# Supersaturation of Even Linear Cycles in Linear Hypergraphs 

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#### Abstract

A classic result of Erdős and, independently, of Bondy and Simonovits 3 says that the maximum number of edges in an $n$-vertex graph not containing $C_{2 k}$, the cycle of length $2 k$, is $O\left(n^{1+1 / k}\right)$. Simonovits established a corresponding supersaturation result for $C_{2 k}$ 's, showing that there exist positive constants $C, c$ depending only on $k$ such that every $n$-vertex graph $G$ with $\mathrm{e}(G) \geq C n^{1+1 / k}$ contains at least $c\left(\frac{\mathrm{e}(G)}{\mathrm{v}(G)}\right)^{2 k}$ many copies of $C_{2 k}$, this number of copies tightly achieved by the random graph (up to a multiplicative constant).

In this paper, we extend Simonovits' result to a supersaturation result of $r$-uniform linear cycles of even length in $r$-uniform linear hypergraphs. Our proof is self-contained and includes the $r=2$ case. As an auxiliary tool, we develop a reduction lemma from general host graphs to almost-regular host graphs that can be used for other supersaturation problems, and may therefore be of independent interest.


## 1 Introduction

One of the central problems in extremal graph theory is the Turán problem, where for a fixed graph $H$ and fixed $n$, one wishes to determine the maximum number of edges an $n$-vertex graph can have without creating a copy of $H$ as a subgraph. This number is called the Turán number of $H$ and denoted by $e x(n, H)$. The celebrated Erdős-Stone-Simonovits [6] theorem says that $e x(n, H)=$ $\left(1-\frac{1}{\chi(H)-1}\right) n^{2}+o\left(n^{2}\right)$, where $\chi(H)$ is the chromatic number of the graph $H$. This solves the Turán problem asymptotically for all non-bipartite graphs $H$. However, asymptotic results or exact results are known only for a handful number of bipartite graphs. While the Turán problem asks for the threshold on the number of edges on $n$ vertices that guarantees at least one copy of $H$, it is natural to ask what is the minimum number of copies of $H$ guaranteed in a host graph once its number of edges exceeds $e x(n, H)$. Such problems are referred to as supersaturation problems. When $H$

[^0]is non-bipartite, we know the correct order of magnitude of the answer. Let $H$ be a graph with $\chi(H)=p \geq 3$ and $\mathrm{v}(H)$ vertices. A simple averaging argument (see, for example, Lemma 2.1 in [17]) can be used to show that for any $\varepsilon>0$ there exist $\delta, n_{0}>0$ such that if $G$ is a graph on $n \geq n_{0}$ vertices with $\mathrm{e}(G) \geq\left(1-\frac{1}{p-1}+\varepsilon\right)\binom{n}{2}$ then $G$ contains at least $\delta\binom{n}{\mathrm{v}(H)}$ copies of $H$. This count is tight up to a multiplicative constant, as shown by the random graph of the same edge density as $G$. The threshold on the number of edges on $G$ for which the count is valid is also asymptotically best possible, as shown by the Turán graph $T_{n, p-1}$, which is defined as the balanced blowup of the complete graph on $(p-1)$ vertices. For the supersaturation problem for bipartite graphs, Erdős and Simonovits 10 made the following conjecture in the 1980s.

Conjecture 1.1 [10] Let $H$ be a bipartite graph with $v$ vertices and e edges. Suppose that ex $(n, H)=$ $O\left(n^{2-\alpha}\right)$ for some real $0<\alpha<1$. Then there exist $\alpha^{\prime} \leq \alpha$ and constants $C, c>0$ such that if $G$ is an n-vertex graph with

$$
\begin{equation*}
\mathrm{e}(G) \geq C n^{2-\alpha^{\prime}} \tag{1}
\end{equation*}
$$

edges then $G$ contains at least $c \frac{(\mathrm{e}(G))^{e}}{n^{2 e-v}}$ copies of $H$.
The Erdős-Rényi random graph $G(n, p)$ with $p=\frac{\mathrm{e}(G)}{\binom{n}{2}}$ shows that if Conjecture 1.1 is true then it is best possible up to a multiplicative constant. Conjecture 1.1 is closely related to the famous Sidorenko's conjecture [24], which says that if $H$ is any bipartite graph then the random graph with edge density $p$ has in expectation the minimum number of homomorphic copies of $H$ over all graphs of the same order and edge density. For dense enough host graphs $G$, i.e. (if we do not worry about finding the optimal $\alpha^{\prime}$ in Conjecture 1.1) then any family of graphs that satisfy Sidorenko's conjecture also satisfy Conjecture 1.1. However, works on Conjecture 1.1 often aim at finding the best possible threshold beyond which the counting statement holds. In fact, in the same paper Erdős and Simonovits made two even stronger conjectures by relaxing condition (11) to e $(G) \geq C \cdot e x(n, H)$ and to $\mathrm{e}(G) \geq(1+\varepsilon) e x(n, H)$, respectively. For details, see [10]. At this stage, resolving these stronger versions seem hopeless since the exact value or even just the order of magnitude of ex $(n, H)$ is only known for very few bipartite graphs $H$.

Now we turn our attention to the main focus of this paper, that is, the supersaturation of linear cycles of even length in linear $r$-uniform hypergraphs, or in short, $r$-graphs. First let us give the background for $r=2$. A classic result of Erdős (unpublished) and of Bondy and Simonovits 3 ] establishes that $\operatorname{ex}\left(n, C_{2 k}\right)=O\left(n^{1+1 / k}\right)$. The explicit upper bound that Bondy and Simonovits gave was ex $\left(n, C_{2 k}\right) \leq 100 k n^{1+1 / k}$. This upper bound was later improved by Verstraëte to $8(k-1) n^{1+1 / k}$ for sufficiently large $n$, by Pikhurko [22] to $(k-1) n^{1+1 / k}+O(n)$, and by Bukh and Jiang [4] to $80 \sqrt{k} \log k n^{1+1 / k}+O(n)$. It is conjectured by Erdős and Simonovits that $e x\left(n, C_{2 k}\right)=\Omega\left(n^{1+1 / k}\right)$ also holds. This is known to be true for $k=2,3,5$.

For supersaturation of even cycles, it was mentioned in [10] that Simonovits proved Conjecture 1.1 with $\alpha=\alpha^{\prime}=1-1 / k$. This proof has not been published at the time, but is expected to appear in an upcoming paper of Faudree and Simonovits [13]. Very recently, Morris and Saxton [21] developed a balanced version of the supersaturation result for even cycles, which they use to obtain a sharp result on the number of $C_{2 k}$-free graphs via the container method. Since Morris and Saxton require a balanced version of supersaturation where the collection of $C_{2 k}$ 's they obtain are, informally speaking, uniformly distributed, their proof is quite involved. In this paper, we extend Simonovits'
supersaturation result of even cycles to supersaturation of even linear cycles in linear $r$-graphs. Our proof is self-contained and includes the $r=2$ case.

Before stating our main result, we need a few definitions. An $r$-graph $G$ is called linear if any two edges share at most one vertex. For instance, all 2-graphs are linear. The linear Turán number of an $r$-graph $H$, denoted by $e x_{l}(n, H)$ is defined to be the the maximum number of edges an $n$-vertex linear $r$-graph can have without creating a copy of $H$. The study of linear Turán numbers of linear $r$-graphs is motivated in part by their similarity to the Turán numbers of 2 -graphs. Also, such studies were implicit in some classic extremal hypergraph problems, such as the famous (6,3)-problem (see [2] and [23]) which is asymptotically equivalent to determining the linear Turán number of a linear 3 -cycle. The ( 6,3 )-problem asks for the maximum size of an $n$-vertex 3 -graph in which no six vertices span three or more edges. Note that the usual Turán number $e x(n, H)$ of a linear $r$-graph $H$ and the linear Turán number $e x_{l}(n, H)$ of $H$ are typically very different. The former is already at least $\binom{n-1}{r-1}$ as long as $H$ contains two disjoint edges while the latter is $O\left(n^{2}\right)$.

An $r$-uniform linear cycle $C_{m}^{(r)}$ of length $m$ is obtained from a 2 -uniform $m$-cycle $v_{1} v_{2} \ldots v_{m} v_{1}$ by extending each $v_{i} v_{i+1}$ (indices taken modulo $m$ ) with an ( $r-2$ )-tuple $I_{i}$ such that the tuples $I_{i}$ are pairwise disjoint for distinct indices. Collier-Cartaino, Graber, and Jiang [5] extended aforementioned result of Bondy and Simonovits on even cycles in 2-graphs to linear cycles in linear $r$-graphs. They showed that for all $r \geq 3$ and $m \geq 4$, $e x_{l}\left(n, C_{m}^{(r)}\right)=O\left(n^{1+1 /\lfloor m / 2\rfloor}\right)$. For even linear cycles, their result also works for uniformity $r=2$. It is interesting to note that when $r \geq 3$ the linear Turán number of an odd linear cycle resembles that of an even linear cycle, which is very different from the situation for $r=2$.

Our main result is the supersaturation version of the linear Turán result for linear cycles, but only for even linear cycles.

Theorem 1.2 Given $k, r \geq 2$, there exist constants $C$, $c$ such that if $G$ is an n-vertex linear $r$-graph with $\mathrm{e}(G) \geq C n^{1+1 / k}$ then $G$ contains at least $c\left(\frac{\mathrm{e}(G)}{n}\right)^{2 k}$ copies of $C_{2 k}^{(r)}$.

It is not hard to see that this lower bound on the number of copies of linear cycles is tight, up to a multiplicative constant. Indeed, for $r=2$, in the Erdős-Rényi graph $G(n, p)$, in expectation there are $\Theta\left(p^{2 k} n^{2 k}\right)$ many $2 k$-cycles. For $r \geq 3$, one may consider random subgraphs of almost complete partial Steiner systems. An $(n, \lambda, r, q)$-Steiner system is defined to be an $r$-graph on $n$ vertices such that every $q$-tuple is in exactly $\lambda$ many $r$-edges. A partial ( $n, \lambda, r, q$ )-Steiner system is defined to be an $r$-graph on $n$ vertices such that every $q$-tuple is in at most $\lambda$ many $r$-edges. It was proved by Rödl [20] that for all $n$, there are partial $(n, 1, r, q)$-Steiner systems with $(1-o(1))\binom{n}{q}$ edges. (Note that this is also implied by recent solution of existence conjecture by Keevash [18], while the $q=2$ case was proved much earlier by Wilson [26, 27, 28].) By taking random subgraphs of such partial Steiner systems with $\lambda=1$ and $q=2$, one can show that for every $n$ and $0 \leq e \leq\binom{ n}{2}$, there is a linear $r$-graph $G$ on $n$ vertices and $\mathrm{e}(G)=e$ in which the number of copies of the linear cycles of length $2 k$ is $O\left(\left(\frac{e}{n}\right)^{2 k}\right)$.

Our Theorem 1.2 includes the $r=2$ case as a special case and has a much simpler proof than the proof of Morris and Saxton of their stronger version of supersaturation. We use an approach developed by Faudree and Simonovits [12] in the study of the Turán numbers of so-called $\Theta$-graphs which in its original form is not well-suited for effective counting of $C_{2 k}$ 's. So we adapt their approach
to facilitate counting. Our proofs are greatly simplified via a reduction tool which allows us to reduce the supersaturation problem in a general host $r$-graph to one which has some regularity property. Our regularization tool is an analogue of a regularization theorem of Erdős and Simonovits for the Turán problem and can be used for supersaturation problems of more general graphs. The exact statement of our reduction lemma for linear $r$-graphs is slightly technical, so we refer the reader to Section 3 for the precise statement (see Theorem 3.3).

We organize the rest of the paper roughly as follows. In Section 2 we present notation and definitions. In Section 3 we develop our reduction results, which as we mentioned, may be of independent interest. In Section 4 and Section 5 we give proofs of the $r=2$ and $r \geq 3$ cases of Theorem 1.2, respectively. Even though we could have proved Theorem 1.2 for the general $r$ directly, we feel that proving the $r=2$ case first helps illustrating the main ideas. However, we will give two slightly different proofs for $r=2$ and $r \geq 3$, modulo the reduction mentioned earlier. Both of these proofs could be written for all $r \geq 2$. The proof we present for the $r=2$ case is more constructive and gives a better bound on constants. The proof we present for $r \geq 3$ follows the approach of Faudree-Simonovits more closely (modulo the reduction arguments) and is perhaps more intuitive and easier to follow for some readers. In Section 6 we give some concluding remarks.

## 2 Notation and Definitions

For an $r$-graph $G$, we let $\mathrm{v}(G)$ and $\mathrm{e}(G)$ denote its number of vertices and edges, respectively. Let $\Delta(G), \delta(G)$ denote the maximum and minimum degree of $G$, respectively. Given a vertex $v \in V(G)$, the link of $v$ in $G$, denoted by $L_{G}(v)$, is defined to be

$$
L_{G}(v)=\left\{I \in[V(G)]^{(r-1)} \mid I \cup\{v\} \in E(G)\right\},
$$

where recall that $[V(G)]^{(r-1)}$ refers to the family of all $(r-1)$-subsets of $V(G)$.
Given a real $q \geq 1$, we say that an $r$-graph $G$ is $q$-almost-regular if $\Delta(G) \leq q \delta(G)$ holds. Given positive constants $C, \gamma$, we say that an $r$-graph $G$ is $(C, \gamma)$-dense if $\mathrm{e}(G) \geq C(\mathrm{v}(G))^{\gamma}$.

We denote by $t_{H}(G)$ the number of copies of an $r$-graph $H$ in an $r$-graph $G$. In this paper we will always assume that $H$ has no isolated vertices, however, it is not hard to see that all the mentioned results work for all graphs. Given a 2-graph $F$, the $r$-expansion of $F$ is the $r$-graph obtained by replacing each edge $e$ of $F$ with $e \cup I_{e}$, where $I_{e}$ is an $(r-2)$-tuple of new vertices, such that the $I_{e}$ 's are pairwise disjoint for distinct edges $e$. Note that $F^{(2)}=F$. If $G$ is an $r$-expansion of a 2 -graph $F$, then we call $F$ a skeleton of $G$. So, for example, the linear $r$-cycle, $C_{m}^{(r)}$ is the $r$-expansion of the 2 -uniform $m$-cycle. Now we define the notion of supersaturation of expansions in linear $r$-graphs.

Definition 2.1 Given a 2-graph $F$ with $v$ vertices and e edges, $r \geq 2$ and $c$ a positive real, we say that a linear r-graph $G c$-supersaturates $F^{(r)}$, if $t_{F^{(r)}}(G) \geq c \frac{(\mathrm{e}(G))^{e}}{(\mathrm{v}(G))^{2 e-v}}$.

Note that for $r=2$ this definition is the usual supersaturation for 2-graphs ( as in Conjecture 1.1), that is, a graph $G c$-supersaturates another graph $F$ with $e$ edges and $v$ vertices if $t_{F}(G) \geq c \frac{(\mathrm{e}(G))^{e}}{(\mathrm{v}(G))^{2 e-v}}$. As we have discussed earlier, for 2-graphs the bound on the number of copies of $F^{(r)}=F$ in Definition 2.1 is achieved up to a multiplicative constant by the random graph of the same edge density.

For general $r \geq 3$, the bound is tight as well, obtained by random subgraphs of appropriate edge density in almost complete partial Steiner systems, which as we discussed in the introduction exist.

Proposition 2.2 Let $F$ be a 2 -uniform graph with $v$ vertices and $e$ edges and let $r \geq 2$. For all $n$ and all $0<E \leq\binom{ n}{2}$ there exist n-vertex linear $r$-graphs $G$ with $\mathrm{e}(G)=E$ in which the number of copies of $F^{(r)}$ is $O\left(\frac{E^{e}}{n^{2 e-v}}\right)$.

The proof of Proposition 2.2 is folklore, so we omit it. One can take a random subgraph of an almost complete Steiner systems and apply standard concentration inequalities.

Let us remark that the bound given in Definition 2.1 is specific to the setting where the host graph is linear and embedded graphs is an expansion. In a different setting, the supersaturation problem typically becomes very different and the expected optimal count is expected to be different. In fact, as mentioned in the introduction, the thresholds for forcing even just one copy of $F^{(r)}$ can be very different depending on whether we require the host graph to be linear or not.

Recall that an $r$-graph $G$ is $r$-partite with an $r$-partition $A_{1}, \ldots, A_{r}$ if each edge of $G$ contains exactly one vertex from each $A_{i}$. Given an $r$-partite $r$-graph $G$ with an $r$-partition $A_{1}, \ldots, A_{r}$, we define a 2-projection of $G$ to be the 2-graph we obtain by taking the projection of the edges of $G$ onto two of its partition classes. More formally, for $1 \leq i<j \leq r$, we define the ( $i, j$ )-projection of $G$, denoted by $P_{i, j}(G)$, to be a 2-graph whose edge set is defined as

$$
E\left(P_{i, j}(G)\right)=\left\{e \cap\left(A_{i} \cup A_{j}\right) \mid e \in E(G)\right\} .
$$

Note that when $G$ is linear, for any $1 \leq i<j \leq r$ the map $e \rightarrow e \cap\left(A_{i} \cup A_{j}\right)$ is a bijection, and in particular, $\mathrm{e}(G)=e\left(P_{i, j}(G)\right)$. Next, we give the following slightly technical definition of what we call projection-restricted supersaturation in linear $r$-partite $r$-graphs. In the next section we show that if we have this form of supersaturation for an expansion $F^{(r)}$ in linear $r$-graphs that have an almost-regular 2-projection, then we also get supersaturation of $F^{(r)}$ in all linear $r$-graphs.

Definition 2.3 Given a 2-graph $F$ with $v$ vertices and e edges, $r \geq 2$ and $c$ a positive real. For a linear, $r$-partite $r$-graph $G$ and any 2 -projection of it, call $P$, we say $(G, P) c$-supersaturates $F^{(r)}$ if

$$
t_{F^{(r)}}(G) \geq c \frac{(\mathrm{e}(P))^{e}}{(\mathrm{v}(P))^{2 e-v}}
$$

Note that for 2-graphs, Definition 2.3 coincides with Definition 2.1,

## 3 Reduction results

Erdős and Simonovits [7] proved the following "regularization" theorem for 2-graphs.
Theorem 3.1 [7] Let $0<\alpha<1$ be a real and $q=20 \cdot 2^{(1 / \alpha)^{2}}$. There exists $n_{0}=n_{0}(\alpha)$ such that if $G$ is a $(1,1+\alpha)$-dense graph on $n \geq n_{0}$ vertices then there exists a $q$-almost-regular subgraph of $G$, say $G^{\prime}$, which is $(2 / 5,1+\alpha)$-dense such that $v\left(G^{\prime}\right)>n^{\alpha \frac{1-\alpha}{1+\alpha}}$.

Theorem 3.1 is a useful tool for the Turán problem for $H$. Indeed, given a dense enough $G$, we may first find an almost-regular subgraph $G^{\prime}$ that has similar density as $G$ and look for a copy of
$H$ in $G^{\prime}$. This theorem itself is not sufficient for establishing supersaturation results for 2-graphs since we look to force many copies of $H$ in $G$. By going into $G^{\prime}$, we might lose many copies of $H$. What we likely need is the existence of a collection of dense enough almost regular subgraphs of $G$ which together supply the number of copies of $H$ that we need. Indeed, this is the rough idea behind the following lemma, which may be viewed as some kind of extension of Theorem 3.1 For any $s, t$ integers, $t \geq s \geq 1$, and a graph $H$, we define $f(H, s, t)=\frac{(\mathrm{e}(H))^{s}}{(\mathrm{v}(H))^{t}}$.

Lemma 3.2 Let $\alpha$ be a real and $s, t$ integers, where $0<\alpha<1$ and $t \geq s \geq 1$, then there exist positive reals $C_{0}=C_{0}(\alpha, s, t)$ and $q=q 3.2(\alpha, s, t)$ such that the following holds. For every $C \geq C_{0}$ if $G$ is a $(C, 1+\alpha)$-dense graph $G$ then it contains a collection of edge-disjoint subgraphs $G_{1}, \ldots, G_{m}$ satisfying

1. $\forall i \in[m], G_{i}$ is $q$-almost-regular and $\left(\frac{1}{4} C, 1+\alpha\right)$-dense,
2. $\sum_{i=1}^{m} f\left(G_{i}, s, t\right) \geq \frac{1}{4^{s}} f(G, s, t)$.

Proof. While we specify the choice of $q$ explicity, we don't do so for $C_{0}$. We assume $C_{0}$ is sufficiently large as a function of $\alpha, s$, and $t$. Let $p=\left\lceil 2^{\max \left\{\frac{4}{\alpha}, \frac{2 s+t}{t-s+1}\right\}}\right\rceil$ and $q=8 p$. By the definition of $p$, we have

$$
\begin{equation*}
p^{\alpha} \geq 16 \text { and } p^{t-s+1} \geq 2^{2 s+t} . \tag{2}
\end{equation*}
$$

Suppose $G$ has $n$ vertices. Let us partition $V(G)$ into $p$ sets $A_{1}, \ldots, A_{p}$ of almost equal sizes, i.e. each of size $\lceil n / p\rceil$ or $\lfloor n / p\rfloor$, such that $A_{1}$ contains vertices of the highest degrees in $G$. For convenience, we will drop the ceilings and floors in our arguments as doing so does not affect the arguments except for the slight changes to constants.

We now prove our statement by induction on $n$. When $n<q$ the claim holds trivially since either $G$ itself is $q$-almost-regular or no $(C, 1+\alpha)$-dense graph on $n<q$ many vertices exists. For the induction step, we consider two cases.

Case 1. The number of edges in $G$ with at least one endpoint in $A_{1}$ is at most $\frac{\mathrm{e}(G)}{2}$.
Let $d=d(G)$ be the average degree of $G$. By our definition of $A_{1}$, for each vertex $v \in V(G) \backslash A_{1}$, we have $d_{G}(v) \leq p d$; otherwise $\sum_{u \in A_{1}} d_{G}(u)>p d(n / p)=n d$, a contradiction. Let $G^{\prime}=G-A_{1}$. Then

$$
\Delta\left(G^{\prime}\right) \leq p d .
$$

and $\mathrm{e}\left(G^{\prime}\right) \geq \frac{\mathrm{e}(G)}{2}$, by initial assumptions. By iteratively deleting vertices whose degree becomes less than $\frac{d}{8}$, we obtain a subgraph $G^{\prime \prime} \subseteq G^{\prime}$ with $e\left(G^{\prime \prime}\right) \geq \mathrm{e}\left(G^{\prime}\right)-\frac{n d}{8} \geq \frac{\mathrm{e}(G)}{4}$ and $\delta\left(G^{\prime \prime}\right) \geq \frac{d}{8}$.

Since $\Delta\left(G^{\prime \prime}\right) \leq \Delta\left(G^{\prime}\right) \leq p d$ and $\delta\left(G^{\prime \prime}\right) \geq d / 8, G^{\prime \prime}$ is $8 p$-almost-regular, that is, $G^{\prime \prime}$ is $q$-almostregular. Also,

$$
e\left(G^{\prime \prime}\right) \geq \frac{1}{4} \mathrm{e}(G) \geq \frac{1}{4} C[\mathrm{v}(G)]^{1+\alpha} \geq \frac{1}{4} C\left[v\left(G^{\prime \prime}\right)\right]^{1+\alpha} .
$$

Thus, $G^{\prime \prime}$ is $\left(\frac{1}{4} C, 1+\alpha\right)$-dense. Now,

$$
f\left(G^{\prime \prime}\right)=\frac{\left[e\left(G^{\prime \prime}\right)\right]^{s}}{\left[v\left(G^{\prime \prime}\right)\right]^{t}} \geq \frac{[\mathrm{e}(G) / 4]^{s}}{[\mathrm{v}(G)]^{t}} \geq \frac{1}{4^{s}} \frac{[\mathrm{e}(G)]^{s}}{[\mathrm{v}(G)]^{t}}=\frac{1}{4^{s}} f(G) .
$$

So the claim holds by letting our collection of subgraphs be $\left\{G^{\prime \prime}\right\}$.

Case 2. The number of edges in $G$ with at least one endpoint in $A_{1}$ is more than $\frac{1}{2} \mathrm{e}(G)$.
For each $i=2, \ldots, p$, let $G_{i}=G\left[A_{1} \cup A_{i}\right], n_{i}=\mathrm{v}\left(G_{i}\right)$ and $e_{i}=\mathrm{e}\left(G_{i}\right)$. Then for each $i \in\{2, \ldots, p\}$, $n_{i}=\frac{2 n}{p}$. Also, $\sum_{i=2}^{p} e_{i} \geq \frac{\mathrm{e}(G)}{2}$. Let $\mathcal{I}=\{2, \ldots, p\}$. Define

$$
\mathcal{I}_{1}=\left\{i \in \mathcal{I}: e_{i} \geq C n_{i}^{1+\alpha}\right\} \quad \text { and } \quad \mathcal{I}_{2}=\mathcal{I} \backslash \mathcal{I}_{1} .
$$

Recall that $p^{\alpha} \geq 16$. By the definition of $\mathcal{I}_{2}$ and the fact that $n_{i}=2 n / p$ for each $i \in[p]$, we have

$$
\sum_{i \in \mathcal{I}_{2}} e_{i} \leq \frac{C}{p^{1+\alpha}} \sum_{i \in \mathcal{I}_{2}}(2 n)^{1+\alpha} \leq \frac{C\left|\mathcal{I}_{2}\right| 2^{1+\alpha} n^{1+\alpha}}{p^{1+\alpha}} \leq \frac{4 C n^{1+\alpha}}{p^{\alpha}} \leq \frac{C}{4} n^{1+\alpha} \leq \frac{\mathrm{e}(H)}{4}=\frac{\mathrm{e}(G)}{4} .
$$

Hence,

$$
\begin{equation*}
\sum_{i \in \mathcal{I}_{1}} e_{i} \geq \frac{\mathrm{e}(G)}{4} \tag{3}
\end{equation*}
$$

For each $i \in \mathcal{I}_{1}$ since $e_{i} \geq C n_{i}^{\gamma}$ and $n_{i}<n$, by the induction hypothesis, $G_{i}$ contains a collection of edge-disjoint subgraphs $G_{i}^{1}, \ldots, G_{i}^{m_{i}}$ each of which is $q$-almost-regular and ( $\frac{1}{4} C, 1+\alpha$ )-dense such that

$$
\sum_{j=1}^{m_{i}} f\left(G_{i}^{j}\right) \geq \frac{1}{4^{s}} f\left(G_{i}\right)
$$

Hence,

$$
\begin{equation*}
\sum_{i \in \mathcal{I}_{1}} \sum_{j=1}^{m_{i}} f\left(G_{i}^{j}\right) \geq \frac{1}{4^{s}} \sum_{i \in \mathcal{I}_{1}} f\left(G_{i}\right)=\frac{1}{4^{s}} \sum_{i \in \mathcal{I}_{1}} \frac{e_{i}^{s}}{n_{i}^{t}}=\frac{p^{t}}{4^{s}(2 n)^{t}} \sum_{i \in \mathcal{I}_{1}} e_{i}^{s} . \tag{4}
\end{equation*}
$$

Hence, by (3), (4), $p^{t-s+1} \geq 2^{2 s+t}$, and the convexity of the function $x^{s}$, we have

$$
\sum_{i \in \mathcal{I}_{1}} \sum_{j=1}^{m_{i}} f\left(G_{i}^{j}\right) \geq \frac{p^{t}}{4^{s}(2 n)^{t}} \frac{\left(\sum_{i \in \mathcal{I}_{1}} e_{i}\right)^{s}}{\left|\mathcal{I}_{1}\right|^{s-1}} \geq \frac{p^{t-s+1}}{4^{2 s} 2^{t}} \frac{e^{s}}{n^{t}} \geq \frac{1}{4^{s}} f(G)
$$

Hence the claims holds by letting $\left\{G_{i}^{j}: i \in \mathcal{I}_{1}, 1 \leq j \leq m_{i}\right\}$ be our collection of subgraphs of $G$. This completes Case 2 and the proof.

Now we are ready to state our main result of the section, which informally says that if projectionrestricted supersaturation holds for those host graphs which have almost-regular projections, then the supersaturation holds for all graphs. The formal statement follows.

Theorem 3.3 Let $\alpha \in(0,1)$ be a real and $r \geq 2$. Let $F$ be a graph with $v$ vertices and e edges, where $e \geq v$. There exists a real $q=q(\alpha, F) \geq 1$ such that the following holds. Suppose $C, c>0$ are constants such that for every linear $r$-partite $r$-graph $G$ that has a $(C, 1+\alpha)$-dense and $q$-almost-regular 2 projection $P,(G, P) c$-supersaturates $F^{(r)}$. Then there exist $C^{\prime}, c^{\prime}$ such that every linear $r$-partite $r$-graph that is $\left(C^{\prime}, 1+\alpha\right)$-dense $c^{\prime}$-supersaturates $F^{(r)}$.

Proof. Let $s=e, t=2 e-v$. By our assumption, $t \geq s \geq 1$. We show that the theorem holds for $q$ to be chosen as $93.2(\alpha, s, t)$, derived from Lemma 3.2 applied with constant $\alpha, s$ and $t$. Finally let $C^{\prime}=4 C$, $c^{\prime}=\frac{c r^{2 e-v}}{2^{4 e-v}}$. Let $G$ be an $r$-partite $r$-graph on $n$ vertices such that $\mathrm{e}(G) \geq 4 C n^{1+\alpha}$. Suppose $G$ has an $r$-partition $\left(A_{1}, A_{2}, \ldots, A_{r}\right)$, such that $\left|A_{1}\right| \geq\left|A_{2}\right| \geq \ldots\left|A_{r}\right|$. It follows that $\left|A_{1} \cup A_{2}\right| \geq 2 n / r$, and
$H=P_{1,2}(G)$ is a $(4 C, 1+\alpha)$-dense graph. By Lemma 3.2, there exists a collection of edge-disjoint subgraphs $H_{1}, \ldots, H_{m}$ of $H$, each of which is $(C, 1+\alpha)$-dense and $q$-almost-regular, such that

$$
\sum_{i=1}^{m} f\left(H_{i}, s, t\right) \geq \frac{1}{4^{s}} f(H, s, t)
$$

For each $i \in[m]$, let $G_{i}$ be the subgraph of $G$ such that $P_{1,2}\left(G_{i}\right)=H_{i}$. For each $i \in[m]$, since $H_{i}$ is $(C, 1+\alpha)$-dense and $q$-almost-regular, by the hypothesis of the theorem, $\left(G_{i}, H_{i}\right) c$-supersaturates $F^{(r)}$. That is,

$$
t_{F^{(r)}}\left(G_{i}\right) \geq c \frac{\left(e\left(H_{i}\right)\right)^{s}}{\left(v\left(H_{i}\right)\right)^{t}}=c f\left(H_{i}, s, t\right)
$$

Since the $H_{i}$ 's are edge-disjoint, $G_{i}^{\prime}$ 's are also edge-disjoint (i.e. there is no edge contained in two different $G_{i}$ 's). Thus, we have

$$
t_{F^{(r)}}(G) \geq \sum_{i=1}^{m} t_{F}\left(G_{i}\right) \geq c \sum_{i=1}^{m} f\left(H_{i}\right) \geq \frac{c}{4^{s}} f(H)=\frac{c}{4^{e}} \cdot \frac{(\mathrm{e}(H))^{e}}{(\mathrm{v}(H))^{2 e-v}} \geq \frac{c r^{2 e-v}}{2^{4 e-v}} \cdot \frac{(\mathrm{e}(G))^{e}}{n^{2 e-v}}
$$

Theorem 3.3 says that to establish supersaturation of $F^{(r)}$ in an $n$-vertex linear $r$-partite $r$-graph $G$, we may assume $G$ has a dense enough almost-regular 2-projection $P$. Our next lemma can be used to show that we may further assume $P$ to have edge density exactly $\Theta\left(\mathrm{v}(P)^{1+\alpha}\right)$, where $\alpha$ is any fixed real for which $\operatorname{ex}\left(n, F^{(r)}\right)=O\left(n^{1+\alpha}\right)$. The proof uses random sampling and the classic Chernoff bound, which we state here for completeness.

Lemma 3.4 (f.e. [16] Corollary 2.3) Given a binomially distributed variable $X \in \operatorname{BIN}(n, p)$ we have $\mathbb{P}(|X-E(X)| \geq a E(X)) \leq 2 e^{-\frac{a^{2}}{3} E(X)}$, as long as $0<a \leq 3 / 2$.

Lemma 3.5 Let $r \geq 2$ be an integer. Let $\alpha \in(0,1)$ be a real. Let $F$ be a graph with vertices and $e$ edges, where $e \geq v$. There is a constant $m_{0}=m_{0}(\alpha)$ such that the following holds for all $M \geq m_{0}$. Suppose $D, q, c>0$ are reals where $q \geq 1$ such that for every linear r-partite r-graph $G^{\prime}$ that has a 2-projection $P^{\prime}$ on $m \geq M$ vertices satisfying $D m^{\alpha} \leq \delta\left(P^{\prime}\right) \leq \Delta\left(P^{\prime}\right) \leq 3 q D m^{\alpha}$ we have that $\left(G^{\prime}, P^{\prime}\right)$ c-supersaturates $F^{(r)}$. Then for every linear r-partite $r$-graph $G$ that has a $(q D, 1+\alpha)$-dense and $q$-almost-regular 2-projection $P$ on at least $M$ vertices, $(G, P) \frac{c}{2^{e+1}}$-supersaturates $F^{(r)}$.

Proof. The choice of $m_{0}$ will be given implicitly in the proof. Let $G$ be a linear $r$-partite graph with an $r$-partition $\left(A_{1}, A_{2}, \ldots, A_{r}\right)$, where without loss of generality $P=P_{1,2}(G)$ is $(q D, 1+\alpha)$-dense, $q$ -almost-regular, and has $m=\mathrm{v}(P) \geq M$ vertices. The idea is to sample randomly a subgraph $G^{\prime}$ of $G$ with an appropriate edge probability, count $F^{(r)}$ in $G^{\prime}$, and then use it to bound the count of $F^{(r)}$ in $G$. Now let $\delta(P), \Delta(P)$ and $d$ denote the minimum, maximum and average degrees in $P$, respectively. Then $d=\frac{2 \mathrm{e}(P)}{\mathrm{v}(P)} \geq \frac{2 q D m^{1+\alpha}}{m}=2 q D m^{\alpha}$. Since $P$ is $q$-almost-regular, we have $\delta(P) \geq d / q \geq 2 D m^{\alpha}$.

Let $p=\frac{2 D m^{\alpha}}{\delta(P)}$ and let $G^{\prime}$ be a random subgraph of $G$, obtained by including each edge of $G$ in $G^{\prime}$ independently with probability $p$. Then

$$
\mathbb{E}\left[\mathrm{e}\left(G^{\prime}\right)\right]=p \mathrm{e}(G) \text { and } \mathbb{E}\left[t_{F^{(r)}}\left(G^{\prime}\right)\right]=t_{F^{(r)}}(G) \cdot p^{e} \text { and } \forall v \in V\left(G^{\prime}\right) \mathbb{E}\left[d_{G^{\prime}}(v)\right]=p d_{G}(v)
$$

Since $G$ is linear and $P$ is a 2-projection, for each $v \in V(P)=A_{1} \cup A_{2}$ we have $d_{G}(v)=d_{P}(v)$. Since $P$ is $q$-almost-regular, $\Delta(P) / \delta(P) \leq q$. So, for each $v \in A_{1} \cup A_{2}$, we have

$$
\mathbb{E}\left[d_{G^{\prime}}(v)\right]=p d_{G}(v)=p d_{P}(v) \leq \frac{2 D m^{\alpha}}{\delta(P)} \Delta(P) \leq 2 q D m^{\alpha}
$$

and similarly $\mathbb{E}\left[d_{G^{\prime}}(v)\right] \geq 2 D m^{\alpha}$.
Now random variables $d_{G^{\prime}}(v)$ and $\mathrm{e}\left(G^{\prime}\right)$ have binomial distributions. Hence using Markov's inequality and Chernoff's inequality, one can show that

$$
\begin{gathered}
\mathbb{P}\left[t_{F^{(r)}}\left(G^{\prime}\right)>2 t_{F^{(r)}}(G) \cdot p^{e}\right]<\frac{1}{2} \\
\mathbb{P}\left[\mathrm{e}\left(G^{\prime}\right)<\frac{p}{2} \mathrm{e}(G)\right]<\frac{1}{4}
\end{gathered}
$$

and

$$
\mathbb{P}\left[\exists v \in A_{1} \cup A_{2} \text { such that } d_{G^{\prime}}(v)<D m^{\alpha} \text { or } d_{G^{\prime}}(v)>3 q D m^{\alpha}\right]<1 / 4
$$

In some of the inequalities above, we used Chernoff (and the union bound for the last one). The desired inequalities hold when $m$ is large enough as a function of $\alpha$ which is guaranteed by choosing $m_{0}$ to be large enough. So there exists a subgraph $G^{\prime}$ of $G$ satisfying

$$
\begin{equation*}
\mathrm{e}\left(G^{\prime}\right) \geq \frac{p}{2} \mathrm{e}(G), t_{F^{(r)}}\left(G^{\prime}\right) \leq 2 t_{F^{(r)}}(G) \cdot p^{e} \tag{5}
\end{equation*}
$$

and that for each $v \in A_{1} \cup A_{2}$

$$
\begin{equation*}
D m^{\alpha} \leq d_{G^{\prime}}(v) \leq 3 q D m^{\alpha} \tag{6}
\end{equation*}
$$

Now, let $P^{\prime}=P_{1,2}\left(G^{\prime}\right)$. Since there is a bijection between $E\left(G^{\prime}\right)$ and $E\left(P^{\prime}\right)$, for each $v \in V(P)$, we have $d_{P^{\prime}}(v)=d_{G^{\prime}}(v)$. By (6), for each $v \in V\left(P^{\prime}\right)=A_{1} \cup A_{2}$, we have

$$
D m^{\alpha} \leq d_{P^{\prime}}(v) \leq 3 q D m^{\alpha} .
$$

Thus, by the hypothesis of our theorem, $\left(G^{\prime}, P^{\prime}\right) c^{\prime}$-supersaturates $F^{(r)}$. By (5), we have

$$
t_{F^{(r)}}(G) \geq \frac{1}{2 p^{e}} t_{F^{(r)}}\left(G^{\prime}\right) \geq \frac{c}{2 p^{e}} \cdot \frac{\left(\mathrm{e}\left(G^{\prime}\right)\right)^{e}}{m^{2 e-v}} \geq \frac{c}{2 p^{e}} \cdot \frac{p^{e}(\mathrm{e}(G))^{e}}{2^{e} m^{2 e-v}}=\frac{c}{2^{e+1}} \cdot \frac{(\mathrm{e}(G))^{e}}{m^{2 e-v}}
$$

Applying Theorem 3.3 an Lemma 3.5 we obtain the following reduction tool for proving supersaturation of expansions. In this paper, we use it on $C_{2 k}^{(r)}$. But it can be applied to other expansions as well and thus is of independent interest.

Corollary 3.6 Let $r \geq 2$ be an integer and $\alpha \in(0,1)$ a real. Let $F$ be a graph with vertices and $e$ edges, where $e \geq v$. Let $q=q 3.3(\alpha, F)$ be given as in Theorem 3.3 and $m_{0}=m_{0}(\alpha)$ be given as in Lemma 3.5. Suppose there exist reals $D, \lambda, M, c>0$, where $\lambda \geq 3 q, M \geq m_{0}$, such that for every linear $r$-partite $r$-graph $G$ that has a 2-projection $P$ on $m \geq M$ vertices satisfying $D m^{\alpha} \leq \delta(P) \leq \Delta(P) \leq \lambda D m^{\alpha}$ we have that $(G, P)$ c-supersaturates $F^{(r)}$. Then there exist $C^{\prime}, c^{\prime}$
such that every $\left(C^{\prime}, 1+\alpha\right)$-dense linear r-graph $G c^{\prime}$-supersaturates $F^{(r)}$.
Proof. Suppose there exist reals $D, \lambda>0$, where $\lambda \geq 3 q$, such that for every linear $r$-partite $r$-graph $G$ that has a 2-projection $P$ on $m \geq M$ vertices satisfying $D m^{\alpha} \leq \delta(P) \leq \Delta(P) \leq \lambda D m^{\alpha}$ we have that $(G, P) c$-supersaturates $F^{(r)}$. By Lemma 3.5, there exists a constant 4.5 such that for every linear $r$-partite $r$-graph $G$ that has a $(q D, 1+\alpha)$-dense $q$-almost-regular 2 -projection $P$ on at least $M$ vertices $(G, P)$ 3.5-supersaturates $F^{(r)}$. Set $C=\max \{q D, M\}$. Then for every linear $r$-partite $r$-graph $G$ that has a $(C, 1+\alpha)$-dense $q$-almost-regular 2-projection $P$ we have that $(G, P) q 3.5$ supersaturates $F^{(r)}$ (as $P$ must have at least $C \geq M$ vertices). Applying Theorem 3.3 with the $C$ above and $c=43.5$, there exist constant $C_{3.3}^{\prime}$ and $q_{3.3}^{\prime}$ such that every linear $r$-partite $r$-graph that is $\left(C_{[3.3)^{\prime}}^{1+\alpha) \text {-dense } q^{\prime}}{ }_{3.3]}\right.$ supersaturates $F^{(r)}$.

Let $C^{\prime}=\frac{r^{r}}{r!} C_{3.3}^{\prime}$. Let $G$ be a linear $\left(C^{\prime}, 1+\alpha\right)$-dense $r$-graph. By a well-known fact, $G$ contains a subgraph $G^{\prime}$ with $\mathrm{e}\left(G^{\prime}\right) \geq \frac{r!}{r^{r}} \mathrm{e}(G)$. Clearly, $G^{\prime}$ is $\left(C_{3.3}^{\prime}, 1+\alpha\right)$-dense. By our discussion above, $G^{\prime}$ $q^{\prime} \overline{3.3}$ supersaturates $F^{(r)}$. Hence $G{ }^{\prime}$ 3.3' supersaturates $F^{(r)}$.

Applying Corollary 3.6 to even linear cycles, we get
Corollary 3.7 Let $r, k \geq 2$ be integers. There exist constants $m_{k}, q_{k}$ depending only on $k$ such that the following holds. Suppose there are reals $D, \lambda, M, c>0$, where $\lambda \geq q_{k} D$ and $M \geq m_{k}$, such that for every n-vertex linear r-partite r-graph $G$ that has a 2-projection $P$ on $m \geq M$ vertices satisfying $D m^{1 / k} \leq \delta(P) \leq \Delta(P) \leq \lambda D m^{1 / k}$ we have $t_{C^{(2 k)}}(G) \geq \mathrm{cm}^{2}$. Then there exist constants $C^{\prime}, c^{\prime}$ such that every $n$-vertex linear $r$-graph $G$ with $\mathrm{e}(G) \geq C^{\prime} n^{1+1 / k}$ satisfies $t_{C_{2 k}^{(r)}}(G) \geq c^{\prime}\left(\frac{\mathrm{e}(G)}{n}\right)^{2 k}$.

## 4 Supersaturation of even cycles in graphs

### 4.1 Preliminary Lemmas

In this short section, we prove a few lemmas. The first simple lemma provides a new ingredient to the usual Faudree-Simonovits approach [12] which enables more efficient counting of $C_{2 k}$ 's in the host graph and gives better constants. On surface, applying it will appear to be essentially equivalent to using the Faudree-Simonovits method directly. However, in the concluding remarks we will point out some subtle differences to illuminate the advantage that comes with applying the lemma.

Lemma 4.1 Let $T$ be a tree of height $h$ with a root $x$. For each $v \in V(T)$, let $T_{v}$ be the subtree of $T$ rooted at $v$. Let $b$ be a positive integer. Let $S$ be a set of at least $b h+1$ vertices in $T$. Then there exists a vertex $y$ at distance $i$ from $x$, for some $0 \leq i \leq h-1$, such that $\left|V\left(T_{y}\right) \cap S\right| \geq|S|-i b$ and that for any child $z$ of $y$ in $T,\left|V\left(T_{z}\right) \cap S\right| \leq\left|V\left(T_{y}\right) \cap S\right|-b$.

Proof. We define a sequence of vertices as follows. Let $x_{0}=x$. Among all the children of $x_{0}$, let $x_{1}$ be one such that $T_{x_{1}}$ contains the maximum number of vertices in $S$. Among all the children of $x_{1}$, let $x_{2}$ be one that contains the maximum number of vertices in $S$, and etc. Suppose the sequence we define this way is $x_{0}, x_{1}, \ldots, x_{p}$, where $p \leq h$ and $\left|V\left(T_{x_{p}}\right) \cap S\right|=1$. Since $\left|V\left(T_{x_{0}}\right) \cap S\right| \geq b h+1$ and $\left|V\left(T_{x_{p}}\right) \cap S\right|=1$, there must exist a smallest index $0 \leq i<p$ such that $\left|V\left(T_{x_{i+1}}\right) \cap S\right| \leq\left|V\left(T_{x_{i}}\right) \cap S\right|-b$. Let $y=x_{i}$. Then $y$ satisfies the claim.

Lemma 4.2 Given a bipartite graph $G$ with a bipartition $(A, B)$. There exists a subgraph $G^{\prime}$ of $G$ with a bipartition $\left(A^{\prime}, B^{\prime}\right)$ where $A^{\prime} \subseteq A, B^{\prime} \subseteq B$ such that $\mathrm{e}\left(G^{\prime}\right) \geq \frac{1}{2} \mathrm{e}(G)$ and that $\delta_{A^{\prime}}\left(G^{\prime}\right) \geq \frac{1}{4} d_{A}(G)$ and $\delta_{B^{\prime}}\left(G^{\prime}\right) \geq \frac{1}{4} d_{B}(G)$.

Proof. Let $d_{A}=d_{A}(G)$ and $d_{B}=d_{B}(G)$. Let us iteratively delete any vertex in $A$ whose degree becomes less than $\frac{1}{4} d_{A}$ and any vertex in $B$ whose degree becomes less than $\frac{1}{4} d_{B}$. We continue until we no longer have such vertices or run out of vertices. Let $G^{\prime}$ denote the remaining graph. Let $A^{\prime}$ denote the set of remaining vertices in $A$ and $B^{\prime}$ the set of remaining vertices in $B$. The number of edges removed in the process is at most

$$
\frac{1}{4} d_{A}|A|+\frac{1}{4} d_{B}|B| \leq \frac{1}{4} \mathrm{e}(G)+\frac{1}{4} \mathrm{e}(G)=\frac{\mathrm{e}(G)}{2} .
$$

Hence, $G^{\prime}$ is non-empty. By the procedure, each vertex in $A^{\prime}$ has degree at least $\frac{1}{4} d_{A}$ and each vertex in $B^{\prime}$ has degree at least $\frac{1}{4} d_{B}$ in $G^{\prime}$.

We also need the following crude bound on the number of paths of a given length in an asymmetric bipartite graph. Even though sharper estimates exist in literature, the lemma suffices for our purposes and is self-contained.

Lemma 4.3 Let $p$ be a positive integer. Let $G$ be a bipartite graph with a bipartition $(A, B)$. Let $d_{A}, d_{B}$ denote the average degrees of vertices in $A$ and in $B$, respectively. Suppose $d_{A}, d_{B} \geq 8 p$. Then the number of paths of length $2 p+1$ in $G$ is at least $\frac{1}{2^{6 p+1}} \mathrm{e}(G)\left(d_{A} d_{B}\right)^{p}$.

Proof. By Lemma 4.2. $G$ contains a subgraph $G^{\prime}$ with a bipartition $\left(A^{\prime}, B^{\prime}\right)$ such that $\mathrm{e}\left(G^{\prime}\right) \geq$ $\frac{1}{2} \mathrm{e}(G)$ and that $d_{A^{\prime}} \geq \frac{1}{4} d_{A}, d_{B^{\prime}} \geq \frac{1}{4} d_{B}$. Considering growing a $(2 p-1)$-path $v_{1} v_{2} \ldots v_{2 p-1}$ where $v_{1} \in A, v_{2 p-1} \in B$. There are $\mathrm{e}\left(G^{\prime}\right)$ ways to pick $v_{1} v_{2}$. Then there are at least $\prod_{i=1}^{p}\left(d_{B^{\prime}}-i\right)\left(d_{A^{\prime}}-i\right)$ ways to pick the remaining vertices one by one. Since $d_{A^{\prime}} \geq \frac{1}{4} d_{A} \geq 2 p$ and $d_{B^{\prime}} \geq \frac{1}{4} d_{B} \geq 2 p$, we have $d_{A^{\prime}}-i \geq \frac{1}{2} d_{A^{\prime}} \geq \frac{1}{8} d_{A}$ and $d_{B^{\prime}} \geq \frac{1}{2} d_{B^{\prime}} \geq \frac{1}{8} d_{B}$ for all $i \in[p]$. Hence the number of ( $2 p+1$ )-paths in $G$ is at least $\frac{1}{2} \mathrm{e}(G) \cdot \frac{1}{2^{6 p}}\left(d_{A} d_{B}\right)^{p}=\frac{1}{2^{6 p+1}} \mathrm{e}(G)\left(d_{A} d_{B}\right)^{p}$.

### 4.2 Proof of Theorem 1.2 for $r=2$

In this section, we reprove the supersaturation result for even cycles in graphs. That is, we prove the $r=2$ case of Theorem 1.2. Here is an outline of the proof. First we develop a lemma that says if the leaves of a rooted tree $T$ of height at most $k$ are attached to one partite set of a bipartite graph with average degrees on each side being a large enough constant depending on $k$, then we get many $C_{2 k}$ 's that contain vertices in some particular level of $T$. The key ingredient to proving this lemma is Lemma 4.1. We then apply this lemma to show that every $n$-vertex almost regular graph $G$ with $\mathrm{e}(G)=\Theta\left(n^{1+1 / k}\right)$ (with adequate bounds on the coefficients) has $\Omega\left(n^{2}\right)$ many $C_{2 k}$ 's. By Corollary 3.7 this proves the $r=2$ case of Theorem 1.2,

Lemma 4.4 Let $h, k$ be positive integers where $h \leq k$. Let $G$ be a graph. Let $T$ be a tree of height $h$ in $G$ with a root $x$. For each $i \in[h]$ let $L_{i}$ be the set of vertices at distance $i$ from $x$ in $T$. Let $W$ be a set of vertices in $V(G) \backslash V(T)$. Let $F$ denote the bipartite subgraph of $G$ containing all the
edges of $G$ between $L_{h}$ and $W$. Let $d_{L}, d_{W}$ be the average degree in $F$ of vertices in $L_{h}$ and in $W$, respectively. Suppose $d_{L}, d_{W} \geq 16 k^{2}$. Then there exists $j \in[h]$ such that the number of $C_{2 k}$ 's in $G$ that contain some vertex in $L_{j}$ is at least $\alpha_{k}\left|L_{h}\right| d_{L}^{k-h+j} d_{W}^{k-h+j-1}$, where $\alpha_{k}=\frac{1}{2^{6 k}}\left(\frac{1}{2 k}\right)^{2 k-2}$.

Proof. For each vertex $y$ in $T$, let $T_{y}$ denote the subtree of $T$ rooted at $y$. We clean up $F$ to get a subgraph $F^{\prime}$ of $F$ as follows. First, we delete vertices $w$ in $W$ with $d_{F}(w) \leq k h$. Let $W^{\prime}$ denote the set of remaining vertices in $W$. Let $w \in W^{\prime}$. Applying Lemma 4.1 to $T$ and $S=N_{F}(w)$, we conclude that there exists some vertex $r(w) \in L_{j}$ for some $j \in[h-1]$ such that there are at least $\left|N_{F}(w)\right|-k j$ members of $N_{F}(w)$ that lie in $T_{r(w)}$. Furthermore, for any child $z$ of $r(w)$ in $T$, there are at least $k$ members of $N_{F}(w) \cap V\left(T_{r(w)}\right)$ that lie outside $T_{z}$. To form $F^{\prime}$, we include edges between $w$ and $N_{F}(w) \cap V\left(T_{r(w)}\right)$ for each $w \in W^{\prime}$. By our assumptions, in forming $F^{\prime}$ from $F$ we have deleted at most $k h$ edges incident to each $w \in W$. Hence,

$$
e\left(F^{\prime}\right) \geq e(F)-k h|W|
$$

For each $j \in[h-1]$, let $W_{j}=\left\{w \in W^{\prime}: r(w) \in L_{j}\right\}$ and let $F_{j}$ be the subgraph of $F^{\prime}$ induced by $L_{h} \cup W_{j}$. Let us choose an $j \in[h-1]$ such that $e\left(F_{j}\right)$ is maximum. Then

$$
\begin{equation*}
e\left(F_{j}\right) \geq \frac{1}{k} e\left(F^{\prime}\right) \geq \frac{1}{k}(e(F)-k h|W|) \geq \frac{1}{2 k} e(F) \tag{7}
\end{equation*}
$$

where the last inequality follows from the fact that $e(F)=d_{W}|W| \geq 16 k^{2}|W| \geq 2 k h|W|$,
Suppose $L_{j}=\left\{z_{1}, z_{2}, \ldots, z_{t}\right\}$. For each $e \in[t]$, let $S_{e}=L_{h} \cap V\left(T_{z_{e}}\right)$. By our definition of $F^{\prime}$ and $F_{j}$, in $F_{j}$ each $w \in W_{j}$ has edges to precisely one $S_{e}$. Let $N_{e}=N_{F_{j}}\left(S_{e}\right)$. Then $N_{1}, \ldots, N_{t}$ partition $W_{j}$ and $F_{j}$ is the vertex disjoint union of $F_{j}\left[S_{e} \cup N_{e}\right]$, for $e \in[t]$. Let $m=k-(h-j)$.

Claim 4.5 Every $(2 m-1)$-path $P$ in $F_{j}$ extends to a $C_{2 k}$ in $G$ that contains a vertex in $L_{j}$.
Proof of Claim. Consider any (2m-1)-path $P$ in $F_{j}$. By our discussion, $P \subseteq F_{j}\left[S_{e} \cup N_{e}\right]$ for some $e \in[t]$. Suppose $Q=v_{1} v_{2} \ldots v_{2 m}$, where $v_{1} \in S_{e}$ and $v_{2 m} \in N_{e}$. Then $r\left(v_{2}\right)=r\left(v_{2 m}\right)=z_{e}$. Let $a$ denote the child of $z_{e}$ in $T_{z_{e}}$ such that $v_{1}$ lies under $a$. Since $r\left(v_{2 m}\right)=z_{e}$, by definition, $v_{2 m}$ has at least $k$ neighbors in $T_{z_{e}}$ that lies outside $T_{a}$. Among them, at least one, say $u$ lies outside $V(P)$. Let $Q, Q^{\prime}$ denote the unique $\left(v_{1}, z_{e}\right)$-path and the unique $\left(u, z_{e}\right)$-path in $T_{z_{e}}$, respectively. Since $v_{1}$ and $u$ lie under different children of $z_{e}, V(Q) \cap V\left(Q^{\prime}\right)=\left\{z_{e}\right\}$. Now, $P \cup v_{2 m} u \cup Q \cup Q^{\prime}$ is a cycle of length $2 m-1+1+2(k-m)=2 k$ in $G$ that contains $z_{e}$.

By Claim 4.5 the number of $C_{2 k}$ 's in $G$ that contain a vertex in $L_{j}$ is at least the number of ( $2 m-1$ )-paths in $F_{j}$. To complete our proof, it suffices to find a corresponding lower bound on the number of $(2 m-1)$-paths in $F_{j}$. For convenience, let $A=L_{h}$ and $B=W_{j}$. Let $d_{A}, d_{B}$ denote the average degrees in $F_{j}$ of vertices in $A$ and $B$, respectively. By (7),

$$
d_{A} \geq \frac{1}{2 k} d_{L} \geq 8 k \geq 8 m \quad \text { and } \quad d_{B} \geq \frac{1}{2 k} d_{W} \geq 8 k \geq 8 m
$$

By Lemma 4.3 with $p=m-1$, the number of $(2 m-1)$-paths in $F_{j}$ is at least

$$
\frac{1}{2^{6(m-1)+1}} e(F)\left[d_{A} d_{B}\right]^{m-1} \geq \frac{1}{2^{6 m}}\left(\frac{1}{2 k}\right)^{2 m-2} e(F) d_{L}^{m-1} d_{W}^{m-1} \geq \alpha_{k}\left|L_{h}\right| d_{L}^{m} d_{W}^{m-1}=\alpha_{k}\left|L_{h}\right| d_{L}^{k-h+j} d_{W}^{k-h+j-1}
$$

where $\alpha=\frac{1}{2^{6 k}}\left(\frac{1}{2 k}\right)^{2 k-2}$. This completes our proof.
In the next theorem, we use Lemma 4.4 to quickly obtain the desired lower bound on the number of $C_{2 k}$ 's in almost regular $n$-vertex graphs whose number of edges is $\Theta\left(n^{1+1 / k}\right)$.

Theorem 4.6 Let $k \geq 2$ be an integer. Let $D, \lambda>0$ be constants where $D \geq 64 k^{2}$ and $\lambda \geq 1$. Let $n_{0}=(8 \lambda)^{k}$. Let $G$ be an n-vertex graph, $n \geq n_{0}$ such that for each $v \in V(G), D n^{1 / k} \leq d(v) \leq \lambda D n^{1 / k}$. Then there exists a positive constant $\beta=\beta(D, \lambda, k)$ such that $t_{C_{2 k}}(G) \geq \beta n^{2}$.

Proof. For each $x \in V(G)$, let $L_{i}(x)$ denote the set of vertices at distance $i$ from $x$. Let $h(x)$ be the minimum $i \leq k-1$ such that $\left|L_{i+1}(x)\right| /\left|L_{i}(x)\right|<n^{1 / k}$. Clearly $h(x)$ exists or else we run out of vertices. Let $h=h(x)$. Let $T$ be a breadth first search tree rooted at $x$ that includes $L_{0}(x), L_{1}(x), \ldots, L_{h}(x)$. By our assumption,

$$
\left|L_{h}(x)\right| \geq n^{h / k} \text { and }\left|L_{h+1}(x)\right|<n^{1 / k}\left|L_{h}(x)\right|
$$

Recall that $\left|L_{i+1}(x)\right| /\left|L_{i}(x)\right| \geq n^{1 / k}$ for all $i=0,1, \ldots, h-1$. Since $n \geq n_{0} \geq(8 \lambda)^{k}, n^{1 / k} \geq 8 \lambda$. By our assumption,

$$
\left|V(T) \backslash L_{h}(x)\right| \leq\left|L_{h}\right| \sum_{i=1}^{h-1}\left(\frac{1}{8 \lambda}\right)^{i} \leq \frac{1}{4 \lambda}\left|L_{h}(x)\right| .
$$

Let $F$ be the bipartite subgraph of $G$ consisting of all the edges of $G$ between $L_{h}(x)$ and $L_{h+1}(x)$. The total number of edges of $G$ incident to $L_{h}(x)$ is at least $D n^{1 / k}\left|L_{h}(x)\right| / 2$. Among them, the number of edges that are incident to $V(T) \backslash L_{h}(x)$ is at most

$$
(1 / 4 \lambda) \lambda D n^{1 / k}\left|L_{h}(x)\right|=(1 / 4) D n^{1 / k}\left|L_{h}(x)\right|
$$

Hence,

$$
e(F) \geq(1 / 4) D n^{1 / k}\left|L_{h}(x)\right|
$$

Let $d_{A}, d_{B}$ denote the average degrees in $F$ of vertices in $L_{h}(x)$ and $L_{h+1}(x)$, respectively. Then

$$
d_{A} \geq(1 / 4) D n^{1 / k} \geq 16 k^{2} n^{1 / k} \geq 16 k^{2}
$$

and

$$
d_{B} \geq(1 / 4) D n^{1 / k}\left|L_{h}(x)\right| /\left|L_{h+1}(x)\right| \geq(1 / 4) D \geq 16 k^{2}
$$

By Lemma4.4, there exists a $j \in[h-1]$ such that the number of $C_{2 k}$ 's in $G$ that contain a vertex in $L_{j}(x)$ is at least

$$
\alpha_{k}\left|L_{h}\right| d_{A}^{k-h+j} \geq \alpha_{k} n^{h / k}\left(n^{1 / k}\right)^{k-h+j}=\alpha_{k} n^{1+\frac{j}{k}} .
$$

Let us denote this $j$ value by $j(x)$. For each $t \in[h-1]$, let $S_{t}=\{x \in V(G): j(x)=t\}$. By the pigeonhole principle, for some $t \in[h-1]$, we have $\left|S_{t}\right| \geq n /(h-1)$. Let us fix such a $t$. By our discussion, for each $x \in S_{t}$, the number of $C_{2 k}$ 's that contain a vertex in $L_{t}\left(T_{x}\right)$ is at least $\alpha_{k} n^{1+\frac{t}{k}}$. On the other hand, a vertex $y$ lies in $L_{t}\left(T_{x}\right)$ for at most $\left[\lambda D n^{1 / k}\right]^{t}$ different $x$. Hence the number of distinct $C_{2 k}$ 's in $G$ is at least

$$
\left|S_{t}\right| \alpha_{k} n^{1+\frac{t}{k}} / \lambda^{t} D^{t} n^{\frac{t}{k}} \geq\left(\alpha_{k} / k \lambda^{k} D^{k}\right) n^{2}
$$

The claim holds by setting $\beta=\alpha_{k} / k \lambda^{k} D^{k}$.

Now we can prove the $r=2$ case of Theorem 1.2.
Proof of the $r=2$ case of Theorem 1.2; Theorem 4.6 applies along as $D \geq 64 k^{2}, \lambda \geq 1$, and $n_{0} \geq(8 \lambda)^{k}$. To apply Corollary 3.7, we set $D=\max \left\{64 k^{2}, m_{k}\right\}, \lambda=q_{k} D$ and $M_{k}=(8 \lambda)^{k}$, where $m_{k}, q_{k}$ are as given in Corollary 3.7. The claim follows readily from Corollary 3.7,

## 5 Supersaturation of even linear cycles in linear hypergraphs

For the supersaturation of linear cycles, we follow the approach of the $r=2$ case. However, instead of using the usual BFS tree, we need an adaption of it to hypergraph case. We define the notion of maximal rainbow rooted tree in Section 5.2.

### 5.1 Notation and Preliminary Results

Let $H$ be a graph and $S$ be a some set of vertices, where possibly $S \cap V(H) \neq \varnothing$. Let $\varphi$ be any colouring of the edges of $H$ using non-empty subsets of $S$. Given any subgraph $F$ of $H$, we let

$$
\mathcal{C}(F)=\bigcup_{e \in E(F)} \varphi(e)
$$

and call it the colour set of $F$ under $\varphi$. We say that $\varphi$ is strongly proper on $H$ if for any $e, e^{\prime} \in E(H)$ that share a vertex we have $\varphi(e) \cap \varphi\left(e^{\prime}\right)=\varnothing$. We say that $\varphi$ is rainbow on $F$ (or that $F$ is rainbow under $\varphi$ ) if for every two edges $e, e^{\prime}$ in $F$ we have $\varphi(e) \neq \varphi\left(e^{\prime}\right)$ and that $\mathcal{C}(F)$ is disjoint from $V(F)$. Note that if $\varphi$ uses $(r-2)$-subsets of $S$ and $F$ is rainbow under $\varphi$ then $F \cup \mathcal{C}(F)$ forms an $r$-expansion of $F$. Observe that if $G$ is an $r$-partite linear $r$-graph with an $r$-partition $\left(A_{1}, \ldots, A_{r}\right)$ then the natural colouring $\varphi$ of $P_{i, j}(G)$, where $\forall f \in E\left(P_{i, j}(G)\right) \varphi(f)$ is the unique $(r-2)$-tuple $I_{f}$ for which $f \cup I_{f} \in E(G)$, is strongly proper on $P_{i, j}(G)$ by the linearity of $G$.

Let $G$ be an $r$-graph and $v \in V(G)$. Recall the definition of $L_{G}(v)$ from the introduction. For any subset $S \subseteq V(G)$ we denote by $\left.L_{G}(v)\right|_{S}$ the restriction of the link of $v$ to $S$, that is,

$$
\left.L_{H}(v)\right|_{S}=\left\{I \subseteq S \mid I \in L_{H}(v)\right\}
$$

We give a very crude analogue of Lemma 4.3, this time counting rainbow paths of a given length in an asymmetric bipartite graph.

Lemma 5.1 Let $p, m$ be positive integers and $H$ be a bipartite graph with a bipartition $(A, B)$. Let $\varphi$ be a strongly proper edge-colouring of $H$ using m-sets. If $\delta_{A}, \delta_{B} \geq 4 p(m+1)$ then the number of rainbow paths of length $2 p+1$ in $H$ is at least $\frac{1}{2^{2 p}} \mathrm{e}(H)\left(\delta_{A} \delta_{B}\right)^{p}$.

Proof. Consider growing a rainbow path $P=v_{1} v_{2} \ldots v_{2 p+2}$ where $v_{1} \in A$ and $v_{2 p+2} \in B$. There are $\mathrm{e}(H)$ choices for $v_{1} v_{2}$. In general, suppose the subpath $v_{1} v_{2} \ldots v_{t}$ has been grown, where $2 \leq t \leq 2 p+1$. If $v_{t} \in A$ then we let $v_{t+1}$ be a neighbor of $v_{t}$ in $B$ such that $\left\{v_{t+1}\right\} \cup \varphi\left(v_{t} v_{t+1}\right)$ is disjoint from
$\left(\left(V\left(P_{t}\right) \backslash\left\{v_{t}\right\}\right) \cup \mathcal{C}\left(P_{t}\right)\right.$. If $v_{t} \in B, v_{t+1}$ is defined symmetrically. Assume first that $v_{t} \in A$. Note that

$$
\left.\mid\left(V\left(P_{t}\right) \backslash\left\{v_{t}\right\}\right) \cup \mathcal{C}\left(P_{t}\right)\right) \mid \leq t-1+(t-1) m \leq 2 p(m+1)
$$

Since $\varphi$ is strongly proper, the set $\left\{u \cup \varphi(u): u \in N_{H}\left(v_{t}\right)\right\}$ is an $(m+1)$-uniform matching of size $d_{H}\left(v_{t}\right)$. At most $2 p(m+1)$ of these members contain a vertex in $\left(V\left(P_{t}\right) \backslash\left\{v_{t}\right\}\right) \cup \mathcal{C}\left(P_{t}\right)$. So there are at least $d_{H}\left(v_{t}\right)-2 p(m+1) \geq \delta_{A}-2 p(m+1) \geq \frac{1}{2} \delta_{A}$ choices for $v_{t+1}$. Similarly, if $v_{t} \in B$, there there are at least $\frac{1}{2} \delta_{B}$ choices for $v_{t+1}$. Hence, the number of ways to grow $P$ is at least

$$
\mathrm{e}(H)\left(\frac{1}{2} \delta_{A}\right)^{p}\left(\frac{1}{2} \delta_{B}\right)^{p}=\frac{1}{2^{2 p}} \mathrm{e}(H)\left(\delta_{A} \delta_{B}\right)^{p}
$$

Lemma 5.2 (Splitting Lemma) Suppose we are given $D \in \mathbb{R}^{+}, \gamma \in(0,1)$ and integers $k, r \geq 2$. There exists $n_{0}=n_{\underline{5.2}}(D, k, r, \gamma)$ such that for all $n \geq n_{0}$ if $G$ is a linear r-partite r-graph such that two of its r-partition classes, say $A$ and $B$, satisfy that $|A \cup B|=n$ and that $\left|L_{G}(v)\right| \geq D n^{\gamma}$ for each $v \in A \cup B$ then there exists a partition of $V(G)$ into $S_{1}, S_{2}, \ldots, S_{k}$ such that for every $v \in A \cup B$ and every $i \in[k]$, we have

$$
\left|L_{G}(v)\right|_{S_{i}} \left\lvert\, \geq \frac{D n^{\gamma}}{2 k^{r-1}}\right.
$$

Proof. Let us independently assign each vertex $x$ in $V(G)$ a colour from [ $k$ ] chosen uniformly at random. Let $S_{i}$ be the vertices of assigned colour $i$. For a vertex $v \in A \cup B$, we denote by $X_{i}(v)$ the number of edges (which are $(r-1)$-sets) in $L_{G}(v)$ that are completely contained in $S_{i}$. For each $I \in L_{G}(v)$

$$
\mathbb{P}\left[I \subseteq S_{i}\right]=\frac{1}{k^{r-1}}
$$

Since $G$ linear, edges in $L_{G}(v)$ are pairwise disjoint. Hence the events $\left\{I \subseteq S_{i}\right\}$, for different $I \in$ $L_{G}(v)$ 's are independent. Therefore $X_{i}(v)$ has binomial distribution $\operatorname{BIN}\left(d_{G}(v), \frac{1}{k^{r-1}}\right)$. Writing $d$ for $d_{G}(v)$, we have $\mathbb{E}\left(X_{i}(v)\right)=\frac{d}{k^{r-1}}$. By the Chernoff bound,

$$
\mathbb{P}\left[X_{i}(v)<\frac{d}{2 k^{r-1}}\right] \leq P\left[\left|X_{i}(v)-\frac{d}{k^{r-1}}\right|<\frac{d}{2 k^{r-1}}\right]<2 e^{-\frac{d}{12 k^{r-1}}}<2 e^{-\frac{D n^{\gamma}}{12 k^{r-1}}}
$$

Therefore the probability that for some vertex $v \in A \cup B$ and some $i \in[k]$ such that the event $\left\{X_{i}(v)<\frac{d}{2 k^{r-1}}\right\}$ occurs is less than

$$
k n \cdot 2 e^{-\frac{D n^{\gamma}}{12 k^{r-1}}}<1
$$

when $n_{2}$ is large enough and $n \geq n_{2}$. Thus there exists some colouring which guarantees for every vertex $v \in A \cup B$ to have

$$
\left|L_{G}(v)\right|_{S_{i}} \left\lvert\, \geq \frac{d}{2 k^{r-1}} \geq \frac{D n^{\gamma}}{2 k^{r-1}}\right.
$$

Before we establish supersaturation of $C_{2 k}^{(r)}$,s in linear $r$-partite $r$-graphs that have an almost regular 2-projection with the right density, we need another lemma. Given an $r$-graph $G$, where
$r \geq 2$, and $S \subseteq V(G), S$ is a vertex cover of $G$ if $S$ contains at least one vertex of each edge of $G$.
Lemma 5.3 Let $r \geq 2$. Let $G$ be an $r$-graph and $S$ a vertex cover of $G$. There exist a subset $S^{\prime} \subseteq S$ and a subgraph $G^{\prime} \subseteq G$ such that $\mathrm{e}\left(G^{\prime}\right) \geq \frac{r}{2^{r}} \mathrm{e}(G)$ and that $\forall e \in E\left(G^{\prime}\right)\left|e \cap S^{\prime}\right|=1$.

Proof. Let $S^{\prime}$ be a random subset of $S$ obtained by including each vertex of $S$ randomly and independently with probability $\frac{1}{2}$. Let $e$ be any edge of $G$. Suppose $|e \cap S|=m$. Then $1 \leq m \leq r$. The probability that exactly one of these $m$ vertices of $e \cap S$ is chosen for $S^{\prime}$ is $\frac{m}{2^{m}} \geq \frac{r}{2^{r}}$. So the expected number of edges of $G$ that meet $S^{\prime}$ in exactly one vertex is at least $\frac{r}{2^{r}}$ e $(G)$. So there exists $S^{\prime} \subseteq S$ such that at least $\frac{r}{2^{r}} \mathrm{e}(G)$ of the edges meet $S^{\prime}$ in exactly one vertex. Let $G^{\prime}$ be the subgraph of $G$ consisting of these edges.

### 5.2 Rainbow Rooted Trees

We now introduce the following adaption of the BFS tree to linear hypergraphs.
Definition 5.4 (Maximal rooted rainbow tree) Given $r \geq 3$, let $G$ be a linear r-partite r-graph with two of its partition classes being $A$ and $B$ and let $t \geq 0$ be integer. Suppose there exists a partition of $V(G)$ into $S_{1}, S_{2}, \ldots, S_{t}$ such that for every $v \in A \cup B$ and for every $i \in[t]$,

$$
\begin{equation*}
\left.L_{G}(v)\right|_{S_{i}} \neq \varnothing \tag{8}
\end{equation*}
$$

For every $x \in A \cup B$, we define a tree $T_{x}$, rooted at $x$ and of height $t$, together with a colouring $\varphi$ of its edges by $(r-2)$-sets as follows. We define the tree by defining its levels $L_{i}$ iteratively. The $L_{i}$ 's will alternate between being completely inside $A$ and being completely inside $B$. Without loss of generality, suppose $x \in A$. The tree $T_{x}$ is defined symmetrically if $x \in B$.
(1) Let $L_{0}=\{x\}$.
(2) Having defined $L_{i}$, we define $L_{i+1}$ as follows. Without loss of generality, suppose $L_{i} \subseteq A$. Let $F=\left.\bigcup_{v \in L_{i}} L_{G}(v)\right|_{S_{i+1}}$. Since $G$ is $r$-partite with $A, B$ being two partite sets and $L_{i} \subseteq A, V(F)$ is disjoint from $A$ and each edge of $F$ contains exactly one vertex in $B$. Let $M_{i+1}$ be a maximum matching in F. By condition (8), $M_{i+1}$ is nonempty. Define

$$
L_{i+1}=\left.M_{i+1}\right|_{B}=\left\{b \in B \mid \exists I \in M_{i+1} \text { such that } b \in I\right\} .
$$

It remains to define how the vertices of $L_{i}$ are connected to $L_{i+1}$. For each $b \in L_{i+1}$, there exists a unique $I_{b} \in M_{i+1}$ which contains $b$, and due to linearity of $G$ there is a unique $v \in L_{i}$ such that $I_{b} \cup\{v\} \in E(G)$. We add the edge vb to $T_{x}$ and let $\varphi(v b)=I_{b} \backslash\{b\}$.
(3) Repeat step (2) until all vertices of $H$ are exhausted or $i>t$.

Proposition 5.5 Under the assumptions of Definition 5.4, $T_{x}$ is a tree of height $t$ rooted at $x$ that is rainbow under the assigned colouring $\varphi$. In particular, if $P$ is a path in $T_{x}$ then $P \cup \mathcal{C}(P)$ is a linear path of the same length in $G$ with $P$ being a skeleton of it.

Proof. That $T_{x}$ is a height $t$ tree rooted at $x$ is clear from the definition. We now show that $T_{x}$ is rainbow under $c$. By the way we define $T_{x}$ and $c, \mathcal{C}\left(T_{x}\right) \cap V\left(T_{x}\right)=\varnothing$. Let $e, e^{\prime}$ be any two edges in $T_{x}$. Suppose $e$ joins a vertex in $L_{i}$ to $L_{i+1}$ and $e^{\prime}$ joins a vertex in $L_{i^{\prime}}$ to $L_{i^{\prime}+1}$. If $i \neq i^{\prime}$, then $\varphi(e) \cap \varphi\left(e^{\prime}\right)=\varnothing$, since $\varphi(e) \subseteq S_{i+1}$ and $\varphi\left(e^{\prime}\right) \subseteq S_{i^{\prime}+1}$ and $S_{i+1} \cap S_{i^{\prime}+1}=\varnothing$. If $i=i^{\prime}$ then $e \subseteq I$ and $e^{\prime} \in I^{\prime}$ for two different members $I, I^{\prime} \in M_{i+1}$. Since $M_{i+1}$ is a matching, $\varphi(e) \cap \varphi\left(e^{\prime}\right)=\varnothing$. So $T_{x}$ is rainbow under $c$.

The second statement follows immediately from our discussion in Subsection 5.1 that a rainbow subgraph $F$ together with it colours form an expansion of $F$.

We are now ready to prove the following analogue of Lemma 4.4. As we mentioned in the introduction, we will give a slightly different proof from that of Lemma 4.4. Instead of using Lemma 4.1, we will use the strong/weak level notion used by Faudree and Simonovits [12] in the study of theta graphs. Let us remark that we could also prove Lemma 5.6 using Lemma 4.1 and Lemma 5.1. But we feel that there is also a benefit to use the strong/weak level notion used by Faudree and Simonovits since this is the original approach we used to solve the problem and also that it is on some level more intuitive.

Lemma 5.6 Let $i, k, m$ be integers where $k \geq i+1 \geq 1, m \geq 1$. Let $b, d$ be reals satisfying $b, d \geq$ $16^{i}(2 m+2) k$. Let $T_{x}$ be a tree of height $i$ rooted at $x$. For each $j=0, \ldots, i$, let $L_{j}$ be the set of vertices in $T_{x}$ at distance $j$ from $x$. Let $W$ be some set of vertices disjoint from $V(T)$ and $H$ be a bipartite graph with bipartition $\left(L_{i}, W\right)$ such that

$$
\mathrm{e}(H) \geq \max \left\{d\left|L_{i}\right|, b|W|\right\}
$$

Suppose $c$ is an edge colouring of $G=T_{x} \cup H$ such that $c$ is rainbow on $T_{x}$ and strongly proper on $H$ and that $\mathcal{C}(G) \cap V(G)=\varnothing$ and $\mathcal{C}\left(T_{x}\right) \cap \mathcal{C}(H)=\varnothing$. Then there exist $0 \leq q \leq i$ and some positive real $a_{i}=a_{i}(i, k)$ such that there are at least $a_{i}(b d)^{k-i-1+q} \mathrm{e}(H)$ many rainbow $C_{2 k}$ 's in $G$ that contain a vertex in $L_{q}$.

Proof. We proceed by induction on the height $i$ of the tree. It holds vacuously for $i=0$. For all $i \geq 1$ we prove the result by splitting the argument into two cases and only in one of the cases we use induction. It is important to point out that when $i=1$ we are in Case 1 and thus need not use the vacuous case of $i=0$ as our induction hypothesis.

Let us denote by $x_{1}, x_{2}, \ldots, x_{p}$ the children of $x$ in $T_{x}$. For each $j \in[p]$, let $T\left(x_{j}\right)$ be the subtree of $T_{x}$ rooted at $x_{j}$. For each $j \in[p]$, we define the $j$ th sector to be $S_{j}=L_{i} \cap V\left(T_{j}\right)$. Note that since $T_{x}$ is a tree, the $S_{j}$ 's are pairwise disjoint. For a vertex $v \in L_{i}$, we denote by $S(v)$ the sector that $v$ lies in.

We say that a sector $S_{j}$ is dominant for a vertex $w \in W$ if

$$
\left|N_{H}(w) \cap S_{j}\right|>\max \left\{\left|N_{H}(w)\right|-2 k m, \frac{\left|N_{H}(w)\right|}{2}\right\}
$$

We say that $w \in W$ is strong if it has no dominant sector and weak otherwise. Note that by our definition if $w \in W$ has a dominant sector then there is only one such dominant sector for $w$.

Let $W_{s}$ be the set of strong vertices and $W_{w}$ be the set of weak vertices, respectively. Let $H_{s}$ denote the subgraph of $H$ induced by $L_{i}$ and $W_{s}, H_{w}$ denote the subgraph of $H$ induced by $L_{i}$ and $W_{w}$. The argument splits into two cases, depending whether the majority of the edges of $H$ lie in $H_{s}$ or in $H_{w}$. In the first case, we build the necessary number of rainbow $2 k$-cycles going through the vertex $x$ (so in the outcome of the theorem we have $j=0$ as $x \in L_{0}$ ). In the second case we use induction to find rainbow $2 k$-cycles in $T\left(x_{j}\right)$ 's for many $j$.

## Case 1.

$$
\begin{equation*}
e\left(H_{s}\right) \geq \mathrm{e}(H) / 2 \tag{9}
\end{equation*}
$$

Let $d_{\text {avg }}\left(L_{i}\right)$ and $d_{\text {avg }}\left(W_{s}\right)$ denote the average degrees in $H$ for vertices in $L_{i}$ and $W_{s}$ respectively. Then by (9), we have $d_{\text {avg }}\left(L_{i}\right) \geq \frac{d}{2}, \quad d_{\text {avg }}\left(W_{s}\right) \geq \frac{b}{2}$. By Lemma 4.2, there is a subgraph $H^{\prime}$ of $H_{s}$ with bipartition $(A, B), A \subseteq L_{i}, B \subseteq W_{s}$, such that

$$
\begin{equation*}
\mathrm{e}\left(H^{\prime}\right) \geq \frac{\mathrm{e}\left(H_{s}\right)}{2} \geq \frac{\mathrm{e}(H)}{4}, \quad \delta_{A}\left(H^{\prime}\right) \geq \frac{d_{a v g}\left(L_{i}\right)}{4} \geq \frac{d}{8}, \quad \delta_{B}\left(H^{\prime}\right) \geq \frac{d_{a v g}\left(W_{i}\right)}{4} \geq \frac{b}{8} \tag{10}
\end{equation*}
$$

Since $b, d \geq 16^{i}(2 m+2) k$, clearly $\frac{b}{8}, \frac{d}{8} \geq(4 m+4) k \geq(4 m+4)(k-i-1)$. Since $c$ is strongly proper on $H$, by Lemma 5.1 with $p=k-i-1$, the number of rainbow paths of length $2(k-i)-1$ in $H^{\prime}$ is at least

$$
\frac{1}{2^{2(k-i-1)}} e\left(H^{\prime}\right)\left(\delta_{A}\left(H^{\prime}\right) \delta_{B}\left(H^{\prime}\right)\right)^{k-i-1} \geq \frac{1}{2^{5(k-i-1)+2}} \mathrm{e}(H)(b d)^{k-i-1}
$$

Claim 5.7 Every rainbow path $P=v_{1} v_{2} \ldots v_{2(k-i)}$ of length $2(k-i)-1$ extends to a rainbow $C_{2 k}$ in $G$ that contains $x$.

Proof of Claim. By symmetry, we may assume that $v_{1} \in A, v_{2(k-i)} \in B$. For convenience, let $t=2(k-i)$. It suffices to show that there exists $u \in N_{H}\left(v_{t}\right)$ (note that $u$ lies in $L_{i}$ but does not necessarily lie in $A$ ) such $P \cup v_{t} u$ is a rainbow path in $H$ and that $S\left(v_{1}\right) \neq S(u)$. Indeed, suppose such $u$ exists. Then since $S\left(v_{1}\right) \neq S(u)$ the unique path $Q_{1}$ in $T_{x}$ from $v_{1}$ to $x$ and the unique path $Q_{2}$ from $u$ to $x$ intersect only at $x$. Since $T_{x}$ is rainbow, $Q_{1} \cup Q_{2}$ is rainbow. By our assumption, $\mathcal{C}\left(T_{x}\right) \cap \mathcal{C}(H)=\varnothing$. Thus, $P, Q_{1}, Q_{2}$ together form a rainbow $C_{2 k}$ in $G$.

Now we show that such $u$ exists. Since $v_{t} \in W_{s}$, by definition, $\left|N_{H}\left(v_{t}\right) \backslash S\left(v_{1}\right)\right| \geq 2 k m$. Since $\varphi$ is a strongly proper edge-colouring using $m$-sets, $\left\{w \cup \varphi(w): w \in N_{H}\left(v_{t}\right) \backslash S\left(v_{1}\right)\right\}$ is an (m+1)uniform matching of size $\left|N_{H}\left(v_{t}\right) \backslash S\left(v_{1}\right)\right| \geq 2 k m$. Since clearly $|V(P) \cup \mathcal{C}(P)|<2 k m$, there exists $u \in N_{H}\left(v_{t}\right) \backslash S\left(v_{1}\right)$ such that $(w \cup \varphi(w)) \cap(V(P) \cup \mathcal{C}(P))=\varnothing$. It is easy to see that $P \cup v_{t} u$ is a rainbow path in $H$. Also, $u \notin S\left(v_{1}\right)$ by choice.

Case 2: $e\left(H_{w}\right) \geq \mathrm{e}(H) / 2$.
In this case, we have

$$
\begin{equation*}
e\left(H_{w}\right) \geq \frac{d}{2}\left|L_{i}\right|, \quad e\left(H_{w}\right) \geq \frac{b}{2}|W| . \tag{11}
\end{equation*}
$$

Recall that $x_{1}, \ldots, x_{p}$ are the children of the root $x$ and for each $j \in[p], S_{j}=V\left(T\left(x_{j}\right)\right) \cap L_{i}$. For each $j \in[p]$, let $W_{j}$ be the set of vertices in $W_{w}$ whose dominant sector is $S_{j}$. Now we run the following "cleaning" procedure. For every vertex $y \in W_{w}$ we only keep those edges in $H_{w}$ joining $y$ to vertices in its dominant sector. Let $H^{\prime \prime}$ denote the resulting subgraph of $H_{w}$. By the definition of $W_{w}$, every
vertex $y \in W_{w}$ satisfies

$$
d_{H^{\prime \prime}}(y) \geq\left|N_{H}(y)\right|-2 k m .
$$

Hence,

$$
\mathrm{e}\left(H^{\prime \prime}\right) \geq e\left(H_{w}\right)-2 k m\left|W_{w}\right| .
$$

Since $c \geq 8 k m$, by (11) $e\left(H_{w}\right) \geq 4 k m|W|$. Therefore

$$
\begin{equation*}
e\left(H^{\prime \prime}\right) \geq \frac{1}{2} e\left(H_{w}\right) \geq \frac{1}{4} \mathrm{e}(H) . \tag{12}
\end{equation*}
$$

For each $j \in[p]$, let $H_{j}$ denote the subgraph of $H^{\prime \prime}$ induced by $S_{j} \cup W_{j}$. Note that the $H_{j}$ 's are pairwise vertex-disjoint. We want to apply induction to those $T\left(X_{j}\right) \cup H_{j}$ where $H_{j}$ is relatively dense from both partite sets. For that purpose we partition the index set $[p]$ as follows. Let

$$
\mathcal{I}_{1}=\left\{j \in[p]: e\left(H_{j}\right) \leq \frac{d}{16}\left|S_{j}\right|\right\}, \quad \mathcal{I}_{2}=\left\{i \in[p]: e\left(H_{j}\right) \leq \frac{b}{16}\left|W_{j}\right|\right\}, \quad \mathcal{I}_{3}=[p] \backslash\left(\mathcal{I}_{1} \cup \mathcal{I}_{2}\right) .
$$

By the definition and disjointness of the $H_{j}$ 's, we have

$$
\sum_{j \in \mathcal{I}_{1} \cup \mathcal{I}_{2}} e\left(H_{j}\right) \leq \frac{d}{16}\left|L_{i}\right|+\frac{b}{16}|W| \leq \frac{1}{8} \mathrm{e}(H) .
$$

Hence,

$$
\begin{equation*}
\sum_{j \in \mathcal{I}_{3}} e\left(H_{j}\right) \geq \frac{1}{8} e(H) . \tag{13}
\end{equation*}
$$

For each $j \in \mathcal{I}_{3}$, by definition, we have $e\left(H_{j}\right) \geq \frac{d}{16}\left|S_{j}\right|$ and $e\left(H_{j}\right) \geq \frac{b}{16}\left|W_{j}\right|$. Since $T\left(x_{j}\right)$ has height $i-1$ and $\frac{d}{16}, \frac{b}{16}>(16)^{i-1}(2 m+2) k$, by the induction hypothesis with $d, b$ replaced with $\frac{d}{16}$ and $\frac{b}{16}$ respectively, there exists $q=q(j)$ such that the number of rainbow $2 k$-cycles in $T\left(x_{j}\right) \cup H_{j}$ that contain a vertex in level $q(j)$ of $T\left(x_{j}\right)$ is at least

$$
a_{i-1}\left(\frac{b d}{16^{2}}\right)^{k-(i-1)-1+q(j)} e\left(H_{j}\right)=a_{i-1}\left(\frac{b d}{16^{2}}\right)^{k-i+q(j)} e\left(H_{j}\right) .
$$

For each $t=0, \ldots, i-2$, let $\mathcal{I}_{3, t}=\left\{j \in \mathcal{I}_{3}: q(j)=t\right\}$. By the pigeonhole principle, there exists $t \in\{0, \ldots, i-2\}$, such that

$$
\sum_{j \in \mathcal{I}_{3, t}} e\left(H_{j}\right) \geq \frac{1}{i-1} \sum_{j \in \mathcal{I}_{3}} e\left(H_{j}\right) \geq \frac{1}{8 k} \mathrm{e}(H) .
$$

Let us fix such a $t$. By our earlier discussion and the fact that vertices in level $t$ of each $T\left(x_{j}\right)$ for $j \in \mathcal{I}_{3, t}$ lie in level $t+1$ of $T_{x}$, the number of rainbow $2 k$-cycles in $G$ that contain a vertex from $L_{t+1}$ is at least

$$
\sum_{j \in \mathcal{I}_{3, t}} a_{i-1}\left(\frac{b d}{16^{2}}\right)^{k-i+t} e\left(H_{j}\right)=a_{i}(b d)^{k-i-1+(t+1)} \mathrm{e}(H)
$$

with the choice of $a_{i}=\frac{a_{i-1}}{2^{8(k-i+l)+3 k}}$. Hence, in this case the lemma holds for $q=t+1$.

### 5.3 Proof of the $r \geq 3$ case of Theorem 1.2

We are finally ready to prove the supersaturation statement of $C_{2 k}^{(r)}$ for linear $r$-partite $r$-graphs $G$ that have a 2-projection on two parts $A, B$ that is almost regular and have number of edges exactly $\Theta\left(|A \cup B|^{1+1 / k}\right)$. By Corollary 3.7 this would imply Theorem 1.2 for all $r \geq 3$. For this we first define an adequate partition $V(G)$ into $S_{1}, \ldots, S_{k}$. From each vertex $x$ we define the maximal rainbow tree $T_{x}$ rooted at $x$ relative to the partition $\left(S_{1}, \ldots, S_{k}\right)$. Then we apply Lemma 5.6 to find many rainbow $2 k$-cycles containing a vertex from some fixed level of $T_{x}$, which corresponds to linear $2 k$-cycles in $G$. Summing over all $x$ and eliminating overcount, we get a lower bound on the number of $2 k$-cycles in $G$.

Theorem 5.8 Let $k, r \geq 2$ be integers. Let $D$ be a constant such that $D \geq 2^{r+1} r k^{r}(16)^{k}$. There exist $n_{0}$ such that if $G$ is a linear $r$-partite $r$-graph with an $r$-partition $A_{1}, \ldots, A_{r}$ such that $\left|A_{1} \cup A_{2}\right|=n \geq n_{0}$ and for every $v \in A_{1} \cup A_{2}$,

$$
D n^{1 / k} \leq\left|L_{G}(v)\right| \leq \lambda D n^{1 / k},
$$

where $\lambda \geq 1$ is a real, then there exists $\alpha=\alpha(k, r, \lambda)$ such that $t_{C_{2 k}^{(r)}}(G) \geq \alpha n^{2}$.
Proof. The choice of $\alpha$ will be specified at the end of the proof. We will choose $n_{0}$ be large enough so that $n_{0} \geq \eta_{5.2}(D, k, r, 1 / k)$, where $\eta_{[5.2}$ is specified in Lemma 5.2. Let $S_{1}, S_{2}, \ldots, S_{k}$ be a partition obtained by applying Lemma 5.2 to $G$. In particular, for each $x \in A_{1} \cup A_{2}$ and $j \in[k]$, we have

$$
\begin{equation*}
\left|L_{G}(x)\right| S_{j} \left\lvert\, \geq \frac{D n^{1 / k}}{2 k^{r-1}} .\right. \tag{14}
\end{equation*}
$$

For each $x \in A_{1} \cup A_{2}$, let $T_{x}$ be a maximal rainbow tree of height $k$ rooted at $x$ relative to the partition $S_{1}, \ldots, S_{k}$, as described in Definition 5.4. The proof is similar to that of Theorem 4.6. For each $x$, we find an $i \in[k]$ such that
(i) there exists some set $W^{\prime}$ and a bipartite subgraph $H_{x}$ induced by $L_{i}$ and $W^{\prime}$ which has high average degree from both partite sets.
(ii) The colouring $c$ on $T_{x}$ is extended to also include a strongly proper edge-colouring of $H_{x}$ such that $\mathcal{C}\left(H_{x}\right) \cap \mathcal{C}\left(T_{x}\right)=\varnothing$.

We then use Lemma 5.6 to find many rainbow $2 k$-cycles that contain some vertex in some fixed level of $T_{x}$. Below are the details.

Fix $x$ and write $T$ for $T_{x}$. For $j=0, \ldots, k$, let $L_{j}$ be defined as in Definition 5.4 and let $\varphi$ be the assigned edge-colouring of $T$ given in Definition 5.4. Since $\left|L_{1}\right| \geq D n^{1 / k}>n^{1 / k}$ and $\left|L_{k}\right| \leq n$, there exists a smallest $i \in[k-1]$ such that for all $1 \leq j \leq i,\left|L_{j}\right|>n^{j / k}$ but

$$
\left|L_{i+1}\right| \leq n^{(i+1) / k} .
$$

Let $T^{\prime}$ be the subtree of $T$ induced by $\bigcup_{j=0}^{i} L_{j}$. Let $F=\left.\bigcup_{v \in L_{i}} L_{G}(v)\right|_{S_{i+1}}$. Since $S_{i+1}$ is disjoint from $S_{1} \cup \cdots \cup S_{i}$ and since $L_{i} \subseteq A_{1}$ where $A_{1}$ is a partite set in an $r$-partition of $G, V(F) \cap V\left(T^{\prime}\right)=\varnothing$. By the construction of $T,\left|L_{i+1}\right|$ is equal to the size of a maximum matching in $F$. Since $F$ is an
$(r-1)$-graph, we have $\tau(F) \leq(r-1) \alpha^{\prime}(F)$, where $\tau(F)$ and $\alpha^{\prime}(F)$ denote the vertex cover number and matching number of $F$, respectively. Let $W$ be a minimum vertex cover of $F$. Then

$$
|W| \leq(r-1)\left|L_{i+1}\right| \leq(r-1) n^{(i+1) / k}
$$

By Lemma 5.3, there exist $W^{\prime} \subseteq W$ and $F^{\prime} \subseteq F$ such that

$$
e\left(F^{\prime}\right) \geq \frac{r-1}{2^{r-1}} e(F) \text { and } \forall e \in e\left(F^{\prime}\right)\left|e \cap W^{\prime}\right|=1
$$

We define a bipartite graph $H_{x}$ between $L_{i}$ and $W^{\prime}$ and extend the edge-colouring $\varphi$ restricted on $T^{\prime}$ to an edge-colouring of $T^{\prime} \cup H_{x}$ as follows. We go through the edges of $F^{\prime}$ one by one. For each $e \in E\left(F^{\prime}\right)$, since $G$ is linear, there is a unique $v \in L_{i}$ such that $v \cup e \in E(G)$. Also by our definition of $F^{\prime}$, $e \cap W^{\prime}$ has exactly one vertex $w$. We include $v w$ in $H_{x}$ and let $\varphi(v w)=e \backslash\{w\}$. By the linearity of $G$ and our discussion so far, each edge of $F^{\prime}$ yields a different edge of $H_{x}$. There is a bijection between $E\left(F^{\prime}\right)$ and $E\left(H_{x}\right)$. Moreover, $\mathcal{C}\left(H_{x}\right) \cap \mathcal{C}\left(T^{\prime}\right)=\varnothing$, since colours used on $H_{x}$ are $(r-2)$-sets in $S_{i+1}$ while $\mathcal{C}\left(T^{\prime}\right) \subseteq S_{1} \cup \cdots \cup S_{i}$.

Since $G$ is linear and $r \geq 3, \forall v, v^{\prime} \in L_{i}$ we have $\left.\left.L_{G}(v)\right|_{S_{i+1}} \cap L_{G}\left(v^{\prime}\right)\right|_{S_{i+1}}=\varnothing$. By (14),

$$
e(F)=\sum_{v \in L_{i}}\left|L_{G}(v) \cap S_{i+1}\right| \geq \frac{D n^{1 / k}}{2 k^{r-1}}\left|L_{i}\right|
$$

Hence, we have

$$
\begin{equation*}
e\left(H_{x}\right)=e\left(F^{\prime}\right) \geq \frac{r-1}{2^{r-1}} e(F) \geq \frac{D(r-1)}{2^{r} k^{r-1}} n^{1 / k}\left|L_{i}\right| \tag{15}
\end{equation*}
$$

Also, by our choice of $i,\left|L_{i+1}\right| \leq n^{1 / k}\left|L_{i}\right|$. Recall also that $\left|W^{\prime}\right| \leq|W| \leq(r-1)\left|L_{i+1}\right|$. Hence,

$$
\begin{equation*}
e\left(H_{x}\right) \geq \frac{D(r-1)}{2^{r} k^{r-1}}\left|L_{i+1}\right| \geq \frac{D}{2^{r} k^{r-1}}|W| \geq \frac{D}{2^{r} k^{r-1}}\left|W^{\prime}\right| \tag{16}
\end{equation*}
$$

Let $b=\frac{D}{2^{r} k^{r-1}}$ and $d=\frac{D(r-1)}{2^{r} k^{r-1}} n^{1 / k}$. Since $D \geq 2^{r+1} r k^{r}(16)^{k}$, we have $d>b \geq(2(r-2)+2) k(16)^{k}$. So $T^{\prime}$ and $H_{x}$ satisfy the conditions of Lemma 5.6 with constants $b, d$ and $m=r-2$. By Lemma 5.6, there exists some $q=q(x)$ with $0 \leq q \leq i$ and some $a_{i}=a_{i}(i, k)>0$ such that there are at least

$$
a_{i}(b d)^{k-i-1+q} e\left(H_{x}\right)
$$

many rainbow $C_{2 k}$ 's in $T^{\prime} \cup H_{x}$ that contain some vertex in level $L_{q}$ of $T^{\prime}$. Now, $\left|L_{i}\right| \geq n^{\frac{i}{k}}$ by definition, $e\left(H_{x}\right) \geq \Omega\left(n^{\frac{i+1}{k}}\right)$ by (15). Also, $d=\Omega\left(n^{\frac{1}{k}}\right)$. Hence, the number of rainbow $C_{2 k}$ 's in $T^{\prime} \cup H_{x}$ that contain some vertex in $L_{q}$ is at least

$$
\beta n^{\frac{k-i-1+q}{k}} \cdot n^{\frac{i+1}{k}}=\beta n^{\frac{k+q}{k}}
$$

for some $\beta=\beta(k, r)>0$. So in $G$ there are at least $\beta n^{\frac{k+q}{k}}$ different linear $2 k$-cycles each of whose skeletons contains some vertex in $L_{q}$.

For each $t \in[k-1]$, let $S_{t}=\{x \in V(G) \mid q(x)=t\}$. By the pigeonhole principle, for some $t \in[k-1]$, $\left|S_{t}\right| \geq n /(k-1)$. Let us fix such a $t$. Let $M$ denote the number of triples $(C, x, y)$, where $x \in S_{t}$, $C$ is
a linear $2 k$-cycle in $G$ whose skeleton contains a vertex in $L_{t}\left(T_{x}\right)$ and $y$ is a vertex on the skeleton of $C$ that lies in $L_{t}\left(T_{x}\right)$. Let $\mu$ denote the number of different linear $2 k$-cycles $C$ in $G$ that are involved in these triples. By our discussion above, for each $x \in S_{t}$, there are at least $\beta n^{\frac{k+q}{k}}$ different $C$. For each such $C$ there is at least one $y$. So

$$
\begin{equation*}
M \geq\left|S_{t}\right| \beta n^{\frac{k+t}{k}}>(\beta / k) n^{2+\frac{t}{k}} \tag{17}
\end{equation*}
$$

On the other hand, for each of the $\mu$ linear $2 k$-cycles $C$ involved, there are at most $2 k$ different choices of $y$. For fixed $y$, there are at most $\left(\lambda D n^{1 / k}\right)^{t}$ choices of $x$ since such $x$ is at distance at most $t$ from $y$ in the $(1,2)$-projection $P_{1,2}(G)$ of $G$, which has maximum degree at most $\lambda D n^{1 / k}$. So,

$$
\begin{equation*}
M \leq \mu(2 k)\left(\lambda D n^{1 / k}\right)^{t} \tag{18}
\end{equation*}
$$

Combining (17) and (18) and solving for $\mu$, we get

$$
\mu \geq \frac{\beta}{2 k^{2}(\lambda D)^{t}} n^{2}
$$

Let $\alpha=\frac{\beta}{2 k^{2}(\lambda D)^{k}}$. Then $\alpha$ is a function of $k, r, \lambda$ and we have $t_{C_{2 k}^{(r)}}(G) \geq \mu \geq \alpha n^{2}$.
We are now ready to prove the $r \geq 3$ case of Theorem 1.2 ,
Proof of the $r \geq 3$ case of Theorem 1.2: First note that Theorem 5.8 can be rephrased as saying that if $G$ is linear $r$-partite $r$-graph that has a 2 -projection $P$ on at least $m \geq n_{0}$ vertices such that $D m^{1 / k} \leq \delta(P) \leq \Delta(P) \leq \lambda D m^{1 / k}$ then $t_{C_{2 k}^{(r)}}(G) \geq \alpha m^{2}$. The statement holds as long as $D \geq 2^{r+1} r k^{r}(16)^{k}, \lambda \geq 1$, and $m \geq n_{0}$. To apply Corollary 3.7. we set

$$
D=\max \left\{2^{r+1} r k^{r}(16)^{k}, m_{k}\right\}, \lambda=q_{k} D, \text { and } M=\max \left\{n_{0}, m_{k}\right\}
$$

where $m_{k}, q_{k}$ are as given in Corollary 3.7. The claim follows readily from Corollary 3.7.

## 6 Concluding remarks

First, we say a few words about the difference between our proofs between the $r=2$ and the $r \geq 3$ cases for almost regular host graphs. The one for $r=2$ uses Lemma 4.1 and the one for $r \geq 3$ uses induction on the height the maximal rainbow tree. As we pointed out both proofs work for both cases. We choose to present one for each to illustrate both methods. The one that avoids induction potentially could be applied in other settings such as when considering odd linear cycles.

Next, we would like to point out that the reduction to proving the supersaturation of $C_{2 k}$ for $n$ vertex host graphs $G$ with density exactly at $\Theta\left(n^{1+1 / k}\right)$ is crucial to the proof of our general theorem. Using the BFS approach, one can indeed find many copies of $C_{2 k}$. However, the approach works perfectly only when $G$ has density $\Theta\left(n^{1+1 / k}\right)$. For denser $G$, one can still get a bound, but the bound becomes worse and worse compared to the optimal $c\left(\frac{\mathrm{e}(G)}{\mathrm{v}(G)}\right)^{2 k}$ as $G$ gets denser. A reason for that is the subgraph of $G$ induced by consecutive levels of a BFS tree is now much denser and no longer resembles a tree structure. If we only use the BFS structure to construct our $C_{2 k}$ 's, we will lose count
on many $C_{2 k}$ 's. It might be possible to make the BFS approach work directly for dense $G$ without a reduction. But the analysis becomes exceedingly complicated.

For all integers $k, p \geq 2$, the theta graph $\Theta_{p, k}$ is the graph consisting of $p$ many internally disjoint paths of length $k$ sharing the same endpoints. It was shown by Faudree and Simonovits 12 that $e x\left(n, \Theta_{p, k}\right)=O\left(n^{1+1 / k}\right)$. The method of our paper can be used to establish the supersaturation of the $r$-expansion $\Theta_{p, k}^{(r)}$ (where $r \geq 2$ ) of $\Theta_{p, k}$ in linear $r$-graphs. When $r=2$ this establishes the truth of Conjecture 1.1 for $H=\Theta_{p, k}$ with $\alpha=\alpha^{\prime}=1-1 / k$. Again, the lower bound is tight up to a multiplicative constant, obtained by taking a random graph of an almost complete Steiner system.

It would be very interesting to establish the supersaturation of odd linear cycles in linear $r$-graphs, for $r \geq 3$. Toward this end, in [5] it is shown that when $r \geq 3$ we have $e x_{l}\left(n, C_{2 k+1}^{(r)}\right)=O\left(n^{1+1 / k}\right)$, which is very different from the 2-uniform case where for all sufficiently large $n$ it is known that $e x\left(n, C_{2 k+1}\right)=\left\lceil\frac{n}{2}\right\rceil\left\lfloor\frac{n}{2}\right\rfloor$. The proof of this theorem is much more involved than its counterpart for even linear cycles. It is unclear if a similar supersaturation statement as Theorem 1.2 holds for $C_{2 k+1}^{(r)}$. At least our methods don't readily give this. We raise this as an open question.

Question 6.1 Let $k, r$ be integers where $k \geq 2, r \geq 3$. Do there exist positive constants $C$ and $c$ depending only on $k$ and $r$ such that every $n$-vertex linear $r$-graph $G$ with $\mathrm{e}(G) \geq C n^{1+1 / k}$ contains at least $c\left(\frac{\mathrm{e}(G)}{\mathrm{v}(G)}\right)^{2 k+1}$ copies of $C_{2 k+1}^{(r)}$ ?

Very recently, Balogh, Narayanan and Skokan [1] obtained a balanced supersaturation result for linear cycles of all lengths in general $r$-graphs. Note that this is a different setting from ours, as in our case host graphs are linear, and hence are sparse, while they are working with dense ones. As Morris and Saxton did for cycles in graphs, Balogh et al. used their supersaturation result to obtain a bound on the number of of $n$-vertex $C_{m}^{(r)}$-free $r$-graphs. It would be interesting to obtain such a balanced version of supersaturation for even linear cycles in linear $r$-graphs as well and hence generalize the result of Morris and Saxton. Our methods a priori do not give such strong supersaturation.

Another problem worth exploring is to sharpen the result of Balogh et al. Unlike for 2-uniform even cycles, when $r \geq 3$ the usual Turán number of the $r$-uniform linear cycle $C_{m}^{(r)}$ has been completely determined in [14] and [19] for all sufficiently large $n$. Asymptotically, ex $\left(n, C_{m}^{(r)}\right) \sim\left\lfloor\frac{m-1}{2}\right\rfloor\binom{ n}{r-1}$. The supersaturation result of Balogh et al. applies to $n$-vertex $r$-graphs $G$ with $\mathrm{e}(G) \geq C \cdot e x\left(n, C_{m}^{(r)}\right)$ for a sufficiently large constant $C$. It is possible that one can establish a similar statement for all $n$-vertex $r$-graphs $G$ with $\mathrm{e}(G) \geq(1+o(1)) e x\left(n, C_{m}^{(r)}\right)$ and hence sharpen the bound on the number of $n$-vertex $C_{m}^{(r)}$-free $r$-graphs. As a supersaturation problem on its own without the application to the count of $C_{m}^{(r)}$-free graphs, it would also be interesting to at least establish supersaturation of $C_{m}^{(r)}$ in all $n$-vertex $r$-graphs $G$ with $\mathrm{e}(G) \geq(1+o(1)) e x\left(n, C_{m}^{(r)}\right)$.

## References

[1] J. Balogh, B. Narayanan, J. Skokan, The number of hypergraphs without linear cycles, arXiv:1706.01207v1.
[2] W.G. Brown, P. Erdős, V. Sós, On the existence of triangulated spheres in 3-graphs and related problems, Period. Math. Hungaria 3 (1973), 221-228.
[3] A. Bondy, M. Simonovits, Cycles of even length in graphs, J. Combin. Theory Ser. B 16 (1974), 97-105.
[4] B. Bukh, Z. Jiang, A bound on the number of edges in graphs without an even cycle, Combin. Probab. Comput., to appear.
[5] C. Collier-Cartaino, N. Graber, T. Jiang, Linear Turán numbers of linear cycles and cyclecomplete Ramsey numbers, Combin. Probab. Comput., to appear.
[6] P. Erdős, M. Simonovits, A limit theorem in graph theory, Studia Sci. Math. Hungar. 1 (1966), 51-57.
[7] P. Erdős, M. Simonovits, Some extremal problems in graph theory, Combinatorial Theory and Its Applications, I. (Proc. Colloq. Baltaonfüred, 1969), North Holland, Amsterdam, 1970, 377390.
[8] P. Erdős, H. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc. 52 (1946), 1087-1091.
[9] P. Erdős, M. Simonovits, Some extremal problems in graph theory, Combinatorial Theory and Its Applications 1 (Proc. Colloq. Balatonfüred, 1969), North Holland, Amsterdam, 1970, 370390.
[10] P. Erdős, M. Simonovits, Cube-supersaturated graphs and related problems, Progress in graph theory (Waterloo, Ont., 1982), pp 203-218, Academic Press, Toronto, ON, 1984.
[11] B. Ergemlidze, E. Györi, A. Methuku, Asymptotics for Turán numbers of cycles in 3-uniform linear hypergraphs, arXiv: 1705.03561 v 2 .
[12] R. Faudree, M. Simonovits, On a class of degenerate extremal graph problems, Combinatorica, 3(1983), 83-93.
[13] R. Faudree, M. Simonovits, Cycle-supersaturated graphs, in preparation.
[14] Z. Füredi, T. Jiang, Hypergraph Turán numbers of linear cycles, J. Combin. Theory Ser. A 123 (2014), 252-270.
[15] Z. Füredi, M. Simonovits, The history of the degenerate (bipartite) extremal graph problems, Erdős centennial, Bolyai Soc. Math. Stud. 25, 169-264, János Bolyai Math. Soc., Budapest, 2013. See also arXiv:1306.5167.
[16] S. Janson, T. Luczak, A. Rucinski, Random Graphs, John Wiley \& Sons, Inc, 2000.
[17] P. Keevash, Hypergraph Turan Problems, Surveys in Combinatorics, Cambridge University Press, 2011, 83-140.
[18] P. Keevash, The existence of designs, arXiv:1401.3665.
[19] A. Kostochka, D. Mubayi, J. Verstraëte, Turán problems and shadows I: Paths and cycles, J. Combin. Theory Ser. A 129 (2015), 57-79.
[20] Rődl, V. On a packing and covering problem, European Journal of Combinatorics 6, (1985), 69-78.
[21] R. Morris, D. Saxton, The number of $C_{2 l}$-free graphs, Advances in Math. 298 (2016), 534-580.
[22] O. Pikhurko, A note on the Turán function of even cycles, Proc. Amer. Math. Soc. 140 no. 11 (2012), 3687-3692.
[23] I. Ruzsa, E. Szemerédi, Triples systems with no six points carrying three triangles, in Combinatorics Keszthely, 1976, Colloq. Math. Soc. J. Bolyai 18, Vol II, 939-945.
[24] A. Sidorenko, Inequalities for functionals generated by bipartite graphs (Russian), Diskret. Mat. 3 (1991), no 3., 50-65. English transl. Discrete Math. Appl. 2 (1992), no 5, 489-504.
[25] J. Verstraëte, On arithmetic progressions of cycle lengths in graphs, Combin. Probab. Comput. 9 number 4(2000), 369-373.
[26] R.Wilson, An existence theory for pairwise balanced designs I. Composition theorems and morphisms, J. Combinatorial Theory Ser. A 13 (1972), 220-245.
[27] R.Wilson, An existence theory for pairwise balanced designs II. The structure of PBD-closed sets and the existence conjectures, J. Combinatorial Theory Ser. A 13 (1972), 246-273.
[28] R.Wilson, An existence theory for pairwise balanced designs III. Proof of the existence conjectures, J. Combinatorial Theory Ser. A 13 (1975), 71-79. Decompositions of complete graphs into subgraphs isomorphic to a given graph, Proc. British Combinatorial Conference, 1975, 647-659.


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