Generalised Majority Colourings of Digraphs

António Girão*

Teeradej Kittipassorn[†]

Kamil Popielarz[‡]

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Abstract

We almost completely solve a number of problems related to a concept called majority colouring recently studied by Kreutzer, Oum, Seymour, van der Zypen and Wood. They raised the problem of determining, for a natural number k, the smallest number m = m(k) such that every digraph can be coloured with m colours where each vertex has the same colour as at most a 1/k proportion of its out-neighbours. We show that $m(k) \in \{2k - 1, 2k\}$. We also prove a result supporting the conjecture that m(2) = 3. Moreover, we prove similar results for a more general concept called majority choosability.

For a natural number $k \geq 2$, a $\frac{1}{k}$ -majority colouring of a digraph is a colouring of the vertices such that each vertex receives the same colour as at most a 1/k proportion of its out-neighbours. We say that a digraph D is $\frac{1}{k}$ -majority m-colourable if there exists a $\frac{1}{k}$ -majority colouring of D using m colours. The following natural question was recently raised by Kreutzer, Oum, Seymour, van der Zypen and Wood [6].

Question 1. Given $k \ge 2$, determine the smallest number m = m(k) such that every digraph is $\frac{1}{k}$ -majority m-colourable.

In particular, they asked whether m(k) = O(k). Let us first observe that $m(k) \ge 2k - 1$. Consider a tournament on 2k - 1 vertices where every vertex has out-degree k - 1. Any $\frac{1}{k}$ -majority colouring of this tournament must be a proper vertex-colouring, and hence it needs at least 2k - 1 colours. Conversely, we prove that $m(k) \le 2k$.

Theorem 2. Every digraph is $\frac{1}{k}$ -majority 2k-colourable for all $k \geq 2$.

This is an immediate consequence of a result of Keith Ball (see [3]) about partitions of matrices. We shall use a slightly more general version proved by Alon [1].

Lemma 3. Let $A = (a_{ij})$ be an $n \times n$ real matrix where $a_{ii} = 0$ for all $i, a_{ij} \ge 0$ for all $i \ne j$, and $\sum_j a_{ij} \le 1$ for all i. Then, for every t and all positive reals c_1, \ldots, c_t whose sum is 1, there is a partition of $\{1, 2, \ldots, n\}$ into pairwise disjoint sets S_1, S_2, \ldots, S_t , such that for every r and every $i \in S_r$, we have $\sum_{j \in S_r} a_{ij} \le 2c_r$.

^{*} Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK; A.Girao@dpmms.cam.ac.uk.

[†]Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro (PUC-Rio), Rua Marquês de São Vicente 225, Gávea, Rio de Janeiro, RJ 22451-900, Brazil; ping41@mat.puc-rio.br.

[‡]Department of Mathematics, University Of Memphis, Memphis, TN 38152, USA; kamil.popielarz@gmail.com

Proof of Theorem 2. Let D be a digraph on n vertices with vertex set $\{v_1, v_2, \ldots, v_n\}$ and write $d^+(v_i)$ for the out-degree of v_i . Let $A = (a_{ij})$ be an $n \times n$ matrix where $a_{ij} = \frac{1}{d^+(v_i)}$ if there is a directed edge from v_i to v_j and $a_{ij} = 0$ otherwise. We apply Lemma 3 with t = 2k and $c_i = \frac{1}{2k}$ for $1 \le i \le 2k$ obtaining a partition of $\{1, 2, \ldots, n\}$ into sets S_1, S_2, \ldots, S_{2k} , such that for every r and every $i \in S_r$, $\sum_{j \in S_r} a_{ij} \le \frac{1}{k}$. Equivalently, the number of out-neighbours of v_i that have the same colour as v_i is at most $\frac{d^+(v_i)}{k}$ where the colouring of D is defined by the partition $S_1 \cup S_2 \cup \cdots \cup S_{2k}$.

Question 1 has now been reduced to whether m(k) is 2k - 1 or 2k.

Question 4. Is every digraph $\frac{1}{k}$ -majority (2k-1)-colourable?

Surprisingly, this is open even for k = 2. Kreutzer, Oum, Seymour, van der Zypen and Wood [6] gave an elegant argument showing that every digraph is $\frac{1}{2}$ -majority 4-colourable and they conjectured that m(2) = 3.

Conjecture 5. Every digraph is $\frac{1}{2}$ -majority 3-colourable.

hence by a Chernoff-type bound, it follows that, for $x \in S_i$,

We provide evidence for this conjecture by proving that tournaments are *almost* $\frac{1}{2}$ -majority 3-colourable.

Theorem 6. Every tournament can be 3-coloured in such a way that all but at most 205 vertices receive the same colour as at most half of their out-neighbours.

Proof. The proof relies on an observation that in a tournament T, the set $S_i = \{x \in V(T) : 2^{i-1} \leq d^+(x) < 2^i\}$ has size at most 2^{i+1} . Indeed, the sum of the out-degrees of the vertices of S_i is at least $\binom{|S_i|}{2}$, the number of edges inside S_i . On the other hand, this sum is at most $(2^i - 1)|S_i|$ by the definition of S_i . Therefore, $\binom{|S_i|}{2} \leq (2^i - 1)|S_i|$ and hence, $|S_i| \leq 2^{i+1} - 1$. We proceed by randomly assigning one of three colours to each vertex independently with probability 1/3. Given a vertex x, let B_x be the number of out-neighbours of x which receive the same colour as x. We say that x is bad if $B_x > d^+(x)/2$. Trivially $\mathbb{E}(B_x) = d^+(x)/3$, and

$$\mathbb{P}(x \text{ is bad}) = \mathbb{P}(B_x > d^+(x)/2) = \mathbb{P}(B_x > (1+1/2)\mathbb{E}(B(x)))$$

$$\leq \exp\left(-\frac{(1/2)^2}{3}\mathbb{E}(B_x)\right) = \exp(-d^+(x)/36) \leq \exp(-2^{i-1}/36)$$

Notice that if $i \ge 11$ then $\mathbb{P}(x \text{ is bad}) \le 2^{-(2i-7)}$. Let X denote the total number of bad vertices. Since the vertices of out-degree 0 cannot be bad,

$$\mathbb{E}(X) = \sum_{i \ge 1} \sum_{x \in S_i} \mathbb{P}(x \text{ is bad}) \le \sum_{i=1}^{10} 2^{i+1} \exp(-2^{i-1}/36) + \sum_{i \ge 11} 2^{i+1} 2^{-(2i-7)} \le 205 + \sum_{i \ge 11} 2^{-i+8} = 205 + \frac{1}{4} < 206.$$

Hence, there is a 3-colouring such that all but at most 205 vertices receive the same colour as at most half of their out-neighbours. $\hfill \Box$

Observe also that the same argument proves a special case of Conjecture 5.

Theorem 7. Every tournament with minimum out-degree at least 2^{10} is $\frac{1}{2}$ -majority 3-colourable.

We remark that Theorem 6 can be strengthened (205 can be replaced by 7) by solving a linear programming problem. Recall that the expected number of bad vertices of out-degree at least 1024 is at most 1/4. We shall use linear programming to show that the expected number of bad vertices of out-degree less than 1024 is less than 7.75. Let V_i be the set of vertices of out-degree *i* for $i \in \{1, 2, ..., 1023\}$ and note that the expected number of bad vertices of out-degree at most 1023 is $f(v_1, v_2, ..., v_{1023}) = \sum_{i=1}^{1023} v_i p_i$ where $v_i = |V_i|$ and $p_i = \sum_{j=\lceil \frac{i+1}{2} \rceil}^{i} {i \choose j} (1/3)^j (2/3)^{i-j}$. As before, observe that the number of vertices of degree at most 2i + 1, and therefore, $\sum_{j=1}^{i} v_i \leq 2i + 1$, leading to the following linear program.

Maximize:
$$f(v_1, v_2, ..., v_{1023})$$

Subject to: $\sum_{j=1}^{i} v_j \le 2i + 1$, for $i \in \{1, 2, ..., 1023\}$
Subject to: $v_i \ge 0$, for $i \in \{1, 2, ..., 1023\}$.

See Appendix A for the source code. Similarly, we can replace 2^{10} in Theorem 7 by 55, by using the same linear program to show that the expected number of bad vertices of out-degree in [55, 1023] is less than 3/4.

Let us now change direction to a more general concept of majority choosability. A digraph is $\frac{1}{k}$ -majority *m*-choosable if for any assignment of lists of *m* colours to the vertices, there exists a $\frac{1}{k}$ -majority colouring where each vertex gets a colour from its list. In particular, a $\frac{1}{k}$ majority *m*-choosable digraph is $\frac{1}{k}$ -majority *m*-colourable. Kreutzer, Oum, Seymour, van der Zypen and Wood [6] asked whether there exists a finite number *m* such that every digraph is $\frac{1}{2}$ -majority *m*-choosable. Anholcer, Bosek and Grytczuk [2] showed that the statement holds with m = 4. We generalise their result as follows.

Theorem 8. Every digraph is $\frac{1}{k}$ -majority 2k-choosable for all $k \ge 2$.

Theorem 8 was independently proved by Fiachra Knox and Robert Šámal [5]. We prove Theorem 8 using a slight modification of Lemma 3 whose proof is very similar to that of Lemma 3.

Lemma 9. Let $A = (a_{ij})$ be an $n \times n$ real matrix where $a_{ii} = 0$ for all $i, a_{ij} \ge 0$ for all $i \ne j$, and $\sum_j a_{ij} \le 1$ for all i. Then, for every m and subsets $L_1, L_2, \ldots, L_n \subset \mathbb{N}$ of size m, there is a function $f : \{1, 2, \ldots, n\} \rightarrow \mathbb{N}$ such that, for every $i, f(i) \in L_i$ and $\sum_{j \in f^{-1}(r)} a_{ij} \le \frac{2}{m}$ where r = f(i).

Proof. By increasing some of the numbers a_{ij} , if needed, we may assume that $\sum_j a_{ij} = 1$ for all *i*. We may also assume, by an obvious continuity argument, that $a_{ij} > 0$ for all $i \neq j$. Thus, by the Perron–Frobenius Theorem, 1 is the largest eigenvalue of A with right eigenvector

(1, 1, ..., 1) and left eigenvector $(u_1, u_2, ..., u_n)$ in which all entries are positive. It follows that $\sum_i u_i a_{ij} = u_j$. Define $b_{ij} = u_i a_{ij}$, then $\sum_i b_{ij} = u_j$ and $\sum_j b_{ij} = u_i \left(\sum_j a_{ij}\right) = u_i$.

Let $f : \{1, 2, ..., n\} \to \mathbb{N}$ be a function such that $f(i) \in L_i$ and f minimises the sum $\sum_{r \in \mathbb{N}} \sum_{i,j \in f^{-1}(r)} b_{ij}$. By minimality, the value of the sum will not decrease if we change f(i) from r to l where $l \in L_i$. Therefore, for any $i \in f^{-1}(r)$ and $l \in L_i$, we have

$$\sum_{i \in f^{-1}(r)} (b_{ij} + b_{ji}) \le \sum_{j \in f^{-1}(l)} (b_{ij} + b_{ji}).$$

Summing over all $l \in L_i$, we conclude that

$$m\sum_{j\in f^{-1}(r)}(b_{ij}+b_{ji})\leq \sum_{j\in f^{-1}(L_i)}(b_{ij}+b_{ji})\leq \sum_{j=1}^n(b_{ij}+b_{ji})=2u_i.$$

Hence, $\sum_{j \in f^{-1}(r)} u_i a_{ij} = \sum_{j \in f^{-1}(r)} b_{ij} \leq \sum_{j \in f^{-1}(r)} (b_{ij} + b_{ji}) \leq \frac{2u_i}{m}$. Dividing by u_i , the desired result follows.

Proof of Theorem 8. The proof is the same as that of Theorem 2, using Lemma 9 instead of Lemma 3. $\hfill \Box$

In fact, the same statement also holds when the size of the lists is odd.

Corollary 10. Every digraph is $\frac{2}{m}$ -majority m-choosable for all $m \geq 2$.

This statement generalises a result of Anholcer, Bosek and Grytczuk [2] where they prove the case m = 3 which says that, given a digraph with colour lists of size three assigned to the vertices, there is a colouring from these lists such that each vertex has the same colour as at most two thirds of its out-neighbours.

We have established that the $\frac{1}{k}$ -majority choosability number is either 2k - 1 or 2k. Let us end this note with an analogue of Question 4.

Question 11. Is every digraph $\frac{1}{k}$ -majority (2k-1)-choosable?

References

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Appendix A Linear program

We use the toolkit [4] to solve the linear program with the following source code:

```
param N := 1024;
param comb 'n choose k' {n in 0..N, k in 0..n} :=
    if k = 0 or k = n then 1 else comb[n-1,k-1] + comb[n-1,k];
param prob 'probability' {n in 0..N} :=
    sum{k in (floor(n/2)+1)..n} comb[n,k]*((1/3)^k)*((2/3)^(n-k));
var x{1..N}, integer, >= 0;
subject to constraint{i in 1..N}: sum{j in 1..i} x[j] <= 2*i+1;
maximize expectation: sum{i in 1..N} x[i]*prob[i];
solve;
end;
```