

ON THE EDIT DISTANCE FUNCTION OF THE RANDOM GRAPH

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ABSTRACT. Given a hereditary property of graphs \mathcal{H} and a $p \in [0, 1]$, the edit distance function $\text{ed}_{\mathcal{H}}(p)$ is asymptotically the maximum proportion of edge-additions plus edge-deletions applied to a graph of edge density p sufficient to ensure that the resulting graph satisfies \mathcal{H} . The edit distance function is directly related to other well-studied quantities such as the speed function for \mathcal{H} and the \mathcal{H} -chromatic number of a random graph.

Let \mathcal{H} be the property of forbidding an Erdős-Rényi random graph $F \sim \mathbb{G}(n_0, p_0)$, and let φ represent the golden ratio. In this paper, we show that if $p_0 \in [1 - 1/\varphi, 1/\varphi]$, then **a.a.s.** as $n_0 \rightarrow \infty$,

$$\text{ed}_{\mathcal{H}}(p) = (1 + o(1)) \frac{2 \log n_0}{n_0} \cdot \min \left\{ \frac{p}{-\log(1 - p_0)}, \frac{1 - p}{-\log p_0} \right\}.$$

Moreover, this holds for $p \in [1/3, 2/3]$ for any $p_0 \in (0, 1)$.

1. INTRODUCTION

All graphs are finite and simple, i.e., without loops and multi-edges. A graph is *nonempty* if it has at least one edge. Denote P_n to be the path graph on n vertices.

For any $p \in [0, 1]$ and any positive integer n , write $\mathbb{G}(n, p)$ to be the distribution of graphs according to the Erdős-Rényi random graph model with edge probability p . That is, $G \sim \mathbb{G}(n, p)$ means that the event that $uv \in E(G)$ for $uv \in \binom{\{1, \dots, n\}}{2}$ are independent and identically distributed (i.i.d.) with common probability p . We write **a.a.s.** to mean that a sequence of events holds with probability approaching 1 under some implied limit. The limit will be clear by the context. All

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logarithms are natural unless explicitly stated otherwise.

1.1. Edit distance results and forbidding a random graph. The edit distance measures the minimum number of “edits” (that is, edge-additions plus edge-deletions) sufficient to turn one graph into another. This metric has been studied in contexts such as property testing and evolutionary biology (see [3, 11]). Formally, for any two n -vertex graphs G, H on the same vertex set,

$$\text{dist}(G, H) = |E(G) \Delta E(H)| \cdot \binom{n}{2}^{-1}$$

where Δ is the symmetric difference operation for sets.

A graph property \mathcal{H} is *hereditary* if \mathcal{H} is closed under isomorphism and vertex deletion. For any family \mathcal{F} of graphs, we may write $\text{Forb}(\mathcal{F})$ for the hereditary property of graphs which do not contain an induced copy of F for any $F \in \mathcal{F}$. Any hereditary property is of the form $\text{Forb}(\mathcal{F})$ for some \mathcal{F} . A hereditary property of the form $\text{Forb}(\{F\})$ for a single graph F is called a *principal hereditary property* and we will write $\text{Forb}(F)$ for simplicity.

A hereditary property \mathcal{H} is *nontrivial* if, for every positive integer n , there exists a graph in \mathcal{H} of order n . All hereditary properties in this paper are nontrivial. If \mathcal{H} is a nontrivial hereditary property, then for all graphs G , we define

$$\text{dist}(G, \mathcal{H}) = \min\{\text{dist}(G, H) : \exists H \in \mathcal{H} \text{ s.t. } V(H) = V(G)\}.$$

An early result that has motivated subsequent research is as follows:

Theorem 1 (Alon-Stav [2]). *For a nontrivial hereditary property \mathcal{H} , there exists a $p^* = p_{\mathcal{H}}^* \in [0, 1]$ so that with $G \sim \mathbb{G}(n, p^*)$,*

$$\lim_{n \rightarrow \infty} \max_{|V(G)|=n} \text{dist}(G, \mathcal{H}) = \mathbb{E}[\text{dist}(G, \mathcal{H})] + o(1).$$

In other words, random graphs of density p^* asymptotically achieve the maximum distance to \mathcal{H} . For any $p \in [0, 1]$ and any property \mathcal{H} nontrivial and hereditary, let

$$(1) \quad \text{ed}_{\mathcal{H}}(p) := \limsup_{n \rightarrow \infty} \max_{\substack{|V(G)|=n, \\ e(G) = \lfloor p \binom{n}{2} \rfloor}} \text{dist}(G, \mathcal{H}).$$

We call $\text{ed}_{\mathcal{H}}$ the *edit distance function* of \mathcal{H} . Theorem 2 below demonstrates that the maximum distance \mathcal{H} among all density- p graphs is achieved asymptotically by Erdős-Rényi random graphs of expected density p .

Theorem 2 (Balogh-Martin [4]). *Let \mathcal{H} be a nontrivial hereditary property. For all $p \in [0, 1]$, if $G \sim \mathbb{G}(n, p)$, then*

$$\text{ed}_{\mathcal{H}}(p) = \lim_{n \rightarrow \infty} \mathbb{E}[\text{dist}(G, \mathcal{H})].$$

Moreover the function $\text{ed}_{\mathcal{H}}$ is continuous and concave-down.

Proposition 3 below has several short proofs and follows from Bollobás' asymptotic result on the chromatic number of a random graph (see [5]), together with established techniques for computing edit distance functions (see [11]).

Proposition 3 (Alon-Stav [2]). *Let $F \sim \mathbb{G}(n_0, 1/2)$ and define $\mathcal{H} := \text{Forb}(F)$. Then **a.a.s.** with $n_0 \rightarrow \infty$,*

$$\text{ed}_{\mathcal{H}}(p) = (1 + o(1)) \frac{2 \log_2 n_0}{n_0} \cdot \min\{p, 1 - p\}.$$

Our main result extends Proposition 3 so that we are able to determine the edit distance function asymptotically for all p_0 in a relatively large open interval around $1/2$. Let $\varphi = (1 + \sqrt{5})/2$ be the golden ratio. Note that $1 - \varphi^{-1} \approx 0.381966$ and $\varphi^{-1} \approx 0.618034$.

Theorem 4. *Fix $p_0 \in (0, 1)$, let $F \sim \mathbb{G}(n_0, p_0)$, and define $\mathcal{H} := \text{Forb}(F)$. If $p_0 \in [1 - \varphi^{-1}, \varphi^{-1}]$ then **a.a.s.** with $n_0 \rightarrow \infty$,*

$$(2) \quad \text{ed}_{\mathcal{H}}(p) = (1 + o(1)) \frac{2 \log n_0}{n_0} \cdot \min \left\{ \frac{p}{-\log(1 - p_0)}, \frac{1 - p}{-\log p_0} \right\}$$

*holds for all $p \in [0, 1]$. If $p_0 \in [0, 1 - \varphi^{-1})$, then **a.a.s.** (2) holds for all $p \in [1/3, 1]$. If $p_0 \in (\varphi^{-1}, 1]$, then **a.a.s.** (2) holds for all $p \in [0, 2/3]$.*

In fact, the $o(1)$ error term depends only on the constant p_0 and holds uniformly for all p in each of the respective intervals.

The first author conjectured (see [11]) that for all $p_0 \in [0, 1]$, (2) holds **a.a.s.** for all $p \in [0, 1]$. Theorem 4 proves this for a range of p_0 of size ≈ 0.236068 .

1.2. Equivalent parameters. The edit distance function is also interesting because of its connection to other parameters involving random graphs. For \mathcal{H} any nontrivial hereditary property and any $p \in (0, 1)$, the *speed* of \mathcal{H} is

$$c_{\mathcal{H}}(p) := \lim_{k \rightarrow \infty} -\log_2 (\mathbb{P} [\mathbb{G}(k, p) \in \mathcal{H}]) \cdot \binom{n}{2}^{-1}$$

Indeed, this limit does exist and a proof of that fact appears in [1] and in [6]. See also the survey [11].

The following observation was made by Thomason but it can be shown to follow from a prior result due to Bollobás and Thomason [6].

Theorem 5 (Thomason [13]). *Let \mathcal{H} be a nontrivial hereditary property. Then for all $p \in (0, 1)$,*

$$c_{\mathcal{H}}(p) = (-\log_2(p(1-p))) \cdot \text{ed}_{\mathcal{H}} \left(\frac{\log(1-p)}{\log(p(1-p))} \right).$$

Note that the function $f : (0, 1) \rightarrow (0, 1)$ defined by

$$f(x) := \frac{\log(1-x)}{\log(x(1-x))}$$

on $x \in (0, 1)$ is invertible. Since $\text{ed}_{\mathcal{H}}$ is continuous, $c_{\mathcal{H}}$ can be computed from $\text{ed}_{\mathcal{H}}$ and vice versa. As a result, combining Theorem 5 with Theorem 4 yields a result on the speed function of hereditary properties defined by random graphs.

Corollary 6. *Fix $p_0 \in (0, 1)$, let $F \sim \mathbb{G}(n_0, p_0)$, and define $\mathcal{H} := \text{Forb}(F)$. If $p_0 \in [1 - \varphi^{-1}, \varphi^{-1}]$ then **a.a.s.** with $n_0 \rightarrow \infty$,*

$$(3) \quad c_{\mathcal{H}}(p) = (1 + o(1)) \frac{2 \log_2 n_0}{n_0} \cdot \min \left\{ \frac{\log(1-p)}{\log(1-p_0)}, \frac{\log p}{\log p_0} \right\}$$

*holds for all $p \in [0, 1]$. If $p_0 \in [0, 1 - \varphi^{-1})$, then **a.a.s.** (3) holds for all $p \in [1 - \varphi^{-1}, 1]$. If $p_0 \in (\varphi^{-1}, 1]$, then **a.a.s.** (3) holds for all $p \in [0, \varphi^{-1}]$.*

For any hereditary property \mathcal{H} and any graph G , let $\chi_{\mathcal{H}}(G)$ be the \mathcal{H} -chromatic number of G . This is the minimum nonnegative integer k for which there exists a partition $V(G) = V_1 \cup \dots \cup V_k$ such that $G[V_i]$ satisfies \mathcal{H} for all $i \in \{1, \dots, k\}$. If \mathcal{H} is the property of being an empty graph, then $\chi_{\mathcal{H}}(G)$ is the chromatic number of G .

Bollobás and Thomason established Theorem 7 for the \mathcal{H} -chromatic number of a random graph.

Theorem 7 (Bollobás-Thomason [6]). *Let $p \in (0, 1)$ and let \mathcal{H} be a nontrivial hereditary property. Then **a.a.s.** with $G \sim \mathbb{G}(n, p)$,*

$$(4) \quad \chi_{\mathcal{H}}(G) = (1 + o(1)) c_{\mathcal{H}}(p) \frac{n}{2 \log_2 n}$$

Bollobás' classic asymptotic result [5] on the chromatic number of the random graph can be derived from Theorem 7 by observing that if \mathcal{H}_{em} is the property of being an empty graph, then $c_{\mathcal{H}_{\text{em}}}(p) = -\log_2(1 - p)$ and so $\chi_{\mathcal{H}_{\text{em}}}(G) = (1 + o(1)) \frac{n}{2 \log_2(1-p)}$ **a.a.s.**

However, the fact that $c_{\mathcal{H}_{\text{em}}}(p) = -\log_2(1 - p)$ can itself be derived from Theorem 5 and the entirely trivial observation that $\text{ed}_{\mathcal{H}_{\text{em}}}(p) = p$. In general, $\chi_{\mathcal{H}}$ has a close relationship with both $c_{\mathcal{H}}$ and $\text{ed}_{\mathcal{H}}$.

The rest of the paper is organized as follows: In Section 2, we discuss colored regularity graphs (CRGs) and prove some basic results that have the potential to apply to a wide variety of edit distance results beyond the scope of this paper. In Section 3, we give the proof of Theorem 4. Section 4 includes a proof of the fact that for all $p \in [1 - \varphi^{-1}, \varphi^{-1}]$, $\text{ed}_{\mathcal{H}}(p)$ can be computed by a set CRGs whose order is bounded by a constant depending only on \mathcal{H} . Section 4 also includes a discussion of the role paths play in CRGs. In Section 5, we discuss open questions and potential future work.

2. COLORED REGULARITY GRAPHS

In this section, we address colored regularity graphs. In Section 2.1, we address background and basic facts about colored regularity graphs. Section 2.2 discusses the new notion of p -prohibited CRGs. Lemma 20

and Lemma 23 are important new results on p -prohibited CRGs. They are proven in Section 2.3 and Section 2.4, respectively.

2.1. Background on CRGs. The key element to studying the edit distance problem is the colored regularity graph, which was defined by Alon and Stav [2] but appeared as *types* in the prior literature by Bollobás and Thomason (see [6]).

Definition 8. A **colored regularity graph** K is a complete graph, together with a partition $V(K) = VW(K) \cup VB(K)$ of the vertex set into white and black vertices, and a partition $E(K) = EW(K) \cup EB(K) \cup EG(K)$ of the edge set into white, black, and gray edges.

A CRG K' is called a **sub-CRG** of CRG K (denoted $K' \subseteq K$) if K' is obtained by deleting some vertices from K and all incident edges.

CRGs approximate large graphs and we want to know whether a forbidden graph F is in a graph approximated by a given CRG. We express this in terms of embeddings of graphs into CRGs.

Definition 9. A graph F **embeds into** CRG K (written $F \mapsto K$) if there exists a function $\phi : V(F) \rightarrow V(K)$ such that:

- If $uv \in E(F)$, then either $\phi(u) = \phi(v) \in VB(K)$, or $\phi(u)\phi(v) \in EB(K) \cup EG(K)$
- If $uv \in E(F^c)$, then either $\phi(u) = \phi(v) \in VW(K)$, or $\phi(u)\phi(v) \in EW(K) \cup EG(K)$.

For any CRG K , we will treat the elements of $\mathbb{R}^{V(K)}$ both as functions on $V(K)$ and as vectors indexed by the vertices of K . For any two such $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{V(K)}$, we define $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{u \in V(K)} \mathbf{x}(u)\mathbf{y}(u)$. We also let $M_K(p) \in \mathbb{R}^{V(K) \times V(K)}$ be the matrix whose uv -th entry is

$$(5) \quad m_{uv} := \begin{cases} p, & u \neq v \text{ and } uv \in EW(K), \text{ or } u = v \text{ and } u \in VW(K); \\ 1 - p, & u \neq v \text{ and } uv \in EB(K), \text{ or } u = v \text{ and } u \in VB(K); \\ 0, & u \neq v \text{ and } uv \in EG(K). \end{cases}$$

The all-ones vector $\mathbf{1} \in \mathbb{R}^{V(K)}$ is defined by $\mathbf{1}(u) = 1$ for all $u \in V(K)$ and the all-zeroes vector is just $\mathbf{0} = 0 \cdot \mathbf{1}$. Furthermore, we let Δ_K be the *standard simplex associated to* K which consists of all $\mathbf{x} \in \mathbb{R}^{V(K)}$ so that $\mathbf{x} \geq \mathbf{0}$ in the component-wise sense and $\langle \mathbf{x}, \mathbf{1} \rangle = 1$. The elements

of Δ_K will be called *weight vectors*.

Now define

$$g_K(p, \mathbf{x}) := \langle \mathbf{x}, M_K(p)\mathbf{x} \rangle \text{ and}$$

$$g_K(p) := \min \{g_K(p, \mathbf{x}) : \mathbf{x} \in \Delta_K\}.$$

A weight vector $\mathbf{x} \in \Delta_K$ is said to be *optimal for K* if $g_K(p, \mathbf{x}) = g_K(p)$. For any $p \in [0, 1]$, a CRG K is said to be *p -core* if for any optimal weight vector \mathbf{x} , $\mathbf{x}(u) > 0$ for all $u \in V(K)$. It follows that for K a p -core CRG, there exists a unique optimal weight vector.

For any hereditary property $\mathcal{H} = \text{Forb}(\mathcal{F})$ we define the following family of CRGs:

$$\mathcal{K}_{\mathcal{H}} := \{K \text{ a CRG} : F \not\rightarrow K \text{ for all } F \in \mathcal{F}\}.$$

Theorem 10 is the main technique for computing $\text{ed}_{\mathcal{H}}(p)$, hence understanding the set $\mathcal{K}_{\mathcal{H}}$ is crucial to understanding $\text{ed}_{\mathcal{H}}(p)$. The first equality was given by Balogh and Martin [4] and the second by Marchant and Thomason [10].

Theorem 10. *Let \mathcal{H} be a nontrivial hereditary property. Then for all $p \in [0, 1]$,*

$$(6) \quad \text{ed}_{\mathcal{H}}(p) = \inf_{K \in \mathcal{K}_{\mathcal{H}}} g_K(p) = \min_{K \in \mathcal{K}_{\mathcal{H}}} g_K(p).$$

It follows by definition that the minimum in Theorem 10 is obtained by a p -core CRG and as Theorem 11 shows, p -core CRGs have a well-defined structure.

Theorem 11 (Marchant-Thomason [10]). *Let $p \in [0, 1]$ and suppose K is a p -core CRG.*

- (a) *If $p \in [0, 1/2]$, then $\text{EB}(K) = \emptyset$ and for all $uv \in \text{EW}(K)$, $u, v \in \text{VB}(K)$.*
- (b) *If $p \in [1/2, 1]$, then $\text{EW}(K) = \emptyset$ and for all $uv \in \text{EB}(K)$, $u, v \in \text{VW}(K)$.*

To summarize, if $p \leq 1/2$, then a p -core CRG has no black edges and all white edges must be between black vertices. If $p \geq 1/2$, then

a p -core CRG has no white edges and all black edges must be between white vertices. As a result, if $p = 1/2$, p -core CRGs have neither black nor white edges.

Remark 12. *A CRG K is $1/2$ -core if and only if all edges of K are gray.*

2.2. p -prohibited CRGs. In this paper, we introduce the notion of a prohibited CRG.

Definition 13. *For any $p \in [0, 1]$ and any CRG J , we say that J is p -prohibited if for any p -core CRG K , J is not a sub-CRG of K .*

For example, Theorem 11 shows that if $p \in [0, 1/2)$, then the only 2-vertex CRGs that are not p -prohibited are those with a gray edge or the CRG with two black vertices and a white edge. See Figure 1.

Remark 14. *There is an abundance of CRGs which are neither p -core nor p -prohibited. For example, consider the CRGs K and K' defined as follows. Let K consist of 3 black vertices with 2 white edges and 1 gray edge. For all $p \in [0, 1]$, K is not p -core. Now let K' be the CRG on 4 black vertices whose white edges induce a P_4 and all other edges are gray. Clearly K' contains K . It is an exercise to see that K' is p -core for all $p \in [0, 1 - \varphi^{-1})$, so K is not p -prohibited on this interval.*

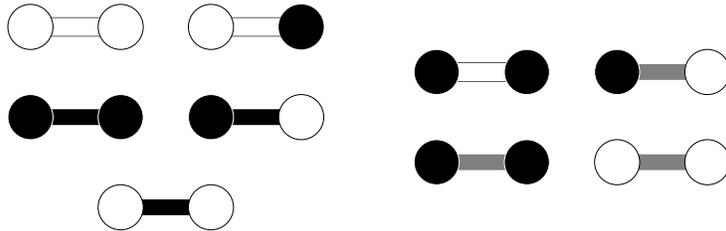


FIGURE 1. All two-vertex CRGs. The five on the left are p -prohibited for all $p \in [0, 1/2]$. The four on the right are p -core for all $p \in [0, 1/2)$.

We also want to introduce the notion of the complement of a CRG.

Definition 15. *If K is a CRG, then the **complement** of K is the unique CRG \overline{K} , such that*

- $\text{VW}(\overline{K}) = \text{VB}(K)$, $\text{VB}(\overline{K}) = \text{VW}(K)$,
- $\text{EW}(\overline{K}) = \text{EB}(K)$, $\text{EB}(\overline{K}) = \text{EW}(K)$, and $\text{EG}(\overline{K}) = \text{EG}(K)$.

For a graph G , the notation G^c is used to denote the graph complement, so as to avoid confusion. There is symmetry in the edit distance function about $p = 1/2$ with respect to complements.

Proposition 16. *If $p \in [0, 1]$ and K is a CRG, then $g_K(p) = g_{\overline{K}}(1-p)$.*

Proof. This follows from the equality of the matrices $M_K(p) = M_{\overline{K}}(1-p)$:

$$\begin{aligned} g_K(p) &= \min \{ \langle \mathbf{x}, M_K(p)\mathbf{x} \rangle : \mathbf{x} \in \Delta_K \} \\ &= \min \{ \langle \mathbf{x}, M_{\overline{K}}(1-p)\mathbf{x} \rangle : \mathbf{x} \in \Delta_K \} = g_{\overline{K}}(1-p). \quad \square \end{aligned}$$

There is also symmetry in the edit distance function about $p = 1/2$ when it comes to p -prohibition.

Proposition 17. *For all $p \in [0, 1]$, a CRG J is p -prohibited if and only if \overline{J} is $(1-p)$ -prohibited.*

Proof. Suppose J is p -prohibited but \overline{J} is not $(1-p)$ -prohibited. Then, there is a $(1-p)$ -core CRG \overline{K} that contains \overline{J} as a sub-CRG. If K is not p -core, then there is a $K' \subseteq K$ such that $g_{K'}(p) = g_K(p)$, but Proposition 16 gives that $g_{\overline{K'}}(1-p) = g_{\overline{K}}(1-p)$, a contradiction to \overline{K} being $(1-p)$ -core. \square

Next, we introduce terminology which is useful in describing the structure of p -core and p -prohibited CRGs.

Definition 18. *Let K be a CRG.*

- The **underlying graph** of K is the graph $G = (V(K), \text{EB}(K) \cup \text{EW}(K))$.
- A **component** of K is a component of the underlying graph of K .
- A **disjoint union** of vertex-disjoint CRGs J, K , denoted $J \oplus K$, is a CRG with vertex set $V_J \oplus V_K$, where the sub-CRG induced on V_J is isomorphic to J , the sub-CRG induced on V_K is isomorphic to K , and every edge incident to a vertex in each of V_J and in V_K has color gray. The disjoint union of k copies of K is $k \cdot K$.
- Let G be a nonempty graph. The CRG, K , **associated** to G is defined as follows: If $p \in [0, 1/2]$, then $\text{VW}(K) = \text{EB}(K) = \emptyset$, $\text{VB}(K) = V(G)$, $\text{EW}(K) = E(G)$, and $\text{EG}(K) = E(\overline{G})$. If $p \in (1/2, 1]$, then $\text{VB}(K) = \text{EW}(K) = \emptyset$, $\text{VW}(K) = V(G)$, $\text{EB}(K) = E(G)$, and $\text{EG}(K) = E(\overline{G})$.

We associate CRGs to graphs for the purposes of discussing p -core CRGs. See Figure 2 for an example. Since $1/2$ -core CRGs are precisely those which have only gray edges, the definition of the CRG associated to a graph for $p = 1/2$ is made purely out of convenience.

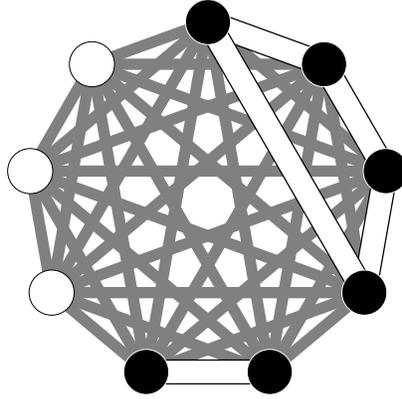


FIGURE 2. A CRG with 5 components. One component is the CRG associated to the cycle C_4 for $p \in [0, 1/2)$. The edges satisfy the necessary conditions from Theorem 11 for a p -core CRG with $p \in [0, 1/2)$.

In order to apply Lemma 20 below, we need the minimum adjacency eigenvalue to be at most -1 . This occurs for all nonempty graphs.

See [7] for a more detailed discussion about eigenvalues associated to graphs.

Proposition 19. *Every nonempty graph that is not disjoint cliques has minimum adjacency eigenvalue at most $-\sqrt{2}$. If a nonempty graph consists of disjoint cliques, its minimum adjacency eigenvalue is -1 . An empty graph has all adjacency eigenvalues zero.*

Lemma 20. *Let G be a nonempty graph and let $\lambda \leq -1$ be the minimum eigenvalue of the adjacency matrix of G . The CRG associated to G is p -prohibited for all*

$$p \in \left[\frac{1}{1 - \lambda}, 1 - \frac{1}{1 - \lambda} \right].$$

In Section 2.3, we prove Lemma 20. First, we need some essential terms.

Definition 21. *For any positive integer t , the t -dalmatian CRG is the CRG, denoted D_t , consisting of t black vertices and all edges white. The ∞ -dalmatian CRG is the CRG, denoted D_∞ , which is a single white vertex. The set of CRGs denoted by \mathcal{D}_p is as follows:*

- *If $p \in [0, 1/2)$, then \mathcal{D}_p is the set of all CRGs whose components are dalmatian CRGs.*
- *If $p \in (1/2, 1]$, then \mathcal{D}_p is the set of all CRGs whose components are complements of dalmatian CRGs.*
- *If $p = 1/2$, then $\mathcal{D}_{1/2}$ is the set of all CRGs whose components are single vertices.*

See Figure 3 for dalmatian CRGs of small order.

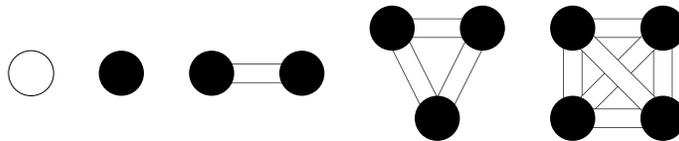


FIGURE 3. The dalmatian CRGs $D_\infty = \overline{D_1}$, $D_1 = \overline{D_\infty}$, D_2 , D_3 , and D_4 .

Remark 22. For all $p \in [0, 1]$ and each $K \in \mathcal{D}_p$, K is a p -core CRG. Moreover, if $p \in [0, 1/2]$, then $g_{D_\infty}(p) = p$ and for each positive integer t ,

$$g_{D_t}(p) = \min_{\mathbf{x} \in \Delta_{D_t}} \langle \mathbf{x}, M_{D_t}(p)\mathbf{x} \rangle = \frac{1}{t^2} \langle \mathbf{1}, M_{D_t}(p)\mathbf{1} \rangle = p + \frac{1-2p}{t}.$$

If $p \in [1/2, 1]$, then $g_{\overline{D}_\infty}(p) = 1 - p$ and for each positive integer t , $g_{\overline{D}_t}(p) = 1 - p + \frac{2p-1}{t}$.

In Lemma 23, we show that for all $p \in [1 - \varphi^{-1}, \varphi^{-1}]$, the *only* p -core CRGs are those that belong to \mathcal{D}_p .

Lemma 23. For $p \in [0, 1]$, the CRG associated to P_3 is p -prohibited if and only if $p \in [1 - \varphi^{-1}, \varphi^{-1}]$. In particular for all $p \in [1 - \varphi^{-1}, \varphi^{-1}]$, a CRG K is p -core if and only if $K \in \mathcal{D}_p$.

Remark 24. Since the minimum eigenvalue of the adjacency matrix of P_3 is $-\sqrt{2}$, Lemma 20 gives that P_3 is prohibited for p in the interval $[\sqrt{2} - 1, 2 - \sqrt{2}] \approx [0.414, 0.586]$. Lemma 23, however, gives that P_3 is prohibited on the larger interval $[1 - \varphi^{-1}, \varphi^{-1}] \approx [0.382, 0.618]$.

In Section 2.4, we prove Lemma 23.

2.3. Proof of Lemma 20. The result from Theorem 25 below was originally shown by Sidorenko [12] using different language and has appeared in several other forms throughout the study of hereditary properties. See [11] for a more detailed history. For convenience, we state it in the language of CRGs.

Theorem 25. Let K be a p -core CRG with optimal weight vector $\mathbf{x} \in \Delta_K$. Then

$$M_K(p)\mathbf{x} = g\mathbf{1}.$$

So, Theorem 25 establishes that the optimal weight vector produces a weighting that is balanced.

Lemma 26. *Let $0 \leq p \leq 1$ and J be a CRG. If there exists a nonzero vector $\boldsymbol{\delta} \in \mathbb{R}^{V(J)}$ so that $\langle \boldsymbol{\delta}, \mathbf{1} \rangle = 0$ and*

$$(7) \quad \langle \boldsymbol{\delta}, M_J(p) \boldsymbol{\delta} \rangle \leq 0,$$

then J is p -prohibited.

Proof. We proceed by contradiction. Suppose J is not p -prohibited and that there exists $\boldsymbol{\delta}$ as above. Then there exists a p -core CRG K containing J and we may let \mathbf{x} denote the optimal weight vector for K . We extend $\boldsymbol{\delta}$ to a vector $\boldsymbol{\delta}' \in \mathbb{R}^{V(K)}$ by letting $\boldsymbol{\delta}'(u) := 0$ if $u \in V(K) \setminus V(J)$ and $\boldsymbol{\delta}'(u) := \boldsymbol{\delta}(u)$ if $u \in V(J)$. Note that $\langle \boldsymbol{\delta}', \mathbf{1} \rangle = 0$.

Since \mathbf{x} is the optimal weight vector for the p -core CRG K , $\mathbf{x}(u) > 0$ for all $u \in V(K)$ and it follows that there exists some $\varepsilon > 0$ so that $\mathbf{x}' := \mathbf{x} + \varepsilon \boldsymbol{\delta}'$ lies in Δ_K and $\mathbf{x}'(u) = 0$ for some $u \in V(K)$. By the definition of $g_K(p)$, the fact that \mathbf{x} is optimal, and Theorem 25,

$$\begin{aligned} 0 &< g_K(p, \mathbf{x}') - g_K(p, \mathbf{x}) \\ &= \langle \mathbf{x} + \varepsilon \boldsymbol{\delta}', M_K(p)(\mathbf{x} + \varepsilon \boldsymbol{\delta}') \rangle - \langle \mathbf{x}, M_K(p)\mathbf{x} \rangle \\ &= 2\varepsilon \langle \boldsymbol{\delta}, M_K(p)\mathbf{x} \rangle + \varepsilon^2 \langle \boldsymbol{\delta}', M_K(p)\boldsymbol{\delta}' \rangle \\ &= 2\varepsilon \langle \boldsymbol{\delta}', g_K(p)\mathbf{1} \rangle + \varepsilon^2 \langle \boldsymbol{\delta}, M_J(p)\boldsymbol{\delta} \rangle \\ &= \varepsilon^2 \langle \boldsymbol{\delta}, M_J(p)\boldsymbol{\delta} \rangle \leq 0, \end{aligned}$$

a contradiction to the assumption that J is not p -prohibited. \square

We include one more well-known fact about p -core CRGs.

Proposition 27 (See [11]). *Let K_1, \dots, K_ℓ be CRGs and let $K = K_1 \oplus \dots \oplus K_\ell$. Then for all $p \in [0, 1]$,*

$$(8) \quad g_K(p)^{-1} = \sum_{i=1}^{\ell} g_{K_i}(p)^{-1}.$$

In particular, K is p -core if and only if each of K_1, \dots, K_ℓ are p -core.

Lemma 28. *Let $p \in [0, 1]$. A CRG J is p -prohibited if and only if for all p -core CRGs K and all positive integers k , the CRG $(k \cdot J) \oplus K$ is p -prohibited.*

Proof. To prove the forward implication, if J is p -prohibited, then no p -core CRG can contain $k \cdot J$ because it would contain J . To prove the reverse implication, if J is not p -prohibited then there exists a p -core CRG, L , containing J . Then $J \oplus K$ is contained in $L \oplus K$, which is p -core by Proposition 27, as desired. \square

With the primary tools of Lemmas 26 and 28 established, we now proceed to prove Lemma 20 itself.

Recall that J is the CRG associated to a nonempty graph G . If $p = 1/2$, then by Theorem 11, J is $1/2$ -prohibited if and only if J has an edge that is not gray. Thus, any nonempty G gives that J is $1/2$ -prohibited, settling the case where $p = 1/2$.

Now suppose $p \in (0, 1/2)$. Write A for the adjacency matrix of G and suppose $A\mathbf{x} = \lambda\mathbf{x}$ for some unit vector \mathbf{x} where λ is the minimum eigenvalue of A . Then J is the the CRG on $V(G)$ with all vertices black, and where edge uv is white if $uv \in E(G)$, and uv is gray if $uv \in E(G^c)$. So $M_J(p) = (1 - p)I + pA$.

For convenience, we write V_1 and V_2 for the vertex sets in $2 \cdot J$ corresponding to each copy of J . Let $\boldsymbol{\delta} \in \mathbb{R}^{V(2 \cdot J)}$ be defined by

$$\boldsymbol{\delta}(u) := \begin{cases} \mathbf{x}(u), & u \in V_1; \\ -\mathbf{x}(u), & u \in V_2. \end{cases}$$

By definition, $\langle \mathbf{1}, \boldsymbol{\delta} \rangle = 0$. Moreover note that

$$\begin{aligned} \langle \boldsymbol{\delta}, M_{2 \cdot J}(p)\boldsymbol{\delta} \rangle &= \langle \mathbf{x}, M_J(p)\mathbf{x} \rangle + \langle -\mathbf{x}, M_J(p)(-\mathbf{x}) \rangle \\ &= 2\langle \mathbf{x}, M_J(p)\mathbf{x} \rangle \\ &= 2\langle \mathbf{x}, ((1 - p)I + pA)\mathbf{x} \rangle \\ &= 2(1 - p)\langle \mathbf{x}, I\mathbf{x} \rangle + 2p\langle \mathbf{x}, A\mathbf{x} \rangle \\ &= 2(1 - p)\langle \mathbf{x}, \mathbf{x} \rangle + 2p\lambda\langle \mathbf{x}, \mathbf{x} \rangle \\ &= 2 \cdot (1 - (1 - \lambda)p) \cdot \langle \mathbf{x}, \mathbf{x} \rangle. \end{aligned}$$

If $p \geq 1/(1 - \lambda)$, then $\langle \boldsymbol{\delta}, M_{2 \cdot J}(p)\boldsymbol{\delta} \rangle \leq 0$. By Lemma 26, the CRG $2 \cdot J$ is p -prohibited and by Lemma 28, J itself is p -prohibited. This

settles the case where $p \in (0, 1/2)$.

Finally, for the case of $p \in (1/2, 1)$, Proposition 17 gives that J is p -prohibited if and only if J is $(1 - p)$ -prohibited. This concludes the proof of Lemma 20. \square

2.4. Proof of Lemma 23. Lemma 29 below is a result in pure graph theory that is reminiscent of the theorem that categorizes $\{P_4, C_4, C_4^c\}$ -free graphs as threshold graphs. A *dominant vertex* in a graph is one for which every other vertex is its neighbor.

Lemma 29. *If G is a connected $\{P_4, C_4\}$ -free graph, then G has a dominant vertex.*

Proof. Let u be a vertex of G which attains the maximum degree $\Delta = \Delta(G)$ and let $A := N_G(u)$ and $B := V(G) \setminus (A \cup \{u\})$. If $B = \emptyset$, then u is the desired vertex, so we assume otherwise. Let $w \in B$. Since G avoids induced P_4 -s, connectivity implies G has diameter at most 2.

In particular, $\text{dist}_G(u, w) = 2$, so there exists some vertex $v \in A$ so that uvw is an induced path on 3 vertices. If v' is any vertex in $A \setminus \{v\}$, then $v'uvw$ is a path on 4 vertices. Since $uw \notin E(G)$ and G avoids both induced P_4 -s and induced C_4 -s, it follows that $vv' \in E(G)$. So v is adjacent to $\{u, w\} \cup (A \setminus \{v\})$ and has degree at least $\Delta + 1$, a contradiction. \square

Lemma 29 yields a very strong structural theorem on CRGs, Lemma 30. Recall the definition of the underlying graph of CRG, K , in Definition 18: the graph whose vertices are the vertices of K and whose edges are the non-gray edges of K .

Lemma 30. *Let $p \in [1/3, 2/3]$. If a p -core CRG has an underlying graph which is $\{P_4, C_4\}$ -free, then every component of the underlying graph is a clique. That is, every component of such a CRG must be a member of \mathcal{D}_p .*

Proof. If $p = 1/2$, then as we saw in Remark 12, a p -core CRG has only gray edges and so the underlying graph is empty.

Let $p \in [1/3, 1/2)$. Every trivial component of the underlying graph is simply a vertex in the CRG. Let K be a nontrivial component of

the CRG. By Theorem 11, the vertices of K must be black and by Lemma 29, the underlying graph of K has a dominant vertex u .

Let $\mathbf{x} \in \Delta_K$ be the optimal weight vector for K and define $g := g_K(p)$. By Theorem 25, $M_K(p)\mathbf{x} = g\mathbf{1}$ and by inspecting the entry indexed by u ,

$$g = (1-p)\mathbf{x}(u) + p \cdot \sum_{u \neq v \in V(K)} \mathbf{x}(v) = p + (1-2p)\mathbf{x}(u) > p.$$

For a contradiction, we now suppose K is not a dalmatian CRG. Hence, there exists some gray edge vw , and the sub-CRG K' on v and w is the disjoint union of two black vertices. Since K is p -core,

$$g < g_{K'}(p) = \min_{\mathbf{y} \in \Delta_{K'}} (1-p)(\mathbf{y}(v)^2 + \mathbf{y}(w)^2) = \frac{1-p}{2}.$$

Altogether, $p < g < (1-p)/2$ which implies that $p < 1/3$, a contradiction.

The case where $p \in (1/2, 2/3]$ follows by symmetry. \square

We now prove Lemma 23 itself with the primary tool being Lemma 30. As mentioned in Remark 14, we leave it as an exercise to verify that the CRG associated with P_4 is p -core for $p \in (0, 1 - \varphi^{-1}) \cup (\varphi^{-1}, 1)$. Hence, P_3 is not p -prohibited in this range, proving the forward implication.

For the reverse implication, let $p \in [1 - \varphi^{-1}, \varphi^{-1}]$. Since the minimum eigenvalues of the adjacency matrices of C_4 and P_4 are -2 and $-\varphi^{-1}$ respectively, Lemma 20 implies that the CRGs associated with C_4 and P_4 are p -prohibited for $p \in [1 - \varphi^{-1}, \varphi^{-1}]$.

Suppose K is a p -core CRG. Since $p \in [1/3, 2/3]$, it follows from Lemma 30 that the components of the underlying graph of K are cliques. No such graph contains an induced P_3 and so P_3 is p -prohibited for all $p \in [1 - \varphi^{-1}, \varphi^{-1}]$, as desired.

For the second statement of the theorem, since P_3 is p -prohibited, the only underlying graphs of a p -core CRG can be disjoint cliques, which is exactly the condition of being in \mathcal{D}_p . As observed in Remark 22, all CRGs in \mathcal{D}_p are p -core. This concludes the proof of Lemma 23. \square

3. PROOF OF THE MAIN RESULT

To proceed with the proof of Theorem 4, we need some preparation. In Section 3.1, Lemma 31 shows that for all $p \in (1/3, 2/3)$ and for any CRG K , there exists a sub-CRG K' of K so that $g_{K'}(p)$ is close to $g_K(p)$ and K' has components whose order is bounded by a function of p and a tolerance term ε .

For the remaining discussion, let $p_0 \in (0, 1)$ and define

$$(9) \quad p^* := \frac{\log(1 - p_0)}{\log(p_0(1 - p_0))}.$$

In Section 3.2, we investigate when a random graph $F \sim \mathbb{G}(n_0, p_0)$ embeds into a CRG (i.e., when $F \mapsto K$). There we show that **a.a.s.**, if a CRG, K , has bounded components in the above sense and the random graph does not map into K , then $g_K(p^*)$ has to be at least the desired value to within a small tolerance. Applying Lemma 31, this is true even if the components of K are not bounded.

Finally in Section 3.3, we put together these ideas to prove our main result.

3.1. Trimming p -core CRGs. The main result in this subsection is Lemma 31, which establishes that, for $p \in (1/3, 2/3)$, a CRG has a sub-CRG with bounded component sizes and a negligible change in the value of the g -function.

Lemma 31. *Fix $p \in (1/3, 2/3)$ and $\varepsilon \in (0, 1)$. There exists a positive integer $B = B(p, \varepsilon)$ such that the following holds: For all CRGs K , there exists a p -core sub-CRG K' whose components have order at most B , and $g_{K'}(p) \leq (1 + \varepsilon)g_K(p)$.*

The first part of the proof is to remove vertices from a CRG K one-by-one in a way which does not affect $g_K(p)$ too much. Once enough vertices are removed, we show that each remaining vertex is incident to a bounded number of non-gray edges. Finally, we use Lemma 20 to bound the diameter of p -core CRGs on the interval $p \in (1/3, 2/3)$. The underlying graph has bounded degree and diameter, thus its connected components have bounded order.

The proof consists of a sequence of propositions:

Proposition 32. *Fix $p \in [0, 1]$ and suppose K is a p -core CRG with least two vertices. If $\mathbf{x} \in \Delta_K$ is the optimal weight vector for K , i.e.,*

$g = g_K(p) = g_K(p, \mathbf{x})$, then for all $u \in V(K)$,

$$g_{K \setminus \{u\}}(p) \leq g + \frac{\mathbf{x}(u)^2}{(1 - \mathbf{x}(u))^2}.$$

Proof. Let $K' := K \setminus \{u\}$. Since K is p -core with at least two vertices, $\mathbf{x}(u) < 1$ and we may define $\mathbf{x}' \in \Delta_K$ by

$$\mathbf{x}'(v) := \begin{cases} 0, & v = u \\ \frac{\mathbf{x}(v)}{1 - \mathbf{x}(u)}, & \text{otherwise} \end{cases}.$$

In other words, if $\mathbf{e}_u \in \mathbb{R}^{V(K)}$ is the indicator vector for the vertex u , then $(1 - \mathbf{x}(u))\mathbf{x}' = \mathbf{x} - \mathbf{x}(u)\mathbf{e}_u$. Recall that $M_K(p)$ denotes the weighted adjacency matrix of K . By Theorem 25, $M_K(p)\mathbf{x} = g\mathbf{1}$ and so

$$\begin{aligned} (1 - \mathbf{x}(u))^2 \langle \mathbf{x}', M_K(p)\mathbf{x}' \rangle &= \langle \mathbf{x} - \mathbf{x}(u)\mathbf{e}_u, M_K(p)(\mathbf{x} - \mathbf{x}(u)\mathbf{e}_u) \rangle \\ &= \langle \mathbf{x}, g\mathbf{1} \rangle - 2\mathbf{x}(u)\langle \mathbf{e}_u, g\mathbf{1} \rangle + \mathbf{x}(u)^2 \langle \mathbf{e}_u, M_K(p)\mathbf{e}_u \rangle \\ &\leq g - 2g\mathbf{x}(u) + \mathbf{x}(u)^2. \end{aligned}$$

By definition of $g_{K'}(p)$ and since $\mathbf{x}'(u) = 0$,

$$\begin{aligned} g_{K'}(p) &\leq g_K(p, \mathbf{x}') \\ &= \frac{\langle \mathbf{x}', M_K(p)\mathbf{x}' \rangle}{(1 - \mathbf{x}(u))^2} \\ &= \frac{g - 2g\mathbf{x}(u) + \mathbf{x}(u)^2}{(1 - \mathbf{x}(u))^2} \\ &= g + \frac{(1 - g)\mathbf{x}(u)^2}{(1 - \mathbf{x}(u))^2} \\ &\leq g + \frac{\mathbf{x}(u)^2}{(1 - \mathbf{x}(u))^2}, \end{aligned}$$

which completes the proof. \square

Proposition 33. *Fix $p \in [0, 1]$ and $\varepsilon \in (0, 1)$. If K is a p -core CRG with $g = g_K(p)$, then there exists a p -core sub-CRG K' of K such that the following holds:*

- (1) $|V(K')| \leq 4/(\varepsilon g)$,
- (2) $g_{K'}(p) \leq (1 + 17\varepsilon)g$, and
- (3) if $\mathbf{x}' \in \Delta_{K'}$ is the optimal weight vector for K' , then

$$\min_{u \in V(K')} \mathbf{x}'(u) \geq \varepsilon g.$$

Proof. Define a finite sequence of sub-CRGs

$$K = K_0 \supset K_1 \supset \cdots \supset K_\ell$$

as follows: First let $K := K_0$ and $g_0 := g_K(p)$. For any $k \geq 0$ so that $|V(K_k)| \geq 2$, do the following:

- (i) Let $\mathbf{x}_k \in \Delta_{K_k}$ be the optimal weight vector for K_k , i.e., $g_{K_k}(p, \mathbf{x}_k) = g_{K_k}(p)$.
- (ii) Let $u_k \in V(K_k)$ so that $\mathbf{x}_k(u_k) = \min \{\mathbf{x}_k(v) : v \in V(K_k)\}$.
- (iii) Let K_{k+1} be any p -core sub-CRG of $K_k \setminus \{u_k\}$.

Since each step removes at least one vertex, $\ell \leq |V(K)|$. For each $k \in \{0, \dots, \ell\}$, denote $g_k := g_{K_k}(p)$.

Let $a \in \{0, \dots, \ell\}$ be the minimum index a so that $|V(K_a)| \leq 4/(\varepsilon g)$. By definition of a and by the fact that $\varepsilon, g < 1$,

$$\mathbf{x}_k(u_k) \leq 1/|V(K_k)| \leq (\varepsilon g)/4 \leq 1/4$$

for all $k \in \{0, \dots, a-1\}$. By Proposition 32,

$$g_{k+1} \leq g_k + \frac{\mathbf{x}_k(u_k)^2}{(1 - \mathbf{x}_k(u_k))^2} \leq g_k + \frac{\mathbf{x}_k(u_k)^2}{(3/4)^2} < g_k + \frac{2}{|V(K_k)|^2}.$$

Because $|V(K_k)| \geq |V(K_{a-1})| + (a-1-k)$ for all $k \in \{0, \dots, a-1\}$,

$$\begin{aligned} g_a &< g + \sum_{k=0}^{a-1} \frac{2}{|V(K_k)|^2} \\ &\leq g + \sum_{k=0}^{a-1} \frac{2}{(|V(K_{a-1})| + (a-1-k))^2} \\ &< g + \sum_{i=|V(K_{a-1})|}^{\infty} \frac{2}{i^2} \\ &< g + \int_{i=|V(K_{a-1})|-1}^{\infty} \frac{2}{x^2} dx \\ &= g + \frac{2}{|V(K_{a-1})| - 1} \\ &\leq g + \frac{2}{\lfloor 4/(\varepsilon g) \rfloor}. \end{aligned}$$

Since $\varepsilon g < 1$, it is the case that $\lfloor 4/(\varepsilon g) \rfloor > 2/(\varepsilon g)$. Consequently,

$$(10) \quad g_a \leq (1 + \varepsilon)g.$$

Let b be the least index $a \leq b \leq \ell$ so that $\mathbf{x}_b(u) \geq \varepsilon g$ for all $u \in V(K_b)$. Note that b is well-defined since $\mathbf{x}_\ell(u) = 1 > \varepsilon g$ where x is the optimal weighting for K_ℓ , a CRG with a single vertex. For any $k \in \{a, \dots, b-1\}$, it is the case that $\mathbf{x}_k(u_k) < \varepsilon g$ and that $\mathbf{x}_k(u_k) \leq 1/|V(K_k)| \leq 1/2$ and again by Proposition 32,

$$g_{k+1} \leq g_k + \frac{\mathbf{x}_k(u_k)^2}{(1 - \mathbf{x}_k(u_k))^2} < g_k + \frac{(\varepsilon g)^2}{(1/2)^2} = g_k + 4\varepsilon^2 g^2.$$

Finally by (10) and since $b - a \leq |V(K_a)| \leq 4/(\varepsilon g)$,

$$g_b \leq g_a + \sum_{k=a}^{b-1} 4\varepsilon^2 g^2 \leq (1 + \varepsilon)g + |V(K_a)| \cdot 4\varepsilon^2 g^2 \leq (1 + 17\varepsilon)g.$$

Letting $K' := K_b$, we have the desired sub-CRG. \square

Proposition 34 below implies that we can decrease the degree of the underlying graph of a CRG K without changing $g_K(p)$ too much.

Proposition 34. *Fix $p \in (0, 1)$ and $\varepsilon \in (0, 1)$. If K is a p -core CRG, then there exists a p -core sub-CRG K' of K so that*

- (1) $g_{K'}(p) \leq (1 + \varepsilon)g_K(p)$, and
- (2) for each $u \in V(K')$, u is incident in K' to at most

$$17\varepsilon^{-1} \cdot \max \left\{ \frac{1}{p}, \frac{1}{1-p} \right\}$$

black or white edges.

Proof. We prove the claim for all $p \in (0, 1/2]$. By duality (that is, by replacing K with \overline{K} and p with $1 - p$) the claim holds also for all $p \in [1/2, 1)$. By Proposition 33 applied to K and $\varepsilon/17$, there exists a sub-CRG K' of K so that $g_{K'}(p) \leq (1 + \varepsilon)g_K$ and whose optimal weight vector $\mathbf{x} \in \Delta_{K'}$ has $\mathbf{x}(u) \geq \varepsilon g/17$ for all $u \in V(K')$. By Theorem 11, the white vertices of K' are incident to no white or black edges. So it suffices to prove the desired inequality for black vertices. Suppose $u \in \text{VB}(K')$. By Theorem 25, $M_{K'}(p)\mathbf{x} = g\mathbf{1}$ and it follows that

$$\frac{g}{p} > \frac{g}{p} - \frac{1-p}{p} \mathbf{x}(u) = \sum_{uv \in \text{EW}(K')} \mathbf{x}(v) \geq \frac{\varepsilon g}{17} \cdot |\{v : uv \in \text{EW}(K')\}|.$$

So the number of vertices adjacent to a vertex via a non-gray edge is at most $17/(\varepsilon p)$, as desired. \square

Next, we uniformly bound the diameter of all p -core CRGs, for each $p \in (1/3, 2/3)$.

Proposition 35. *For all positive integers d , the CRG associated to the path P_d on d vertices is p -prohibited for all*

$$(11) \quad p \in \left[\frac{1}{1 + 2 \cos(\pi/(d+1))}, 1 - \frac{1}{1 + 2 \cos(\pi/(d+1))} \right].$$

Proof. It is a well-known fact from spectral graph theory that the spectrum of the adjacency matrix of P_d , the path on d vertices is the multi-set $\{2 \cos(\frac{\pi k}{d+1}) : k \in \{1, \dots, d\}\}$. See, for example, [8]. In particular, the minimum such eigenvalue is $-2 \cos(\pi/(d+1))$. Lemma 20 gives that P_d is p -prohibited for p in the stated range. \square

Finally, we prove Lemma 31.

Proof of Lemma 31. Since the sequence $\left\{ \frac{1}{1+2 \cos(\pi/(d+1))} \right\}$ is monotone decreasing and converges to $1/3$, there exists a positive integer $d = d_p$ so that P_d is p -prohibited. Let K' be the sub-CRG from Proposition 34 and write G for its underlying graph. Then by construction, G has degree at most

$$D = 17\varepsilon^{-1} \cdot \max \left\{ \frac{1}{p}, \frac{1}{1-p} \right\}.$$

Let C be any component of G . Since P_d is p -prohibited, C has diameter at most $d-1$. It follows that

$$|C| \leq 1 + D + D(D-1) + \dots + D(D-1)^{d-1} =: B(p, \varepsilon),$$

as desired. \square

3.2. Forbidding a random graph. Now we proceed to prove our main result, Theorem 4. First, recall that Theorem 7 says that if $F' \sim \mathbb{G}(n'_0, p_0)$ then **a.a.s.**,

$$\chi_H(F') = (1 + o(1)) c_{\mathcal{H}}(p_0) \frac{n'_0}{2 \log_2 n'_0}.$$

This is useful because in the proof of Lemma 37, we repeatedly decompose induced subgraphs of a random graph of order $n_0 \gg n'_0$. An essential tool is the following restatement of a theorem of Bollobás and Thomason:

Lemma 36 (Lemma 5.1 from [6]). *Let K be a CRG and define \mathcal{H} to be the hereditary property of graphs, G , such that $G \mapsto K$. For all $p_0 \in (0, 1)$,*

$$c_{\mathcal{H}}(p_0) = -\log_2(p_0(1-p_0)) \cdot g_K(p^*)$$

Lemma 37. *Let p_0 and ε be fixed such that $p_0 \in (0, 1)$ and $\varepsilon \in (0, 1)$. Moreover, fix \mathcal{B} to be a finite set of CRGs. If $F \sim \mathbb{G}(n_0, p_0)$, the following holds **a.a.s.** as $n_0 \rightarrow \infty$: For all CRGs K such that all components of K lie in \mathcal{B} and $F \not\mapsto K$, then*

$$(12) \quad (1 + \varepsilon) g_K(p^*) \geq \frac{2 \log n_0}{-\log(p_0(1-p_0)) \cdot n_0}.$$

Proof. For any function $\mu : \mathcal{B} \rightarrow \{0, 1, \dots\}$, define the CRG, K_μ as follows:

$$K_\mu := \bigoplus_{B \in \mathcal{B}} \mu(B) \cdot B.$$

That is, K_μ consists of a disjoint union of $\mu(B)$ copies of B , for all $B \in \mathcal{B}$. For any induced subgraph G of the random graph $F \sim \mathbb{G}(n_0, p_0)$ and any CRG K , we will denote the event that G embeds into K by $[G \mapsto K]$. Additionally, let

$$E_\mu := [F \mapsto K_\mu].$$

Let

$$\mathcal{B}_0 := \left\{ \mu : \mathcal{B} \rightarrow \{0, 1, \dots\} : (1 + \varepsilon) g_{K_\mu}(p^*) < \frac{2 \log n_0}{-\log(p_0(1-p_0)) \cdot n_0} \right\}.$$

To prove the desired claim, it is equivalent to show that the probability that $F \not\mapsto K_\mu$ for all $\mu \in \mathcal{B}_0$ goes to zero. That is,

$$(13) \quad \lim_{n_0 \rightarrow \infty} \mathbb{P} \left[\bigcup_{\mu \in \mathcal{B}_0} \overline{E_\mu} \right] = 0.$$

Recall $F \sim \mathbb{G}(n_0, p_0)$. We will partition $V(F)$ by setting

$$C := \left\lceil 2|\mathcal{B}| \cdot \frac{1 + \varepsilon/2}{\varepsilon/2} \right\rceil$$

and let I_1, \dots, I_C be an equipartition of $V(F)$, i.e., $|I_k| \in \{\lfloor n_0/C \rfloor, \lceil n_0/C \rceil\}$ for $k \in \{1, \dots, C\}$. Let $n'_0 = \lceil n_0/C \rceil$.

For any $B \in \mathcal{B}$, set

$$m_B := \left\lfloor \frac{(1 + \varepsilon/2) \cdot (n_0/C) \cdot (-\log(p_0(1 - p_0))) \cdot g_B(p^*)}{2 \log(n_0/C)} \right\rfloor.$$

We expect to be able to $(m_B \cdot B)$ -color a $\mathbb{G}(n'_0, p_0)$ graph (hence a $\mathbb{G}(n'_0 - 1, p_0)$ graph as well). Now for any $k \in \{1, \dots, C\}$ and any $B \in \mathcal{B}$, we define the event

$$E_{k,B} := [F[I_k] \mapsto m_B \cdot B].$$

In other words, $E_{k,B}$ is the event that the subgraph of F that is induced by vertices in I_k is colorable by m_B copies of the CRG B .

The induced subgraphs $F[I_1], \dots, F[I_C]$ are each independently sampled according to the Erdős-Rényi random graph model $\mathbb{G}(n'_0, p_0)$. Moreover, the number of events of the form $E_{k,B}$ is equal to $C \cdot |\mathcal{B}|$, which is bounded as a function of the constants p_0, ε , and $|\mathcal{B}|$.

Then by Theorem 7 and Lemma 36, since the number of vertices in each I_k uniformly tends to ∞ ,

$$(14) \quad \lim_{n_0 \rightarrow \infty} \mathbb{P} \left[\bigcup_{k \in \{1, \dots, C\}} \bigcup_{B \in \mathcal{B}} \overline{E_{k,B}} \right] = 0.$$

It suffices to show that

$$\bigcup_{\mu \in \mathcal{B}_0} \overline{E_\mu} \subseteq \bigcup_{k \in \{1, \dots, C\}} \bigcup_{B \in \mathcal{B}} \overline{E_{k,B}},$$

because then (14) will imply (13) and complete the proof.

Indeed, suppose $\mu \in \mathcal{B}_0$ and suppose $\varphi_{k,B}$ are embeddings defined by the events $E_{k,B}$, for all $k \in \{1, \dots, C\}$ and all $B \in \mathcal{B}$. Further, for any $B \in \mathcal{B}$, let

$$M_B := \left\lfloor \frac{\mu(B)}{m_B} \right\rfloor.$$

We will show that if $\sum_{B \in \mathcal{B}} M_B \geq C$, then F can be colored by M_B copies of $m_B \cdot B$, over all $B \in \mathcal{B}$, and so $F \mapsto K_\mu$. We begin by summing the M_B 's.

$$\begin{aligned} \sum_{B \in \mathcal{B}} M_B &= \sum_{B \in \mathcal{B}} \left\lfloor \frac{\mu(B)}{m_B} \right\rfloor \\ &\geq \sum_{B \in \mathcal{B}} \left(\frac{\mu(B)}{m_B} - 1 \right) \\ &= -|\mathcal{B}| + \sum_{B \in \mathcal{B}} \frac{\mu(B) \cdot 2 \log(n_0/C)}{(1 + \varepsilon/2) \cdot (n_0/C) \cdot (-\log(p_0(1 - p_0))) \cdot g_B(p^*)} \\ &= -|\mathcal{B}| + \frac{2C \log(n_0/C)}{(1 + \varepsilon/2)n_0} \cdot \sum_{B \in \mathcal{B}} \frac{\mu(B)}{-\log(p_0(1 - p_0)) \cdot g_B(p^*)}. \end{aligned}$$

By Proposition 27,

$$\sum_{B \in \mathcal{B}} M_B = -|\mathcal{B}| + \frac{2C \log(n_0/C)}{(1 + \varepsilon/2)n_0} \cdot \frac{1}{-\log(p_0(1 - p_0)) \cdot g_{K_\mu}(p^*)}.$$

By (12),

$$\begin{aligned} \sum_{B \in \mathcal{B}} M_B &\geq -|\mathcal{B}| + \frac{2C \log(n_0/C)}{(1 + \varepsilon/2)n_0} \cdot \frac{(1 + \varepsilon)n_0}{2 \log n_0} \\ &= -|\mathcal{B}| + C \left(1 - \frac{\log C}{\log n_0} \right) \left(1 + \frac{\varepsilon}{2 + \varepsilon} \right) \\ &= C + C \left(\frac{\varepsilon}{2 + \varepsilon} - \frac{2 + 2\varepsilon}{2 + \varepsilon} \cdot \frac{\log C}{\log n_0} \right) - |\mathcal{B}| \end{aligned}$$

With ε fixed and $n_0 \gg C \gg |\mathcal{B}|$, we have $\sum_{B \in \mathcal{B}} M_B \geq C$, as desired. Thus, the index set $\{1, \dots, C\}$ may be partitioned as

$$\{1, \dots, C\} = \bigcup_{B \in \mathcal{B}} S_B$$

so that $|S_B| \leq M_B$, for each $B \in \mathcal{B}$. Also, for each $B \in \mathcal{B}$, we combine the embeddings $\varphi_{B,k}$ for all $k \in S_B$ to form an embedding

$$F \left[\bigcup_{k \in S_B} I_k \right] \mapsto (|S_B| \cdot m_B) \cdot B \subseteq \mu(B) \cdot B.$$

Combining all such embeddings, $F \mapsto K_\mu$, as desired. So

$$\bigcap_{k \in \{1, \dots, C\}} \bigcap_{B \in \mathcal{B}} E_{k,B} \subseteq \bigcap_{\mu \in \mathcal{B}_0} E_\mu \iff \bigcup_{\mu \in \mathcal{B}_0} \overline{E_\mu} \subseteq \bigcup_{k \in \{1, \dots, C\}} \bigcup_{B \in \mathcal{B}} \overline{E_{k,B}}.$$

This completes the proof of the desired claim. \square

Finally, we are ready to prove Theorem 4.

3.3. Proof of Theorem 4. Formally, our goal is to show that for each $\varepsilon \in (0, 1)$, the following occurs **a.a.s.** as $n_0 \rightarrow \infty$:

(15)

$$\sup_{p \in I} \left| \text{ed}_{\mathcal{H}}(p) \left(\frac{2 \log n_0}{n_0} \cdot \min \left\{ \frac{p}{-\log(1-p_0)}, \frac{1-p}{-\log p_0} \right\} \right)^{-1} - 1 \right| < \varepsilon,$$

where

$$I = \begin{cases} [0, 1], & \text{if } p^* \in [1 - \varphi^{-1}, \varphi^{-1}]; \\ [1/3, 1], & \text{if } p^* \in [0, 1 - \varphi^{-1}); \\ [0, 2/3], & \text{if } p^* \in (\varphi^{-1}, 1]. \end{cases}$$

We first establish an upper bound for $\text{ed}_{\mathcal{H}}(p)$ for all $p \in [0, 1]$. By the main result in [5], if $F \sim \mathbb{G}(n_0, p_0)$, then **a.a.s.** as $n_0 \rightarrow \infty$,

$$\begin{aligned} \chi(F) - 1 &\geq (1 - \varepsilon/2) \frac{n_0}{2 \log_{1/(1-p_0)} n_0}, \\ \chi(F^c) - 1 &\geq (1 - \varepsilon/2) \frac{n_0}{2 \log_{1/p_0} n_0}. \end{aligned}$$

Clearly F does not map into $\chi(F) - 1$ white vertices or $\chi(F^c) - 1$ black vertices. By, for example Theorem 10, and the fact that $1/(1 - \varepsilon/2) < 1 + \varepsilon$, then the following occurs **a.a.s.**:

$$\begin{aligned} \text{ed}_{\mathcal{H}}(p) &\leq (1 + \varepsilon) \min \left\{ \frac{p}{n_0 / (2 \log_{1/(1-p_0)} n_0)}, \frac{1-p}{n_0 / (2 \log_{1/p_0} n_0)} \right\} \\ (16) \quad &= (1 + \varepsilon) \frac{2 \log n_0}{n_0} \cdot \min \left\{ \frac{p}{-\log(1-p_0)}, \frac{1-p}{-\log p_0} \right\}. \end{aligned}$$

We now find a lower bound to match Inequality (16) over the interval I as stated above. We will do this by finding the lower bound for $p \in (1/3, 2/3)$ and then use concavity show how this extends to I in

the various cases.

We will choose a $\tilde{p} \in (1/3, 2/3)$ depending on the case:

$$\tilde{p} := \begin{cases} p^*, & \text{if } p^* \in (1/3, 2/3); \\ 1/3 + \varepsilon/9, & \text{if } p^* \in (0, 1/3]; \\ 2/3 - \varepsilon/9, & \text{if } p^* \in [2/3, 1). \end{cases}$$

Let $K = K(\tilde{p}) \in \mathcal{K}_{\mathcal{H}}$ be a \tilde{p} -core CRG that satisfies $\text{ed}_{\mathcal{H}}(\tilde{p}) = g_K(\tilde{p})$, as guaranteed by Theorem 10.

Since $\tilde{p} \in (1/3, 2/3)$, Lemma 31 gives that there exists a sub-CRG $K' = K'(\tilde{p}, \varepsilon/4)$ of K so that

$$(17) \quad g_K(\tilde{p}) \leq g_{K'}(\tilde{p}) \leq (1 + \varepsilon/4) g_K(\tilde{p})$$

and whose components lie in some finite set $\mathcal{B} = \mathcal{B}(\tilde{p}, \varepsilon/4)$ of CRGs.

The function $g_{K'} : [0, 1] \rightarrow [0, 1]$ is concave-down. To see this, let $M_{K'}(p)$ be the matrix defined by the CRG K' . Let $p_1, p_2 \in [0, 1]$, $t \in [0, 1]$, and let $\mathbf{x} \in \Delta_{K'}$ be the vector that witnesses the value of $g_{K'}$ at $tp_1 + (1-t)p_2$,

$$\begin{aligned} g_{K'}(tp_1 + (1-t)p_2) &= \langle \mathbf{x}, M_{K'}(tp_1 + (1-t)p_2) \mathbf{x} \rangle \\ &= t \cdot \langle \mathbf{x}, M_{K'}(p_1) \mathbf{x} \rangle + (1-t) \cdot \langle \mathbf{x}, M_{K'}(p_2) \mathbf{x} \rangle \\ &\geq t \cdot g_{K'}(p_1) + (1-t) \cdot g_{K'}(p_2), \end{aligned}$$

establishing the concavity of $g_{K'}$. Since $g_{K'}(0), g_{K'}(1) \geq 0$, the graph of $g_{K'}$ lies above the line segment from $(0, 0)$ to $(\tilde{p}, g_{K'}(\tilde{p}))$ to $(1, 0)$. So

$$(18) \quad g_{K'}(p) \geq g_{K'}(p^*) \cdot \min \left\{ \frac{p}{p^*}, \frac{1-p}{1-p^*} \right\}$$

Also by Lemma 37 (recall the definition of p^* from (9)), the following is true **a.a.s.**:

$$(19) \quad g_{K'}(p^*) \geq (1 - \varepsilon/4) \cdot \frac{2 \log n_0}{-\log(p_0(1-p_0)) \cdot n_0}.$$

Combining (17), (18), and (19),

$$\begin{aligned}
 \text{ed}_{\mathcal{H}}(\tilde{p}) &= g_K(\tilde{p}) \\
 &\geq \frac{1}{1 + \varepsilon/4} g_{K'}(\tilde{p}) \\
 &\geq \frac{1}{1 + \varepsilon/4} g_{K'}(p^*) \cdot \min \left\{ \frac{\tilde{p}}{p^*}, \frac{1 - \tilde{p}}{1 - p^*} \right\} \\
 &\geq \frac{1 - \varepsilon/4}{1 + \varepsilon/4} \cdot \frac{2 \log n_0}{-\log(p_0(1 - p_0)) \cdot n_0} \cdot \min \left\{ \frac{\tilde{p}}{p^*}, \frac{1 - \tilde{p}}{1 - p^*} \right\} \\
 (20) \quad &\geq (1 - \varepsilon/2) \frac{2 \log n_0}{-\log(p_0(1 - p_0)) \cdot n_0} \cdot \min \left\{ \frac{\tilde{p}}{p^*}, \frac{1 - \tilde{p}}{1 - p^*} \right\}
 \end{aligned}$$

Case 1. $p_0 \in (1 - \varphi^{-1}, \varphi^{-1}) \iff p^* \in (1/3, 2/3)$.

In this case, $\tilde{p} = p^*$. By (20) and the concavity of $\text{ed}_{\mathcal{H}}(p)$,

$$\begin{aligned}
 \text{ed}_{\mathcal{H}}(p^*) &\geq (1 - \varepsilon/2) \frac{2 \log n_0}{-\log(p_0(1 - p_0)) \cdot n_0} \\
 \text{ed}_{\mathcal{H}}(p) &\geq (1 - \varepsilon/2) \frac{2 \log n_0}{-\log(p_0(1 - p_0)) \cdot n_0} \cdot \min \left\{ \frac{p}{p^*}, \frac{1 - p}{1 - p^*} \right\} \\
 &= (1 - \varepsilon/2) \frac{2 \log n_0}{n_0} \cdot \min \left\{ \frac{p}{-\log(1 - p_0)}, \frac{1 - p}{-\log p_0} \right\}.
 \end{aligned}$$

This satisfies (15) for all $p \in [0, 1]$, completing the proof in this case.

Case 2. $p_0 \in (0, 1 - \varphi^{-1}] \iff p^* \in (0, 1/3]$.

In this case, $\tilde{p} = 1/3 + \varepsilon/9 > p^*$. By (20) and the concavity of $\text{ed}_{\mathcal{H}}$,

$$\begin{aligned}
 \text{ed}_{\mathcal{H}}(\tilde{p}) &\geq (1 - \varepsilon/2) \frac{2 \log n_0}{-\log(p_0(1 - p_0)) \cdot n_0} \cdot \frac{1 - \tilde{p}}{1 - p^*} \\
 \text{ed}_{\mathcal{H}}(p) &\geq (1 - \varepsilon/2) \frac{2 \log n_0}{-\log(p_0(1 - p_0)) \cdot n_0} \cdot \frac{1 - \tilde{p}}{1 - p^*} \cdot \min \left\{ \frac{p}{\tilde{p}}, \frac{1 - p}{1 - \tilde{p}} \right\}.
 \end{aligned}$$

Substituting $\tilde{p} = 1/3 + \varepsilon/9$ and $p = 1/3$,

$$\begin{aligned}
 \text{ed}_{\mathcal{H}}(1/3) &\geq (1 - \varepsilon/2) \frac{2 \log n_0}{-\log(p_0(1 - p_0)) \cdot n_0} \cdot \frac{2/3 - \varepsilon/9}{1 - p^*} \cdot \frac{1/3}{1/3 + \varepsilon/9} \\
 &= \frac{(1 - \varepsilon/2)(1/3 - \varepsilon/18)}{1/3 + \varepsilon/9} \cdot \frac{2 \log n_0}{n_0} \cdot \frac{2/3}{-\log p_0} \\
 &\geq (1 - \varepsilon) \frac{2 \log n_0}{n_0} \cdot \frac{2/3}{-\log p_0}.
 \end{aligned}$$

Again by concavity,

$$\text{ed}_{\mathcal{H}}(p) \geq (1 - \varepsilon) \frac{2 \log n_0}{n_0} \cdot \frac{2/3}{-\log p_0} \cdot \min \left\{ \frac{p}{1/3}, \frac{1-p}{2/3} \right\}.$$

This matches the upper bound (16) for all $p \in [1/3, 1]$ and, in fact, if $p^* = 1/3$, then it matches the upper bound for all $p \in [0, 1]$. This completes the proof in this case.

Case 3. $p_0 \in [\varphi^{-1}, 1) \iff p^* \in [2/3, 1)$.

This case may be shown with a similar argument as Case 2. In this case, $\tilde{p} = 2/3 - \varepsilon/9 < p^*$. By (20) and the concavity of $\text{ed}_{\mathcal{H}}$,

$$\text{ed}_{\mathcal{H}}(p) \geq (1 - \varepsilon) \frac{2 \log n_0}{n_0} \cdot \frac{1/3}{-\log p_0} \cdot \min \left\{ \frac{p}{2/3}, \frac{1-p}{1/3} \right\}.$$

This matches the upper bound (16) for all $p \in [0, 2/3]$ and if $p^* = 2/3$, then it matches the upper bound for all $p \in [0, 1]$. This completes the proof in this case and the proof of Theorem 4. \square

4. DISCUSSION

In the process of proving the main result, Theorem 4, we have developed a number of observations that apply generally to computing edit distance functions. Lemma 20 gives a general condition for which a CRG is p -prohibited and Lemma 23 shows that for $p \in [1 - \varphi^{-1}, \varphi^{-1}]$, the only CRGs that need to be considered are dalmatian sets and their complements.

In this section, we discuss some other general results.

4.1. Defining the edit distance function with a finite set of CRGs. The following was conjectured by the first author:

Conjecture 38 ([11]). *Let \mathcal{H} be a nontrivial hereditary property. For every $\varepsilon > 0$ there exists a finite set of CRGs $\mathcal{K}' = \mathcal{K}'(\varepsilon, \mathcal{H})$ such that*

$$\text{ed}_{\mathcal{H}}(p) = \min \{g_K(p) : K \in \mathcal{K}'\}, \quad \text{for all } p \in (\varepsilon, 1 - \varepsilon).$$

Note that this is stronger than the Marchant-Thomason result in Theorem 10 which says that, for every p there is a finite set of CRGs that define $\text{ed}_{\mathcal{H}}(p)$. Conjecture 38 asserts that a single finite set will define $\text{ed}_{\mathcal{H}}(p)$ for all p an arbitrary open interval in $(0, 1)$. In Theorem 39, we provide a partial answer by showing that the conjecture is

true for $\varepsilon \geq 1 - \varphi^{-1}$.

Theorem 39. *Let \mathcal{H} be a nontrivial hereditary property. There exists a finite set of CRGs, $\mathcal{K}' = \mathcal{K}'(\mathcal{H})$, such that*

$$\text{ed}_{\mathcal{H}}(p) = \min \{g_K(p) : K \in \mathcal{K}'\}, \quad \text{for all } p \in [1 - \varphi^{-1}, \varphi^{-1}].$$

Proof. By Lemma 23, the only p -core CRGs are denoted \mathcal{D}_p and consist of components which are dalmatian CRGs if $p \in [1 - \varphi^{-1}, 1/2)$, complements of dalmatian CRGs if $p \in (1/2, \varphi^{-1}]$, and CRGs with only gray edges if $p = 1/2$.

It is easy to see that if $p = 1/2$ and $\mathcal{H} = \text{Forb}(\mathcal{F})$, then for any $F \in \mathcal{F}$ such that $F \not\mapsto K$, the number of vertices of K is bounded by $\chi(F) + \chi(F^c)$, so the number of such CRGs is finite. We will now show that a finite set of CRGs suffice for $p \in [1 - \varphi^{-1}, 1/2)$. The case $p \in (1/2, \varphi^{-1}]$ follows by symmetry. Note that \mathcal{D}_p is the same for all $p \in [0, 1/2)$ (that is, CRGs whose components are all dalmatian CRGs) so we denote $\mathcal{D}_{0,\mathcal{H}} := \mathcal{D}_p \cap \mathcal{K}_{\mathcal{H}}$.

Write $\mathcal{H} = \text{Forb}(\mathcal{F})$ and, for a contradiction, let $\{p_k\}_{k=1}^{\infty} \subset [1 - \varphi^{-1}, 1/2)$ be an infinite set and let $\mathcal{K}' := \{K_k\}_{k=1}^{\infty} \subset \mathcal{D}_{0,\mathcal{H}}$ an infinite set of CRGs such that K_k is p_k -core and $g_{K_k}(p_k) = \text{ed}_{\mathcal{H}}(p_k)$.

For all $k \geq 1$, with D_i denoting a dalmatian CRG of order i , we may write

$$(21) \quad K_k = D_{c_k^{(1)}} \oplus \cdots \oplus D_{c_k^{(\ell_k)}} \oplus (w_k \cdot D_{\infty})$$

where $w_k, \ell_k, c_k^{(1)}, \dots, c_k^{(\ell_k)}$ are nonnegative integers and $c_k^{(1)} \geq \cdots \geq c_k^{(\ell_k)}$.

For all $k \geq 1$, $w_k + \ell_k \leq |V(F)|$ for any $F \in \mathcal{F}$ because otherwise $F \mapsto K_k$ by embedding each vertex of F into a different component of K_k . Thus, ℓ_k is bounded by an absolute constant $\ell = \ell(\mathcal{H})$. So we associate each K_k in (21) with the $(\ell + 1)$ -tuple

$$\left(c_k^{(1)}, \dots, c_k^{(\ell)}; w_k \right),$$

where $c_k^{(\ell_k+1)} = \cdots = c_k^{(\ell)} = 0$ if $\ell > \ell_k$.

Because \mathcal{K}' is infinite, there exists a maximum $m \in \{1, \dots, \ell\}$ such that $\sup_k \{c_k^{(1)}\} = \cdots = \sup_k \{c_k^{(m)}\} = \infty$. That is, if $m < \ell$, then

$\sup_k \{c_k^{(m+1)}\} < \infty$. Thus, there is a fixed (possibly empty) tuple $(c_*^{(m+1)}, \dots, c_*^{(\ell)}; w_*)$ and an infinite subsequence k_1, k_2, \dots such that K_{k_i} is associated with $(\ell + 1)$ -tuple $(c_{k_i}^{(1)}, \dots, c_{k_i}^{(m)}, c_*^{(m+1)}, \dots, c_*^{(\ell)}; w_*)$.

With this choice of $(c_*^{(m+1)}, \dots, c_*^{(\ell)}; w_*)$, if ℓ' is the largest entry such that $c_*^{(\ell')} \geq 1$, then define

$$K_* = D_{c_k^{(m+1)}} \oplus \dots \oplus D_{c_k^{(\ell')}} \oplus ((m + w_*) \cdot D_\infty).$$

We claim that $K_* \in \mathcal{D}_{0, \mathcal{H}} \subseteq \mathcal{K}_{\mathcal{H}}$.

If not, then there exists some $F \in \mathcal{F}$ and some embedding $\phi : V(F) \rightarrow V(K_*)$. Let A_1, \dots, A_m be the preimages of the first m copies of D_∞ under ϕ , respectively. Then A_1, \dots, A_m are independent sets in F .

Let k be sufficiently large so that $c_1^k, \dots, c_m^k \geq |V(F)|$. Then we define $\phi' : V(F) \rightarrow V(K_k)$ by instead sending the vertices of A_i to distinct vertices of the dalmatian set $D_{c_i^k}$, for each $i \in \{1, \dots, m\}$. As a result, ϕ' is an embedding of F into K_k , a contradiction.

Finally by Proposition 27 and Remark 22, for all $p \in (0, 1/2)$ and k chosen as above,

$$\begin{aligned} g_{K_k}(p)^{-1} &= \frac{w}{p} + \sum_{i=1}^m \frac{1}{p + (1 - 2p)/c_i^k} + \sum_{i=1}^{\ell-m} \frac{1}{p + (1 - 2p)/c_i} \\ &< \frac{w + m}{p} + \sum_{i=1}^{\ell-m} \frac{1}{p + (1 - 2p)/c_i} \\ &= g_{K_*}(p)^{-1} \end{aligned}$$

The fact that $g_{K_*}(p_k) < g_{K_k}(p_k) = \text{ed}_{\mathcal{H}}(p_k)$ contradicts $K_* \in \mathcal{K}_{\mathcal{H}}$, hence the original assumption that \mathcal{K}' is infinite. \square

4.2. Paths. In Proposition 35, it is established that P_d is p -prohibited for $p \in \left[\frac{1}{1+2 \cos(\pi/(d+1))}, 1 - \frac{1}{1+2 \cos(\pi/(d+1))} \right]$. In the case of $d = 3$, P_3 is p -prohibited for p in the interval

$$[\sqrt{2} - 1, 2 - \sqrt{2}] \approx [0.414214, 0.585786].$$

However, Lemma 23 establishes that P_3 is p -prohibited if and only if p is in the interval

$$[1 - \varphi^{-1}, \varphi^{-1}] = \left[\frac{3 - \sqrt{5}}{2}, \frac{\sqrt{5} - 1}{2} \right] \approx [0.381966, 0.618034].$$

We ask whether P_d is p -prohibited over a larger interval than given in (11). See Table 1 for small values.

d	$(1 + 2 \cos(\pi/(d + 1)))^{-1}$	$1 - (1 + 2 \cos(\pi/(d + 1)))^{-1}$
3	$\sqrt{2} - 1 \approx 0.414214$	$2 - \sqrt{2} \approx 0.585786$
4	$(3 - \sqrt{5})/2 \approx 0.381966$	$(\sqrt{5} - 1)/2 \approx 0.618034$
5	$(\sqrt{3} - 1)/2 \approx 0.366025$	$(3 - \sqrt{3})/2 \approx 0.633975$
6	≈ 0.356896	≈ 0.643104
7	≈ 0.351153	≈ 0.648847
8	≈ 0.347296	≈ 0.652704
9	≈ 0.344577	≈ 0.655423
10	≈ 0.342585	≈ 0.657415
11	≈ 0.341081	≈ 0.658919
12	≈ 0.339918	≈ 0.660082
13	≈ 0.339000	≈ 0.661000
14	≈ 0.338261	≈ 0.661739
15	≈ 0.337659	≈ 0.662341

TABLE 1. Table for endpoints of an interval where P_d is prohibited.

Question 40. For $d \geq 4$, what is the largest interval over which P_d is p -prohibited?

5. QUESTIONS AND FUTURE WORK

5.1. **p -core CRGs.** Lemma 23 classifies all p -core CRGs on the interval $[1 - \varphi^{-1}, \varphi^{-1}]$.

Question 41. For which $a \in (0, 1 - \varphi^{-1})$ does there exist an elementary classification of all p -core CRGs for all $p \in [a, 1 - a]$? Additionally, are all sufficiently large connected p -core CRGs either dalmatian CRGs (if $p \leq 1/2$) or the complement of a dalmatian CRG (if $p \geq 1/2$)?

A crucial part of the proof of Theorem 4 is Lemma 31, which establishes that, for $p \in (1/3, 2/3)$, a p -core CRG can be approximated so that the g function does not increase by much, but the components are bounded.

Question 42. *Does Lemma 31 hold if the interval $(1/3, 2/3)$ is widened to $(a, 1 - a)$ for some $a \in (0, 1/3)$?*

5.2. Inhomogeneous random graphs. Since the development of graph limits, inhomogeneous generalizations $\mathbb{G}(n, W)$ of the Erdős-Rényi random graph models have emerged as a topic of research interest (see [9]). Here, $W : \Omega^2 \rightarrow [0, 1]$ is a *graphon*, which is a symmetric measurable function where Ω is a probability space, frequently $[0, 1]$ equipped with the Lebesgue measure. To form a W -random graph $G \sim \mathbb{G}(n, W)$, sample n elements $x_1, \dots, x_n \sim \Omega$ independently and form a graph on $\{1, \dots, n\}$ by adding edge ij independently with probability $W(x_i, x_j)$. We may also generate a sequence of W -random graphs $(G_n)_{n=1}^\infty \sim \mathbb{G}(\mathbb{N}, W)$ adding the vertices corresponding to x_1, x_2, \dots , one at a time.

There are several questions we may ask related to the edit distance problem and inhomogeneous random graphs.

First, note that Theorem 4 implies that with $p_0 \in [1 - \varphi^{-1}, \varphi^{-1}]$ and $(F_n) \sim \mathbb{G}(\mathbb{N}, p_0)$, then **a.a.s.**,

$$(22) \quad \lim_{n \rightarrow \infty} \sup_{p \in [0, 1]} \frac{\text{ed}_{\text{Forb}(F_{n+1})}(p)}{\text{ed}_{\text{Forb}(F_n)}(p)} = 1.$$

Question 43. *For what graphons W does Equation (22) **a.a.s.** hold for $(F_n) \sim \mathbb{G}(\mathbb{N}, W)$?*

For the next question, we note the following expression for the distance from a homogeneous random graph to a hereditary property. Combining Theorems 2 and 10, we see that if \mathcal{H} is a (fixed) nontrivial hereditary property and $p \in [0, 1]$, then with $G_n \sim \mathbb{G}(n, p)$,

$$(23) \quad \lim_{n \rightarrow \infty} \mathbb{E}[\text{dist}(G_n, \mathcal{H})] = \inf_{K \in \mathcal{K}_{\mathcal{H}}} g_K(p) = \min_{K \in \mathcal{K}_{\mathcal{H}}} g_K(p).$$

Question 44. *If instead $G_n \sim \mathbb{G}(n, W)$, is there a similar expression for $\limsup_{n \rightarrow \infty} \mathbb{E}[\text{dist}(G_n, \mathcal{H})]$? In particular, can we extend the functions $g_K(\cdot)$ from $[0, 1]$ to the set of all graphons so that Equation (23) holds?*

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