EXACT DISTANCE COLORING IN TREES

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ABSTRACT. For an integer $q \geq 2$ and an even integer d, consider the graph obtained from a large complete q-ary tree by connecting with an edge any two vertices at distance exactly d in the tree. This graph has clique number q+1, and the purpose of this short note is to prove that its chromatic number is $\Theta(\frac{d \log q}{\log d})$. It was not known that the chromatic number of this graph grows with d. As a simple corollary of our result, we give a negative answer to a problem of Van den Heuvel and Naserasr, asking whether there is a constant C such that for any odd integer d, any planar graph can be colored with at most C colors such that any pair of vertices at distance exactly d have distinct colors. Finally, we study interval coloring of trees (where vertices at distance at least d and at most cd, for some real c>1, must be assigned distinct colors), giving a sharp upper bound in the case of bounded degree trees.

1. Introduction

Given a metric space X and some real d>0, let $\chi(X,d)$ be the minimum number of colors in a coloring of the elements of X such that any two elements at distance exactly d in X are assigned distinct colors. The classical Hadwiger-Nelson problem asks for the value of $\chi(\mathbb{R}^2,1)$, where \mathbb{R}^2 is the Euclidean plane. It is known that $5 \leq \chi(\mathbb{R}^2,1) \leq 7$ [1] and since the Euclidean plane \mathbb{R}^2 is invariant under homothety, $\chi(\mathbb{R}^2,1)=\chi(\mathbb{R}^2,d)$ for any real d>0. Let \mathbb{H}^2 denote the hyperbolic plane. Kloeckner [3] proved that $\chi(\mathbb{H}^2,d)$ is at most linear in d (the multiplicative constant was recently improved by Parlier and Petit [6]), and observed that $\chi(\mathbb{H}^2, d) > 4$ for any d > 0. He raised the question of determining whether $\chi(\mathbb{H}^2, d)$ grows with d or can be bounded independently of d. As noticed by Kahle (see [3]), it is not known whether $\chi(\mathbb{H}^2, d) > 5$ for some real d > 0. Parlier and Petit [6] recently suggested to study infinite regular trees as a discrete analog of the hyperbolic plane. Note that any graph G can be considered as a metric space (whose elements are the vertices of G and whose metric is the graph distance in G), and in this context $\chi(G,d)$ is precisely the minimum number of colors in a vertex coloring of G such that vertices at distance dapart are assigned different colors. Note that $\chi(G,d)$ can be equivalently defined as the chromatic number of the exact d-th power of G, that is, the graph with the same vertex-set as G in which two vertices are adjacent if and only if they are at distance exactly d in G.

Let T_q denote the infinite q-regular tree. Parlier and Petit [6] observed that when d is odd, $\chi(T_q, d) = 2$ and proved that when d is even, $q \leq \chi(T_q, d) \leq (d+1)(q-1)$. A

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similar upper bound can also be deduced from the results of Van den Heuvel, Kierstead, and Quiroz [2], while the lower bound is a direct consequence of the fact that when d is even, the clique number of the exact d-th power of T_q is q (note that it does not depend on d). In this short note, we prove that when $q \geq 3$ is fixed,

$$\frac{d \log(q-1)}{4 \log(d/2) + 4 \log(q-1)} \le \chi(T_q, d) \le (2 + o(1)) \frac{d \log(q-1)}{\log d},$$

where the asymptotic o(1) is in terms of d. A simple consequence of our main result is that for any even integer d, the exact d-th power of a complete binary tree of depth d is of order $\Theta(d/\log d)$ (while its clique number is equal to 3).

The following problem (attributed to Van den Heuvel and Naserasr) was raised in [4] (see also [2] and [5]).

Problem 1.1 (Problem 11.1 in [4]). Is there a constant C such that for every odd integer d and every planar graph G we have $\chi(G,d) \leq C$?

We will show that our result on large complete binary trees easily implies a negative answer to Problem 1.1. More precisely, we will prove that the graph U_3^d obtained from a complete binary tree of depth d by adding an edge between any two vertices with the same parent gives a negative answer to Problem 1.1 (in particular, for odd d, the chromatic number of the exact d-th power of U_3^d grows as $\Theta(d/\log d)$). We will also prove that the exact d-th power of a specific subgraph Q_3^d of U_3^d grows as $\Omega(\log d)$. Note that U_3^d and Q_3^d are outerplanar (and thus, planar) and chordal (see Figure 2).

Kloeckner [3] proposed the following variant of the original problem: For a metric space X, an integer d and a real c > 1, we denote by $\chi(X, [d, cd])$ the smallest number of colors in a coloring of the elements of X such that any two elements of X at distance at least d and at most cd apart have distinct colors. Considering as above the natural metric space defined by the infinite q-regular tree T_q , Parlier and Petit [6] proved that

$$q(q-1)^{\lfloor cd/2\rfloor - \lfloor d/2\rfloor} \le \chi(T_q, [d, cd]) \le (q-1)^{\lfloor cd/2 + 1\rfloor} (\lfloor cd\rfloor + 1).$$

We will show that $\chi(T_q, [d, cd]) \leq \frac{q}{q-2}(q-1)^{\lfloor cd/2 \rfloor - d/2 + 1} + cd + 1$, which implies that the lower bound of Parlier and Petit [6] (which directly follows from a clique size argument) is asymptotically sharp.

2. Exact distance coloring

Throughout the paper, we assume that the infinite q-regular tree T_q is rooted in some vertex r. This naturally defines the children and descendants of a vertex and the parent and ancestors of a vertex distinct from r. In particular, given a vertex u, we define the ancestors u^0, u^1, \ldots of u inductively as follows: $u^0 = u$ and for any i such that u^i is not the root, u^{i+1} is the parent of u^i . With this notation, u^d can be equivalently defined as the ancestor of u at distance d from u (if such a vertex exists). For a given vertex u in T_q , the depth of u, denoted by depth(u), is the distance between u and r in T_q . For a vertex v and an integer ℓ , we define $L(v,\ell)$ as the set of descendants of v at distance exactly ℓ from v in T_q .

We first prove an upper bound on $\chi(T_q, d)$.

Theorem 2.1. For any integer $q \geq 3$, any even integer d, and any integer $k \geq 1$ such that $k(q-1)^{k-1} \leq d$, we have $\chi(T_q,d) \leq (q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k} + 1$. In particular, $\chi(T_q,d) \leq d+q+1$, and when q is fixed and d tends to infinity, $\chi(T_q,d) \leq (2+o(1)) \frac{d \log(q-1)}{\log d}$.

Proof. A vertex of T_q distinct from r and whose depth is a multiple of k is said to be a special vertex. Let v be a special vertex. Every special vertex v distinct from v such that $v = v^k$ is called a cousin of v. Note that v has at most $q(q-1)^{k-1} - 1$ cousins (at most $(q-1)^k - 1$ if $v^k \neq r$). A special vertex v is said to be a relative of v if v is either a cousin of v, or v has the property that v and v have the same depth and are at distance at most v apart in v and v are depth must satisfy v and so the number of vertices v such that v and v have the same depth and are at distance at most v apart in v has at most v

The first step is to define a coloring C of the special vertices of T_q . This will be used later to define the desired coloring of T_q , i.e. a coloring such that vertices of T_q at distance d apart are assigned distinct colors (in this second coloring, the special vertices will not retain their color from C).

We greedily assign a color C(v) to each special vertex v of T_q as follows: we consider the vertices of T_q in a breadth-first search starting at r, and for each special vertex v we encounter, we assign to v a color distinct from the colors already assigned to its relatives, and from the set of ancestors v^{ik} of v, where $2 \le i \le \frac{d}{k} + 1$ (there are at most $\frac{d}{k}$ such vertices). Note that if $v^k = r$, the number of colors forbidden for v is at most $q(q-1)^{k-1} - 1$ and if $v^k \ne r$ the number of colors forbidden for v is at most $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k} - 1$. Since $k(q-1)^{k-1} \le d$, in both cases v has at most $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k} - 1$ forbidden colors, therefore we can obtain the coloring C by using at most $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k}$ colors.

For any special vertex v, the set of descendants of v at distance at least d/2 - k and at most d/2 - 1 from v is denoted by K(v, k). We now define the desired coloring of T_q as follows: for each special vertex v, all the vertices of K(v, k) are assigned the color C(v). Finally, all the vertices at distance at most d/2 - 1 from r are colored with a single new color (note that any two vertices in this set lie at distance less than d apart). The resulting vertex-coloring of T_q is called c. Note that c uses at most $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + \frac{d}{k} + 1$ colors, and indeed every vertex of T_q gets exactly one color.

We now prove that vertices at distance d apart in T_q are assigned distinct colors in c. Assume for the sake of contradiction that two vertices x and y at distance d apart were assigned the same color. Then the depth of both x and y is at least d/2. We can assume by symmetry that the difference t between the depth of x and the depth of y is such that $0 \le t \le d$ since otherwise they would be at distance more than d. Let u be the unique (special) vertex of T_q such that $x \in K(u, k)$ and v be the unique (special) vertex such that $y \in K(v, k)$. By the definition of our coloring c, we have C(u) = C(v). Note that u and v are distinct; indeed, otherwise x and y would not be at distance d in T_q . Assume first that

u and v have the same depth. Then since u and x (resp. v and y) are distance at least d/2 - k apart, u and v are cousins (and thus, relatives), which contradicts the definition of the vertex-coloring C. We may, therefore, assume that the depths of u and v are distinct. Moreover, since u and v are special vertices, we may assume that their depths differ by at least k. In particular, u lies deeper than v in T_q .

First assume that the depths of u and v differ by at least 2k. Then v is not an ancestor of u in T_q . Indeed, for otherwise we would have $v = u^{ik}$ for some integer $2 \le i \le \frac{d}{k} + 1$, which would contradict the definition of C. This implies that the distance between x and y is at least d/2 - k + d/2 - k + 2k + 2 = d + 2, which is a contradiction. Therefore, we can assume that the depths of u and v differ by precisely k. Since v is not a relative of u, we have that $v \ne u^k$ and the distance between u^k and v is more than k. Moreover, since u and v (resp. v and v) are at distance at least v0 apart, this implies that the distance between v1 and v2 is more than v3.

Thus, c is a proper coloring.

By taking k=1 we obtain a coloring c using at most $(q-1)^1+(q-1)^{\lfloor 1/2\rfloor}+\frac{d}{1}+1=q+d+1$ colors, and by taking $k=\lfloor \frac{\log d-\log\log d+\log\log(q-1)}{\log(q-1)}\rfloor$, we obtain a coloring c using at most

$$\frac{d \log(q-1)}{\log d} + \sqrt{\frac{d \log(q-1)}{\log d}} + \frac{d \log(q-1)}{\log d - \log \log d + \log \log(q-1) - \log(q-1)} + 1 = (2 + o(1)) \frac{d \log(q-1)}{\log d}$$
 colors.

For k=1, the proof above can be optimized to show that $\chi(T_q,d) \leq q + \frac{d}{2}$ (by simply noting that vertices at even depth and vertices at odd depth can be colored independently). Since we are mostly interested in the asymptotic behaviour of $\chi(T_q,d)$ (which is of order $O(\frac{d}{\log d})$), we omit the details.

We now prove a simple lower bound on $\chi(T_q, d)$. Let T_q^d be the rooted complete (q-1)ary tree of depth d, with root r. Note that each node has q-1 children, so this graph is
a subtree of T_q .

Theorem 2.2. For any integer $q \geq 3$ and any even d, $\chi(T_q^d, d) \geq \log_2(\frac{d}{4} + q - 1)$.

Proof. Consider any coloring of T_q^d with colors $1, 2, \ldots, C$, such that vertices at distance precisely d apart have distinct colors. For any vertex v at depth at most $\frac{d}{2} + 1$ in T_q^d , the set of colors appearing in $L(v, \frac{d}{2} - 1)$ is denoted by S_v . Observe that if v and w have the same parent, then S_v and S_w are disjoint since for any $x \in L(v, \frac{d}{2} - 1)$ and $y \in L(w, \frac{d}{2} - 1)$, x and y are at distance d.

Fix some vertex u at depth at most $\frac{d}{2}$ in T_q^d and some child v of u. We claim that:

Claim 2.3. For any integer $1 \le k \le \frac{\operatorname{depth}(u)}{2}$, there is a color of $S_{u^{2k-1}}$ that does not appear in S_v .

To see that Claim 2.3 holds, observe that in the subtree of T_q^d rooted in u^k , there is a vertex of $L(u^{2k-1}, \frac{d}{2} - 1)$ at distance d from all the elements of $L(v, \frac{d}{2} - 1)$. The color of such a vertex does not appear in S_v , therefore Claim 2.3 holds.

In particular, Claim 2.3 implies that all the sets $\{S_{u^{2k-1}} | 1 \le k \le d/4\}$ and $\{S_w | w \text{ is a child of } u\}$ are pairwise distinct. Since there are $\frac{d}{4} + q - 1$ such sets, we have $\frac{d}{4} + q - 1 \le 2^C$ and therefore $C \ge \log_2(\frac{d}{4} + q - 1)$, as desired.

It was observed by Stéphan Thomassé that the proof of Theorem 2.2 only uses a small fraction of the graph T_q^d . Consider for simplicity the case q=3, and define P_3^d as the graph obtained from a path $P=v_0,v_1,\ldots,v_d$ on d edges, by adding, for each $1 \leq i \leq d$, a path on i edges ending at v_i (see Figure 1). This graph is an induced subgraph of T_q^d and the proof of Theorem 2.2 directly shows the following¹.

Corollary 2.4. For any even integer d, $\chi(P_3^d, d) \ge \log_2(d+8) - 2$.

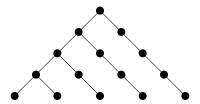


FIGURE 1. The graph P_3^4 .

The proof of Theorem 2.2 can be refined to prove the following better estimate for T_q^d , showing that the upper bound of Theorem 2.1 is (asymptotically) tight within a constant multiplicative factor of 8.

Theorem 2.5. For any integer $q \ge 3$ and every even integer $d \ge 2$, $\chi(T_q^d, d) \ge \frac{d \log(q-1)}{4 \log(d/2) + 4 \log(q-1)}$.

Proof. Consider any coloring of T_q^d with colors $1, 2, \ldots, C$, such that vertices at distance precisely d apart have distinct colors. We perform a random walk v_0, v_1, \ldots, v_d in T_q^d as follows: we start with $v_0 = r$, and for each $i \geq 1$, we choose a child of v_i uniformly at random and set it as v_{i+1} . Note that the depth of each vertex v_i is precisely i.

From now on we fix a color $c \in \{1, ..., C\}$. For any vertex v of T_q^d , the set of vertices contained in the subtree of T_q^d rooted in v is denoted by V_v , and we set $A_v = \{\text{depth}(u) \mid u \in V_v \text{ and } u \text{ has color } c\}$. When $v = v_i$, for some integer $0 \le i \le d$, we write A_i instead of A_{v_i} .

Claim 2.6. Assume that for some even integers i and j with $2 \le i < j \le d$, and for some vertex v at depth $\frac{i+j-d}{2}$, the set A_v contains both i and j. Then v has precisely one child u such that A_u contains i and j, and moreover all the children w of v distinct from u are such that A_w contains neither i nor j.

¹Stéphan Thomassé noticed that this can also be deduced from the fact that the vertices at depth at least $\frac{d}{2}$ and at most d in the exact d-th power of P_3^d induce a *shift graph*.

To see that Claim 2.6 holds, simply note that $\frac{i+j-d}{2} < i < j$ and if two vertices u_1, u_2 colored c are respectively at depths i and j, and their common ancestor is v, then they are at distance d in T_q^d (which contradicts the fact that they were assigned the same color). Indeed, the distance of u_1 to v is $i - \frac{i+j-d}{2}$ and the distance of u_2 to v is $j - \frac{i+j-d}{2}$. This proves the claim.

We now define a family of graphs $(G_k)_{0 \le k \le d/2}$ as follows. For any $0 \le k \le \frac{d}{2}$, the vertexset $V(G_k)$ of G_k is the set $A_k \cap 2\mathbb{N} \cap (d/2, d]$, and two (distinct) even integers $i, j \in A_k$ are adjacent in G_k if and only if $\frac{i+j-d}{2} < k$. For each $0 \le k \le \frac{d}{2}$ we define the energy \mathcal{E}_k of G_k as follows: $\mathcal{E}_k = \sum_{i \in V(G_k)} (q-1)^{\deg(i)}$, where $\deg(i)$ denotes the degree of the vertex i in G_k .

Note that each graph G_k depends on the (random) choice of v_1, v_2, \ldots, v_k .

Claim 2.7. For any
$$0 \le k \le \frac{d}{2} - 1$$
, $\mathbb{E}(\mathcal{E}_{k+1}) \le \mathbb{E}(\mathcal{E}_k)$.

Assume that v_1, v_2, \ldots, v_k (and therefore also G_k) are fixed. Observe that G_{k+1} is obtained from G_k by possibly removing some vertices and adding some edges. Thus, \mathcal{E}_{k+1} can be larger than \mathcal{E}_k only if G_{k+1} contains edges that are not in G_k . Therefore, it suffices to consider the contributions of those pairs of nonadjacent vertices in G_k which could become adjacent in G_{k+1} (since these correspond to pairs i,j with $k=\frac{i+j-d}{2}$, these pairs are pairwise disjoint), and prove that these contributions are, in expectation, equal to 0. Fix a pair of even integers i < j in $V(G_k)$ with $k = \frac{i+j-d}{2}$ (and note that i and j are not adjacent in G_k). By Claim 2.6, either v_{k+1} is such that A_{k+1} contains i and j (this event occurs with probability $\frac{1}{q-1}$), or A_{k+1} contains neither i nor j (with probability $1 - \frac{1}{q-1}$). As a consequence, for any i < j in $V(G_k)$ with $k = \frac{i+j-d}{2}$, with probability $\frac{1}{q-1}$ we add the edge ij in G_{k+1} and with probability $1 - \frac{1}{q-1}$ we remove vertices i and j from G_{k+1} . This implies that for any $i, j \in V(G_k)$, i < j, with $k = \frac{i+j-d}{2}$, with probability $\frac{1}{q-1}$ we have contribution at most $(q-1)^{\deg(i)+1}+(q-1)^{\deg(j)+1}-(q-1)^{\deg(i)}-(q-1)^{\deg(j)}=(q-2)((q-1)^{\deg(i)}+(q-1)^{\deg(j)})$ to \mathcal{E}_{k+1} (where deg refers to the degree in G_k) and with probability $1 - \frac{1}{q-1}$ we have a contribution of at most $-(q-1)^{\deg(i)}-(q-1)^{\deg(i)}+(q-1)^{\deg(j)})-\frac{q-2}{q-1}((q-1)^{\deg(i)}+(q-1)^{\deg(j)})=0$. Summing over all such pairs i,j, we obtain $\mathbb{E}(\mathcal{E}_{k+1}) \leq \mathbb{E}(\mathcal{E}_k)$. This proves Claim 2.7.

Since $2 \leq i < j \leq d$, we have $\frac{i+j-d}{2} \leq \frac{d}{2} - 1$, and in particular it follows that $G_{d/2}$ is a (possibly empty) complete graph, whose number of vertices is denoted by $\omega \geq 0$. Note that the energy \mathcal{E} of a complete graph on ω vertices is equal to $\omega(q-1)^{\omega-1}$, while the energy \mathcal{E}_0 of G_0 is equal to $|A_0 \cap 2\mathbb{N} \cap (d/2, d]| \leq \frac{d}{4}$. For a vertex $u \in L(r, \frac{d}{2})$, let $\omega_u = |A_u \cap 2\mathbb{N} \cap (d/2, d]|$ (this is the number of distinct even depths at which a vertex colored c appears in the subtree of height $\frac{d}{2}$ rooted in u). It follows from Claim 2.7 that the average of $\omega_u(q-1)^{\omega_u-1}$, over all vertices $u \in L(r, \frac{d}{2})$, is at most $\frac{d}{4}$. Let a be the average of ω_u , over all vertices $u \in L(r, \frac{d}{2})$. By Jensen's inequality and the convexity of the function $x \mapsto x(q-1)^{x-1}$ for $x \geq 0$, we have that $a(q-1)^{a-1} \leq \frac{d}{4}$, and thus $a \leq \frac{\log(d/2)}{\log(q-1)} + 1$.

Note that a depends on the color c under consideration (to make this more explicit, let us now write a_c instead of a). Since there are $\frac{d}{4}$ even depths between depth $\frac{d}{2}$ and depth d, there is a color $c \in \{1, \ldots, C\}$ such that $a_c \cdot C \ge \frac{d}{4}$ and thus, $C \ge \frac{d}{4a_c} \ge \frac{d \log(q-1)}{4 \log(d/2) + 4 \log(q-1)}$, as desired.

We now explain how the results proved above give a negative answer to Problem 1.1. Let U_3^d (resp. Q_3^d) be obtained from T_3^d (resp. P_3^d) by adding an edge uv for any pair of vertices u, v having the same parent. Note that for any d, U_3^d and Q_3^d are outerplanar (and thus, planar) and chordal, and Q_3^d has pathwidth 2 (U_3^3 and Q_3^5 are depicted in Figure 2) and the original copies of T_3^d and P_3^d are spanning trees of U_3^d and Q_3^d , respectively. In the remainder of this section, whenever we write T_3^d , we mean the original copy of T_3^d in U_3^d .

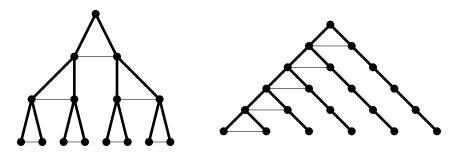


FIGURE 2. The graphs U_3^3 (left) and Q_3^5 (right). The bold edges represent the original copies of T_3^3 and P_3^5 , respectively.

Observe that for any two vertices u and v distinct from the root of T_3^d , u and v are at distance d in T_3^d if and only if they are at distance d-1 in U_3^d (since the depth of T_3^d is d, the fact that u and v differ from the root and are at distance d apart implies that none of the two vertices is an ancestor of the other). The same property holds for Q_3^d and P_3^d . As a consequence, for any odd integer d, $\chi(U_3^{d+1}, d)$ and $\chi(T_3^{d+1}, d+1)$ differ by at most one, and $\chi(Q_3^{d+1}, d)$ and $\chi(P_3^{d+1}, d+1)$ also differ by at most one. Using this observation, we immediately obtain the following corollary of Theorem 2.5 and Corollary 2.4, which gives a negative answer to Problem 1.1.

Corollary 2.8. For any odd integer d,

$$\chi(U_3^{d+1},d) \geq \frac{(d+1)\log(2)}{4\log((d+1)/2) + 4\log(2)} - 1 \ \ and \ \ \chi(Q_3^{d+1},d) \geq \log_2(d+8) - 3.$$

The graphs U_3^{d+1} and its exact d-th power have $n=2^{d+2}$ vertices, and thus the chromatic number of the exact d-th power of U_3^{d+1} grows as $\Omega\left(\frac{\log n}{\log\log n}\right)$. The graphs Q_3^{d+1} and its exact d-th power have $n=\binom{d+2}{2}$ vertices, and thus the chromatic number of the exact d-th power of Q_3^{d+1} grows as $\Omega(\log n)$. It is not difficult (using Theorem 2.1 for U_3^{d+1}) to show that these bounds are asymptotically tight.

It was recently proved by Quiroz [8] that if G is a chordal graph of clique number at most $t \geq 2$, and d is an odd number, then $\chi(G, d) \leq {t \choose 2}(d+1)$. By Corollary 2.8, the

graph U_3^d shows that this is asymptotically best possible (as d tends to infinity), up to a log d factor.

3. Interval coloring

For an integer d and a real c > 1, recall that $\chi(T_q, [d, cd])$ denotes the smallest number of colors in a coloring of the vertices of T_q such that any two vertices of T_q at distance at least d and at most cd apart have distinct colors. Parlier and Petit [6] proved that

$$q(q-1)^{\lfloor cd/2\rfloor - \lfloor d/2\rfloor} \leq \chi(T_q,[d,cd]) \leq (q-1)^{\lfloor cd/2+1\rfloor} (\lfloor cd\rfloor + 1).$$

In this final section, we prove that their lower bound (which is proved by finding a set of vertices of this cardinality that are pairwise at distance at least d and at most cd apart in T_a) is asymptotically tight.

Theorem 3.1. For any integers $q \geq 3$ and d and any real c > 1, $\chi(T_q, [d, cd]) \leq \frac{q}{q-2}(q-1)^{\lfloor cd/2 \rfloor - d/2 + 1} + cd + 1$.

Proof. The proof is similar to the proof of Theorem 2.1. We consider any ordering e_1, e_2, \ldots of the edges of T_q obtained from a breadth-first search starting at r. Then, for any $i=1,2,\ldots$ in order, we assign a color $c(e_i)$ to the edge e_i as follows. Let $e_i=uv$, with u being the parent of v, and let $\ell=\lfloor cd/2\rfloor-d/2$. We assign to uv a color c(uv) distinct from the colors of all the edges xy (with x being the parent of y) such that x is at distance at most ℓ from u^k (where k is the minimum of ℓ and the depth of u), or x is an ancestor of u at distance at most cd from u (and u lies on the path from u to u). There are at most u0 and u1 are u2 and u3 are u4 and u5 are u5 and u6 are u6 are u6 are all the edges following this procedure by using a total of at most u6 are u7 and u8 are colors.

As in the proof of Theorem 2.1, we now define our coloring of the vertices of T_q as follows: first color all the vertices at distance at most $\frac{d}{2} - 1$ from r with a new color that does not appear on any edge of T_q , then for each vertex v with parent u, we color all the vertices of $L(v, \frac{d}{2} - 1)$ with color c(uv). In this vertex-coloring, at most $\frac{q}{q-2}(q-1)^{\ell+1} + cd + 1$ colors are used.

Assume that two vertices s and t, at distance at least d and at most cd apart, were assigned the same color. This implies that $c(s^{d/2-1}s^{d/2}) = c(t^{d/2-1}t^{d/2})$. Assume without loss of generality that the depth of s is at least the depth of t, and consider first the case where $t^{d/2-1}$ is an ancestor of s. Then $t^{d/2}$ is an ancestor of $s^{d/2}$ at distance at most cd from $s^{d/2}$ (and $t^{d/2-1}$ lies on the path from $s^{d/2}$ to $t^{d/2}$), which contradicts the definition of our edge-coloring c. Thus, we can assume that $t^{d/2-1}$ is not an ancestor of s. This implies that $t^{d/2-1}t^{d/2}$ lies on the path between s and t, and therefore $t^{d/2}$ is at distance at most $\ell = \lfloor cd/2 \rfloor - d/2$ from the ancestor of $s^{d/2}$ at distance ℓ from $s^{d/2}$ (or simply from r, if the depth of $s^{d/2}$ is at most ℓ). Again, this contradicts the definition of our coloring c. We obtained a coloring of the vertices of T_q with at most $\frac{q}{q-2}(q-1)^{\ell+1} + cd + 1$ colors in which each pair of vertices at distance at least d and at most cd apart have distinct colors, as desired.

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