# On Turán exponents of bipartite graphs

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#### Abstract

A long-standing conjecture of Erdős and Simonovits asserts that for every rational number  $r \in (1, 2)$  there exists a bipartite graph H such that  $ex(n, H) = \Theta(n^r)$ . So far this conjecture is known to be true only for rationals of form 1 + 1/k and 2 - 1/k, for integers  $k \ge 2$ . In this paper we add a new form of rationals for which the conjecture is true; 2 - 2/(2k+1), for  $k \ge 2$ . This in its turn also gives an affirmative answer to a question of Pinchasi and Sharir on cube-like graphs.

Recently, a version of Erdős and Simonovits's conjecture where one replaces a single graph by a family, was confirmed by Bukh and Conlon. They proposed a construction of bipartite graphs which should satisfy Erdős and Simonovits's conjecture. Our result can also be viewed as a first step towards verifying Bukh and Conlon's conjecture.

We also prove the an upper bound on the Turán's number of  $\theta$ -graphs in an asymmetric setting and employ this result to obtain yet another new rational exponent for Turán exponents; r = 7/5.

# 1 Introduction

Given a family  $\mathcal{H}$  of graphs, a graph G is called  $\mathcal{H}$ -free if it contains no member of  $\mathcal{H}$  as a subgraph. The Turán number  $ex(n, \mathcal{H})$  of  $\mathcal{H}$  is the maximum number of edges in an *n*-vertex  $\mathcal{H}$ -free graph. When  $\mathcal{H}$  consists of a single graph H, we write ex(n, H) for  $ex(n, \{H\})$ . The study of Turán numbers plays a central role in extremal graph theory. The celebrated Erdős-Simonovits-Stone theorem [10, 12] states that if  $\chi(\mathcal{H})$  denotes the minimum chromatic number of a graph in  $\mathcal{H}$ , then

$$ex(n, \mathcal{H}) = \left(1 - \frac{1}{\chi(\mathcal{H}) - 1}\right) \binom{n}{2} + o(n^2).$$

Thus, the function is asymptotically determined if  $\chi(\mathcal{H}) \geq 3$ . If  $\chi(\mathcal{H}) = 2$ , that is, if  $\mathcal{H}$  contains a bipartite graph, then this only gives  $ex(n, \mathcal{H}) = o(n^2)$ . We will refer to  $ex(n, \mathcal{H})$  with  $\chi(\mathcal{H}) = 2$ 

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as degenerate Turán numbers, as in [18]. Concerning degenerate Turán numbers, there are several general conjectures (see [18]). First, Erdős and Simonovits conjectured that if  $\mathcal{H}$  is a finite family with  $\chi(\mathcal{H}) = 2$  then there is a rational  $r \in [1, 2)$  and a constant c > 0 such that  $\lim_{n\to\infty} \exp(n, \mathcal{H})/n^r = c$ . (See Conjecture 1.6 of [18]). This conjecture is still wide open. In fact the order of magnitude of  $\exp(n, \mathcal{H})$  where  $\chi(\mathcal{H}) = 2$  is known only for very few families  $\mathcal{H}$ . Another conjecture, which may be viewed as the inverse extremal problem of the previous one, is that for very rational  $r \in [1, 2)$ there exists a finite family  $\mathcal{H}$  of graphs such that  $c_1n^r < \exp(n, \mathcal{H}) < c_2n^r$  for some constants  $c_1, c_2$ . (See Conjecture 2.37 of [18].) In a recent breakthrough work by Bukh and Conlon [4], this second conjecture has been verified, using a random algebraic method (developed earlier in [2, 3, 7]).

However, the following analogous problem on the Turán number of a single bipartite graph, raised by Erdős and Simonovits [8], on the other hand, is still wide open.

**Question 1.1** ([8]) Is it true that for every rational number r in (1,2) there exists a single bipartite graph  $H_r$  such that  $ex(n, H_r) = \Theta(n^r)$ ?

We will refer to a rational r for which Problem 1.1 has an affirmative answer as a *Turán exponent* for a single graph. The only known Turán exponents for single graphs from the literature are rational numbers of the forms  $1 + \frac{1}{s}$  and  $2 - \frac{1}{s}$  for all integers  $s \ge 2$ . Specifically it is known that  $ex(n, K_{s,t}) = \Theta(n^{2-1/s})$  when t > (s-1)! (by [23, 22, 1]). Let  $\theta_{s,p}$  denote the graph obtained by taking the union of p internally disjoint paths of length s between a pair of vertices. Faudree and Simnovits [13] showed that  $ex(n, \theta_{s,p}) = O(n^{1+1/s})$  for all  $p \ge 2$  (see [5] for a recent improvement on the specific bound) while Conlon [7] showed that for every  $s \ge 2$  there exists a  $p_0$  such that for all  $p \ge p_0$  we have  $ex(n, \theta_{s,p}) = \Omega(n^{1+1/s})$ . Hence for each s and sufficiently large p we have  $ex(n, \theta_{s,p}) = \Theta(n^{1+1/s})$ . For a more thorough introduction to degenerate Turán numbers, the reader is referred to the recent survey by Füredi and Simonvits [18].

Our main theorem is as follows, which in particular establishes an infinite sequence of new Turán exponents.

**Theorem 1.2** For any rational number  $r = 2 - \frac{2}{2s+1}$ , where  $s \ge 2$  is an integer, or  $r = \frac{7}{5}$ , there exists a single bipartite graph  $H_r$  such that  $ex(n, H_r) = \Theta(n^r)$ .

In establishing the first part of our main theorem, we establish a stronger result concerning the Turán numbers of cube-like graphs, which answers a question of Pinchasi and Sharir [25]. This result may be of independent interest. To establish the second part of our main result, we develop an asymmetric Turán bound on  $\theta_{s,p}$  which may be viewed as a common generalization (in a general sense) of [13] and [24], and may also be of independent interest.

To describe our results, we need some more detailed background, which we discuss over several subsections.

#### 1.1 The theorem of Bukh and Conlon and a conjecture

To describe Bukh and Conlon's results, we need some definitions. Given a tree T together with an independent set  $R \subseteq V(T)$ , we call (T, R) a rooted tree and R the root set. Given any  $S \subseteq V(T) \setminus R$ , let e(S) denote the number of edges of T with at least one endpoint in S. Let  $\rho_S = e(S)/|S|$ . Let

 $\rho_T = \rho(V(T) \setminus R)$ . We say that the rooted tree (T, R) is balanced if  $\rho_S \ge \rho_T$  for all  $S \subseteq V(T) \setminus R$ . Given a rooted tree (T, R) and a positive integer p, let  $\mathcal{T}_R^p$  denote the family of graphs consisting of all possible union of p distinct labelled copies of T, each of which agree on the root set R. We call  $\mathcal{T}_R^p$  the *pth power* of (T, R). The key result of Bukh and Conlon [4] is the following

**Theorem 1.3** ([4]) For any balanced rooted tree (T, R), there exists a  $p_0$  such that for all  $p \ge p_0$ ,

$$\operatorname{ex}(n, \mathcal{T}_R^p) = \Omega(n^{2-1/\rho_T})$$

A straightforward counting argument shows that  $ex(n, \mathcal{T}_R^p) = O(n^{2-1/\rho_T})$  and thus implies that  $ex(n, \mathcal{T}_R^p) = \Theta(n^{2-1/\rho_T})$  for sufficiently large p. Bukh and Conlon [4] also showed that for each rational r in (1,2), there exists a balanced rooted tree (T, R) with  $\rho_T = \frac{1}{2-r}$ , thereby establishing the existence of a family  $\mathcal{H}_r$  with  $ex(n, \mathcal{H}_r) = \Theta(n^r)$  for each rational  $r \in (1, 2)$ .

Let (T, R) be a balanced rooted tree. Let  $T_R^p$  denote the unique member of  $\mathcal{T}_R^p$  in which the p labelled copies of T are pairwise vertex disjoint outside R. By Theorem 1.3,

$$\exp(n, T_R^p) \ge \exp(n, \mathcal{T}_R^p) = \Omega(n^{2-1/\rho_T})$$

If there exists a matching upper bound on  $ex(n, T_R^p)$ , then together with earlier discussion this would answer Question 1.1 in the affirmative in a very strong way. Indeed Bukh and Conlon conjectured that a matching upper bound indeed exists.

**Conjecture 1.4** ([4]) If (T, R) is a balanced rooted tree, then

$$\operatorname{ex}(n, T_R^p) = O(n^{2-1/\rho_T})$$

Let  $D_s$  be the tree obtained by taking two disjoint stars with s leaves and joining the two central vertices by an edge, and R the set of all the leaves in  $D_s$ . It is easy to check that  $(D_s, R)$  is balanced with  $\rho_{D_s} = \frac{2s+1}{2}$ . Let  $T_{s,t} = D_s^R$ . We will show that (in Corollary 1.8) that  $\exp(n, T_{s,t}) = O(n^{2-\frac{2}{2s+1}})$  for all  $t \ge s \ge 2$  and thereby verifying Conjecture 1.4 for  $T = D_s$ , R be its set of leaves, making a first step towards Conjecture 1.4.

#### 1.2 The cube and its generalization

Let  $Q_8$  denote the 3-dimensional cube, that is, the graph obtained from two vertex-disjoint  $C_4$ 's by adding a perfect matching between them. The well-known cube theorem of Erdős and Simonovits [11] states that

$$ex(n, Q_8) = O(n^{8/5}).$$
 (1)

Pinchasi and Sharir [25] gave a new proof of this and extended to the bipartite setting. More recently, Füredi [17] showed that  $ex(n, Q_8) \le n^{8/5} + (2n)^{3/2}$ , giving another new proof of the cube theorem.

Pinchasi and Sharir's approach is motivated by certain geometric incidence problems. In their approach it is more convenient to view  $Q_8$  as a special case of the graph  $H_{s,t}$  defined as follows. Let  $t \ge s \ge 2$  be positive integers. Let M be an s-matching  $a_1b_1, a_2b_2, \ldots, a_sb_s$ , and N a t-matching  $c_1d_1, c_2d_2, \ldots, c_td_t$ , where M and N are vertex disjoint. Let  $H_{s,t}$  be obtained from  $M \cup N$  by adding edges  $a_id_j$  and  $b_ic_j$  over all  $i \in [s]$  and  $j \in [t]$ .

Alternatively, we may view  $H_{s,t}$  as being obtained from two vertex disjoint copies of  $K_{s,t}$  by adding a matching that joins the two images of every vertex in  $K_{s,t}$ . In particular, we see that  $Q_8 = H_{2,2}$ . In addition to giving a new proof of (1), Pinchasi and Sharir [25] proved that if G is an n-vertex graph that contains neither a copy of  $H_{s,t}$  nor a copy of  $K_{s+1,s+1}$ , then  $e(G) \leq O(n^{2-2/(2s+1)})$ .

**Question 1.5** [25] Is it true that for all  $t \ge s \ge 2$ ,

$$ex(n, H_{s,t}) = O(n^{2-\frac{2}{2s+1}})?$$

This was answered affirmatively if s = t in [20].

In this paper, we answer Pinchasi and Sharir's question affirmatively as follows.

**Theorem 1.6** For any  $t \ge s \ge 2$ ,  $ex(n, H_{s,t}) = O(n^{2-2/(2s+1)})$ .

Note that  $T_{s,t} \subseteq H_{s,t}$ . Hence, by Theorem 1.3 we have the following.

**Proposition 1.7** There exists a function  $\ell$  such that for all  $s \geq 2$  and  $t \geq \ell(s)$ ,

$$\exp(n, H_{s,t}) \ge \exp(n, T_{s,t}) \ge \exp(n, \mathcal{T}_R^t) \ge \Omega\left(n^{2-2/(2s+1)}\right).$$

Theorem 1.6 and Proposition 1.7 now give

**Corollary 1.8** There exists a function  $\ell$  such that for all  $s \geq 2$  and  $t \geq \ell(s)$ ,

$$ex(n, H_{s,t}) = \Theta(n^{2-2/(2s+1)})$$
 and  $ex(n, T_{s,t}) = \Theta(n^{2-2/(2s+1)}).$ 

#### **1.3** Theta graphs and 3-comb-pastings

For the second part of our work, we give another new Turán exponent of 7/5 for a bipartite graph  $S_p$  which we define below.

By a 3-comb  $T_3$ , we denote the tree obtained from a 3-vertex path P = abc by adding three new vertices a', b', c' and three new edges aa', bb', cc'. For each  $p \ge 2$ , a 3-comb-pasting, denoted by  $S_p$ , is the graph obtained by first taking p vertex disjoint copies of  $T_3$  and then combining the images of a' into one vertex, the images of b' into one vertex, and the images of c' into one vertex.

Let R denote the set of leaves of  $T_3$ . It is easy to see that  $(T_3, R)$  is balanced with density 5/3, while the 3-comb-pasting  $S_p$  is just a member in the pth power of  $(T_3, R)$ . Hence, by Theorem 1.3, there exists  $p_0$  such that for all  $p \ge p_0$ ,  $ex(n, S_p) \ge \Omega(n^{7/5})$ .

We prove a matching upper bound as follows.

**Theorem 1.9** For all  $p \ge 2$ , it holds that  $ex(n, S_p) = O(n^{7/5})$ .

**Corollary 1.10** There exists a positive integer  $p_0$  such that for all  $p \ge p_0$ , it holds that

$$\operatorname{ex}(n, S_p) = \Theta(n^{7/5}).$$

A key step in the proof of Theorem 1.9 is to study Turán numbers of theta graphs in the bipartite setting, which continue the line of work of Faudree and Simonovits [13] and of Naor and Verstraete [24] and may be of independent interest.

Given a family  $\mathcal{H}$  of graphs and positive integers m, n, the asymmetric bipartite Turán number  $z(m, n, \mathcal{H})$  of  $\mathcal{H}$  denote the maximum number of edges in an m by n bipartite graph that does not contain any member of  $\mathcal{H}$  as a subgraph. If  $\mathcal{H}$  has just one member H, we write z(m, n, H) for  $z(m, n, \{H\})$ . The function  $z(m, n, C_{2k})$  had been studied in the context of number theoretic problems and geometric problems. Naor and Verstraëte [24] proved that for  $m \leq n$  and  $k \geq 2$ ,

$$z(m, n, C_{2k}) \leq \begin{cases} (2k-3) \cdot [(mn)^{\frac{k+1}{2k}} + m + n] & \text{if } k \text{ is odd,} \\ (2k-3) \cdot [m^{\frac{k+2}{2k}}n^{\frac{1}{2}} + m + n] & \text{if } k \text{ is even.} \end{cases}$$

A different form of upper bounds on  $z(m, n, C_{2k})$  can be found in [19].

Recall the theta graph  $\theta_{k,p}$ , that is the graph consisting of the union of p internally disjoint paths of length k joining a pair of vertices. In particular,  $\theta_{k,2} = C_{2k}$ . The following result can be viewed as a common generalization of the results in [13, 24].

**Theorem 1.11** Let  $m, n, k, p \ge 2$  be integers, where  $m \le n$ . There exists a positive constant c = c(k, p) such that

$$z(m, n, \theta_{k, p}) \leq \begin{cases} c \cdot [(mn)^{\frac{k+1}{2k}} + m + n] & \text{if } k \text{ is odd,} \\ c \cdot [m^{\frac{k+2}{2k}}n^{\frac{1}{2}} + m + n] & \text{if } k \text{ is even.} \end{cases}$$

The rest of the paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we prove Theorem 1.6. In Section 4, we prove Theorem 1.11. In Section 5, we prove Theorem 1.9.

### 2 Preliminaries

In this section we present some of the auxiliary lemmas which are used in the proofs of main results. The first three are folklore, the proofs of the other two can also be found in [20].

**Lemma 2.1** If G is a graph with average degree d then it contains a bipartite subgraph  $G_1$  with  $e(G_1) \geq \frac{1}{2}e(G)$  and a subgraph  $G_2$  with minimum degree  $\delta(G_2) \geq \frac{1}{2}d$ .

**Lemma 2.2** Let G be a bipartite graph with a bipartition (A, B). Let  $d_A = e(G)/|A|$  and  $d_B = e(G)/|B|$ . There exists a subgraph G' of G with  $e(G') \ge \frac{1}{2}e(G)$  such that each vertex in  $V(G') \cap A$  has degree at least  $\frac{1}{4}d_A$  in G' and each vertex in  $V(G') \cap B$  has degree at least  $\frac{1}{4}d_B$  in G'.

**Lemma 2.3** Let k be a positive integer and T be a rooted tree with k edges. If G is a graph with minimum degree at least k and v is any vertex in G, then G contains a copy of T rooted at v.

**Lemma 2.4 ([20], Lemma 5.3)** Let t be a positive integer and G be an n-vertex bipartite graph with at least 4tn edges. Then the number of t-matchings in G is at least  $\frac{e(G)^t}{2^t t!}$ .

**Lemma 2.5 ([20], Lemma 5.5)** Let t be a positive integer and G be an n-vertex bipartite graph with a bipartition (A, B). Suppose G has at least  $4\sqrt{2tn^{3/2}}$  edges. Then the number of  $H_{1,t}$ 's in G is at least

$$\frac{1}{2^{5t+2}t!} \cdot \frac{e(G)^{3t+1}}{|A|^{2t}|B|^{2t}}.$$

We also need the following regularization theorem of Erdős and Simonovits which is an important tool for Turán-type problems of sparse graphs. Recently, the first and third author have developed a version of this result for linear hypergraphs [21]. For a positive real  $\lambda$ , G is called  $\lambda$ -almost-regular if  $\Delta(G) \leq \lambda \cdot \delta(G)$ .

**Theorem 2.6** ([11]) Let  $\alpha$  be any real in (0,1),  $\lambda = 20 \cdot 2^{(1/\alpha)^2}$ , and n be a sufficiently large integer depending only on  $\alpha$ . Suppose G is an n-vertex graph with  $e(G) \ge n^{1+\alpha}$ . Then G has a  $\lambda$ -almost-regular subgraph on m vertices, where  $m > n^{\alpha \frac{1-\alpha}{1+\alpha}}$  such that  $e(G') > \frac{2}{5}m^{1+\alpha}$ .

## 3 Turán numbers of generalized cubes

In this section we prove Theorem 1.6. Our proof is partly based on the ideas of Pinchasi and Sharir [25]. The key new idea is Lemma 3.1. To state the lemma, we need some notation.

In a graph G, for any  $S \subseteq V(G)$ , the common neighbourhood of S in G is defined by  $N_G(S) = \bigcap_{v \in S} N_G(v)$ , and the common degree of S in G is  $d_G(S) = |N_G(S)|$ . When G is clear from the context, we will drop the subscripts. For a matching M in the bipartite graph G with bipartition (A, B), we define  $A_M = V(M) \cap A$ ,  $B_M = V(M) \cap B$ . We call the subgraph induced by the vertex sets  $N(B_M) \setminus V(M)$  and  $N(A_M) \setminus V(M)$  the neighbourhood graph of M and with some abuse of notation, for brevity, we denote it by N(M).

Let M and L be two matchings in G. We write  $M \sim L$  if L is a subgraph in N(M). For t non-negative integer, we say that an ordered pair (M, L) of matchings is t-correlated if  $M \sim L$  and there exists a vertex v in V(M) such that  $d_{N(L)}(v) \geq t$ .

**Lemma 3.1** Let G be an  $H_{s,t}$ -free bipartite graph and M be an (s-1)-matching in G. Then the number of s-matchings L in N(M) such that (M,L) is 2t-correlated is at most  $(s-1)(t-1) \cdot e(N(M))^{s-1}v(N(M))$ .

*Proof.* It suffices to prove that for any  $x \in A_M$ , L' an (s-1)-matching in N(M), and  $y \in (V(N(M)) \cap B) \setminus V(L')$ , the number of s-matchings L in N(M) that contain L' and y and satisfy  $d_{N(L)}(x) \geq 2t$  is at most t-1.

Suppose otherwise, let  $L' = \{c_1d_1, \ldots, c_{s-1}d_{s-1}\}$ , where  $c_1, \ldots, c_{s-1} \in A$  and  $d_1, \ldots, d_{s-1} \in B$ for which the claim fails. Then there exist t distinct s-matchings  $L_1, \ldots, L_t$  in N(M) containing L'and y that satisfy  $d_{N(L_i)}(x) \ge 2t$ . Let  $u_1, u_2, \ldots, u_t$  be distinct vertices such that  $L_i = L' \cup \{u_iy\}$ . For each  $i \in [t]$ , since  $d_{N(L_i)}(x) \ge 2t$ , we have  $|N_{N(L_i)}(x) \setminus B_M| \ge t$ . We can therefore find t distinct vertices  $v_1, \ldots, v_t$  such that for each  $i \in [t]$   $v_i \in N_{N(L_i)}(x) \setminus B_M$ .

Let  $B^* = \{b_1, \ldots, b_{s-1}, y\}, C^* = \{x, c_1, \ldots, c_{s-1}\}, U^* = \{u_1, \ldots, u_t\}, \text{ and } V^* = \{v_1, \ldots, v_t\}.$  It is easy to see that  $G_1 := G[B^* \cup U^*], G_2 := G[C^* \cup V^*]$  are both copies of  $K_{s,t}$ . Let  $M_1 :=$ 

 $\{u_1v_1,\ldots,u_tv_t\}, M_2 = \{b_1c_1,\ldots,b_{s-1}c_{s-1},xy\}.$  One can easily check that  $G_1 \cup G_2 \cup M_1 \cup M_2$  is a copy of  $H_{s,t}$  in G, contradicting G being  $H_{s,t}$ -free.

**Proof of Theorem 1.6.** Our choice of constant C here will be explicit. Let  $\alpha = \frac{2s-1}{2s+1}$ . As  $s \ge 2$ , we have  $\frac{3}{5} \le \alpha < 1$ . Let  $\lambda$  be the constant derived from Theorem 2.6 applied for  $\alpha$ . By Theorem 2.6, it suffices to show that there is a constant C = C(s,t) > 0 such that the following holds for sufficiently large n: if G is a  $\lambda$ -almost-regular graph with n vertices and  $m \ge Cn^{1+\alpha}$  edges, then G contains a copy of  $H_{s,t}$ . By Proposition 2.1, we may further assume that G is bipartite with a bipartition (A, B). Let  $\mathcal{M}$  be the collection of all (s - 1)-matchings in G. Denote

$$\mathcal{M}_1 = \{ M : M \in \mathcal{M}, e(N(M)) \le 2^{s+1} s! (s-1)(t-1)v(N(M)) \},$$
  
$$\mathcal{M}_2 = \mathcal{M} \setminus \mathcal{M}_1,$$
  
$$\mathcal{M}_2^{2t,s} = \{ (M,L) : M \in \mathcal{M}_2, L \in \mathcal{L}, L \text{ is an } s\text{-matching}, M \sim L, (M,L) \text{ is not } 2t\text{-correlated} \}.$$

We suppose that G is  $H_{s,t}$ -free and derive a contradiction on the number of edges of the graph G. For doing so, we will use upper and lower bounds on the size of the set  $\mathcal{M}_2^{2t,s}$ .

**Claim 1.6.1.**  $\sum_{M \in \mathcal{M}_2} e(N(M)) = \Omega(\frac{m^{3s-2}}{n^{4s-4}}).$ 

Proof of Claim. Let us call a tree obtained from  $K_{1,p}$  by subdividing each edge once a *p*-spider of height 2. Note that  $\sum_{M \in \mathcal{M}} v(N(M))$  counts the number of (s-1)-spiders of height 2 in G. Since G is  $\lambda$ -almost-regular,  $\Delta := \Delta(G) \leq \lambda \cdot \delta(G) \leq \lambda \cdot 2m/n$ . Thus,

$$\sum_{M \in \mathcal{M}} v(N(M)) \le n\Delta^{2s-2} = O\left(\frac{m^{2s-2}}{n^{2s-3}}\right)$$

By the definition of  $\mathcal{M}_1$ , we have  $\sum_{M \in \mathcal{M}_1} e(N(M)) = O\left(\sum_{M \in \mathcal{M}_1} v(N(M))\right) = O\left(\frac{m^{2s-2}}{n^{2s-3}}\right)$ . On the other hand,  $\sum_{M \in \mathcal{M}} e(N(M))$  counts the number of  $H_{1,s-1}$ 's in G. So by Lemma 2.5, we have  $\sum_{M \in \mathcal{M}} e(N(M)) = \Omega\left(\frac{m^{3s-2}}{n^{4s-4}}\right)$ . Since  $m \ge Cn^{4s/(2s+1)}$  and n is sufficiently large, we have  $\frac{m^{3s-2}}{n^{4s-4}} \gg \frac{m^{2s-2}}{n^{2s-3}}$  thus the claim follows.

Now consider a matching  $M \in \mathcal{M}_2$ . By Lemma 2.4, the number of s-matchings L in N(M) is at least  $(1/2^s s!)e(N(M))^s$ . By Lemma 3.1 and the definition of  $\mathcal{M}_2$ , the number of s-matchings L in N(M) such that (M, L) is 2t-correlated is at most

$$(s-1)(t-1)e(N(M))^{s-1}v(N(M)) \le \frac{e(N(M))^s}{2^{s+1}s!}$$

Hence the number of s-matchings L in N(M) such that (M, L) is not 2t-correlated is at least  $(1/2)(1/2^s s!)e(N(M))^s$ .

By Claim 1.6.1, the convexity of the function  $f(x) = x^s$  and the fact that  $|\mathcal{M}_2| \leq m^{s-1}$ ,

$$|\mathcal{M}_{2}^{2t,s}| \ge (1/2^{s+1}s!) \sum_{M \in \mathcal{M}_{2}} e(N(M))^{s} = \Omega\left(\frac{(\sum_{M \in \mathcal{M}_{2}} e(N(M)))^{s}}{|\mathcal{M}_{2}|^{s-1}}\right) = \Omega\left(\frac{m^{2s^{2}-1}}{n^{4s^{2}-4s}}\right)$$

Claim 1.6.2.  $|\mathcal{M}_2^{2t,s}| \leq {t-1 \choose s-1}(2t-1)^{s-1}m^s.$ 

Proof of Claim. Let L be an s-matching in G. Since G is  $H_{s,t}$ -free, N(L) has matching number at most t-1. Since N(L) is bipartite, by the König-Egerváry theorem it has a vertex cover Q of size at most t-1. Let  $Q^+$  denote the set of vertices in Q that have degree at least 2t in N(L)and  $Q^- = Q \setminus Q^+$ . If M is an (s-1)-matching in G that satisfies  $M \sim L$  and that (M, L) is not 2t-correlated, then M is contained in N(L) and could not contain any vertex in  $Q^+$ . Since  $Q = Q^+ \cup Q^-$  is a vertex cover in N(L), each edge of M must contain a vertex in  $Q^-$ . Thus,

$$|\mathcal{M}_2^{2t,s}| \le \binom{|Q^-|}{s-1} (2t-1)^{s-1} m^s \le \binom{t-1}{s-1} (2t-1)^{s-1} m^s.$$

Combining the lower and upper bounds on  $|\mathcal{M}_2^{2t,s}|$ , we get that  $\frac{m^{2s^2-1}}{n^{4s^2-4s}} = O(m^s)$ , which implies that  $m = O(n^{4s/(2s+1)})$ , where the constant factor in  $O(\cdot)$  only depends on s and t. This contradicts that  $m \ge Cn^{4s/(2s+1)}$ , assuming C is chosen to be sufficiently large.

# 4 Asymmetric bipartite Turán numbers of Theta graphs

In this section we establish a upper bound (i.e., Theorem 1.11) of the asymmetric bipartite Turán numbers of theta graphs  $\theta_{k,p}$ . This, in turn, will be crucial in the proof of Theorem 1.9.

Our proof, in a conspectus, employs the standard breadth-first-search tree (BFS-tree) approach and thus the major challenge is to show that the distance levels of the BFS-tree should grow in magnitude rapidly. This will be essentially unravelled by the following lemma, where we adopt a modification of the so-called "blowup method" by Faudree and Simonovits [13]. A similar lemma was proved in [21].

**Lemma 4.1** Let k, p, t be positive integers, where  $k, p \ge 2$  and  $t \le k-1$ . Let T be a tree of height t rooted at a vertex x. Let A be the set of vertices at distance t from x in T. Let B be set of vertices disjoint from V(T). Let G be a bipartite graph with a bipartition (A, B). If  $T \cup G$  is  $\theta_{k,p}$ -free then  $e(G) \le 2ktp^t \cdot (|A| + |B|)$ .

*Proof.* We use induction on t. For the basis case t = 1, let

$$B^+ = \{y \in B : d_G(y) \ge pk\}$$
 and  $B^- = B \setminus B^+$ .

By definition,  $e(G[A \cup B^-]) \leq pk|B^-|$ . We show that  $e(G[A \cup B^+]) \leq pk|A \cup B^+|$ . Suppose that is not the case. Then  $G[A \cup B^+]$  has average degree at least 2pk and hence (by Proposition 2.1) contains a subgraph H with minimum degree at least pk. If k is odd then let v be a vertex in  $V(H) \cap A$ . If kis even then let v be a vertex in  $V(H) \cap B$ . Let F denote the union of p paths of length k-2 that share a common endpoint u but are otherwise vertex disjoint, and view u as the root of the tree F. By Lemma 2.3, H contains a copy F' of F which has v as its root. Let  $v_1, \ldots, v_p$  denote the leaves of F'. By our choice of v, we have  $v_1, \ldots, v_p \in V(H) \cap B^+$ . By the definition of  $B^+$ ,  $d_G(v_i) \geq pk$ for each  $i \in [p]$ . Hence we can find distinct vertices  $w_1, \ldots, w_p$  in A that lie outside V(F') such that  $v_i w_i \in E(G)$  for each  $i \in [p]$ . Now,  $F' \cup \{v_1 w_1, \ldots, v_p w_p\} \cup \{xw_1, \ldots, xw_p\}$  forms a copy of  $\theta_{k,p}$  in  $T \cup G$ , a contradiction. Hence, we have  $e(G[A \cup B^+]) \leq pk|A \cup B^+|$ . Putting everything together, we have  $e(G) \leq pk(|A| + |B|)$ . So the basis case holds.

For the induction step, let us consider  $t \ge 2$ . Let  $x_1, \ldots, x_q$  denote all children of x in T. For each  $i \in [q]$ , let  $T_i$  denote the subtree of T - x that contains  $x_i$  and let  $S_i = V(T_i) \cap A$ . Then  $S_1, \ldots, S_q$  partition A. For each  $u \in A$ , let  $P_u$  denote the unique path from u to x in T. Then for each  $u \in A$ ,  $P_u$  has length t, and if u, v lie in different  $S_i$ 's then  $V(P_u) \cap V(P_v) = \{x\}$ . Let

$$B^* = \{ y \in B : d_G(y) < kp^2 \}.$$

By definition, it is clear that we have

$$e(G[A \cup B^*]) \le kp^2 \cdot |B^*|. \tag{2}$$

We further partition  $B \setminus B^*$  into the following two sets. Let

 $B^+ = \{y \in B \setminus B^* : \text{ for } \forall i \in [q], y \text{ has no more than } d_G(y)/p \text{ neighbours in } S_i\}$ 

and  $B^- = B \setminus (B^* \cup B^+)$ . We now observe the following property for  $B^+$ .

Claim 1. For each  $y \in B^+$  and any subset  $I \subseteq [q]$  of size p-1, we have  $|N_G(y) \setminus \bigcup_{i \in I} S_i| > kp$ . *Proof of Claim 1.* Since  $y \in B^+$ , for any  $i \in I$ , we have  $|N_G(y) \cap S_i| < d_G(y)/p$ . Hence,

$$\left| N_G(y) \setminus \bigcup_{i \in I} S_i \right| \ge \left( 1 - \frac{p-1}{p} \right) \cdot d_G(y) = \frac{d_G(y)}{p} \ge kp,$$

proving the claim.

We prove two more claims, which bounds  $e(G[A \cup B^+])$  and  $e(G[A \cup B^-])$ , respectively.

Claim 2.  $e(G[A \cup B^+]) \le kp \cdot |A \cup B^+|.$ 

Proof of Claim 2. Let F be the tree consisting of p paths of length k - t - 1 that share a common endpoint u but are otherwise vertex disjoint; also view u as the root of F. So F has p(k - t - 1)edges. Suppose for a contradiction that  $e(G[A \cup B^+]) > kp \cdot |A \cup B^+|$ . Then by Proposition 2.1,  $G[A \cup B^+]$  contains a subgraph H with minimum degree more than kp. If k - t - 1 is odd, then let v be a vertex in  $V(H) \cap A$ ; and if k - t - 1 is even, then let v be a vertex in  $V(H) \cap B^+$ . By Lemma 2.3, H contains a copy F' of F which has v as its root. Let  $v_1, \ldots, v_p$  denote the leaves of F'. By our choice of v, we have  $v_1, \ldots, v_p \in V(H) \cap B^+$ . By Claim 1, we can find vertices  $w_1, \ldots, w_p$ outside V(F') such that they all lie in different  $S_i$ 's and  $v_1w_1, v_2w_2, \ldots, v_pw_p \in E(G)$ . Indeed, for any  $\ell \leq p$ , suppose we have found  $w_1, \ldots, w_{\ell-1}$ . By Claim 1,  $v_\ell$  has at least kp neighbours that lie outside the  $S_j$ 's that contain vertices in  $\{w_1, \ldots, w_{\ell-1}\}$ . Among these neighbours we can find one that also does not lie in V(F'). We let  $w_\ell$  be such a vertex. Since  $w_1, \ldots, w_p$  all lie in different  $S_i$ 's, the paths  $P_{w_1}, \ldots, P_{w_p}$  pairwise intersect only in vertex x. Now  $F' \cup \{v_1w_1, \ldots, w_pw_p\} \cup \bigcup_{i=1}^k P_{w_i}$ forms a copy of  $\theta_{k,p}$  in G, a contradiction. Hence we must have  $e(G[A \cup B^+]) \leq kp \cdot |A \cup B^+|$ .

Claim 3.  $e(G[A \cup B^{-}]) \leq 2k(t-1)p^{t} \cdot |A \cup B^{-}|.$ 

Proof of Claim 3. For each  $y \in B^-$  by definition there exists  $i(y) \in [q]$  such that  $|N_G(y) \cap S_{i(y)}| \geq d_G(y)/p$ ; let us fix such an i(y). We then define a subgraph H obtained from  $G[A \cup B^-]$  by only taking the edges from every  $y \in B^-$  to  $N_G(y) \cap S_{i(y)}$ . By the definition of H, we see that  $e(H) \geq \frac{1}{p}e(G[A \cup B^-])$ . Now, for each  $j \in [q]$  let  $B_j = \{y \in B^- : i(y) = j\}$ . Then in fact H is the vertex-disjoint union of  $H[S_1 \cup B_1], H[S_2 \cup B_2], \ldots, H[S_q \cup B_q]$ . Let  $j \in [q]$ . Note that  $T_j$  is tree of height t-1 rooted at  $x_j$  and  $B_j$  is the set of vertices at distance t-1 from  $x_j$ . Also,  $B_j$  is vertex disjoint from  $T_j$  and  $H[S_j \cup B_j]$  is a bipartite graph with a bipartition  $(S_j, B_j)$ . Since  $T_j \cup H[S_j \cup B_j]$  is  $\theta_{k,p}$ -free, by the induction hypothesis,  $e(H[S_j, B_j]) \leq 2k(t-1)p^{t-1} \cdot |S_j \cup B_j|$ . Hence,

$$e(H) = \sum_{j=1}^{p} e(H[S_j, B_j]) \le 2k(t-1)p^{t-1} \cdot \sum_{j=1}^{p} |S_j \cup B_j| \le 2k(t-1)p^{t-1} \cdot |A \cup B^-|,$$

implying that  $e(G[A \cup B^-]) \le p \cdot e(H) \le 2k(t-1)p^t \cdot |A \cup B^-|$ . This proves Claim 3.

Finally, combining (2) with Claims 2 and 3, we have that

$$\begin{split} e(G) &= e(G[A \cup B^*]) + e(G[A, B^+]) + e(G[A, B^-]) \\ &\leq kp^2 \cdot |B^*| + kp \cdot (|A| + |B^+|) + 2k(t-1)p^t \cdot (|A| + |B^-|) < 2ktp^t \cdot (|A| + |B|), \end{split}$$

finishing the proof of Lemma 4.1.

We are ready to show Theorem 1.11.

**Proof of Theorem 1.11:** Let G be a  $\theta_{k,p}$ -free bipartite graph with a bipartition (A, B) where |A| = m and |B| = n. Let  $c = 16k^2p^k$ . If k is odd, then we assume  $e(G) \ge c \cdot (mn)^{\frac{1}{2} + \frac{1}{2k}} + c \cdot (m+n)$ . If k is even, then assume  $e(G) \ge c \cdot m^{\frac{1}{2} + \frac{1}{k}} n^{\frac{1}{2}} + c \cdot (m+n)$ . Let  $d_A = e(G)/|A|$  and  $d_B = e(G)/|B|$ . So each of  $d_A, d_B$  is more than  $c = 16k^2p^k$ .

By Lemma 2.2, G contains a subgraph G' with  $e(G') \ge \frac{1}{2}e(G)$  such that each vertex in  $V(G') \cap A$ has degree at least  $\frac{1}{4}d_A$  in G' and that each vertex in  $V(G') \cap B$  has degree at least  $\frac{1}{4}d_B$  in G'. Fix a vertex  $x \in V(G') \cap A$ . For each integer  $i \ge 0$ , let  $L_i$  denote the set of vertices at distance i from x in G', and let  $d_i = d_A$  if i is odd and  $d_i = d_B$  if i is even. So we see that every vertex in  $L_{i-1}$  has degree at least  $\frac{1}{4}d_i$  in G'.

Using Lemma 4.1, we show that the (growth) ratio of two consecutive levels must be large in the following

**Claim.** For each  $i \in [k]$ , we have  $|L_i|/|L_{i-1}| \ge \frac{d_i}{16kip^i}$ . In particular,  $|L_i| \ge |L_{i-1}|$  holds.

Proof of Claim. Since  $d_i \ge 16k^2p^k$  for each *i*, we observe that the second statement follows easily by the first statement. So it suffices to prove the first statement, which we will prove by induction on *i*. If i = 1, then we have  $\frac{|L_1|}{|L_0|} \ge \frac{1}{4}d_A \ge \frac{d_1}{16kp}$ . So the claim holds for the basis step.

For the inductive step, consider  $i \ge 2$ . Let  $T_{i-1}$  be a breadth-first-search tree in G' rooted at x with vertex set  $L_0 \cup L_1 \cup \cdots \cup L_{i-1}$ . Applying Lemma 4.1 to  $T_{i-1}$  and  $G'[L_{i-1} \cup L_i]$ , we get

$$e(G'[L_{i-1}, L_i]) \le 2k(i-1)p^{i-1}(|L_{i-1}| + |L_i|).$$

Similarly, it holds that

$$e(G'[L_{i-2} \cup L_{i-1}]) \le 2k(i-2)p^{i-2}(|L_{i-2}| + |L_{i-1}|) \le 4k(i-2)p^{i-2}|L_{i-1}|,$$

where the last step holds because  $|L_{i-1}| \ge |L_{i-2}|$  by the induction hypothesis. All edges in  $G'[L_{i-2} \cup L_{i-1} \cup L_i]$  are either in  $(L_{i-2}, L_{i-1})$  or in  $(L_{i-1}, L_i)$ , so we get that

$$e(G'[L_{i-2} \cup L_{i-1} \cup L_i]) = e(G'[L_{i-2} \cup L_{i-1}]) + e(G'[L_{i-1}, L_i]) \le 2kip^i \cdot (|L_{i-1}| + |L_i|).$$

On the other hand, each vertex in  $L_{i-1}$  has degree at least  $\frac{1}{4}d_i$  in G', and all edges of G' incident to  $L_{i-1}$  lie in  $G'[L_{i-2} \cup L_{i-1} \cup L_i]$ . Hence, we have

$$\frac{1}{4}d_i \cdot |L_{i-1}| \le e(G'[L_{i-2} \cup L_{i-1} \cup L_i]) \le 2kip^i \cdot (|L_{i-1}| + |L_i|).$$

Solving for  $|L_i|$ , we get  $|L_i| \ge \left(\frac{d_i}{8kip^i} - 1\right) \cdot |L_{i-1}| \ge \frac{d_i}{16kip^i} \cdot |L_{i-1}|$ , proving the claim.

By the claim, we have

$$|L_k| \ge \alpha \cdot \prod_{i=1}^k d_i \cdot |L_0| = \alpha \cdot \prod_{i=1}^k d_i,$$

where  $\alpha = \prod_{i=1}^{k} \frac{1}{16kip^{i}}$ . Recall that  $c = 16k^2p^k$ . So  $\alpha c^k > 1$ . Suppose first that k is odd, say k = 2s + 1. Then it follows that  $L_k \subseteq B$  and

$$|L_k| \ge \alpha \cdot d_A^{s+1} d_B^s = \alpha \cdot \frac{e(G)^k}{m^{s+1} n^s}.$$

By the assumption, we have  $e(G) > c \cdot (mn)^{\frac{1}{2} + \frac{1}{2k}}$ , which shows that  $|L_k| \ge \alpha c^k n > n$ . This is a contradiction, since  $L_k \subseteq B$  and |B| = n. Now consider that k is even, say k = 2s. Then we have

$$|L_k| \geq \alpha \cdot d_A^s d_B^s = \alpha \cdot \frac{e(G)^k}{m^s n^s}$$

In this case  $e(G) > c \cdot m^{\frac{1}{2} + \frac{1}{k}} n^{\frac{1}{2}}$ . This gives that  $|L_k| \ge \alpha c^k \cdot \left(m^{\frac{1}{2} + \frac{1}{k}} n^{\frac{1}{2}}\right)^k / m^s n^s = \alpha c^k \cdot m > m$ , again a contradiction, since  $L_k \subseteq A$  and |A| = m. This completes the proof of Theorem 1.11.

One can promptly derive the following special case of Theorem 1.11, which will play an important role in the proof of Theorem 1.9.

**Corollary 4.2** Let  $m, n \geq 2$  be integers. Then it holds that

$$z(m, n, \theta_{3,p}) \le 144p^3 \cdot \left( (mn)^{2/3} + m + n \right).$$

# 5 The Turán exponent of 7/5

Here we prove the existence of the Turán exponent of 7/5. This is achieved by the combination of Theorem 1.9, which states that  $ex(n, S_p) = O(n^{7/5})$  for all  $p \ge 2$ , and the matched lower bound of

this function for sufficiently large p from [4].

By considering a supergraph<sup>1</sup> of  $S_p$ , in fact we will prove a slightly stronger result than Theorem 1.9. We start with a definition introduced by Faudree and Simonovits [13]. Let H be a bipartite graph with an ordered pair (A, B) of partite sets and  $t \ge 2$  be an integer. Define  $L_t(H)$  to be the graph obtained from H by adding a new vertex u and joining u to all vertices of A by internally disjoint paths of length t - 1 such that the vertices of these paths are disjoint from V(H).

We observe that the theta graph  $\theta_{3,p}$  are symmetric between its two partite sets. So  $L_3(\theta_{3,p})$  is uniquely defined. The following proposition can be verified easily.

**Proposition 5.1** For each  $p \ge 2$ , we have  $S_p \subseteq L_3(\theta_{3,p})$  and thus  $ex(n, S_p) \le ex(n, L_3(\theta_{3,p}))$ .

Note that as a special case, the graph  $L_3(\theta_{3,2})$  also denotes the subdivision of  $K_4$ , where each edge of  $K_4$  is replaced by an internally disjoint path of length two.

We are now in a position to prove the following strengthening of Theorem 1.9.

**Theorem 5.2** For each  $p \ge 2$ , there exists a positive constant  $c_p$  such that

$$\exp(n, L_3(\theta_{3,p})) \le c_p n^{7/5}$$

*Proof.* We will show that it suffices to choose  $c_p = 12^4 p^6$ . Suppose for a contradiction that there exists an *n*-vertex  $L_3(\theta_{3,p})$ -free graph G with  $e(G) > c_p n^{7/5}$ . By Proposition 2.1, G contains a bipartite subgraph  $G_1$  with

$$d := \delta(G_1) \ge d(G)/4 \ge (c_p/2) \cdot n^{2/5} > (4 \cdot 12^3 p^6) \cdot n^{2/5}.$$
(3)

Let x be a vertex of minimum degree in  $G_1$ . For each  $i \ge 0$ , let  $L_i$  denote the set of vertices at distance i from x in  $G_1$ . Then  $|L_1| = |\delta(G_1)| = d$ . Let  $L_2^+$  denote the set of vertices v in  $L_2$  such that  $|N_{G_1}(v) \cap L_1| \ge 2p + 2$ , and  $L_2^- = L_2 \setminus L_2^+$ .

Claim 1.  $G_1[L_1 \cup L_2^+]$  is  $\theta_{3,p}$ -free.

Proof of Claim 1. Suppose for contradiction that  $G_1[L_1 \cup L_2^+]$  contains a copy F of  $\theta_{3,p}$ . Let A, B denote the two partite sets of F where  $A \subseteq L_1$  and  $B \subseteq L_2^+$ . Then |A| = |B| = p + 1. Suppose  $B = \{b_1, \ldots, b_{p+1}\}$ . Since each vertex in  $L_2^+$  has at least 2p+2 neighbours in  $L_1$ , we can find distinct vertices  $c_1, \ldots, c_{p+1}$  in  $L_1 \setminus A$  such that  $b_1c_1, \ldots, b_{p+1}c_{p+1} \in E(G_1)$ . Now F together with the paths  $b_1c_1x, \ldots, b_{p+1}c_{p+1}x$  form a copy of  $L_3(\theta_{3,p})$  in G, a contradiction.

Claim 2.  $|L_2| \ge d^2/(24^3 p^{9/2}).$ 

Proof of Claim 2. By Claim 1 and Corollary 4.2, we have

$$e(G_1[L_1 \cup L_2^+]) \le 144p^3 \cdot \left( |L_1|^{2/3} |L_2^+|^{2/3} + |L_1| + |L_2^+| \right);$$

and by the definition of  $L_2^-$ ,  $e(G_1[L_1 \cup L_2^-]) \leq (2p+2) \cdot |L_2^-|$ . Adding these inequalities up, we have

$$e(G_1[L_1, L_2]) = e(G_1[L_1 \cup L_2^+]) + e(G_1[L_1 \cup L_2^-]) \le 144p^3 \cdot \left(|L_1|^{2/3}|L_2|^{2/3} + |L_1| + |L_2|\right).$$
(4)

<sup>&</sup>lt;sup>1</sup>i.e., a graph containing  $S_p$  as its subgraph.

Since every vertex in  $L_1$  has at least  $d-1 \ge 3d/4$  neighbours in  $L_2$ , it follows that

$$(3d/4)|L_1| \le e(G_1[L_1, L_2]) \le 144p^3 \cdot \left(|L_1|^{2/3}|L_2|^{2/3} + |L_1| + |L_2|\right).$$

Since  $d \ge 4 \cdot 12^3 p^6$ , we see  $144p^3 |L_1| \le (d/4)|L_1|$ . Thus it follows that either  $144p^3 |L_1|^{2/3} |L_2|^{2/3} \ge (d/4)|L_1|$  or  $144p^3 |L_2| \ge (d/4)|L_1|$ . Using  $|L_1| = d$ , we get that

$$|L_2| \ge \min\left\{\frac{d^2}{24^3p^{9/2}}, \frac{d^2}{24^2p^3}\right\} = \frac{d^2}{24^3p^{9/2}},$$

proving this claim.

Next we consider the subgraph H of  $G_1$  induced on  $L_2 \cup L_3$ , i.e.,

$$H = G_1[L_2 \cup L_3].$$

Our goal in the rest of the proof is to reach a contradiction by showing that H can not contain theta graphs  $\theta_{3,s}$  for large s, which in turn shows that  $|L_3|$  must be  $\Omega(d^{5/2})$  and thus exceed the total number of vertices in G.

Let T be a breadth-first search tree rooted at x with vertex set  $\{x\} \cup L_1 \cup L_2$ . Let  $x_1, \ldots, x_m$  be the children of x in T. For each  $i \in [m]$ , let  $S_i$  be the set of children of  $x_i$  in T. Then  $S_1, \ldots, S_m$ partition  $L_2$ . Since each vertex in  $L_2$  has degree at least d in  $G_1$ , we have

$$e(G_1[L_1 \cup L_2]) + e(G_1[L_2 \cup L_3]) \ge d|L_2|.$$

On the other hand, by (3) and Claim 2, we have  $d \ge 4 \cdot 12^3 p^6$  and thus  $(d|L_2|)^{1/3} \ge 4 \cdot 144 \cdot p^3$ , which together with (4) imply that

$$e(G_1[L_1 \cup L_2]) \le d|L_2|/4 + 144p^3(|L_1| + |L_2|) \le d|L_2|/2.$$

Hence

$$e(H) = e(G_1[L_2 \cup L_3]) \ge d|L_2|/2.$$
(5)

Given a vertex  $u \in L_3$  and some  $S_i$ , we say the pair  $(u, S_i)$  is rich, if u has at least 2p + 1 neighbours of H in  $S_i$ . Let  $E_H(u, S_i)$  denote the set of all edges in H between u and  $S_i$ . We now partition H into two (spanning) subgraphs  $H_1, H_2$  such that

$$E(H_1) = \bigcup E_H(u, S_i)$$
 and  $E(H_2) = E(H) \setminus E(H_1),$ 

where the union in  $E(H_1)$  is over all rich pairs  $(u, S_i)$ . Note that by this definition, any  $u \in L_3$  has at most 2p neighbours of  $H_2$  in any  $S_i$ , i.e.,  $|E_{H_2}(u, S_i)| \leq 2p$ . Let  $H_3$  be a subgraph of  $H_2$  obtained by including exactly one edge in  $E_{H_2}(u, S_i)$  over all pairs  $(u, S_i)$  with  $|E_{H_2}(u, S_i)| \geq 1$ . By the above discussion, it follows that

$$e(H_3) \ge e(H_2)/(2p),$$
 (6)

and for any  $u \in L_3$ , all its neighbours in  $H_3$  belong to distinct  $S_i$ 's.

Claim 3.  $H_1$  is  $\theta_{3,p^2}$ -free.

Proof of Claim 3. Suppose for contradiction that  $H_1$  contains a copy F of  $\theta_{3,p^2}$ . Suppose F consists of  $p^2$  internally disjoint paths of length three between u and v where  $u \in L_3$  and  $v \in L_2$ . Let these paths be  $ua_1b_1v, ua_2b_2v, \ldots, ua_{p^2}b_{p^2}v$ , where  $a_1, \ldots, a_{p^2} \in L_2$  and  $b_1, \ldots, b_{p^2} \in L_3$ .

We consider two cases. First, suppose that there exists some  $S_i$  which contains p different  $a_j$ 's. Without loss of generality, suppose that  $S_1$  contains  $a_1, \ldots a_p$ . For each  $j \in [p]$ , since  $b_j a_j \in E(H_1)$ , by definition  $(b_j, S_1)$  is a rich pair, i.e., there are at least 2p + 1 edges of H from  $b_j$  to  $S_1$ . Similarly as  $ua_1 \in E(H_1)$ , there are at least 2p + 1 edges of H from u to  $S_1$ . Hence we can find distinct vertices  $u', a'_1, \ldots, a'_p \in S_1 \setminus \{a_1, \ldots, a_p\}$  such that  $uu', a'_1 b_1, \ldots, a'_p b_p \in E(H)$ . Now  $F \cup \{uu', a'_1 b_1, \ldots, a'_p b_p\} \cup \{x_1 u', x_1 a'_1, \ldots, x_1 a'_p\}$  forms a copy of  $L_3(\theta_{3,p})$  in G, a contradiction.

Next, suppose that each  $S_i$  contains at most p-1 different  $a_j$ 's. Then among  $a_1, \ldots, a_{p^2}$  we can find p+1 of them, say  $a_1, \ldots, a_{p+1}$  that all lie in different  $S_i$ 's. Furthermore, we may assume that  $a_1, \ldots, a_p$  are outside the  $S_i$ 's that contains v. Now F together with the paths in T from x to  $a_1, \ldots, a_p, v$  form a copy of  $L_3(\theta_{3,p})$  in G, a contradiction. Hence  $H_1$  must be  $\theta_{3,p^2}$ -free.

Claim 4.  $H_3$  is  $\theta_{3,p}$ -free.

Proof of Claim 4. Suppose for contradiction that  $H_3$  contains a copy F of  $\theta_{3,p}$ . Suppose F consists of p internally disjoint paths of length three between u and v, where  $u \in L_3$  and  $v \in L_2$ . Suppose these paths are  $ua_1b_1v, \ldots, ua_pb_pv$ , where  $a_1, \ldots, a_p \in L_2$  and  $b_1, \ldots, b_p \in L_3$ . By the definition of  $H_3$ , since  $ua_1, \ldots, ua_p \in E(H_3)$ ,  $a_1, \ldots, a_p$  must all lie in different  $S_i$ 's. Also, for each  $j \in [p]$  since  $b_ja_j, b_jv \in E(H_3), a_j$  and v must lie in different  $S_i$ . So  $a_1, \ldots, a_p$  and v all lie in different  $S_i$ 's. Now, F together with the paths in T from x to  $a_1, \ldots, a_p, v$  respectively form a copy of  $L_3(\theta_{3,p})$  in G, a contradiction.

Now, we consider two cases.

**Case 1.**  $e(H_1) \ge e(H)/2$ . In this case, by (5), we have  $e(H_1) \ge d|L_2|/4$ . On the other hand, by Claim 3, we see that  $H_1$  is  $\theta_{3,p^2}$ -free, so by Corollary 4.2, we have

$$d|L_2|/4 \le e(H_1) \le 144p^6 \cdot \left( |L_2|^{2/3} |L_3|^{2/3} + |L_2| + |L_3| \right).$$
(7)

Since  $144p^6|L_2| \leq d|L_2|/12$ , we have either

$$144p^6|L_2|^{2/3}|L_3|^{2/3} \ge d|L_2|/12$$
 or  $144p^6|L_3| \ge d|L_2|/12$ .

Using this and Claim 2 that  $|L_2| \ge d^2/(24^3 p^{9/2})$ , we can get

$$|L_3| \ge \min\left\{\frac{d^{3/2}|L_2|^{1/2}}{12^{9/2}p^9}, \frac{d|L_2|}{12^3p^6}\right\} = \frac{d^{3/2}|L_2|^{1/2}}{12^{9/2}p^9} \ge \frac{d^{5/2}}{2^{3/2}12^6p^{45/4}}$$

Since  $d \ge (4 \cdot 12^3 p^6) \cdot n^{2/5}$ , this yields  $|L_3| > n$ , a contradiction.

**Case 2.**  $e(H_2) \ge e(H)/2$ . Then by (5) and (6), we have  $e(H_3) \ge e(H)/4p \ge d|L_2|/8p$ . By Claim

4,  $H_3$  is  $\theta_{3,p}$ -free. Thus, by Corollary 4.2 we get

$$d|L_2|/8p \le e(H_3) \le 144p^3 \cdot \left(|L_2|^{2/3}|L_3|^{2/3} + |L_2| + |L_3|\right)$$

Since  $p \ge 2$ , the above inequality would also imply (7). So we can apply the same analysis as in Case 1 to get a contradiction.

This completes the proof of Theorem 5.2 (and thus of Theorem 1.9).

We proved in Theorem 5.2 that  $ex(n, L_3(\theta_{3,p})) \leq O(n^{7/5})$ . An important idea in this proof is to use the asymmetric bipartite Turán number of  $\theta_{3,p}$ , which help showing that the BFS-tree grow rapidly. The use of asymmetric bipartite Turán numbers may find applications in other Turán type extremal problems.

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