# EXACT DISTANCE COLORING IN TREES 

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#### Abstract

For an integer $q \geq 2$ and an even integer $d$, consider the graph obtained from a large complete $q$-ary tree by connecting with an edge any two vertices at distance exactly $d$ in the tree. This graph has clique number $q+1$, and the purpose of this short note is to prove that its chromatic number is $\Theta\left(\frac{d \log q}{\log d}\right)$. It was not known that the chromatic number of this graph grows with $d$. As a simple corollary of our result, we give a negative answer to a problem of Van den Heuvel and Naserasr, asking whether there is a constant $C$ such that for any odd integer $d$, any planar graph can be colored with at most $C$ colors such that any pair of vertices at distance exactly $d$ have distinct colors. Finally, we study interval coloring of trees (where vertices at distance at least $d$ and at most $c d$, for some real $c>1$, must be assigned distinct colors), giving a sharp upper bound in the case of bounded degree trees.


## 1. Introduction

Given a metric space $X$ and some real $d>0$, let $\chi(X, d)$ be the minimum number of colors in a coloring of the elements of $X$ such that any two elements at distance exactly $d$ in $X$ are assigned distinct colors. The classical Hadwiger-Nelson problem asks for the value of $\chi\left(\mathbb{R}^{2}, 1\right)$, where $\mathbb{R}^{2}$ is the Euclidean plane. It is known that $5 \leq \chi\left(\mathbb{R}^{2}, 1\right) \leq 7$ [1] and since the Euclidean plane $\mathbb{R}^{2}$ is invariant under homothety, $\chi\left(\mathbb{R}^{2}, 1\right)=\chi\left(\mathbb{R}^{2}, d\right)$ for any real $d>0$. Let $\mathbb{H}^{2}$ denote the hyperbolic plane. Kloeckner [3] proved that $\chi\left(\mathbb{H}^{2}, d\right)$ is at most linear in $d$ (the multiplicative constant was recently improved by Parlier and Petit [6]), and observed that $\chi\left(\mathbb{H}^{2}, d\right) \geq 4$ for any $d>0$. He raised the question of determining whether $\chi\left(\mathbb{H}^{2}, d\right)$ grows with $d$ or can be bounded independently of $d$. As noticed by Kahle (see [3]), it is not known whether $\chi\left(\mathbb{H}^{2}, d\right) \geq 5$ for some real $d>0$. Parlier and Petit [6] recently suggested to study infinite regular trees as a discrete analog of the hyperbolic plane. Note that any graph $G$ can be considered as a metric space (whose elements are the vertices of $G$ and whose metric is the graph distance in $G$ ), and in this context $\chi(G, d)$ is precisely the minimum number of colors in a vertex coloring of $G$ such that vertices at distance $d$ apart are assigned different colors. Note that $\chi(G, d)$ can be equivalently defined as the chromatic number of the exact $d$-th power of $G$, that is, the graph with the same vertex-set as $G$ in which two vertices are adjacent if and only if they are at distance exactly $d$ in $G$.

Let $T_{q}$ denote the infinite $q$-regular tree. Parlier and Petit [6] observed that when $d$ is odd, $\chi\left(T_{q}, d\right)=2$ and proved that when $d$ is even, $q \leq \chi\left(T_{q}, d\right) \leq(d+1)(q-1)$. A

[^0] 16-CE40-0009-01) and LabEx PERSYVAL-Lab (ANR-11-LABX-0025-01) and LabEx CIMI.
similar upper bound can also be deduced from the results of Van den Heuvel, Kierstead, and Quiroz [2], while the lower bound is a direct consequence of the fact that when $d$ is even, the clique number of the exact $d$-th power of $T_{q}$ is $q$ (note that it does not depend on $d$ ). In this short note, we prove that when $q \geq 3$ is fixed,
$$
\frac{d \log (q-1)}{4 \log (d / 2)+4 \log (q-1)} \leq \chi\left(T_{q}, d\right) \leq(2+o(1)) \frac{d \log (q-1)}{\log d}
$$
where the asymptotic $o(1)$ is in terms of $d$. A simple consequence of our main result is that for any even integer $d$, the exact $d$-th power of a complete binary tree of depth $d$ is of order $\Theta(d / \log d)$ (while its clique number is equal to 3 ).

The following problem (attributed to Van den Heuvel and Naserasr) was raised in [4] (see also [2] and [5]).

Problem 1.1 (Problem 11.1 in [4]). Is there a constant $C$ such that for every odd integer $d$ and every planar graph $G$ we have $\chi(G, d) \leq C$ ?

We will show that our result on large complete binary trees easily implies a negative answer to Problem 1.1. More precisely, we will prove that the graph $U_{3}^{d}$ obtained from a complete binary tree of depth $d$ by adding an edge between any two vertices with the same parent gives a negative answer to Problem 1.1 (in particular, for odd $d$, the chromatic number of the exact $d$-th power of $U_{3}^{d}$ grows as $\Theta(d / \log d)$ ). We will also prove that the exact $d$-th power of a specific subgraph $Q_{3}^{d}$ of $U_{3}^{d}$ grows as $\Omega(\log d)$. Note that $U_{3}^{d}$ and $Q_{3}^{d}$ are outerplanar (and thus, planar) and chordal (see Figure 24).

Kloeckner [3] proposed the following variant of the original problem: For a metric space $X$, an integer $d$ and a real $c>1$, we denote by $\chi(X,[d, c d])$ the smallest number of colors in a coloring of the elements of $X$ such that any two elements of $X$ at distance at least $d$ and at most $c d$ apart have distinct colors. Considering as above the natural metric space defined by the infinite $q$-regular tree $T_{q}$, Parlier and Petit [6] proved that

$$
q(q-1)^{\lfloor c d / 2\rfloor-\lfloor d / 2\rfloor} \leq \chi\left(T_{q},[d, c d]\right) \leq(q-1)^{\lfloor c d / 2+1\rfloor}(\lfloor c d\rfloor+1)
$$

We will show that $\chi\left(T_{q},[d, c d]\right) \leq \frac{q}{q-2}(q-1)^{\lfloor c d / 2\rfloor-d / 2+1}+c d+1$, which implies that the lower bound of Parlier and Petit [6] (which directly follows from a clique size argument) is asymptotically sharp.

## 2. Exact distance coloring

Throughout the paper, we assume that the infinite $q$-regular tree $T_{q}$ is rooted in some vertex $r$. This naturally defines the children and descendants of a vertex and the parent and ancestors of a vertex distinct from $r$. In particular, given a vertex $u$, we define the ancestors $u^{0}, u^{1}, \ldots$ of $u$ inductively as follows: $u^{0}=u$ and for any $i$ such that $u^{i}$ is not the root, $u^{i+1}$ is the parent of $u^{i}$. With this notation, $u^{d}$ can be equivalently defined as the ancestor of $u$ at distance $d$ from $u$ (if such a vertex exists). For a given vertex $u$ in $T_{q}$, the depth of $u$, denoted by depth $(u)$, is the distance between $u$ and $r$ in $T_{q}$. For a vertex $v$ and an integer $\ell$, we define $L(v, \ell)$ as the set of descendants of $v$ at distance exactly $\ell$ from $v$ in $T_{q}$.

We first prove an upper bound on $\chi\left(T_{q}, d\right)$.
Theorem 2.1. For any integer $q \geq 3$, any even integer $d$, and any integer $k \geq 1$ such that $k(q-1)^{k-1} \leq d$, we have $\chi\left(T_{q}, d\right) \leq(q-1)^{k}+(q-1)^{\lfloor k / 2\rfloor}+\frac{d}{k}+1$. In particular, $\chi\left(T_{q}, d\right) \leq d+q+1$, and when $q$ is fixed and $d$ tends to infinity, $\chi\left(T_{q}, d\right) \leq(2+o(1)) \frac{d \log (q-1)}{\log d}$.
Proof. A vertex of $T_{q}$ distinct from $r$ and whose depth is a multiple of $k$ is said to be a special vertex. Let $v$ be a special vertex. Every special vertex $u$ distinct from $v$ such that $u^{k}=v^{k}$ is called a cousin of $v$. Note that $v$ has at most $q(q-1)^{k-1}-1$ cousins (at most $(q-1)^{k}-1$ if $\left.v^{k} \neq r\right)$. A special vertex $u$ is said to be a relative of $v$ if $u$ is either a cousin of $v$, or $u$ has the property that $u$ and $v^{k}$ have the same depth and are at distance at most $k$ apart in $T_{q}$. Two vertices $a, b$ at distance at most $k$ apart and at the same depth must satisfy $a^{\lfloor k / 2\rfloor}=b^{\lfloor k / 2\rfloor}$, and so the number of vertices $u$ such that $u$ and $v^{k}$ have the same depth and are at distance at most $k$ apart in $T_{q}$ is $(q-1)^{\lfloor k / 2\rfloor}$. It follows that if $v^{k}=r$, then $v$ has at most $q(q-1)^{k-1}-1$ relatives and otherwise $v$ has at most $(q-1)^{k}+(q-1)^{\lfloor k / 2\rfloor}-1$ relatives.

The first step is to define a coloring $C$ of the special vertices of $T_{q}$. This will be used later to define the desired coloring of $T_{q}$, i.e. a coloring such that vertices of $T_{q}$ at distance $d$ apart are assigned distinct colors (in this second coloring, the special vertices will not retain their color from $C$ ).

We greedily assign a color $C(v)$ to each special vertex $v$ of $T_{q}$ as follows: we consider the vertices of $T_{q}$ in a breadth-first search starting at $r$, and for each special vertex $v$ we encounter, we assign to $v$ a color distinct from the colors already assigned to its relatives, and from the set of ancestors $v^{i k}$ of $v$, where $2 \leq i \leq \frac{d}{k}+1$ (there are at most $\frac{d}{k}$ such vertices). Note that if $v^{k}=r$, the number of colors forbidden for $v$ is at most $q(q-1)^{k-1}-1$ and if $v^{k} \neq r$ the number of colors forbidden for $v$ is at most $(q-1)^{k}+(q-1)^{\lfloor k / 2\rfloor}+\frac{d}{k}-1$. Since $k(q-1)^{k-1} \leq d$, in both cases $v$ has at most $(q-1)^{k}+(q-1)^{\lfloor k / 2\rfloor}+\frac{d}{k}-1$ forbidden colors, therefore we can obtain the coloring $C$ by using at most $(q-1)^{k}+(q-1)^{\lfloor k / 2\rfloor}+\frac{d}{k}$ colors.

For any special vertex $v$, the set of descendants of $v$ at distance at least $d / 2-k$ and at most $d / 2-1$ from $v$ is denoted by $K(v, k)$. We now define the desired coloring of $T_{q}$ as follows: for each special vertex $v$, all the vertices of $K(v, k)$ are assigned the color $C(v)$. Finally, all the vertices at distance at most $d / 2-1$ from $r$ are colored with a single new color (note that any two vertices in this set lie at distance less than $d$ apart). The resulting vertex-coloring of $T_{q}$ is called $c$. Note that $c$ uses at most $(q-1)^{k}+(q-1)^{\lfloor k / 2\rfloor}+\frac{d}{k}+1$ colors, and indeed every vertex of $T_{q}$ gets exactly one color.

We now prove that vertices at distance $d$ apart in $T_{q}$ are assigned distinct colors in $c$. Assume for the sake of contradiction that two vertices $x$ and $y$ at distance $d$ apart were assigned the same color. Then the depth of both $x$ and $y$ is at least $d / 2$. We can assume by symmetry that the difference $t$ between the depth of $x$ and the depth of $y$ is such that $0 \leq t \leq d$ since otherwise they would be at distance more than $d$. Let $u$ be the unique (special) vertex of $T_{q}$ such that $x \in K(u, k)$ and $v$ be the unique (special) vertex such that $y \in K(v, k)$. By the definition of our coloring $c$, we have $C(u)=C(v)$. Note that $u$ and $v$ are distinct; indeed, otherwise $x$ and $y$ would not be at distance $d$ in $T_{q}$. Assume first that
$u$ and $v$ have the same depth. Then since $u$ and $x$ (resp. $v$ and $y$ ) are distance at least $d / 2-k$ apart, $u$ and $v$ are cousins (and thus, relatives), which contradicts the definition of the vertex-coloring $C$. We may, therefore, assume that the depths of $u$ and $v$ are distinct. Moreover, since $u$ and $v$ are special vertices, we may assume that their depths differ by at least $k$. In particular, $u$ lies deeper than $v$ in $T_{q}$.

First assume that the depths of $u$ and $v$ differ by at least $2 k$. Then $v$ is not an ancestor of $u$ in $T_{q}$. Indeed, for otherwise we would have $v=u^{i k}$ for some integer $2 \leq i \leq \frac{d}{k}+1$, which would contradict the definition of $C$. This implies that the distance between $x$ and $y$ is at least $d / 2-k+d / 2-k+2 k+2=d+2$, which is a contradiction. Therefore, we can assume that the depths of $u$ and $v$ differ by precisely $k$. Since $v$ is not a relative of $u$, we have that $v \neq u^{k}$ and the distance between $u^{k}$ and $v$ is more than $k$. Moreover, since $u$ and $x$ (resp. $v$ and $y$ ) are at distance at least $d / 2-k$ apart, this implies that the distance between $x$ and $y$ is more than $d / 2-k+k+k+d / 2-k=d$, a contradiction.

Thus, $c$ is a proper coloring.
By taking $k=1$ we obtain a coloring $c$ using at most $(q-1)^{1}+(q-1)^{\lfloor 1 / 2\rfloor}+\frac{d}{1}+1=q+d+1$ colors, and by taking $k=\left\lfloor\frac{\log d-\log \log d+\log \log (q-1)}{\log (q-1)}\right\rfloor$, we obtain a coloring $c$ using at most

$$
\frac{d \log (q-1)}{\log d}+\sqrt{\frac{d \log (q-1)}{\log d}}+\frac{d \log (q-1)}{\log d-\log \log d+\log \log (q-1)-\log (q-1)}+1=(2+o(1)) \frac{d \log (q-1)}{\log d}
$$

colors.
For $k=1$, the proof above can be optimized to show that $\chi\left(T_{q}, d\right) \leq q+\frac{d}{2}$ (by simply noting that vertices at even depth and vertices at odd depth can be colored independently). Since we are mostly interested in the asymptotic behaviour of $\chi\left(T_{q}, d\right)$ (which is of order $\left.O\left(\frac{d}{\log d}\right)\right)$, we omit the details.

We now prove a simple lower bound on $\chi\left(T_{q}, d\right)$. Let $T_{q}^{d}$ be the rooted complete $(q-1)$ ary tree of depth $d$, with root $r$. Note that each node has $q-1$ children, so this graph is a subtree of $T_{q}$.
Theorem 2.2. For any integer $q \geq 3$ and any even $d$, $\chi\left(T_{q}^{d}, d\right) \geq \log _{2}\left(\frac{d}{4}+q-1\right)$.
Proof. Consider any coloring of $T_{q}^{d}$ with colors $1,2, \ldots, C$, such that vertices at distance precisely $d$ apart have distinct colors. For any vertex $v$ at depth at most $\frac{d}{2}+1$ in $T_{q}^{d}$, the set of colors appearing in $L\left(v, \frac{d}{2}-1\right)$ is denoted by $S_{v}$. Observe that if $v$ and $w$ have the same parent, then $S_{v}$ and $S_{w}$ are disjoint since for any $x \in L\left(v, \frac{d}{2}-1\right)$ and $y \in L\left(w, \frac{d}{2}-1\right)$, $x$ and $y$ are at distance $d$.

Fix some vertex $u$ at depth at most $\frac{d}{2}$ in $T_{q}^{d}$ and some child $v$ of $u$. We claim that:
Claim 2.3. For any integer $1 \leq k \leq \frac{\operatorname{depth}(u)}{2}$, there is a color of $S_{u^{2 k-1}}$ that does not appear in $S_{v}$.

To see that Claim 2.3 holds, observe that in the subtree of $T_{q}^{d}$ rooted in $u^{k}$, there is a vertex of $L\left(u^{2 k-1}, \frac{d}{2}-1\right)$ at distance $d$ from all the elements of $L\left(v, \frac{d}{2}-1\right)$. The color of such a vertex does not appear in $S_{v}$, therefore Claim 2.3 holds.

In particular, Claim 2.3 implies that all the sets $\left\{S_{u^{2 k-1}} \mid 1 \leq k \leq d / 4\right\}$ and $\left\{S_{w} \mid w\right.$ is a child of $\left.u\right\}$ are pairwise distinct. Since there are $\frac{d}{4}+q-1$ such sets, we have $\frac{d}{4}+q-1 \leq 2^{C}$ and therefore $C \geq \log _{2}\left(\frac{d}{4}+q-1\right)$, as desired.

It was observed by Stéphan Thomassé that the proof of Theorem 2.2 only uses a small fraction of the graph $T_{q}^{d}$. Consider for simplicity the case $q=3$, and define $P_{3}^{d}$ as the graph obtained from a path $P=v_{0}, v_{1}, \ldots, v_{d}$ on $d$ edges, by adding, for each $1 \leq i \leq d$, a path on $i$ edges ending at $v_{i}$ (see Figure 11). This graph is an induced subgraph of $T_{q}^{d}$ and the proof of Theorem 2.2 directly shows the following ${ }^{11}$.

Corollary 2.4. For any even integer $d, \chi\left(P_{3}^{d}, d\right) \geq \log _{2}(d+8)-2$.


Figure 1. The graph $P_{3}^{4}$.

The proof of Theorem 2.2 can be refined to prove the following better estimate for $T_{q}^{d}$, showing that the upper bound of Theorem 2.1 is (asymptotically) tight within a constant multiplicative factor of 8 .

Theorem 2.5. For any integer $q \geq 3$ and every even integer $d \geq 2, \chi\left(T_{q}^{d}, d\right) \geq$ $\frac{d \log (q-1)}{4 \log (d / 2)+4 \log (q-1)}$.

Proof. Consider any coloring of $T_{q}^{d}$ with colors $1,2, \ldots, C$, such that vertices at distance precisely $d$ apart have distinct colors. We perform a random walk $v_{0}, v_{1}, \ldots, v_{d}$ in $T_{q}^{d}$ as follows: we start with $v_{0}=r$, and for each $i \geq 1$, we choose a child of $v_{i}$ uniformly at random and set it as $v_{i+1}$. Note that the depth of each vertex $v_{i}$ is precisely $i$.

From now on we fix a color $c \in\{1, \ldots, C\}$. For any vertex $v$ of $T_{q}^{d}$, the set of vertices contained in the subtree of $T_{q}^{d}$ rooted in $v$ is denoted by $V_{v}$, and we set $A_{v}=\{\operatorname{depth}(u) \mid u \in$ $V_{v}$ and $u$ has color $\left.c\right\}$. When $v=v_{i}$, for some integer $0 \leq i \leq d$, we write $A_{i}$ instead of $A_{v_{i}}$.

Claim 2.6. Assume that for some even integers $i$ and $j$ with $2 \leq i<j \leq d$, and for some vertex $v$ at depth $\frac{i+j-d}{2}$, the set $A_{v}$ contains both $i$ and $j$. Then $v$ has precisely one child $u$ such that $A_{u}$ contains $i$ and $j$, and moreover all the children $w$ of $v$ distinct from $u$ are such that $A_{w}$ contains neither $i$ nor $j$.

[^1]To see that Claim 2.6 holds, simply note that $\frac{i+j-d}{2}<i<j$ and if two vertices $u_{1}, u_{2}$ colored $c$ are respectively at depths $i$ and $j$, and their common ancestor is $v$, then they are at distance $d$ in $T_{q}^{d}$ (which contradicts the fact that they were assigned the same color). Indeed, the distance of $u_{1}$ to $v$ is $i-\frac{i+j-d}{2}$ and the distance of $u_{2}$ to $v$ is $j-\frac{i+j-d}{2}$. This proves the claim.

We now define a family of graphs $\left(G_{k}\right)_{0 \leq k \leq d / 2}$ as follows. For any $0 \leq k \leq \frac{d}{2}$, the vertexset $V\left(G_{k}\right)$ of $G_{k}$ is the set $A_{k} \cap 2 \mathbb{N} \cap(d / 2, d]$, and two (distinct) even integers $i, j \in A_{k}$ are adjacent in $G_{k}$ if and only if $\frac{i+j-d}{2}<k$. For each $0 \leq k \leq \frac{d}{2}$ we define the energy $\mathcal{E}_{k}$ of $G_{k}$ as follows: $\mathcal{E}_{k}=\sum_{i \in V\left(G_{k}\right)}(q-1)^{\operatorname{deg}(i)}$, where $\operatorname{deg}(i)$ denotes the degree of the vertex $i$ in $G_{k}$.

Note that each graph $G_{k}$ depends on the (random) choice of $v_{1}, v_{2}, \ldots, v_{k}$.
Claim 2.7. For any $0 \leq k \leq \frac{d}{2}-1, \mathbb{E}\left(\mathcal{E}_{k+1}\right) \leq \mathbb{E}\left(\mathcal{E}_{k}\right)$.
Assume that $v_{1}, v_{2}, \ldots, v_{k}$ (and therefore also $G_{k}$ ) are fixed. Observe that $G_{k+1}$ is obtained from $G_{k}$ by possibly removing some vertices and adding some edges. Thus, $\mathcal{E}_{k+1}$ can be larger than $\mathcal{E}_{k}$ only if $G_{k+1}$ contains edges that are not in $G_{k}$. Therefore, it suffices to consider the contributions of those pairs of nonadjacent vertices in $G_{k}$ which could become adjacent in $G_{k+1}$ (since these correspond to pairs $i, j$ with $k=\frac{i+j-d}{2}$, these pairs are pairwise disjoint), and prove that these contributions are, in expectation, equal to 0 . Fix a pair of even integers $i<j$ in $V\left(G_{k}\right)$ with $k=\frac{i+j-d}{2}$ (and note that $i$ and $j$ are not adjacent in $G_{k}$ ). By Claim 2.6, either $v_{k+1}$ is such that $A_{k+1}$ contains $i$ and $j$ (this event occurs with probability $\frac{1}{q-1}$ ), or $A_{k+1}$ contains neither $i$ nor $j$ (with probability $1-\frac{1}{q-1}$ ). As a consequence, for any $i<j$ in $V\left(G_{k}\right)$ with $k=\frac{i+j-d}{2}$, with probability $\frac{1}{q-1}$ we add the edge $i j$ in $G_{k+1}$ and with probability $1-\frac{1}{q-1}$ we remove vertices $i$ and $j$ from $G_{k+1}$. This implies that for any $i, j \in V\left(G_{k}\right), i<j$, with $k=\frac{i+j-d}{2}$, with probability $\frac{1}{q-1}$ we have contribution at most $(q-1)^{\operatorname{deg}(i)+1}+(q-1)^{\operatorname{deg}(j)+1}-(q-1)^{\operatorname{deg}(i)}-(q-1)^{\operatorname{deg}(j)}=(q-2)\left((q-1)^{\operatorname{deg}(i)}+(q-1)^{\operatorname{deg}(j)}\right)$ to $\mathcal{E}_{k+1}$ (where deg refers to the degree in $G_{k}$ ) and with probability $1-\frac{1}{q-1}$ we have a contribution of at most $-(q-1)^{\operatorname{deg}(i)}-(q-1)^{\operatorname{deg}(j)}$ to $\mathcal{E}_{k+1}$. Thus, the expected contribution of such a pair $i, j$ is at most $\frac{1}{q-1}(q-2)\left((q-1)^{\operatorname{deg}(i)}+(q-1)^{\operatorname{deg}(j)}\right)-\frac{q-2}{q-1}\left((q-1)^{\operatorname{deg}(i)}+(q-1)^{\operatorname{deg}(j)}\right)=0$.

Summing over all such pairs $i, j$, we obtain $\mathbb{E}\left(\mathcal{E}_{k+1}\right) \leq \mathbb{E}\left(\mathcal{E}_{k}\right)$. This proves Claim 2.7.
Since $2 \leq i<j \leq d$, we have $\frac{i+j-d}{2} \leq \frac{d}{2}-1$, and in particular it follows that $G_{d / 2}$ is a (possibly empty) complete graph, whose number of vertices is denoted by $\omega \geq 0$. Note that the energy $\mathcal{E}$ of a complete graph on $\omega$ vertices is equal to $\omega(q-1)^{\omega-1}$, while the energy $\mathcal{E}_{0}$ of $G_{0}$ is equal to $\left|A_{0} \cap 2 \mathbb{N} \cap(d / 2, d]\right| \leq \frac{d}{4}$. For a vertex $u \in L\left(r, \frac{d}{2}\right)$, let $\omega_{u}=\left|A_{u} \cap 2 \mathbb{N} \cap(d / 2, d]\right|$ (this is the number of distinct even depths at which a vertex colored $c$ appears in the subtree of height $\frac{d}{2}$ rooted in $u$ ). It follows from Claim 2.7 that the average of $\omega_{u}(q-1)^{\omega_{u}-1}$, over all vertices $u \in L\left(r, \frac{d}{2}\right)$, is at most $\frac{d}{4}$. Let $a$ be the average of $\omega_{u}$, over all vertices $u \in L\left(r, \frac{d}{2}\right)$. By Jensen's inequality and the convexity of the function $x \mapsto x(q-1)^{x-1}$ for $x \geq 0$, we have that $a(q-1)^{a-1} \leq \frac{d}{4}$, and thus $a \leq \frac{\log (d / 2)}{\log (q-1)}+1$.

Note that $a$ depends on the color $c$ under consideration (to make this more explicit, let us now write $a_{c}$ instead of $a$ ). Since there are $\frac{d}{4}$ even depths between depth $\frac{d}{2}$ and depth $d$, there is a color $c \in\{1, \ldots, C\}$ such that $a_{c} \cdot C \geq \frac{d}{4}$ and thus, $C \geq \frac{d}{4 a_{c}} \geq \frac{d \log (q-1)}{4 \log (d / 2)+4 \log (q-1)}$, as desired.

We now explain how the results proved above give a negative answer to Problem 1.1. Let $U_{3}^{d}$ (resp. $Q_{3}^{d}$ ) be obtained from $T_{3}^{d}$ (resp. $P_{3}^{d}$ ) by adding an edge $u v$ for any pair of vertices $u, v$ having the same parent. Note that for any $d, U_{3}^{d}$ and $Q_{3}^{d}$ are outerplanar (and thus, planar) and chordal, and $Q_{3}^{d}$ has pathwidth $2\left(U_{3}^{3}\right.$ and $Q_{3}^{5}$ are depicted in Figure 2) and the original copies of $T_{3}^{d}$ and $P_{3}^{d}$ are spanning trees of $U_{3}^{d}$ and $Q_{3}^{d}$, respectively. In the remainder of this section, whenever we write $T_{3}^{d}$, we mean the original copy of $T_{3}^{d}$ in $U_{3}^{d}$.


Figure 2. The graphs $U_{3}^{3}$ (left) and $Q_{3}^{5}$ (right). The bold edges represent the original copies of $T_{3}^{3}$ and $P_{3}^{5}$, respectively.

Observe that for any two vertices $u$ and $v$ distinct from the root of $T_{3}^{d}, u$ and $v$ are at distance $d$ in $T_{3}^{d}$ if and only if they are at distance $d-1$ in $U_{3}^{d}$ (since the depth of $T_{3}^{d}$ is $d$, the fact that $u$ and $v$ differ from the root and are at distance $d$ apart implies that none of the two vertices is an ancestor of the other). The same property holds for $Q_{3}^{d}$ and $P_{3}^{d}$. As a consequence, for any odd integer $d, \chi\left(U_{3}^{d+1}, d\right)$ and $\chi\left(T_{3}^{d+1}, d+1\right)$ differ by at most one, and $\chi\left(Q_{3}^{d+1}, d\right)$ and $\chi\left(P_{3}^{d+1}, d+1\right)$ also differ by at most one. Using this observation, we immediately obtain the following corollary of Theorem 2.5 and Corollary 2.4, which gives a negative answer to Problem 1.1.

Corollary 2.8. For any odd integer $d$,

$$
\chi\left(U_{3}^{d+1}, d\right) \geq \frac{(d+1) \log (2)}{4 \log ((d+1) / 2)+4 \log (2)}-1 \text { and } \chi\left(Q_{3}^{d+1}, d\right) \geq \log _{2}(d+8)-3
$$

The graphs $U_{3}^{d+1}$ and its exact $d$-th power have $n=2^{d+2}$ vertices, and thus the chromatic number of the exact $d$-th power of $U_{3}^{d+1}$ grows as $\Omega\left(\frac{\log n}{\log \log n}\right)$. The graphs $Q_{3}^{d+1}$ and its exact $d$-th power have $n=\binom{d+2}{2}$ vertices, and thus the chromatic number of the exact $d$-th power of $Q_{3}^{d+1}$ grows as $\Omega(\log n)$. It is not difficult (using Theorem 2.1 for $U_{3}^{d+1}$ ) to show that these bounds are asymptotically tight.

It was recently proved by Quiroz [8] that if $G$ is a chordal graph of clique number at most $t \geq 2$, and $d$ is an odd number, then $\chi(G, d) \leq\binom{ t}{2}(d+1)$. By Corollary 2.8, the
graph $U_{3}^{d}$ shows that this is asymptotically best possible (as $d$ tends to infinity), up to a $\log d$ factor.

## 3. Interval coloring

For an integer $d$ and a real $c>1$, recall that $\chi\left(T_{q},[d, c d]\right)$ denotes the smallest number of colors in a coloring of the vertices of $T_{q}$ such that any two vertices of $T_{q}$ at distance at least $d$ and at most $c d$ apart have distinct colors. Parlier and Petit [6] proved that

$$
q(q-1)^{\lfloor c d / 2\rfloor-\lfloor d / 2\rfloor} \leq \chi\left(T_{q},[d, c d]\right) \leq(q-1)^{\lfloor c d / 2+1\rfloor}(\lfloor c d\rfloor+1)
$$

In this final section, we prove that their lower bound (which is proved by finding a set of vertices of this cardinality that are pairwise at distance at least $d$ and at most $c d$ apart in $T_{q}$ ) is asymptotically tight.

Theorem 3.1. For any integers $q \geq 3$ and $d$ and any real $c>1, \chi\left(T_{q},[d, c d]\right) \leq \frac{q}{q-2}(q-$ 1) ${ }^{\lfloor c d / 2\rfloor-d / 2+1}+c d+1$.

Proof. The proof is similar to the proof of Theorem 2.1. We consider any ordering $e_{1}, e_{2}, \ldots$ of the edges of $T_{q}$ obtained from a breadth-first search starting at $r$. Then, for any $i=$ $1,2, \ldots$ in order, we assign a color $c\left(e_{i}\right)$ to the edge $e_{i}$ as follows. Let $e_{i}=u v$, with $u$ being the parent of $v$, and let $\ell=\lfloor c d / 2\rfloor-d / 2$. We assign to $u v$ a color $c(u v)$ distinct from the colors of all the edges $x y$ (with $x$ being the parent of $y$ ) such that $x$ is at distance at most $\ell$ from $u^{k}$ (where $k$ is the minimum of $\ell$ and the depth of $u$ ), or $x$ is an ancestor of $u$ at distance at most $c d$ from $u$ (and $y$ lies on the path from $u$ to $x$ ). There are at most $c d+\sum_{j=0}^{\ell} q(q-1)^{j} \leq \frac{q}{q-2}(q-1)^{\ell+1}+d-1$ such edges, so we can color all the edges following this procedure by using a total of at most $\frac{q}{q-2}(q-1)^{\ell+1}+c d$ colors.

As in the proof of Theorem 2.1, we now define our coloring of the vertices of $T_{q}$ as follows: first color all the vertices at distance at most $\frac{d}{2}-1$ from $r$ with a new color that does not appear on any edge of $T_{q}$, then for each vertex $v$ with parent $u$, we color all the vertices of $L\left(v, \frac{d}{2}-1\right)$ with color $c(u v)$. In this vertex-coloring, at most $\frac{q}{q-2}(q-1)^{\ell+1}+c d+1$ colors are used.

Assume that two vertices $s$ and $t$, at distance at least $d$ and at most $c d$ apart, were assigned the same color. This implies that $c\left(s^{d / 2-1} s^{d / 2}\right)=c\left(t^{d / 2-1} t^{d / 2}\right)$. Assume without loss of generality that the depth of $s$ is at least the depth of $t$, and consider first the case where $t^{d / 2-1}$ is an ancestor of $s$. Then $t^{d / 2}$ is an ancestor of $s^{d / 2}$ at distance at most $c d$ from $s^{d / 2}$ (and $t^{d / 2-1}$ lies on the path from $s^{d / 2}$ to $t^{d / 2}$ ), which contradicts the definition of our edge-coloring $c$. Thus, we can assume that $t^{d / 2-1}$ is not an ancestor of $s$. This implies that $t^{d / 2-1} t^{d / 2}$ lies on the path between $s$ and $t$, and therefore $t^{d / 2}$ is at distance at most $\ell=\lfloor c d / 2\rfloor-d / 2$ from the ancestor of $s^{d / 2}$ at distance $\ell$ from $s^{d / 2}$ (or simply from $r$, if the depth of $s^{d / 2}$ is at most $\ell$ ). Again, this contradicts the definition of our coloring $c$. We obtained a coloring of the vertices of $T_{q}$ with at most $\frac{q}{q-2}(q-1)^{\ell+1}+c d+1$ colors in which each pair of vertices at distance at least $d$ and at most $c d$ apart have distinct colors, as desired.

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[^1]:    ${ }^{1}$ Stéphan Thomassé noticed that this can also be deduced from the fact that the vertices at depth at least $\frac{d}{2}$ and at most $d$ in the exact $d$-th power of $P_{3}^{d}$ induce a shift graph.

