IMPROPER COLORING OF GRAPHS WITH NO ODD CLIQUE MINOR

DONG YEAP KANG AND SANG-IL OUM

Abstract. As a strengthening of Hadwiger's conjecture, Gerards and Seymour conjectured that every graph with no odd K_t minor is $(t-1)$-colorable. We prove two weaker variants of this conjecture. Firstly, we show that for each $t \geq 2$, every graph with no odd K_t minor has a partition of its vertex set into $6t-9$ sets $V_1, \dots, V_{6t-9}$ such that each V_i induces a subgraph of bounded maximum degree. Secondly, we prove that for each $t \geq 2$, every graph with no odd K_t minor has a partition of its vertex set into $10t-13$ sets $V_1, \dots, V_{10t-13}$ such that each V_i induces a subgraph with components of bounded size. The second theorem improves a result of Kawarabayashi (2008), which states that the vertex set can be partitioned into $496t$ such sets.

2010 *Mathematics Subject Classification*: Primary: 05C15; Secondary: 05C83

1. Introduction

Every graph in this paper is finite and simple. For a nonnegative integer k, a graph G is (properly) k-colorable if there are k pairwise disjoint sets $V_1, \dots, V_k$ with $V(G) = \bigcup_{i=1}^{k} V_i$ such that V_i induces a subgraph of maximum degree 0 for $1 \leq i \leq k$.

In 1943, Hadwiger [11] proposed the following question, which is called "Hadwiger's conjecture", one of the deepest conjectures in graph theory. For more on this conjecture and its variants, the readers are referred to the recent survey of Seymour [26].

Conjecture 1.1 (Hadwiger [11]). *For each integer $t \geq 1$, every graph with no K_t minor is $(t-1)$-colorable.*

Robertson, Seymour, and Thomas [25] proved that the conjecture is true for $t \leq 6$, but the conjecture remains open for $t \geq 7$. Kostochka [19, 20] and Thomason [27, 28] proved that graphs with no K_t minor are $O(t\sqrt{\log t})$-colorable, by showing that these graphs contain a vertex of degree $O(t\sqrt{\log t})$. It is still open whether every graph with no K_t minor is ct-colorable for some $c > 0$ independent of t.

Gerards and Seymour (see [14, Section 6.5]) proposed the following odd-minor variant of Hadwiger's conjecture.

Conjecture 1.2 (Gerards and Seymour (see [14, Section 6.5])). *For each integer $t \geq 1$, every graph with no odd K_t minor is $(t-1)$-colorable.*

Catlin [2] proved this conjecture for $t = 4$, and Guenin [10] announced a proof for $t = 5$, but the proof has not been written. Geelen, Gerards, Reed, Seymour, and Vetta [8] proved that every graph with no odd K_t minor is $O(t\sqrt{\log t})$-colorable.

Date: September 17, 2018.

Key words and phrases. Odd minor, Hadwiger's conjecture, Defective coloring, Improper coloring, Chromatic number.

Supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. NRF-2017R1A2B4005020).

The first author has been supported by TJ Park Science Fellowship of POSCO TJ Park Foundation.

A *defective coloring* (see [3, 31]) is a coloring that relaxes the degree condition. A graph G is *k-colorable with defect d* if there are k pairwise disjoint sets $V_1, \dots, V_k$ with $V(G) = \bigcup_{i=1}^k V_i$ such that every V_i induces a subgraph of maximum degree at most d. Note that G is k-colorable if and only if G is k-colorable with defect 0. A *clustered coloring* is a coloring that relaxes the size of monochromatic components. A graph G is *k-colorable with clustering M* if there are k pairwise disjoint sets $V_1, \dots, V_k$ with $V(G) = \bigcup_{i=1}^k V_i$ such that every V_i induces a subgraph having no component with more than M vertices. For a class $\mathcal{C}$ of graphs, the *defective chromatic number* of $\mathcal{C}$ is the minimum k such that for some d, all graphs in $\mathcal{C}$ are k-colorable with defect d. Similarly the *clustered chromatic number* of $\mathcal{C}$ is the minimum k such that for some M, all graphs in $\mathcal{C}$ are k-colorable with clustering M.

We present two theorems, both of which are relaxations of Conjecture 1.2 for graphs with no odd K_t minor. Our first theorem is about defective coloring.

Theorem 1.3. *For each integer $t \geq 2$, there exists an integer $s = s(t)$ such that every graph G with no odd K_t minor is $(6t-9)$-colorable with defect s.*

We remark that the number $6t - 9$ of colors cannot be reduced to the number less than $t - 1$ (see Theorem 2.1).

Our second theorem is about clustered coloring. Kawarabayashi [16] proved that the class of graphs with no odd K_t minor has clustered chromatic number at most $496t$.

Theorem 1.4 (Kawarabayashi [16])**.** *For each integer $t \geq 2$, there is an integer $C = C(t)$ such that every graph G with no odd K_t minor is $496t$-colorable with clustering C.*

We improve $496t$ to $10t - 13$ as follows.

Theorem 1.5. *For each integer $t \geq 2$, there exists an integer $C = C(t)$ such that every graph G with no odd K_t minor is $(10t-13)$-colorable with clustering C.*

We also remark that $10t - 13$ cannot be reduced to the number less than $t - 1$. Both Theorems 1.3 and 1.5 cannot be extended for list-colorings, which we will discuss in Section 6.

The paper is organized as follows. In Section 2 we review related results on minors, some of which will be used in our proof. We briefly introduce some basic notions in Section 3, discuss the structure of graphs with no odd K_t minor in Section 4, and prove Theorems 1.3 and 1.5 in Section 5. In Section 6, we make some further remarks, including extension of our main results to a slightly larger class of graphs. An appendix reviews elementary concepts of signed graphs and minors.

2. Previous results on improper coloring and forbidden minors

There are many studies regarding improper colorings of graphs with forbidden minors. Kawarabayashi and Mohar [17] proved that the clustered chromatic number of the class of graphs with no K_t minor is at most $\lceil \frac{31}{2}t \rceil$. This was improved to $\lceil \frac{7t-3}{2} \rceil$ by Wood [30]. Edwards, Kang, Kim, Oum, and Seymour [6] investigated defective coloring of graphs with no K_t minor, and proved that the defective chromatic number of the graphs with no K_t minor equals $t - 1$.

Theorem 2.1 (Edwards, Kang, Kim, Oum, and Seymour [6])**.** *For each integer $t \geq 1$, there exists an integer $s(t) = O(t^2 \log t)$ such that every graph G with no K_t minor is $(t-1)$-colorable with defect $s(t)$. Moreover, this is sharp in the sense that we cannot reduce the number $t - 1$ of sets to $t - 2$.*

They also proved that the clustered chromatic number of the class of graphs with no K_t minor is at most $4t-4$. Liu and Oum [21] proved that for every graph H, every graph G with no H-minor and maximum degree at most Δ is 3-colorable with clustering $f(H,\Delta)$ for some function f, which generalizes the result of Esperet and Joret [7] for graphs embeddable on surfaces of bounded Euler genus. Combined with Theorem 2.1, this implies that the clustered chromatic number of the class of graphs with no K_t minor is at most $3t-3$. Van den Heuvel and Wood [29] proved that every graph with no K_t minor is $(2t-2)$-colorable with clustering $\lceil \frac{t-2}{2} \rceil$, using different proofs that do not rely on the excluded minor structure theorem. Dvořák and Norin [5] proved that the clustered chromatic number of the class of graphs with no K_t minor and treewidth at most w is at most $t-1$, and announced that the clustered chromatic number of the class of graphs with no K_t minor equals $t-1$.

What happens if the forbidden graph H is not complete? Let I_t be a graph on t vertices with no edges, and for graphs G and H, let $G+H$ be a graph obtained from the disjoint union of G and H by adding an edge between each vertex of G and each vertex of H. For positive integers s and t, let $K^*_{s,t}$ be a graph obtained from K_s+I_t by subdividing every edge joining vertices of the subgraph K_s once. Recently, Ossona de Mendez, Oum, and Wood [24] investigated defective coloring for various graph classes. One of their results implies the following, which extends Theorem 2.1 to a larger class of graphs.

Theorem 2.2 (Ossona de Mendez, Oum, and Wood [24])**.** *For integers $s,t\geq 1$ and real numbers $\delta_1,\delta_2>0$, there exists $M=M(s,t,\delta_1,\delta_2)$ such that every graph G satisfying the following three conditions is s-colorable with defect M.*

(1) G contains no $K^*_{s,t}$ as a subgraph.
(2) Every subgraph of G has average degree at most δ_1.
(3) For every graph H whose 1*-subdivision is a subgraph of G, the average degree of H is at most δ_2.*

Moreover, this is sharp in the sense that we cannot reduce the number s to $s-1$.

Since $K^*_{s,t}$ is a bipartite K_s+I_t subdivision, Theorem 2.2 implies that the defective chromatic number of the class of graphs with no bipartite K_s+I_t subdivision equals s as follows. (The lower bound is obtained by Theorem 2.1.)

Corollary 2.3. *For positive integers s and t, there is an integer $N=N(s,t)$ such that every graph G with no bipartite K_s+I_t subdivision is s-colorable with defect N.*

Proof. By [1, 18], there is $c_0>0$ such that for each integer $p\geq 1$, every n-vertex graph with average degree at least c_0p^2 contains K_p as a topological minor. Since G contains no bipartite K_s+I_t subdivision, the graph G contains no $K^*_{s,t}$ as a subgraph, and no bipartite K_{s+t} subdivision. If there is a subgraph H of G with average degree at least $2c_0(s+t)^2$, then let H_0 be a bipartite spanning subgraph of H with at least $|E(H)|/2$ edges. Since H_0 has average degree at least $c_0(s+t)^2$, it follows that H contains a bipartite K_{s+t} subdivision, contradicting the assumption on G. Hence every subgraph of G has average degree at most $2c_0(s+t)^2$. If there is a graph H whose 1-subdivision is a subgraph of G, then the average degree of H is at most $c_0(s+t)^2$, because otherwise H contains K_{s+t} as a topological minor, and its 1-subdivision is bipartite.

By Theorem 2.2, G is s-colorable with defect $M(s,t,2c_0(s+t)^2,c_0(s+t)^2)$. □

Since K_t+I_1 is isomorphic to K_{t+1}, Theorem 2.1 can be extended to graphs with no bipartite clique subdivision.

Corollary 2.4. *For each integer $t \geq 1$, the defective chromatic number of the class of graphs with no bipartite K_{t+1} subdivision equals t.*

Mohar, Reed, and Wood [22] studied clustered colorings of graphs with no C_{k+1} minor, where C_{k+1} denotes a cycle of length $k+1$. They proved that for every integer $k \geq 2$, every graph with no C_{k+1} minor is $\lfloor 3\log_2 k \rfloor$-colorable with clustering k, and the number of colors is asymptotically tight. Norin, Scott, Seymour, and Wood [23] proved that for every graph H, the clustered chromatic number of the class of H-minor-free graphs is tied to the tree-depth of H, giving a partial answer to a conjecture in [24].

Liu and Oum [21] proved that for every graph H and every integer Δ, the class of graphs with no odd H minor and maximum degree at most Δ has the clustered chromatic number at most 3.

Theorem 2.5 (Liu and Oum [21])**.** *For every graph H and every integer $\Delta \geq 0$, there is $C = C(H, \Delta)$ such that for every graph G with maximum degree at most Δ and no odd H minor, there are pairwise disjoint subsets V_1, V_2, V_3 of $V(G)$ such that $V_1 \cup V_2 \cup V_3 = V(G)$ and every component of $G[V_i]$ has at most C vertices for $i = 1, 2, 3$.*

3. Preliminaries

We follow the definitions in [4] unless stated otherwise. In this section, G and H always denote graphs. For each integer $N \geq 0$, let $[N]$ be the set $\{1, \ldots, N\}$. If $N = 0$, then $[N]$ is an empty set. For a graph H, let $\Delta(H)$ be the maximum degree of vertices in H. For $S \subseteq V(H)$, let $H[S]$ be the subgraph of H induced by S.

A subset $S \subseteq V(G)$ is *stable* if no two vertices in S are adjacent. Let $G \cup H$ and $G \cap H$ be graphs $(V(G) \cup V(H), E(G) \cup E(H))$ and $(V(G) \cap V(H), E(G) \cap E(H))$, respectively. A pair (A, B) of subgraphs of G is a *separation* of G if $G = A \cup B$. The *order* of a separation (A, B) of G is $|V(A \cap B)|$. A *bipartition* $\{X, Y\}$ of a bipartite graph H is a set of two disjoint stable subsets such that $X \cup Y = V(G)$.

Coloring. For a subset S of vertices of G and a set T of *colors*, a function $\alpha : S \to T$ is a *coloring* on S. A *color class* of a coloring $\alpha : S \to T$ is $\alpha^{-1}(i)$ for some $i \in T$. A subgraph H of G is *monochromatic* if every vertex of H has the same α value.

A coloring α on S is *proper* if $\alpha(u) \neq \alpha(v)$ for every $uv \in E(G[S])$; equivalently, every color class of α is stable. For a nonnegative integer k, a graph G is *k-colorable* if there is a proper coloring $\alpha : V(G) \to [k]$. For integers $k, d \geq 0$, a graph G is *k-colorable* with *defect d* if there is a coloring $\alpha : V(G) \to [k]$ such that every color class induces a graph with maximum degree at most d, and G is *k-colorable* with *clustering* d if there is a coloring $\alpha : V(G) \to [k]$ such that every color class induces a subgraph with no connected components having more than d vertices.

Paths. A path is a graph that consists of k vertices $v_1, \ldots, v_k$ for some integer $k \geq 1$ and $k-1$ edges $v_1v_2, \ldots, v_{k-1}v_k$. The vertices v_1 and v_k are called *ends*, and all other vertices $v_2, \ldots, v_{k-1}$ are called *internal* vertices.

The path P *joins* $u, v \in V(G)$ if u and v are ends of P. For vertices $v, w \in V(P)$, $P(v, w)$ denotes the subpath of P with the ends v and w.

A path P *joins* two sets $V_1, V_2 \subseteq V(G)$ if it joins a vertex in V_1 and a vertex in V_2. The *length* of a path is its number of edges. The *parity* of a path P is the parity of its length.

Two paths P and Q in G are *vertex-disjoint* if $V(P) \cap V(Q) = \emptyset$. They are *internally disjoint*[1] if every common vertex of P and Q is an end of both P and Q.

For $S \subseteq V(G)$, an *S-path* is a path in G that joins two *distinct* vertices in S. For a coloring $\alpha : S \to \{1, 2\}$, an S-path P from u to v in G is *parity-breaking with respect to α* if

$$|E(P)| \not\equiv \alpha(u) - \alpha(v) \pmod 2.$$

For a connected bipartite subgraph H of G with a proper coloring $\beta : V(H) \to \{1, 2\}$, a $V(H)$-path P in G is *parity-breaking with respect to H* if P is parity-breaking with respect to β. This is well defined since a proper coloring of H is unique up to permuting colors. We will use the following observation in Section 4.

Observation 3.1.

(1) *For $S \subseteq V(G)$ and a coloring $\alpha : S \to \{1, 2\}$, let P, Q be internally disjoint S-paths sharing precisely one end. Then the S-path $P \cup Q$ is parity-breaking with respect to α if and only if exactly one of P and Q is parity-breaking with respect to α.*

(2) *For a connected bipartite subgraph H of G, no path in H is parity-breaking with respect to H.*

Minors. A graph H is a *minor* of G if a graph isomorphic to H can be obtained from G by deleting vertices or edges and contracting edges. If there are edges wu and wv for some vertex $w \notin \{u, v\}$ and we contract an edge uv, then one of these two edges is removed after contraction to avoid parallel edges. A graph G *contains an H-minor* (or H as a *minor*) if H is a minor of G.

Topological minors. An *H-subdivision* is a graph obtained from H by subdividing edges, where edges may be subdivided more than once. A graph G *contains an H-subdivision* (or H as a *topological minor*), if G contains a subgraph isomorphic to an H-subdivision.

Since every H-subdivision H' is built from H by replacing all edges of H with internally disjoint paths called the *linking paths*, there are vertices of H' that correspond to vertices of H, which we call *branch vertices*.

Odd minors. For $S \subseteq V(G)$ and a coloring $\alpha : S \to \{1, 2\}$, an edge $uv \in E(G[S])$ is *bichromatic* if $\alpha(u) \neq \alpha(v)$, and is *monochromatic* otherwise.

For graphs G and H, G contains H as an *odd minor* if there exist vertex-disjoint subgraphs $\{T_u\}_{u \in V(H)}$ in G which are trees, and a coloring $\alpha : \bigcup_{u \in V(H)} V(T_u) \to \{1, 2\}$ such that for every $u \in V(H)$, every edge in T_u is bichromatic, and for every edge $vw \in E(H)$, there is a monochromatic edge $e \in E(G)$ that joins $V(T_v)$ and $V(T_w)$.

We will use the following alternative definition in Section 4.

Observation 3.2. *For graphs G and H, G contains H as an odd minor if and only if there exist vertex-disjoint subgraphs $\{T_u\}_{u \in V(H)}$ in G which are trees, and a coloring $\alpha : \bigcup_{u \in V(H)} V(T_u) \to \{1, 2\}$ such that*

(1) *for every $u \in V(H)$, every edge in T_u is bichromatic,*

(2) *there are internally disjoint paths $\{P_e\}_{e \in E(H)}$ in G,*

(3) *for every $e = vw \in E(H)$, P_e joins $V(T_v)$ and $V(T_w)$, has no internal vertex in $\bigcup_{u \in V(H)} V(T_u)$, and is parity-breaking with respect to α.*

We remark that for two graphs G and H, G contains H as an odd minor if and only if a signed graph $(G, E(G))$ contains a signed graph $(H, E(H))$ as a minor, which we will discuss in Appendix A.

[1]It is called *independent* in [4].

4. The structure of graphs with no odd clique minor

The proof of Theorem 2.1 is based on the fact that every graph with no K_t minor has a vertex of degree at most c_t for some c_t. In contrast to graphs with no K_t minor, graphs with no odd K_t minor may have arbitrarily large minimum degree; for example, complete bipartite graphs have no odd K_3 minor.

To prove Theorems 1.3 and 1.5, we use the following strategy similar to the one by Geelen, Gerards, Reed, Seymour, and Vetta [8]. If a graph G has no bipartite subdivision of some graph, then we apply Corollary 2.3. Otherwise, we will show in Theorem 4.5 that G contains a bipartite block after removing few vertices, which allows us to use precoloring arguments in the following section.

First of all, we describe how to find an odd K_t minor in a graph G if the graph contains a bipartite $K_{2t-2}+I_t$ subdivision with many vertex-disjoint parity-breaking paths between branch vertices of the subdivision.

Lemma 4.1. *For $t \geq 2$, let G be a graph that contains a bipartite $K_{2t-2}+I_t$ subdivision H, and C be the set of all branch vertices of $K_{2t-2}+I_t$ in H. If there are $t-1$ vertex-disjoint parity-breaking C-paths with respect to H, then G contains an odd K_t minor.*

Proof. For convenience, we identify each vertex in C with its corresponding vertex of $K_{2t-2}+I_t$.

For a collection $\mathcal{Q}$ of paths, let $\ell(\mathcal{Q}) = \sum_{P\in\mathcal{Q}}|E(P)|$. Let $\mathcal{P}$ be a collection of $t-1$ vertex-disjoint parity-breaking C-paths with respect to H, satisfying the following.

(a) $\sum_{P\in\mathcal{P}}|E(P)\setminus E(H)|$ is minimum, and
(b) subject to (a), $\ell(\mathcal{P})$ is minimum.

Note that every vertex in C is not an internal vertex of a path in $\mathcal{P}$. To see this, if a vertex in C is an internal vertex of a path $Q\in\mathcal{P}$, then Q contains a proper subpath Q' that is a parity-breaking C-path with respect to H. For $\mathcal{Q} := (\mathcal{P}\setminus\{Q\})\cup\{Q'\}$ it follows that $\sum_{P\in\mathcal{Q}}|E(P)\setminus E(H)| \leq \sum_{P\in\mathcal{P}}|E(P)\setminus E(H)|$ and $\ell(\mathcal{Q}) < \ell(\mathcal{P})$, contradicting our choice of $\mathcal{P}$.

For distinct $u, v\in C$, if $uv\in E(K_{2t-2}+I_t)$, then let $Q_{u,v}$ be the linking path from u to v in H. If $uv\notin E(K_{2t-2}+I_t)$, then let $Q_{u,v}$ be a graph with two vertices u and v and no edges. Note that there are $2t-2$ branch vertices that appear in paths in $\mathcal{P}$, and t unused branch vertices. Let $C_0\subseteq C$ be the set of those unused t branch vertices.

Claim 1. *Let u, v be distinct vertices in C.*

1. *If $u, v\in C_0$, then no path $P\in\mathcal{P}$ intersects $Q_{u,v}$.*[2]
2. *If $u\in C_0$, then $Q_{u,v}$ intersects at most one path P in $\mathcal{P}$, and if so, then the intersection is a subpath of both $Q_{u,v}$ and P and contains v.*

Proof of Claim 1. We may assume $uv\in E(K_{2t-2}+I_t)$, otherwise $V(Q_{u,v})\subseteq C$ and the claim is trivial. Let $u\in C_0$, and suppose $Q_{u,v}$ intersects a path in $\mathcal{P}$. Since no path in $\mathcal{P}$ intersects u, starting from u and following $Q_{u,v}$ we arrive at the first vertex $w\in V(Q_{u,v})$ on some path $P\in\mathcal{P}$.

Write $P = A\cup B$, where A and B are two subpaths of P with the only common vertex w. Since $w\in V(H)$, either A or B is parity-breaking with respect to H, and we may assume A is parity-breaking with respect to H. By Observation 3.1, a path $R = A\cup Q_{u,v}(w,u)$ is parity-breaking with respect to H, and it intersects no path in $\mathcal{P}$ other than P. Therefore, we conclude that $(\mathcal{P}\setminus\{P\})\cup\{R\}$ is a set of $t-1$ vertex-disjoint parity-breaking C-paths with

[2] Two subgraphs *intersect* if they share at least one common vertex.

respect to H. By our assumption on $\mathcal{P}$, P does not have more edges not in H than R. This implies $E(B) \subseteq E(H)$, and thus $B = Q_{u,v}(w, v)$ since one of the ends of B is in C and no path in $\mathcal{P}$ intersects u. Note that A intersects $Q_{u,v}$ only at w, since $B = Q_{u,v}(w, v)$.

Since P and $Q_{u,v}$ share a common subpath from w to v and w is the only vertex that belongs to both $Q_{u,v}(w, u)$ and some path in $\mathcal{P}$, P is the only path that intersects $Q_{u,v}$. In particular, $B = P \cap Q_{u,v}$ is a path that contains v, and $v \notin C_0$. ■

Let $\mathcal{P} = \{P_1, \ldots, P_{t-1}\}$, and for $1 \leq i \leq t-1$, let x_i and y_i be the ends of P_i. Let $C_1 = \{x_1, \ldots, x_{t-1}\}$ and $C_2 = \{y_1, \ldots, y_{t-1}\}$.

For $v \in C$, if v corresponds to a vertex in the subgraph K_{2t-2} of $K_{2t-2} + I_t$, then we call v *Type-A*. Otherwise we call v *Type-B*. Let q be the number of i's ($1 \leq i \leq t-1$) such that both x_i and y_i are Type-A, and r be the number of i's such that exactly one of x_i and y_i is Type-A, and $s = t - 1 - q - r$. Then there are $(2t-2) - (2q + r)$ vertices of Type-A in C_0. Since $q + r + s = t - 1$, it follows that $(2t-2) - (2q + r) \geq r + s$. Therefore, the number of vertices in C_0 of Type-A is at least $r + s$. Thus, we choose an ordering $z_1, \ldots, z_t$ of the vertices in C_0 such that for $1 \leq i \leq t-1$ if x_i or y_i is Type-B, then z_i is a vertex of Type-A.

In summary, $z_i x_i, z_i y_i \in E(K_{2t-2} + I_t)$ for $1 \leq i \leq t-1$. Equivalently, if z_i is Type-B, then both x_i and y_i are Type-A.

Let $\beta : V(H) \to \{1, 2\}$ be a proper coloring of H unique up to permuting colors. In order to find an odd K_t minor, we now aim to construct vertex-disjoint subgraphs $M_1, \ldots, M_t$ in G which are trees and a coloring $\alpha : \bigcup_{i=1}^{t} V(M_i) \to \{1, 2\}$ as in Observation 3.2.

For $1 \leq i \leq t-1$, let $M_i = P_i \cup Q_{z_i,y_i}$ if z_i is Type-A, and $M_i = P_i \cup Q_{z_i,x_i}$ if z_i is Type-B. Let M_t be a graph with the only vertex z_t. If $i < t$, then by Claim 1, $P_i \cap Q_{z_i,y_i}$ or $P_i \cap Q_{z_i,x_i}$ is a subpath of P_i and so M_i is a tree with maximum degree at most 3 and at most one vertex of degree 3. We choose a coloring $\alpha : \bigcup_{i=1}^{t} V(M_i) \to \{1, 2\}$ such that

(1) α on $V(M_i)$ is a proper coloring of M_i and $\alpha(x_i) = \beta(x_i)$ for all $1 \leq i \leq t-1$, and
(2) $\alpha(z_t) = \beta(z_t)$ if and only if z_t is Type-B.

Observation 3.1 implies that, for $1 \leq i \leq t-1$, $\alpha(y_i) \neq \beta(y_i)$ since P_i is parity-breaking with respect to H and $\alpha(z_i) = \beta(z_i)$ if and only if z_i is Type-B.

For $1 \leq i < j \leq t$, we are now ready to construct a path $P_{i,j}$ joining $V(M_i)$ and $V(M_j)$ that is parity-breaking with respect to α and satisfies Observation 3.2. This will show that G contains an odd K_t minor. The structure of $P_{i,j}$ depends on the types of z_i and z_j.

Case 1. Both z_i and z_j are Type-A.

Following Q_{z_j,x_i} from z_j to x_i, we arrive at the first vertex $a_{i,j}$ in $V(P_i) \cap V(Q_{z_j,x_i})$. By Claim 1, Q_{z_j,x_i} and P_i share the subpath from $a_{i,j}$ to x_i. Since $P_i(x_i, a_{i,j})$ is in H and $\alpha(x_i) = \beta(x_i)$, it follows that $\alpha(a_{i,j}) = \beta(a_{i,j})$ by Observation 3.1. Let us define $P_{i,j} = Q_{z_j,x_i}(z_j, a_{i,j})$. Since $\alpha(z_j) \neq \beta(z_j)$, $\alpha(a_{i,j}) = \beta(a_{i,j})$ and $P_{i,j}$ is a subpath of Q_{z_j,x_i}, we conclude that $P_{i,j}$ is parity-breaking with respect to α by Observation 3.1.

Case 2. z_i and z_j are of different types.

Let us define $P_{i,j} := Q_{z_i,z_j}$. By Claim 1, $P_{i,j}$ intersects no path in $\mathcal{P}$. Since $\alpha(z_i) = \beta(z_i)$, $\alpha(z_j) \neq \beta(z_j)$, and Q_{z_i,z_j} is in H, $P_{i,j}$ is parity-breaking with respect to α by Observation 3.1.

Case 3. Both z_i and z_j are Type-B.

Since z_i is Type-B, y_i is Type-A. Following Q_{z_j,y_i} from z_j to y_i, we arrive at the first vertex $a_{i,j}$ in $V(P_i) \cap V(Q_{z_j,y_i})$. Claim 1 implies that Q_{z_j,y_i} and P_i share the subpath from $a_{i,j}$ to y_i. Since $P_i(y_i, a_{i,j})$ is in H and $\alpha(y_i) \neq \beta(y_i)$, it follows that $\alpha(a_{i,j}) \neq \beta(a_{i,j})$. Let us define $P_{i,j} := Q_{z_j,y_i}(z_j, a_{i,j})$. Since $\alpha(z_j) = \beta(z_j)$, $\alpha(a_{i,j}) \neq \beta(a_{i,j})$ and $P_{i,j}$ is in H, $P_{i,j}$ is parity-breaking with respect to α by Observation 3.1. □

We use the following lemma, which asserts that the family of S-paths of odd length satisfies the Erdős-Pósa property.

Lemma 4.2 (Geelen, Gerards, Reed, Seymour, and Vetta [8, Lemma 11])**.** *Let G be a graph and $S \subseteq V(G)$. For every integer $\ell \geq 1$, G contains ℓ vertex-disjoint S-paths of odd length, or there is $X \subseteq V(G)$ with $|X| \leq 2\ell - 2$ such that $G \setminus X$ contains no S-path of odd length.*

Observation 4.3. *Let G and H be graphs and $X \subseteq V(G)$. If G contains an H-subdivision K, then $G \setminus X$ contains an H'-subdivision K' such that $H' = H \setminus Y$ for some $Y \subseteq V(H)$ with $|Y| \leq |X|$ and K' is a subgraph of K.*

Proof. It is easy to see that if G has an H-subdivision K and v is a vertex of K, then there is a vertex w of H such that $G \setminus v$ has a $(H \setminus w)$-subdivision. $\square$

The following lemma is a variation of [8, Lemma 15].

Lemma 4.4. *Let ℓ be a positive integer and G be a graph. Let H be a bipartite $K_s + I_t$ subdivision in G for integers $s \geq 2\ell$ and $t \geq 1$, and C be the set of all branch vertices in H. At least one of the following holds.*

- *There exists $X \subseteq V(G)$ with $|X| \leq 2\ell - 2$ such that $G - X$ has a bipartite block U that contains at least $s + t - |X|$ vertices in $C \setminus X$ and all linking paths in H between them.*
- *G has ℓ vertex-disjoint parity-breaking C-paths with respect to H.*

Proof. (1) *We claim that either there are ℓ vertex-disjoint parity-breaking C-paths in G with respect to H, or there is $X \subseteq V(G)$ with $|X| \leq 2\ell - 2$ such that $G \setminus X$ contains no parity-breaking C-path with respect to H.*

Let $\{L, R\}$ be the unique bipartition of H. Without loss of generality, we may assume that every linking path corresponding to an edge in $K_s + I_t$ has even length, because otherwise, for every branch vertex $v \in C \cap L$, we subdivide each edge $e \in E(G)$ incident with v once. This gives an H-subdivision H' and a G-subdivision G' such that H' is a bipartite subgraph of G'. We may assume $V(H) \subseteq V(H')$ and $V(G) \subseteq V(G')$, and then every vertex in $V(G') \setminus V(G)$ has degree 2. Note that all vertices in C are in the same part of the bipartition of H', and thus every path in H' between vertices in C has even length. It is easy to check the following.

- A C-path of odd length in G' corresponds to a parity-breaking C-path in G with respect to H.
- If there is $X' \subseteq V(G')$ with $|X'| \leq 2\ell - 2$ such that $G' \setminus X'$ contains no C-path of odd length, then we may assume $X' \subseteq V(G)$ since every vertex in $V(G') \setminus V(G)$ has degree 2.

Lemma 4.2 claims that either G' contains ℓ vertex-disjoint C-paths of odd length, or there is $X' \subseteq V(G')$ with $|X'| \leq 2\ell - 2$ such that $G' \setminus X'$ contains no C-path of odd length. This proves (1).

Suppose G contains no ℓ vertex-disjoint parity-breaking C-paths with respect to H. By (1), there is $X \subseteq V(G)$ with $|X| \leq 2\ell - 2$ such that $G \setminus X$ contains no parity-breaking C-path with respect to H. For convenience, we identify each vertex in C with its corresponding vertex of $K_s + I_t$. For distinct $u, v \in C$, if $uv \in E(K_s + I_t)$ then let $Q_{u,v}$ be the linking path from u to v in H. If $uv \notin E(K_s + I_t)$, then let $Q_{u,v}$ be a graph with two vertices u and v and no edges.

(2) *We claim that there is a block U in $G \setminus X$ containing at least $s + t - |X|$ vertices in $C \setminus X$ and all linking paths in H between them.*

By Observation 4.3, $G\backslash X$ contains a (K_a+I_b)-subdivision K such that $K_a+I_b = (K_s+I_t)\backslash Y$ for some $Y \subseteq V(K_s + I_t)$ with $|Y| \leq |X|$ and K is a subgraph of H. Let T be the set of all branch vertices in K, where $|T| \geq a + b = s + t - |Y| \geq s + t - |X|$.

Since $a \geq s - |Y| \geq 2$ and $a + b = s + t - |Y| \geq 3$, $K_a + I_b$ is 2-connected. Therefore, K is 2-connected and all vertices in T are in the same block of $G \setminus X$.

(3) *We claim that U is bipartite.*

Suppose U contains an odd-length cycle D. For two distinct vertices $u, v \in C \cap V(U)$, there are two vertex-disjoint paths in U joining $\{u, v\}$ and $V(D)$ by Menger's theorem. Using these paths, we obtain both an odd-length path and an even-length path from u to v in U. One of those paths is a parity-breaking C-path with respect to H, contradicting that $G \setminus X$ has no parity-breaking C-path with respect to H. □

Now we are ready to prove the main theorem of this section.

Theorem 4.5. *Let $t \geq 2$ be an integer, and G be a graph. If G contains no odd K_t minor and contains a bipartite $K_{2t-2} + I_t$ subdivision, then there is $X \subseteq V(G)$ with $|X| \leq 2t - 4$ such that $G \setminus X$ contains a bipartite block U having at least $t + 3$ vertices.*

Proof. Let H be a bipartite $K_{2t-2} + I_t$ subdivision of G, and $C = \{v_1, \ldots, v_{3t-2}\}$ be the set of all branch vertices in H. For convenience, we identify each vertex in C with its corresponding vertex in $V(K_{2t-2} + I_t)$. Let $C_1 \subseteq C$ be the set of branch vertices corresponding to vertices in K_{2t-2}, and $C_2 = C \setminus C_1$ be the set of branch vertices corresponding to vertices in I_t.

By Lemmas 4.1 and 4.4, there exists $X \subseteq V(G)$ with $|X| \leq 2t - 4$ such that $G \setminus X$ has a bipartite block U containing at least $(3t - 2) - |X|$ vertices in $C \setminus X$ and all linking paths between them. Let $C' \subseteq C$ be those $(3t - 2) - |X| \geq t + 2$ branch vertices in U, and H' be the union of all linking paths between vertices in C', which is a subgraph of U.

Recall that we identified $C \subseteq V(H)$ with $V(K_{2t-2} + I_t)$. Since vertices in C_1 form a clique of $K_{2t-2} + I_t$ and $|C' \cap C_1| \geq |C'| - |C_2| \geq 2$, the subgraph of $K_{2t-2} + I_t$ induced by C' is not bipartite. To obtain H' from the induced subgraph of $K_{2t-2} + I_t$, we should subdivide edges at least once, because H' is bipartite. Thus H' contains a vertex other than vertices in C', implying $|V(U)| \geq |V(H')| \geq |C'| + 1 \geq t + 3$. □

5. Proofs of Theorems 1.3 and 1.5

For a class $\mathcal{F}$ of graphs and an integer $d \geq 0$, a graph G has a *$(d, \mathcal{F})$-coloring* if there is $f : V(G) \to [d]$ such that $G[f^{-1}(\{i\})]$ is in $\mathcal{F}$ for all $i \in [d]$. A class $\mathcal{F}$ of graphs is *closed under isomorphisms* if for all $G \in \mathcal{F}$, every graph isomorphic to G is in $\mathcal{F}$. A class $\mathcal{F}$ of graphs is *closed under taking disjoint unions* if for all $G, H \in \mathcal{F}$, the disjoint union of G and H is in $\mathcal{F}$.

Now we are ready to prove the following lemma, following the idea of Kawarabayashi and Mohar [17].

Lemma 5.1. *Let $t \geq 2$ and $d \geq 3$ be integers and $\mathcal{F}$ be a class of graphs closed under isomorphisms and taking disjoint unions, which satisfies the following.*

(i) *$\mathcal{F}$ contains every graph with at most $4t - 7$ vertices.*
(ii) *If a graph H contains no odd K_t minor and no bipartite $K_{2t-2} + I_t$ subdivision, then H has a $(d, \mathcal{F})$-coloring.*

Then every graph with no odd K_t minor has a $(d + 4t - 7, \mathcal{F})$-coloring.

Proof. We prove the following stronger claim.

Claim. *Let G be a graph with no odd K_t minor, $Z \subseteq V(G)$ with $|Z| \leq 4t-7$, and $f : Z \to [d+4t-7]$ be a coloring. Then G has a $(d+4t-7,\mathcal{F})$-coloring g that satisfies the following.*

(a) For every $z \in Z$, $f(z) = g(z)$.
(b) For every $v \in Z$ and its neighbor $w \notin Z$, $g(v) \neq g(w)$.

Let G be a counterexample with the minimum $|V(G)| + |E(G)|$. As the claim is true for graphs with at most $4t-7$ vertices by giving distinct colors to each vertex not in Z, $|V(G)| \geq 4t-6$.

(1) *Z is stable.*

Suppose there are adjacent $z_1, z_2 \in Z$. Applying the claim on $G' = G \setminus z_1z_2$ with the same Z and f, G' has a $(d+4t-7,\mathcal{F})$-coloring g that satisfies the claim. We claim that every component of $G[g^{-1}(\{i\})]$ for some $i \in [d+4t-7]$ is in $\mathcal{F}$. Let C be a component of $G[g^{-1}(\{i\})]$ for some $i \in [d+4t-7]$. If $V(C) \cap Z \neq \emptyset$ then $V(C) \subseteq Z$ by (b), implying $C \in \mathcal{F}$ as $|V(C)| \leq |Z| \leq 4t-7$. If $V(C) \cap Z = \emptyset$ then C is a component of $G'[g^{-1}(\{i\})]$, which implies $C \in \mathcal{F}$. Therefore, g is a $(d+4t-7,\mathcal{F})$-coloring of G satisfying (a) and (b), contradicting our assumption.

(2) *For every separation (A, B) of order at most $2t-3$, either $V(A) \setminus V(B) \subseteq Z$ or $V(B) \setminus V(A) \subseteq Z$.*

Suppose G has a separation (A, B) of order at most $2t-3$ such that both $V(A) \setminus V(B) \setminus Z$ and $V(B) \setminus V(A) \setminus Z$ are nonempty. Since $|Z| = |V(A) \cap Z| + |(V(B) \setminus V(A)) \cap Z|$, we may assume $|(V(B) \setminus V(A)) \cap Z| \leq \lfloor \frac{|Z|}{2} \rfloor \leq 2t-4$. Note that $V(B) \setminus V(A) \setminus Z \neq \emptyset$ implies that $|V(A) \cup Z| < |V(G)|$ and we can apply the claim on $A \cup G[Z]$ with Z and f. Let g_1 be a $(d+4t-7,\mathcal{F})$-coloring of $A \cup G[Z]$ satisfying (a) and (b). Let $Z' = V(A \cap B) \cup (V(B) \cap Z)$. Since $|Z'| = |V(A \cap B)| + |(V(B) \setminus V(A)) \cap Z| \leq 4t-7$, we can apply the claim on B with Z' and $g_1|_{Z'}$. Let g_2 be a $(d+4t-7,\mathcal{F})$-coloring of B satisfying (a) and (b).

Let g be a coloring on $V(G)$ such that for each vertex v of G,

$$g(v) = \begin{cases} g_1(v) & \text{for } v \in V(A), \text{ and} \\ g_2(v) & \text{for } v \in V(B). \end{cases}$$

This is well defined since g_1 is identical to g_2 on Z'. We claim that g is a $(d+4t-7,\mathcal{F})$-coloring of G satisfying (a) and (b), which contradicts our assumption.

By the definition of g_1, it follows that $g(z) = g_1(z) = f(z)$ for every $z \in Z$. For every $vw \in E(G)$ with $v \in Z$ and $w \notin Z$, $g(v) \neq g(w)$ since $g_1(v) = g(v) \neq g(w) = g_2(w)$ if $w \in V(A)$ and $g_2(v) = g(v) \neq g(w) = g_2(w)$ if $w \in V(B)$. This verifies (a) and (b).

Let C be a component of $G[g^{-1}(\{i\})]$ for some $i \in [d+4t-7]$. If $V(C) \cap Z \neq \emptyset$ then $V(C) \cap (V(A) \setminus Z) = \emptyset$ by the definition of g_1 and (b), and $V(C) \cap (V(B) \setminus V(A) \setminus Z) = \emptyset$ by the definition of g_2 and (b). This implies $V(C) \subseteq Z$ and thus $C \in \mathcal{F}$ as $|V(C)| \leq |Z| \leq 4t-7$. If $V(C) \cap Z = \emptyset$ and $V(A) \cap V(B) \cap V(C) \neq \emptyset$ then $V(C) \subseteq Z' \setminus Z \subseteq V(A) \cap V(B)$ by the definition of g_2 and (b). Thus C is a component of $G[g_1^{-1}(\{i\})]$, implying $C \in \mathcal{F}$. Finally, if $V(C) \cap Z = \emptyset$ and $V(A) \cap V(B) \cap V(C) = \emptyset$, then either $V(C) \subseteq V(A) \setminus V(B) \setminus Z$ or $V(C) \subseteq V(B) \setminus V(A) \setminus Z$, which implies that $C \in \mathcal{F}$ as C is a component of either $G[g_1^{-1}(\{i\})]$ or $G[g_2^{-1}(\{i\})]$.

(3) *$G \setminus Z$ contains a bipartite $K_{2t-2} + I_t$ subdivision.*

Since $|Z| \leq 4t-7$, we may assume $f(Z) \subseteq \{d+1, \ldots, d+4t-7\}$ by permuting colors. Suppose $G \setminus Z$ does not contain a bipartite $K_{2t-2} + I_t$ subdivision. Let g_0 be a $(d,\mathcal{F})$-coloring of $G \setminus Z$. Let $g : V(G) \to [d + 4t - 7]$ be a coloring such that for each vertex v of G,

$$g(v) = \begin{cases} g_1(v) & \text{for every } v \in V(G) \setminus Z \text{ and,} \\ f(z) & \text{for every } z \in Z. \end{cases}$$

We claim that g is a $(d+4t-7, \mathcal{F})$-coloring of G satisfying (a) and (b), which contradicts our assumption. Let C be a component of $G[g^{-1}(\{i\})]$ for some $i \in [d+4t-7]$. Since g is identical to g_1 on $V(G) \setminus Z$ and $g(u) \neq g(v)$ for every $u \in V(G) \setminus Z$ and $v \in Z$, C is a component of either $G[g_1^{-1}(\{i\})]$ or $G[f^{-1}(\{i\})]$. This implies $C \in \mathcal{F}$. This proves (3).

Since G contains a bipartite $K_{2t-2} + I_t$ subdivision, Theorem 4.5 implies that there exists $X \subseteq V(G)$ with $|X| \leq 2t - 4$ such that $G \setminus X$ admits a block decomposition with a bipartite block U having at least $t + 3$ vertices.

(4) *Every component of $G \setminus X \setminus V(U)$ is a subgraph of $G[Z]$.*

Let C be a component of $G \setminus X \setminus V(U)$. Let V_C be the set of vertices in U adjacent to a vertex in C. As U is a block and C is a component of $G \setminus X \setminus V(U)$, it follows that $|V_C| \leq 1$. If $|V_C| = 1$ then let v_C be the unique vertex in V_C. Let $A_C = G[V(C) \cup X \cup V_C]$ and $B_C = G \setminus V(C)$. Note that $V(A_C) \cap V(B_C) = X \cup V_C$ and (A_C, B_C) is a separation of G of order at most $2t - 3$, since $|X| + |V_C| \leq 2t - 3$. By (2), either $V(A_C) \setminus V(B_C)$ or $V(B_C) \setminus V(A_C)$ is in Z. Since Z is stable, $V(U) \setminus V_C \subseteq V(B_C) \setminus V(A_C)$ and U is 2-connected as $|V(U)| \geq t+3$, $V(B_C) \setminus V(A_C)$ is not a subset Z. Therefore, $V(A_C) \setminus V(B_C) = V(C)$ is a subset of Z. This proves (4).

Since $U \setminus Z$ is a bipartite subgraph of G, let $\{X_1, X_2\}$ be its bipartition. By (4), it follows that $V(G) = Z \cup (X \setminus Z) \cup X_1 \cup X_2$. Let us choose three colors $\{c_1, c_2, c_3\} \subseteq [4t - 4] \setminus f(Z)$. Let $g : V(G) \to [4t - 4] \subseteq [d + 4t - 7]$ be a coloring defined as follows:

$$g(x) = \begin{cases} f(x) & \text{for } x \in Z, \\ c_1 & \text{for } x \in X \setminus Z, \\ c_2 & \text{if } x \in X_1, \\ c_3 & \text{if } x \in X_2. \end{cases}$$

We claim that g is a $(d + 4t - 7, \mathcal{F})$-coloring of G satisfying (a) and (b), which contradicts our assumption.

Let C be a component of $G[g^{-1}(\{i\})]$ for some $i \in [d + 4t - 7]$. One of the following cases hold: either $V(C) \subseteq Z$ or $V(C) \subseteq X \setminus Z$ or $V(C) \subseteq X_1$ or $V(C) \subseteq X_2$. Since $|Z|$ and $|X|$ are at most $4t - 7$ and both X_1 and X_2 are stable in G, C is in $\mathcal{F}$. $\square$

Now we present proofs of our main theorems.

Proof of Theorem 1.3. Let $\mathcal{F}$ be the set of graphs of maximum degree at most $\max(N(2t-2, t), 4t - 8)$ where N is in Corollary 2.3. Corollary 2.3 implies that every graph with no bipartite $K_{2t-2} + I_t$ subdivision has a $(2t-2, \mathcal{F})$-coloring. By Lemma 5.1, G has a $(6t-9, \mathcal{F})$-coloring, implying that G is $(6t - 9)$-colorable with defect $\max(N(2t - 2, t), 4t - 8)$. $\square$

Proof of Theorem 1.5. Let $u(t) := C(t, N(2t - 2, t))$ where C is in Theorem 2.5 and N is in Corollary 2.3. Let $\mathcal{F}$ be the set of graphs that every component has at most $\max(u(t), 4t-7)$ vertices. By Corollary 2.3 and Theorem 2.5, every graph with no odd K_t minor and no bipartite $K_{2t-2} + I_t$ subdivision has a $(3(2t - 2), \mathcal{F})$-coloring. By Lemma 5.1, G has a $(10t - 13, \mathcal{F})$-coloring, implying that G is $(10t - 13)$-colorable with clustering $\max(u(t), 4t - 7)$. $\square$

6. Concluding Remarks

6.1. **List-coloring variant.** We may consider a list-coloring variant of defective coloring. For integers $s, N \geq 0$, a graph G is *s-choosable* with *defect* N if for every set of lists $\{L_v\}_{v\in V(G)}$ with $|L_v| \geq s$ for every $v \in V(G)$, there is a map $f : V(G) \to \bigcup_{v\in V(G)} L_v$ with $f(v) \in L_v$ for each $v \in V(G)$ such that $G[f^{-1}(\{i\})]$ has maximum degree at most N for every $i \in \bigcup_{v\in V(G)} L_v$.

As we remarked in Section 1, Theorems 2.1 and 2.2 can be extended for list-colorings. For instance, Ossona de Mendez, Oum, and Wood [24] showed that for integers $s, t \geq 1$ and every graph G with no $K^*_{s,t}$ subgraph, there is $N = N(s, t)$ such that G is s-choosable with defect N. It follows that for $t \geq 1$, every graph with no K_{t+1} minor is t-choosable with defect M for some constant $M = M(t)$, which is also implied by the proof of Edwards, Kang, Kim, Oum, and Seymour [6].

Note that every n-vertex graph with no K_t minor contains $O(t\sqrt{\log t}\, n)$ edges [19, 20, 27, 28]. In contrast to graphs with no K_t minor, an n-vertex graph with no odd K_3 minor may contain $\Omega(n^2)$ edges. For example, complete bipartite graphs have no odd K_3 minor.

Theorem 6.1 (Kang [15])**.** *For each integer $N \geq 0$, there is a function $s = s(d) = (1/2 + o(1)) \log_2 d$ as $d \to \infty$ such that if a graph G has minimum degree at least d, G is not s-choosable with defect N.*

By Theorem 6.1, it follows that for integers $t \geq 1$ and $s, N \geq 0$, there are graphs with no odd K_t minor not s-choosable with defect N.

6.2. **Extending Theorems 1.3 and 1.5.** We extend our main results to a slightly larger class of graphs. As we mentioned in Section 3, G contains H as an odd minor if and only if a signed graph $(G, E(G))$ contains a signed graph $(H, E(H))$ as a minor. We review the concepts of signed graphs and their minors in Appendix A.

Given $\Sigma \subseteq E(H)$, we provide alternative characterization for signed graphs $(G, E(G))$ containing (H, Σ) as a minor. A signed graph $(G, E(G))$ contains a signed graph (H, Σ) as a *minor* if and only if

(1) there exist vertex-disjoint subgraphs $\{T_u\}_{u\in V(H)}$ in G which are trees, and
(2) a coloring $\alpha : \bigcup_{u\in V(H)} V(T_u) \to \{1, 2\}$ such that for every $u \in V(H)$, every edge in T_u is bichromatic, and for every edge $vw \in E(H)$, there is an edge $e \in E(G)$ that joins $V(T_v)$ and $V(T_w)$ where e is monochromatic if and only if $vw \in \Sigma$.

Note that for every $\Sigma \subseteq E(K_t)$, a signed graph $(K_{2t}, E(K_{2t}))$ contains (K_t, Σ) as a minor. Replacing t by $2t$, Theorem 1.3 implies that for every $t \geq 2$ and every $\Sigma \subseteq E(K_t)$, if $(G, E(G))$ contains no (K_t, Σ) as a minor, then G is $(12t - 9)$-colorable with defect $s(2t)$. Theorem 1.5 also implies that for every $t \geq 2$ and every $\Sigma \subseteq E(K_t)$, if $(G, E(G))$ contains no (K_t, Σ) as a minor, then G is $(20t - 13)$-colorable with clustering $C(2t)$.

By modifying the proofs in Section 4, we can improve these bounds further. In the proof of Lemma 4.1, we join $V(M_i)$ and $V(M_j)$ with a parity-breaking path with respect to α for $1 \leq i < j \leq t$. Because $\alpha(x_i) = \beta(x_i)$ and $\alpha(y_i) \neq \beta(y_i)$, we can also join $V(M_i)$ and $V(M_j)$ with a path that is not parity-breaking with respect to α. In particular, Lemma 4.1 forces not only an odd K_t minor, but also a signed (K_t, Σ) minor for *every* $\Sigma \subseteq E(K_t)$. This extends Theorems 1.3 and 1.5 as follows.

Corollary 6.2. *For each integer $t \geq 2$, there exists an integer $s = s(t)$ such that for every $\Sigma \subseteq E(K_t)$ and $(G, E(G))$ with no (K_t, Σ) minor, the graph G is $(6t - 9)$-colorable with defect s.*

Corollary 6.3. *For each integer $t \geq 2$, there exists an integer $C = C(t)$ such that for every $\Sigma \subseteq E(K_t)$ and $(G, E(G))$ with no (K_t, Σ) minor, the graph G is $(10t-13)$-colorable with clustering C.*

6.3. **Upper bound of maximum degree.** In Theorem 2.2, the function $M(s,t,\delta_1,\delta_2)$ is defined as follows.

$$M(s,t,\delta_1,\delta_2) = \begin{cases} t-1 & \text{if } s=1 \\ \frac{\delta_2 t(\delta_1-2)}{2} + \delta_1 & \text{if } s=2 \\ (\delta_1 - s)\left(\binom{\lfloor \delta_2 \rfloor}{s-1}(t-1) + \frac{\delta_2}{2}\right) + \delta_1 & \text{if } s>2 \end{cases}$$

Therefore, it follows that $N(2t-2,t) = M(2t-2,t,O(t^2),O(t^2)) = \exp(O(t\log t))$ in Corollary 2.3. Hence we have the upper bound of $s(t) = \exp(O(t\log t))$ in Theorem 1.3.

If one replaces a bipartite $K_{2t-2} + I_t$ subdivision of Lemma 4.1 with a bipartite K_{3t-2} subdivision, one may set $s(t) = O(t^4)$ since $N(3t-3,1) = O(t^4)$. However, this will increase the number of colors from $6t-9$ to $7t-10$ of Theorem 1.3, as graphs with no bipartite K_{3t-2} subdivision are defectively colored with $3t-3$ colors, which is more than $2t-2$ colors in defective coloring of graphs with no bipartite $K_{2t-2} + I_t$ subdivision.

References

[1] B. Bollobás and A. Thomason. Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs. *European J. Combin.*, 19(8):883–887, 1998.

[2] P. A. Catlin. Hajós' graph-coloring conjecture: variations and counterexamples. *J. Combin. Theory Ser. B*, 26(2):268–274, 1979.

[3] L. Cowen, W. Goddard, and C. E. Jesurum. Defective coloring revisited. *J. Graph Theory*, 24(3):205–219, 1997.

[4] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, fourth edition, 2010.

[5] Z. Dvořák and S. Norin. Islands in minor-closed classes. I. Bounded treewidth and separators. arXiv:1710.02727, 2017.

[6] K. Edwards, D. Y. Kang, J. Kim, S. Oum, and P. Seymour. A relative of Hadwiger's conjecture. *SIAM J. Discrete Math.*, 29(4):2385–2388, 2015.

[7] L. Esperet and G. Joret. Colouring planar graphs with three colours and no large monochromatic components. *Combin. Probab. Comput.*, 23(4):551–570, 2014.

[8] J. Geelen, B. Gerards, B. Reed, P. Seymour, and A. Vetta. On the odd-minor variant of Hadwiger's conjecture. *J. Combin. Theory Ser. B*, 99(1):20–29, 2009.

[9] J. Geelen and T. Huynh. Colouring graphs with no odd-K_n minor. Manuscript, `http://www.math.uwaterloo.ca/~jfgeelen/Publications/colour.pdf`, 2004.

[10] B. Guenin. Odd-K_5-free graphs are 4-colourable. In *Oberwolfach Report No. 3/2005*, pages 176–178, 2005. `https://www.mfo.de/document/0503/OWR_2005_03.pdf`.

[11] H. Hadwiger. Über eine Klassifikation der Streckenkomplexe. *Vierteljschr. Naturforsch. Ges. Zürich*, 88:133–142, 1943.

[12] F. Harary. On the notion of balance of a signed graph. *Michigan Math. J.*, 2:143–146 (1955), 1953–54.

[13] T. Huynh, S. Oum, and M. Verdian-Rizi. Even-cycle decompositions of graphs with no odd-K_4-minor. *European J. Combin.*, 65:1–14, 2017.

[14] T. R. Jensen and B. Toft. *Graph coloring problems*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1995. A Wiley-Interscience Publication.

[15] R. J. Kang. Improper choosability and Property B. *J. Graph Theory*, 73(3):342–353, 2013.

[16] K.-i. Kawarabayashi. A weakening of the odd Hadwiger's conjecture. *Combin. Probab. Comput.*, 17(6):815–821, 2008.

[17] K.-i. Kawarabayashi and B. Mohar. A relaxed Hadwiger's conjecture for list colorings. *J. Combin. Theory Ser. B*, 97(4):647–651, 2007.

[18] J. Komlós and E. Szemerédi. Topological cliques in graphs. II. *Combin. Probab. Comput.*, 5(1):79–90, 1996.

[19] A. V. Kostochka. The minimum Hadwiger number for graphs with a given mean degree of vertices. *Metody Diskret. Analiz.*, 38:37–58, 1982.
[20] A. V. Kostochka. Lower bound of the Hadwiger number of graphs by their average degree. *Combinatorica*, 4(4):307–316, 1984.
[21] C.-H. Liu and S. Oum. Partitioning H-minor free graphs into three subgraphs with no large components. *J. Combin. Theory Ser. B*, 128:114–133, 2018.
[22] B. Mohar, B. Reed, and D. R. Wood. Colourings with bounded monochromatic components in graphs of given circumference. *Australas. J. Combin.*, 69(2):236–242, 2017.
[23] S. Norin, A. Scott, P. Seymour, and D. R. Wood. Clustered colouring in minor-closed classes. arXiv:1708.02370, 2017.
[24] P. Ossona de Mendez, S. Oum, and D. R. Wood. Defective colouring of graphs excluding a subgraph or minor. *Combinatorica*, 2018. `https://doi.org/10.1007/s00493-018-3733-1`.
[25] N. Robertson, P. Seymour, and R. Thomas. Hadwiger's conjecture for K_6-free graphs. *Combinatorica*, 13(3):279–361, 1993.
[26] P. Seymour. Hadwiger's conjecture. In J. Nash and M. Rassias, editors, *Open problems in mathematics*. Springer, 2016.
[27] A. Thomason. An extremal function for contractions of graphs. *Math. Proc. Cambridge Philos. Soc.*, 95(2):261–265, 1984.
[28] A. Thomason. The extremal function for complete minors. *J. Combin. Theory Ser. B*, 81(2):318–338, 2001.
[29] J. van den Heuvel and D. R. Wood. Improper colourings inspired by Hadwiger's conjecture. *Journal of the London Mathematical Society*, 98(1):129–148.
[30] D. R. Wood. Contractibility and the Hadwiger conjecture. *European J. Combin.*, 31(8):2102–2109, 2010.
[31] D. R. Wood. Defective and clustered graph colouring. *Electron. J. Combin.*, Dynamic Survey(#DS23), April 2018.

Appendix A. signed graphs

We review elementary concepts of signed graphs, following definitions in [12] and [13] unless stated otherwise.

A *signed graph* (G, Σ) is a graph $G = (V, E)$ equipped with a *signature* $\Sigma \subseteq E$. To avoid confusion, graphs always denote unsigned graphs. Every signed graph is assumed to be simple; parallel edges and loops are not allowed. If an edge $e \in E(G)$ is in Σ, e is *negative*. Otherwise, e is *positive*.

For two sets A and B, $A\Delta B$ denotes the set $(A\setminus B)\cup(B\setminus A)$. For a graph G and $X \subseteq V(G)$, let $\delta_G(X)$ be the set of edges joining X and $V(G) \setminus X$. For a signature Σ, a *re-signing* on X is an operation that replaces Σ with another signature $\Sigma\Delta\delta_G(X)$ for some $X \subseteq V(G)$. Note that for $X, Y \subseteq V(G)$, applying re-signing on X and Y is identical to applying re-signing on $X\Delta Y$. In particular, *re-signing at* v is the operation that replaces Σ with $\Sigma\Delta\delta_G(\{v\})$. Note that applying a re-signing operation on X is identical to applying re-signing operations at all vertices in X.

Two signatures Σ and Σ' are *equivalent* if Σ' can be obtained from Σ by re-signing operations; Σ and Σ' are equivalent if and only if there is $X \subseteq V(G)$ such that $\Sigma' = \Sigma\Delta\delta_G(X)$. Two signed graphs (G, Σ) and (G, Σ') are *equivalent* if Σ is equivalent to Σ'.

A cycle C is called *balanced* if it contains an even number of negative edges. Two signed graphs (G, Σ) and (G, Σ') have the same set of balanced cycles if and only if Σ and Σ' are equivalent (see [12]).

A map $f : V(G) \to V(H)$ is an *isomorphism* from (G, Σ) and (H, Σ') if f is an isomorphism from G to H, and $uv \in \Sigma$ if and only if $f(u)f(v) \in \Sigma'$. If there is an isomorphism from (G, Σ) to (H, Σ'), (G, Σ) is *isomorphic* to (H, Σ').

For two signed graphs (G, Σ) and (H, Σ'), (H, Σ') is a *minor* of (G, Σ) if a signed graph isomorphic to (H, Σ') can be obtained from (G, Σ) by deleting vertices, deleting edges, applying re-signing operations, and contracting positive edges.

To avoid parallel edges, if we contract a positive edge uv such that there exists a vertex $w \notin \{u,v\}$ and edges $wu, wv \in E(G)$ of different signs, then we should remove either wu or wv before contracting uv.

When applying a series of operations to find minors, we may assume that deleting vertices and edges always precede re-signing operations; contracting a positive edge uv into a new vertex t and re-signing at t is identical to re-signing on $\{u,v\}$ and contracting a positive edge uv into a vertex t. This implies the following, which can be found in [9].

Lemma A.1. *For graphs G, H and a signature $\Sigma \subseteq E(H)$, a signed graph $(G, E(G))$ contains a signed graph (H, Σ) as a minor if and only if*

(1) there are vertex-disjoint subgraphs $\{T_u\}_{u \in V(H)}$ of G assigned to vertices in $V(H)$,
(2) for every $u \in V(H)$, T_u is a tree and has a proper 2-coloring $c_u : V(T_u) \to \{1,2\}$, and
(3) for every edge $uv \in E(H)$, there is an edge $e = ab \in E(G)$ that joins T_u and T_v such that $c_u(a) = c_v(b)$ if and only if $uv \in \Sigma$.

Department of Mathematical Sciences, KAIST, 291 Daehak-ro Yuseong-gu Daejeon, 34141 South Korea
E-mail address: dyk90@kaist.ac.kr
E-mail address: sangil@kaist.edu