\
& \text { Subject to: } \sum_{j=1}^{i} v_{j} \leq 2 i+1, \text { for } i \in\{1,2, \ldots, 1023\} \\
& \text { Subject to: } v_{i} \geq 0, \text { for } i \in\{1,2, \ldots, 1023\}
\end{aligned}
$$

See Appendix A for the source code. Similarly, we can replace $2^{10}$ in Theorem 7 by 55, by using the same linear program to show that the expected number of bad vertices of out-degree in $[55,1023]$ is less than $3 / 4$.

Let us now change direction to a more general concept of majority choosability. A digraph is $\frac{1}{k}$-majority $m$-choosable if for any assignment of lists of $m$ colours to the vertices, there exists a $\frac{1}{k}$-majority colouring where each vertex gets a colour from its list. In particular, a $\frac{1}{k}$ majority $m$-choosable digraph is $\frac{1}{k}$-majority $m$-colourable. Kreutzer, Oum, Seymour, van der Zypen and Wood [6] asked whether there exists a finite number $m$ such that every digraph is $\frac{1}{2}$-majority m-choosable. Anholcer, Bosek and Grytczuk [2] showed that the statement holds with $m=4$. We generalise their result as follows.

Theorem 8. Every digraph is $\frac{1}{k}$-majority $2 k$-choosable for all $k \geq 2$.
Theorem 8 was independently proved by Fiachra Knox and Robert Šámal [5]. We prove Theorem 8 using a slight modification of Lemma 3 whose proof is very similar to that of Lemma 3 ,

Lemma 9. Let $A=\left(a_{i j}\right)$ be an $n \times n$ real matrix where $a_{i i}=0$ for all $i, a_{i j} \geq 0$ for all $i \neq j$, and $\sum_{j} a_{i j} \leq 1$ for all $i$. Then, for every $m$ and subsets $L_{1}, L_{2}, \ldots, L_{n} \subset \mathbb{N}$ of size $m$, there is a function $f:\{1,2, \ldots, n\} \rightarrow \mathbb{N}$ such that, for every $i, f(i) \in L_{i}$ and $\sum_{j \in f^{-1}(r)} a_{i j} \leq \frac{2}{m}$ where $r=f(i)$.

Proof. By increasing some of the numbers $a_{i j}$, if needed, we may assume that $\sum_{j} a_{i j}=1$ for all $i$. We may also assume, by an obvious continuity argument, that $a_{i j}>0$ for all $i \neq j$. Thus, by the Perron-Frobenius Theorem, 1 is the largest eigenvalue of $A$ with right eigenvector
$(1,1, \ldots, 1)$ and left eigenvector $\left(u_{1}, u_{2}, \ldots, u_{n}\right)$ in which all entries are positive. It follows that $\sum_{i} u_{i} a_{i j}=u_{j}$. Define $b_{i j}=u_{i} a_{i j}$, then $\sum_{i} b_{i j}=u_{j}$ and $\sum_{j} b_{i j}=u_{i}\left(\sum_{j} a_{i j}\right)=u_{i}$.
Let $f:\{1,2, \ldots, n\} \rightarrow \mathbb{N}$ be a function such that $f(i) \in L_{i}$ and $f$ minimises the sum $\sum_{r \in \mathbb{N}} \sum_{i, j \in f^{-1}(r)} b_{i j}$. By minimality, the value of the sum will not decrease if we change $f(i)$ from $r$ to $l$ where $l \in L_{i}$. Therefore, for any $i \in f^{-1}(r)$ and $l \in L_{i}$, we have

$$
\sum_{j \in f^{-1}(r)}\left(b_{i j}+b_{j i}\right) \leq \sum_{j \in f^{-1}(l)}\left(b_{i j}+b_{j i}\right)
$$

Summing over all $l \in L_{i}$, we conclude that

$$
m \sum_{j \in f^{-1}(r)}\left(b_{i j}+b_{j i}\right) \leq \sum_{j \in f^{-1}\left(L_{i}\right)}\left(b_{i j}+b_{j i}\right) \leq \sum_{j=1}^{n}\left(b_{i j}+b_{j i}\right)=2 u_{i}
$$

Hence, $\sum_{j \in f^{-1}(r)} u_{i} a_{i j}=\sum_{j \in f^{-1}(r)} b_{i j} \leq \sum_{j \in f^{-1}(r)}\left(b_{i j}+b_{j i}\right) \leq \frac{2 u_{i}}{m}$. Dividing by $u_{i}$, the desired result follows.

Proof of Theorem 8. The proof is the same as that of Theorem 2, using Lemma 9 instead of Lemma 3 .

In fact, the same statement also holds when the size of the lists is odd.
Corollary 10. Every digraph is $\frac{2}{m}$-majority $m$-choosable for all $m \geq 2$.
This statement generalises a result of Anholcer, Bosek and Grytczuk [2] where they prove the case $m=3$ which says that, given a digraph with colour lists of size three assigned to the vertices, there is a colouring from these lists such that each vertex has the same colour as at most two thirds of its out-neighbours.
We have established that the $\frac{1}{k}$-majority choosability number is either $2 k-1$ or $2 k$. Let us end this note with an analogue of Question 4.
Question 11. Is every digraph $\frac{1}{k}$-majority $(2 k-1)$-choosable?

## References

[1] N. Alon, Splitting digraphs, Combin. Probab. Comput., 15 (2006), pp. 933-937.
[2] M. Anholcer, B. Bosek, and J. Grytczuk, Majority choosability of digraphs, arXiv: 1608.06912, (2016).
[3] J. Bourgain and L. Tzafriri, Restricted invertibility of matrices and applications, in Analysis at Urbana, Vol. II (Urbana, IL, 1986-1987), vol. 138 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 1989, pp. 61-107.
[4] GLPK, GNU Linear Programming Kit. https://www.gnu.org/software/glpk/.
[5] F. Knox and R. Šámal, Linear Bound for Majority Colourings of Digraphs, arXiv: 1701.05715, (2017).
[6] S. Kreutzer, S. Oum, P. D. Seymour, D. van der Zypen, and D. R. Wood, Majority colourings of digraphs, Electr. J. Comb., 24 (2017), p. P2.25.

## Appendix A Linear program

We use the toolkit [4] to solve the linear program with the following source code:

```
param N := 1024;
param comb 'n choose k' {n in 0..N, k in 0..n} :=
    if k = 0 or k = n then 1 else comb[n-1,k-1] + comb[n-1,k];
param prob 'probability' {n in 0..N} :=
    sum{k in (floor(n/2)+1)..n} comb[n,k]*((1/3)^k)*((2/3)^(n-k));
var x{1..N}, integer, >= 0;
subject to constraint{i in 1..N}: sum{j in 1..i} x[j] <= 2*i+1;
maximize expectation: sum{i in 1..N} x[i]*prob[i];
solve;
end;
```


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