# Clustered Colouring of Graph Classes with Bounded Treedepth or Pathwidth 


#### Abstract

Sergey Norin ${ }^{\dagger} \quad$ Alex Scott ${ }^{\ddagger} \quad$ David R. Wood ${ }^{〔}$ Abstract. The clustered chromatic number of a class of graphs is the minimum integer $k$ such that for some integer $c$ every graph in the class is $k$-colourable with monochromatic components of size at most $c$. We determine the clustered chromatic number of any minor-closed class with bounded treedepth, and prove a best possible upper bound on the clustered chromatic number of any minor-closed class with bounded pathwidth. As a consequence, we determine the fractional clustered chromatic number of every minorclosed class.


## 1 Introduction

This paper studies improper vertex colourings of graphs with bounded monochromatic degree or bounded monochromatic component size. This topic has been extensively studied recently [1-6, 8, 10, 12-21, 23-25]; see [26] for a survey.

A $k$-colouring of a graph $G$ is a function that assigns one of $k$ colours to each vertex of $G$. In a coloured graph, a monochromatic component is a connected component of the subgraph induced by all the vertices of one colour.

A colouring has defect $d$ if each monochromatic component has maximum degree at most $d$. The defective chromatic number of a graph class $\mathscr{G}$, denoted by $\chi_{\Delta}(\mathscr{\varphi})$, is the minimum integer $k$ such that, for some integer $d$, every graph in $\mathscr{G}$ is $k$-colourable with defect $d$.

A colouring has clustering $c$ if each monochromatic component has at most $c$ vertices. The clustered chromatic number of a graph class $\mathscr{G}$, denoted by $\chi_{\star}(\varphi)$, is the minimum integer $k$ such that, for some integer $c$, every graph in $\mathscr{G}$ has a $k$-colouring with clustering $c$. We shall consider such colourings, where the goal is to minimise the number of colours, without optimising the clustering value.

Every colouring of a graph with clustering $c$ has defect $c-1$. Thus $\chi_{\Delta}(\mathscr{G}) \leqslant \chi_{\star}(\mathscr{G})$ for every class $\mathscr{G}$.

The following is a well-known and important example in defective and clustered graph colouring. Let $T$ be a rooted tree. The depth of $T$ is the maximum number of vertices

[^0]on a root-to-leaf path in $T$. The closure of $T$ is obtained from $T$ by adding an edge between every ancestor and descendant in $T$. For $h, k \geqslant 1$, let $C\langle h, k\rangle$ be the closure of the complete $k$-ary tree of depth $h$, as illustrated in Figure 1.


Figure 1: The standard example $C\langle 4,2\rangle$.
It is well known and easily proved (see [26]) that there is no ( $h-1$ )-colouring of $C\langle h, k\rangle$ with defect $k-1$, which implies there is no $(h-1)$-colouring of $C\langle h, k\rangle$ with clustering $k$. This says that if a graph class $\mathscr{G}$ includes $C\langle h, k\rangle$ for all $k$, then the defective chromatic number and the clustered chromatic number are at least $h$. Put another way, define the tree-closure-number of a graph class $\mathscr{G}$ to be

$$
\operatorname{tcn}(\mathscr{G}):=\min \{h: \exists k C\langle h, k\rangle \notin \mathscr{G}\}=\max \{h: \forall k C\langle h, k\rangle \in \mathscr{G}\}+1 ;
$$

then

$$
\chi_{\star}(\mathscr{G}) \geqslant \chi_{\Delta}(\mathscr{G}) \geqslant \operatorname{tcn}(\mathscr{G})-1
$$

Our main result, Theorem 1 below, establishes a converse result for minor-closed classes with bounded treedepth. First we explain these terms. A graph $H$ is a minor of a graph $G$ if a graph isomorphic to $H$ can be obtained from some subgraph of $G$ by contracting edges. A class of graphs $\mathcal{M}$ is minor-closed if for every graph $G \in \mathscr{M}$ every minor of $G$ is in $\mathscr{M}$, and $\mathscr{M}$ is proper minor-closed if, in addition, some graph is not in $\mathcal{M}$. The connected treedepth of a graph $H$, denoted by $\overline{\operatorname{td}}(H)$, is the minimum depth of a rooted tree $T$ such that $H$ is a subgraph of the closure of $T$. This definition is a variant of the more commonly used definition of the treedepth of $H$, denoted by $\operatorname{td}(H)$, which equals the maximum connected treedepth of the connected components of $H$. (See [22] for background on treedepth.) If $H$ is connected, then $\operatorname{td}(H)=\overline{\operatorname{td}}(H)$. In fact, $\operatorname{td}(H)=\overline{\operatorname{td}}(H)$ unless $H$ has two connected components $H_{1}$ and $H_{2}$ with $\operatorname{td}\left(H_{1}\right)=\operatorname{td}\left(H_{2}\right)=\operatorname{td}(H)$, in which case $\overline{\operatorname{td}}(H)=\operatorname{td}(H)+1$. It is convenient to work with connected treedepth to avoid this distinction. A class of graphs has bounded treedepth if there exists a constant $c$ such that every graph in the class has treedepth at most $c$.

Theorem 1. For every minor-closed class $\mathscr{G}$ with bounded treedepth,

$$
\chi_{\Delta}(\mathscr{G})=\chi_{\star}(\mathscr{G})=\operatorname{tcn}(\mathscr{G})-1 .
$$

Our second result concerns pathwidth. A path-decomposition of a graph $G$ consists of a sequence ( $B_{1}, \ldots, B_{n}$ ), where each $B_{i}$ is a subset of $V(G)$ called a bag, such that for every vertex $v \in V(G)$, the set $\left\{i \in[1, n]: v \in B_{i}\right\}$ is an interval, and for every edge $v w \in E(G)$ there is a bag $B_{i}$ containing both $v$ and $w$. Here $[a, b]:=\{a, a+1, \ldots, b\}$. The width of a path decomposition $\left(B_{1}, \ldots, B_{n}\right)$ is $\max \left\{\left|B_{i}\right|: i \in[1, n]\right\}-1$. The pathwidth of a graph $G$ is the minimum width of a path-decomposition of $G$. Note that paths (and more generally caterpillars) have pathwidth 1. A class of graphs has bounded pathwidth if there exists a constant $c$ such that every graph in the class has pathwidth at most $c$.

Theorem 2. For every minor-closed class $\mathscr{G}$ with bounded pathwidth,

$$
\chi_{\Delta}\left(\varphi_{\xi}\right) \leqslant \chi_{\star}(\varphi) \leqslant 2 \operatorname{tcn}(\varphi)-2 .
$$

Theorems 1 and 2 are respectively proved in Sections 2 and 3. These results are best possible and partially resolve a number of conjectures from the literature, as we now explain.

Ossona de Mendez et al. [24] studied the defective chromatic number of minor-closed classes. For a graph $H$, let $\mathscr{M}_{H}$ be the class of $H$-minor-free graphs (that is, not containing $H$ as a minor). Ossona de Mendez et al. [24] proved the lower bound, $\chi_{\Delta}\left(\mathscr{M}_{H}\right) \geqslant \overline{\operatorname{td}}(H)-1$ and conjectured that equality holds.

Conjecture 3 ([24]). For every graph $H$,

$$
\chi_{\Delta}\left(\mathscr{M}_{H}\right)=\overline{\operatorname{td}}(H)-1 .
$$

Conjecture 3 is known to hold in some special cases. Edwards et al. [10] proved it if $H=K_{t}$; that is, $\chi_{\Delta}\left(\mathcal{M}_{K_{t}}\right)=t-1$, which can be thought of as a defective version of Hadwiger's Conjecture; see [25] for an improved bound on the defect in this case. Ossona de Mendez et al. [24] proved Conjecture 3 if $\overline{\mathrm{td}}(H) \leqslant 3$ or if $H$ is a complete bipartite graph. In particular, $\chi_{\Delta}\left(\mathcal{M}_{K_{s, t}}\right)=\min \{s, t\}$.
Norin et al. [23] studied the clustered chromatic number of minor-closed classes. They showed that for each $k \geqslant 2$, there is a graph $H$ with treedepth $k$ and connected treedepth $k$ such that $\chi_{\star}\left(\mathcal{M}_{H}\right) \geqslant 2 k-2$. Their proof in fact constructs a set $\mathscr{X}$ of graphs in $\mathscr{M}_{H}$ with bounded pathwidth (at most $2 k-3$ to be precise) such that $\chi_{\star}(\mathscr{X}) \geqslant 2 k-2$. Thus the upper bound on $\chi_{\star}(\mathscr{\varphi})$ in Theorem 2 is best possible.

Norin et al. [23] conjectured the following converse upper bound (analogous to Conjecture 3):

Conjecture 4 ([23]). For every graph $H$,

$$
\chi_{\star}\left(\mathcal{M}_{H}\right) \leqslant 2 \overline{\operatorname{td}}(H)-2 .
$$

While Conjectures 3 and 4 remain open, Norin et al. [23] showed in the following theorem that $\chi_{\Delta}\left(\mathscr{M}_{H}\right)$ and $\chi_{\star}\left(\mathscr{M}_{H}\right)$ are controlled by the treedepth of $H$ :

Theorem 5 ([23]). For every graph $H, \chi_{\star}\left(\mathcal{M}_{H}\right)$ is tied to the (connected) treedepth of H. In particular,

$$
\overline{\operatorname{td}}(H)-1 \leqslant \chi_{\star}\left(\mathscr{M}_{H}\right) \leqslant 2^{\overline{\operatorname{td}}(H)+1}-4
$$

Theorem 1 gives a much more precise bound than Theorem 5 under the extra assumption of bounded treedepth.

Our third main result concerns fractional colourings. For real $t \geqslant 1$, a graph $G$ is fractionally $t$-colourable with clustering $c$ if there exist $Y_{1}, Y_{2}, \ldots, Y_{s} \subseteq V(G)$ and $\alpha_{1}, \ldots, \alpha_{s} \in[0,1]$ such that ${ }^{1}$ :

- Every component of $G\left[Y_{i}\right]$ has at most $c$ vertices,
- $\sum_{i=1}^{s} \alpha_{i} \leqslant t$,
- $\sum_{i: v \in Y_{i}} \alpha_{i} \geqslant 1$ for every $v \in V(G)$.

The fractional clustered chromatic number $\chi_{\star}^{f}(\mathscr{G})$ of a graph class $\mathscr{G}$ is the infimum of $t>0$ such that there exists $c=c(t, \mathscr{G})$ such that every $G \in \mathscr{G}$ is fractionally $t$-colourable with clustering $c$.
Fractionally $t$-colourable with defect $d$ and fractional defective chromatic number $\chi_{\Delta}^{f}\left(\varphi_{)}\right)$ are defined in exactly the same way, except the condition on the component size of $G\left[Y_{i}\right]$ is replaced by "the maximum degree of $G\left[Y_{i}\right]$ is at most $d$ ".

The following theorem determines the fractional clustered chromatic number and fractional defective chromatic number of any proper minor-closed class.

Theorem 6. For every proper minor-closed class $\mathscr{G}$,

$$
\chi_{\Delta}^{f}(\mathscr{G})=\chi_{\star}^{f}(\mathscr{G})=\operatorname{tcn}(\mathscr{G})-1
$$

This result is proved in Section 4.
We now give an interesting example of Theorem 6.
Corollary 7. For every surface $\Sigma$, if $\mathscr{G}_{\Sigma}$ is the class of graphs embeddable in $\Sigma$, then

$$
\chi_{\Delta}^{f}\left(\mathscr{G}_{\Sigma}\right)=\chi_{\star}^{f}\left(\mathscr{G}_{\Sigma}\right)=3
$$

Proof. Note that $C\langle 3, k\rangle$ is planar for all $k$. Thus $\operatorname{tcn}\left(\mathscr{G}_{\Sigma}\right) \geqslant 4$. Say $\Sigma$ has Euler genus g. It follows from Euler's formula that $K_{3,2 g+3} \notin \mathscr{G}_{\Sigma}$. Since $K_{3,2 g+3} \subseteq C\langle 4,2 g+3\rangle$, we have $C\langle 4,2 g+3\rangle \notin \mathscr{G}_{\Sigma}$. Thus $\operatorname{tcn}\left(\mathscr{G}_{\Sigma}\right)=4$. The result follows from Theorem 6 .

In contrast to Corollary 7, Dvořák and Norin [8] proved that $\chi_{\star}\left(\mathscr{G}_{\Sigma}\right)=4$. Note that Archdeacon [2] proved that $\chi_{\Delta}\left(\mathscr{G}_{\Sigma}\right)=3$; see [5] for an improved bound on the defect.

[^1]
## 2 Treedepth

Say $G$ is a subgraph of the closure of some rooted tree $T$. For each vertex $v \in V(T)$, let $T_{v}$ be the maximal subtree of $T$ rooted at $v$ (consisting of $v$ and all its descendants), and let $G\left[T_{v}\right]$ be the subgraph of $G$ induced by $V\left(T_{v}\right)$.

The weak closure of a rooted tree $T$ is the graph $G$ with vertex set $V(T)$, where two vertices $v, w \in V(T)$ are adjacent in $G$ whenever $v$ is a leaf of $T$ and $w$ is an ancestor of $v$ in $T$. As illustrated in Figure 2, let $W\langle h, k\rangle$ be the weak closure of the complete $k$-ary tree of height $h$.


Figure 2: The weak closure $W\langle 4,2\rangle$.
Note that $W\langle h, k\rangle$ is a proper subgraph of $C\langle h, k\rangle$ for $h \geqslant 3$. On the other hand, Norin et al. [23] showed that $W\langle h, k\rangle$ contains $C\langle h, k-1\rangle$ as a minor for all $h, k \geqslant 2$. Therefore Theorem 1 is an immediate consequence of the following lemma.

Lemma 8. For all $d, k, h \in \mathbb{N}$ there exists $c=c(d, k, h) \in \mathbb{N}$ such that for every graph $G$ with treedepth at most $d$, either $G$ contains $a W\langle h, k\rangle$-minor or $G$ is $(h-1)$-colourable with clustering $c$.

Proof. Throughout this proof, $d, k$ and $h$ are fixed, and we make no attempt to optimise c.

We may assume that $G$ is connected. So $G$ is a subgraph of the closure of some rooted tree of depth at most $d$. Choose a tree $T$ of depth at most $d$ rooted at some vertex $r$, such that $G$ is a subgraph of the closure of $T$, and subject to this, $\sum_{v \in V(T)} \operatorname{dist}_{T}(v, r)$ is minimal. Suppose that $G\left[T_{v}\right]$ is disconnected for some vertex $v$ in $T$. Choose such a vertex $v$ at maximum distance from $r$. Since $G$ is connected, $v \neq r$. By the choice of $v$, for each child $w$ of $v$, the subgraph $G\left[T_{w}\right]$ is connected. Thus, for some child $w$ of $v$, there is no edge in $G$ joining $v$ and $G\left[T_{w}\right]$. Let $u$ be the parent of $v$. Let $T^{\prime}$ be obtained from $T$ by deleting the edge $v w$ and adding the edge $u w$, so that $w$ is a child of $u$ in $T^{\prime}$. Note that $G$ is a subgraph of the closure of $T^{\prime}$ (since $v$ has no neighbour in $G\left[T_{w}\right]$ ). Moreover,
$\operatorname{dist}_{T^{\prime}}(x, r)=\operatorname{dist}_{T}(x, r)-1$ for every vertex $x \in V\left(T_{w}\right), \operatorname{and}_{\operatorname{dist}_{T^{\prime}}}(y, r)=\operatorname{dist}_{T}(y, r)$ for every vertex $y \in V(T) \backslash V\left(T_{w}\right)$. Hence $\sum_{v \in V\left(T^{\prime}\right)} \operatorname{dist}_{T^{\prime}}(v, r)<\sum_{v \in V(T)} \operatorname{dist}_{T}(v, r)$, which contradicts our choice of $T$. Therefore $G\left[T_{v}\right]$ is connected for every vertex $v$ of $T$.

Consider each vertex $v \in V(T)$. Define the level $\ell(v):=\operatorname{dist}_{T}(r, v) \in[0, d-1]$. Let $T_{v}^{+}$be the subtree of $T$ consisting of $T_{v}$ plus the $v r$-path in $T$, and let $G\left[T_{v}^{+}\right]$be the subgraph of $G$ induced by $V\left(T_{v}^{+}\right)$. For a subtree $X$ of $T$ rooted at vertex $v$, define the level $\ell(X):=\ell(v)$.

A ranked graph (for fixed $d$ ) is a triple $(H, L, \preceq)$ where:

- $H$ is a graph,
- $L: V(H) \rightarrow[0, d-1]$ is a function,
- $\preceq$ is a partial order on $V(H)$ such that $L(v)<L(w)$ whenever $v \prec w$.

Here and throughout this proof, $v \prec w$ means that $v \preceq w$ and $v \neq w$. Up to isomorphism, the number of ranked graphs on $n$ vertices is at most $2\binom{n}{2} d^{n} 3^{\binom{n}{2}}$. For a vertex $v$ of $T$, a ranked graph $(H, L, \preceq)$ is said to be contained in $G\left[T_{v}^{+}\right]$if there is an isomorphism $\phi$ from $H$ to some subgraph of $G\left[T_{v}^{+}\right]$such that:
(A) for each vertex $v \in V(H)$ we have $L(v)=\ell(\phi(v))$, and
(B) for all distinct vertices $v, w \in V(H)$ we have that $v \prec w$ if and only if $\phi(v)$ is an ancestor of $\phi(w)$ in $T$.

Say $(H, L, \preceq)$ is a ranked graph and $i \in[0, d-1]$. Below we define the $i$-splice of ( $H, L, \preceq$ ) to be a particular ranked graph $\left(H^{\prime}, L^{\prime}, \preceq^{\prime}\right)$, which (intuitively speaking) is obtained from ( $H, L, \preceq$ ) by copying $k$ times the subgraph of $H$ induced by the vertices $v$ with $L(v)>i$. Formally, let

$$
\begin{aligned}
& V\left(H^{\prime}\right):=\{(v, 0): v \in V(H), L(v) \in[0, i]\} \cup \\
&\{(v, j): v \in V(H), L(v) \in[i+1, d], j \in[1, k]\} \\
& E\left(H^{\prime}\right):=\{(v, 0)(w, 0): v w \in E(H), L(v) \in[0, i], L(w) \in[0, i]\} \cup \\
&\{(v, 0)(w, j): v w \in E(H), L(v) \in[0, i], L(w) \in[i+1, d], j \in[1, k]\} \cup \\
&\{(v, j)(w, j): v w \in E(H), L(v) \in[i+1, d], L(w) \in[i+1, d], j \in[1, k]\} .
\end{aligned}
$$

Define $L^{\prime}((v, j)):=L(v)$ for every vertex $(v, j) \in V\left(H^{\prime}\right)$. Now define the following partial order $\preceq^{\prime}$ on $V\left(H^{\prime}\right)$ :

- $(v, j) \preceq^{\prime}(v, j)$ for all $(v, j) \in V\left(H^{\prime}\right)$;
- if $v \prec w$ and $L(v), L(w) \in[0, i]$, then $(v, 0) \prec^{\prime}(w, 0)$;
- if $v \prec w$ and $L(v) \in[0, i]$ and $L(w) \in[i+1, d]$, then $(v, 0) \prec^{\prime}(w, j)$ for all $j \in[1, k]$; and
- if $v \prec w$ and $L(v), L(w) \in[i+1, d]$, then $(v, j) \prec^{\prime}(w, j)$ for all $j \in[1, k]$.

Note that if $(v, a) \prec^{\prime}(w, b)$, then $a \leqslant b$ and $v \prec w$ (implying $(L(v)<L(w)$ ). It follows that $\prec^{\prime}$ is a partial order on $V\left(H^{\prime}\right)$ such that $L^{\prime}((v, a))<L^{\prime}((w, b))$ whenever $(v, a) \prec^{\prime}(w, b)$. Thus $\left(H^{\prime}, L^{\prime}, \preceq^{\prime}\right)$ is a ranked graph.

For $\ell \in[0, d-1]$, let

$$
N_{\ell}:=(d+1)(h-1)(k+1)^{d-1-\ell} .
$$

For each vertex $v$ of $T$, define the profile of $v$ to be the set of all ranked graphs ( $H, L, \preceq$ ) contained in $G\left[T_{v}^{+}\right]$such that $|V(H)| \leqslant N_{\ell(v)}$. Note that if $v$ is a descendant of $u$, then the profile of $v$ is a subset of the profile of $u$. For $\ell \in[0, d-1]$, if $N=N_{\ell}$ then let

$$
M_{\ell}:=2^{2^{\binom{N}{2}} d^{N} 3^{N}\binom{N}{2}} .
$$

Then there are at most $M_{\ell}$ possible profiles of a vertex at level $\ell$.
We now partition $V(T)$ into subtrees. Each subtree is called a group. (At the end of the proof, vertices in a single group will be assigned the same colour.) We assign vertices to groups in non-increasing order of their distance from the root. Initialise this process by placing each leaf $v$ of $T$ into a singleton group. We now show how to determine the group of a non-leaf vertex. Let $v$ be a vertex not assigned to a group at maximum distance from $r$. So each child of $v$ is assigned to a group. Let $Y_{v}$ be the set of children $y$ of $v$, such that the number of children of $v$ that have the same profile as $y$ is in the range $[1, k-1]$. If $Y_{v}=\emptyset$ start a new singleton group $\{v\}$. If $Y_{v} \neq \emptyset$ then merge all the groups rooted at vertices in $Y_{v}$ into one group including $v$. This defines our partition of $V(T)$ into groups. Each group $X$ is rooted at the vertex in $X$ closest to $r$ in $T$. A group $Y$ is above a distinct group $X$ if the root of $Y$ is on the path in $T$ from the root of $X$ to $r$.

The next claim is the key to the remainder of the proof.
Claim 1. Let $u v \in E(T)$ where $u$ is the parent of $v$, and $u$ is in a different group to $v$. Then for every ranked graph $(H, L, \preceq)$ in the profile of $v$, the $\ell(u)$-splice of $(H, L, \preceq)$ is in the profile of $u$.

Proof. Since $(H, L, \preceq)$ is in the profile of $v$, there is an isomorphism $\phi$ from $H$ to some subgraph of $G\left[T_{v}^{+}\right]$such that for each vertex $x \in V(H)$ we have $L(x)=\ell(\phi(x))$, and for all distinct vertices $x, y \in V(H)$ we have that $x \prec y$ if and only if $\phi(x)$ is an ancestor of $\phi(y)$ in $T$.

Since $u$ and $v$ are in different groups, there are $k$ children $y_{1}, \ldots, y_{k}$ of $u$ (one of which is $v$ ) such that the profiles of $y_{1}, \ldots, y_{k}$ are equal. Thus $(H, L, \preceq)$ is in the profile of each of $y_{1}, \ldots, y_{k}$. That is, for each $j \in[1, k]$, there is an isomorphism $\phi_{j}$ from $H$ to some subgraph of $G\left[T_{y_{j}}^{+}\right]$such that for each vertex $x \in V(H)$ we have $L(x)=\ell\left(\phi_{j}(x)\right)$, and for all distinct vertices $x, y \in V(H)$ we have that $x \prec y$ if and only if $\phi_{j}(x)$ is an ancestor of $\phi_{j}(y)$ in $T$.

Let ( $H^{\prime}, L^{\prime}, \preceq^{\prime}$ ) be the $\ell(u)$-splice of ( $H, L, \preceq$ ). We now define a function $\phi^{\prime}$ from $V\left(H^{\prime}\right)$ to $V\left(G\left[T_{u}^{+}\right]\right)$. For each vertex $(x, 0)$ of $H^{\prime}$ (thus with $x \in V(H)$ and $L(x) \in[0, \ell(u)]$ ), define $\phi^{\prime}((x, 0)):=\phi(x)$. For every other vertex $(x, j)$ of $H^{\prime}$ (thus with $x \in V(H)$ and $L(x) \in[\ell(u)+1, d-1]$ and $j \in[1, k])$, define $\phi^{\prime}((x, j)):=\phi_{j}(x)$.
We now show that $\phi^{\prime}$ is an isomorphism from $H^{\prime}$ to a subgraph of $G\left[T_{u}^{+}\right]$. Consider an edge $(x, a)(y, b)$ of $H^{\prime}$. Thus $x y \in E(H)$. It suffices to show that $\phi^{\prime}((x, a)) \phi^{\prime}((y, b)) \in$
$E\left(G\left[T_{u}^{+}\right]\right)$. First suppose that $a=b=0$. So $L(x) \in[0, \ell(u)]$ and $L(y) \in[0, \ell(u)]$. Thus $\phi^{\prime}((x, a))=\phi(x)$ and $\phi^{\prime}((y, b))=\phi(y)$. Since $\phi$ is an isomorphism to a subgraph of $G\left[T_{v}^{+}\right]$, we have $\phi(x) \phi(y) \in E\left(G\left[T_{v}^{+}\right]\right)$, which is a subgraph of $G\left[T_{u}^{+}\right]$. Hence $\phi^{\prime}((x, a)) \phi^{\prime}((y, b)) \in E\left(G\left[T_{u}^{+}\right]\right)$, as desired. Now suppose that $a=0$ and $b \in[1, k]$. Thus $\phi^{\prime}((x, a))=\phi(x)$ and $\phi^{\prime}((y, b))=\phi_{b}(y)$. Moreover, both $\ell(\phi(x))$ and $\ell\left(\phi_{b}(x)\right)$ equal $L(x) \in[0, \ell(u)]$. There is only vertex $z$ in $T_{v}^{+}$with $\ell(z)$ equal to a specific number in $[0, \ell(u)]$. Thus $\phi^{\prime}((x, a))=\phi(x)=\phi_{b}(x)(=z)$. Since $\phi_{b}$ is an isomorphism to a subgraph of $G\left[T_{y_{b}}^{+}\right]$, we have $\phi_{b}(x) \phi_{b}(y) \in E\left(G\left[T_{y_{b}}^{+}\right)\right.$, which is a subgraph of $G\left[T_{u}^{+}\right]$. Hence $\phi^{\prime}((x, a)) \phi^{\prime}((y, b)) \in E\left(G\left[T_{u}^{+}\right]\right)$, as desired. Finally, suppose that $a=b \in[1, k]$. Thus $\phi^{\prime}((x, a))=\phi_{a}(x)$ and $\phi^{\prime}((y, b))=\phi_{b}(y)=\phi_{a}(y)$. Since $\phi_{a}$ is an isomorphism to a subgraph of $G\left[T_{y_{a}}^{+}\right]$, we have $\phi_{a}(x) \phi_{a}(y) \in E\left(G\left[T_{y_{a}}^{+}\right]\right)$, which is a subgraph of $G\left[T_{u}^{+}\right]$. Hence $\phi^{\prime}((x, a)) \phi^{\prime}((y, b)) \in E\left(G\left[T_{u}^{+}\right]\right)$, as desired. This shows that $\phi^{\prime}$ is an isomorphism from $H^{\prime}$ to a subgraph of $G\left[T_{u}^{+}\right]$.

We now verify property (A) for $\left(H^{\prime}, L^{\prime}, \preceq^{\prime}\right)$. For each vertex $(x, 0)$ of $H^{\prime}$ (thus with $x \in V(H)$ and $L(x) \in[0, \ell(u)])$ we have $L^{\prime}((x, 0))=L(x)=\ell(\phi(x))=\ell\left(\phi^{\prime}((x, 0))\right)$, as desired. For every other vertex $(x, j)$ of $H^{\prime}$ (thus with $x \in V(H)$ and $L(x) \in$ $[\ell(u)+1, d-1]$ and $j \in[1, k])$ we have $L^{\prime}((x, j))=L(x)=\ell\left(\phi_{j}(x)\right)=\ell\left(\phi^{\prime}((x, j))\right)$, as desired. Hence property $(\mathrm{A})$ is satisfied for $\left(H^{\prime}, L^{\prime}, \preceq^{\prime}\right)$.

We now verify property (B) for ( $\left.H^{\prime}, L^{\prime}, \preceq^{\prime}\right)$. Consider distinct vertices $(x, a),(y, b) \in$ $V\left(H^{\prime}\right)$. First suppose that $a=0$ and $b=0$. Then $(x, a) \prec^{\prime}(y, b)$ if and only if $x \prec y$ if and only if $\phi(x)$ is an ancestor of $\phi(y)$ in $T$ if and only if $\phi^{\prime}((x, a))$ is an ancestor of $\phi^{\prime}((y, b))$ in $T$, as desired. Now suppose that $a=0$ and $b \in[1, k]$. Then $(x, a) \prec^{\prime}(y, b)$ if and only if $x \prec y$ if and only if $\phi(x)$ is an ancestor of $\phi_{b}(y)$ in $T$ if and only if $\phi^{\prime}((x, a))$ is an ancestor of $\phi^{\prime}((y, b))$ in $T$, as desired. Now suppose that $a=b \in[1, k]$. Then $(x, a) \prec^{\prime}(y, b)$ if and only if $x \prec y$ if and only if $\phi_{a}(x)$ is an ancestor of $\phi_{b}(y)$ in $T$ if and only if $\phi^{\prime}((x, a))$ is an ancestor of $\phi^{\prime}((y, b))$ in $T$, as desired. Finally, suppose that $a, b \in[1, k]$ and $a \neq b$. Then $(x, a)$ and $(y, b)$ are incomparable under $\prec^{\prime}$, and $\phi^{\prime}((x, a))$ and $\phi^{\prime}((y, b))$ in $T$ are unrelated in $T$, as desired. Hence property $(\mathrm{B})$ is satisfied for ( $H^{\prime}, L^{\prime}, \preceq^{\prime}$ ).

So $\phi^{\prime}$ is an isomorphism from $H^{\prime}$ to a subgraph of $G\left[T_{u}^{+}\right]$satisfying properties (A) and (B). Thus ( $H^{\prime}, L^{\prime}, \preceq^{\prime}$ ) is contained in $G\left[T_{u}^{+}\right]$, as desired. Since $(H, L, \preceq)$ is in the profile of $v$, we have $|V(H)| \leqslant(d+1)(h-1)(k+1)^{h-\ell(v)}$. Since $\left|V\left(H^{\prime}\right)\right| \leqslant(k+1)|V(H)|$ and $\ell(u)=$ $\ell(v)-1$, we have $\left|V\left(H^{\prime}\right)\right| \leqslant(d+1)(h-1)(k+1)^{h+1-\ell(v)}=(d+1)(h-1)(k+1)^{h-\ell(u)}$. Thus ( $H^{\prime}, L^{\prime}, \preceq^{\prime}$ ) is in the profile of $u$.

The proof now divides into two cases. If some group $X_{0}$ is adjacent in $G$ to at least $h-1$ other groups above $X_{0}$, then we show that $G$ contains $W\langle h, k\rangle$ as a minor. Otherwise, every group $X$ is adjacent in $G$ to at most $h-2$ other groups above $X$, in which case we show that $G$ is $(h-1)$-colourable with bounded clustering.

## Finding the Minor

Suppose that some group $X_{0}$ is adjacent in $G$ to at least $h-1$ other groups $X_{1}, \ldots, X_{h-1}$ above $X_{0}$. We now show that $G$ contains $W\langle h, k\rangle$ as a minor; refer to Figure 3. For $i \in[1, h-1]$, since $X_{i}$ is above $X_{0}$, the root $v_{i}$ of $X_{i}$ is on the $v_{0} r$-path in $T$. Without loss of generality, $v_{0}, v_{1}, \ldots, v_{h-1}$ appear in this order on the $v_{0} r$-path in $T$. For $i \in[1, h-1]$, let $w_{i}$ be a vertex in $X_{i}$ adjacent to some vertex $z_{i}$ in $X_{0}$; since $G$ is a subgraph of the closure of $T, w_{i}$ is on the $v_{0} r$-path in $T$. For $i \in[0, h-2]$, let $u_{i}$ be the parent of $v_{i}$ in $T$ (which exists since $v_{h-2} \neq r$ ). So $u_{i}$ is not in $X_{i}$ (but may be in $X_{i+1}$ ). Note that $v_{0}, u_{0}, w_{1}, v_{1}, u_{1}, \ldots, w_{h-2}, v_{h-2}, u_{h-2}, w_{h-1}, v_{h-1}$ appear in this order on the $v_{0} r$-path in $T$, where $v_{0}, v_{1}, \ldots, v_{h-1}$ are distinct (since they are in distinct groups).


Figure 3: Construction of a $W\langle 4, k\rangle$ minor (where $u_{i}$ might be in $X_{i+1}$ ).
Let $P_{j}$ be the $z_{j} r$-path in $T$ for $j \in[1, h-1]$. Let $H_{0}$ be the graph with $V\left(H_{0}\right):=$ $V\left(P_{1} \cup \cdots \cup P_{h-1}\right)$ and $E\left(H_{0}\right):=\left\{z_{j} w_{j}: j \in[1, h-1]\right\}$. Define the function $L_{0}:$ $V\left(H_{0}\right) \rightarrow[0, d-1]$ by $L_{0}(x):=\ell(x)$ for each $x \in V\left(H_{0}\right)$. Define the partial order $\preceq_{0}$ on $V\left(H_{0}\right)$, where $x \prec_{0} y$ if and only if $x$ is ancestor of $y$ in $T$. Thus ( $H_{0}, L_{0}, \preceq_{0}$ ) is a ranked graph. By construction, $\left(H_{0}, L_{0}, \preceq_{0}\right)$ is contained in $G\left[T_{v_{0}}^{+}\right]$. Since $H_{0}$ has less than $(d+1)(h-1)$ vertices, $H_{0}$ is in the profile of $v_{0}$. For $i=0,1, \ldots, h-2$, let ( $H_{i+1}, L_{i+1}, \prec_{i+1}$ ) be the $\ell\left(u_{i}\right)$-splice of ( $H_{i}, L_{i}, \prec_{i}$ ).

By induction on $i$, using Claim 1 at each step and since $G\left[T_{u_{i}}^{+}\right] \subseteq G\left[T_{v_{i+1}}^{+}\right]$, we conclude that for each $i \in[0, h-1]$, the ranked graph $\left(H_{i}, L_{i}, \preceq_{i}\right)$ is in the profile of $v_{i}$. In particular, $\left(H_{h-1}, L_{h-1}, \prec_{h-1}\right)$ is in the profile of $v_{h-1}$, and $H_{h-1}$ is isomorphic to a subgraph
of $G$. Note that each vertex of $H_{h-1}$ is of the form $\left(\left(\left(\ldots\left(x, d_{1}\right), d_{2}\right), \ldots\right), d_{h-1}\right)$ for some $x \in V\left(H_{0}\right)$ and $d_{1}, \ldots, d_{h-1} \in[0, k]$. For brevity, call such a vertex $x\left\langle d_{1}, \ldots, d_{h-1}\right\rangle$. Note that if $x=w_{j}$ for some $j \in[1, h-1]$, then $d_{1}=\cdots=d_{j}=0$ (since $w_{j}$ is above $u_{i}$ whenever $i<j$, and ( $H_{i+1}, L_{i+1}, \prec_{i+1}$ ) is the $\ell\left(u_{i}\right)$-splice of $\left(H_{i}, L_{i}, \preceq_{i}\right)$ ).

For $x \in V\left(H_{0}\right)$, let $\Lambda_{x}$ be the set of vertices $x\left\langle d_{1}, \ldots, d_{h-1}\right\rangle$ in $H_{h-1}$. By construction, no two vertices in $\Lambda_{x}$ are comparable under $\preceq_{h-1}$. Therefore, by property (B), $V\left(T_{a}\right) \cap$ $V\left(T_{b}\right)=\emptyset$ for all distinct $a, b \in \Lambda_{x}$. In particular, $V\left(T_{a}\right) \cap V\left(T_{b}\right)=\emptyset$ for all distinct $a, b \in \Lambda_{v_{0}}$. As proved above, $G\left[T_{a}\right]$ is connected for each $a \in V(T)$. Let $G^{\prime}$ be the graph obtained from $G$ by contracting $G\left[T_{a}\right]$ into a single vertex $\alpha\left\langle d_{1}, \ldots, d_{h-1}\right\rangle$, for each $a=v_{0}\left\langle d_{1}, \ldots, d_{h-1}\right\rangle \in \Lambda_{v_{0}}$. So $G^{\prime}$ is a minor of $G$.

Let $U$ be the tree with vertex set

$$
\left\{\left\langle d_{1}, \ldots, d_{h-1}\right\rangle: \exists j \in[0, h-1] d_{1}=\cdots=d_{j}=0 \text { and } d_{j+1}, \ldots, d_{h-1} \in[1, k]\right\}
$$

where the parent of $\left(0, \ldots, 0, d_{j+1}, d_{j+2}, \ldots, d_{h-1}\right)$ is $\left(0, \ldots, 0, d_{j+2}, \ldots, d_{h-1}\right)$. Then $U$ is isomorphic to the complete $k$-tree of height $h$ rooted at $\langle 0, \ldots, 0\rangle$. We now show that the weak closure of $U$ is a subgraph of $G^{\prime}$, where each vertex $\left\langle 0, \ldots, 0, d_{j+1}, \ldots, d_{h-1}\right\rangle$ of $U$ with $j \in[1, h-1]$ is mapped to vertex $w_{j}\left\langle 0, \ldots, 0, d_{j+1}, \ldots, d_{h-1}\right\rangle$ of $G^{\prime}$, and each other vertex $\left\langle d_{1}, \ldots, d_{h-1}\right\rangle$ of $U$ is mapped to $\alpha\left\langle d_{1}, \ldots, d_{h-1}\right\rangle$ of $G^{\prime}$. For all $d_{1}, \ldots, d_{h-1} \in[1, k]$ and $j \in[1, h-1]$ the vertex $z_{j}\left\langle d_{1}, \ldots, d_{h-1}\right\rangle$ of $G$ is contracted into the vertex $\alpha\left\langle d_{1}, \ldots, d_{h-1}\right\rangle$ of $G^{\prime}$. By construction, $z_{j}\left\langle d_{1}, \ldots, d_{h-1}\right\rangle$ is adjacent to $w_{j}\left\langle 0, \ldots, 0, d_{j+1}, \ldots, d_{h-1}\right\rangle$ in $G$. So $\alpha\left\langle d_{1}, \ldots, d_{h-1}\right\rangle$ is adjacent to $w_{j}\left\langle 0, \ldots, 0, d_{j+1}, \ldots, d_{h-1}\right\rangle$ in $G^{\prime}$. This implies that the weak closure of $U$ (that is, $W\langle h, k\rangle$ ) is isomorphic to a subgraph of $G^{\prime}$, and is therefore a minor of $G$.

## Finding the Colouring

Now assume that every group $X$ is adjacent in $G$ to at most $h-2$ other groups above $X$. Then $(h-1)$-colour the groups in order of distance from the root, such that every group $X$ is assigned a colour different from the colours assigned to the neighbouring groups above $X$. Assign each vertex within a group the same colour as that assigned to the whole group. This defines an $(h-1)$-colouring of $G$.

Consider the function $s:[0, d-1] \rightarrow \mathbb{N}$ recursively defined by

$$
s(\ell):= \begin{cases}1 & \text { if } \ell=d-1 \\ (k-1) \cdot M_{\ell+1} \cdot s(\ell+1) & \text { if } \ell \in[0, d-2] .\end{cases}
$$

Then every group at level $\ell$ has at most $s(\ell)$ vertices. By construction, our $(h-1)$ colouring of $G$ has clustering $s(0)$, which is bounded by a function of $d, k$ and $h$, as desired.

## 3 Pathwidth

The following lemma of independent interest is the key to proving Theorem 2. Note that Eppstein [11] independently discovered the same result (with a slightly weaker bound
on the path length). The decomposition method in the proof has been previously used, for example, by Dujmović, Joret, Kozik, and Wood [7, Lemma 17].

Lemma 9. Every graph with pathwidth at most $w$ has a vertex 2-colouring such that each monochromatic path has at most $(w+3)^{w}$ vertices.

Proof. We proceed by induction on $w \geqslant 1$. Every graph with pathwidth 1 is a caterpillar, and is thus properly 2 -colourable. Now assume $w \geqslant 2$ and the result holds for graphs with pathwidth at most $w-1$. Let $G$ be a graph with pathwidth at most $w$. Let $\left(B_{1}, \ldots, B_{n}\right)$ be a path-decomposition of $G$ with width at most $w$. Let $t_{1}, t_{2}, \ldots, t_{m}$ be a maximal sequence such that $t_{1}=1$ and for each $i \geqslant 2, t_{i}$ is the minimum integer such that $B_{t_{i}} \cap B_{t_{i-1}}=\emptyset$. For odd $i$, colour every vertex in $B_{t_{i}}$ 'red'. For even $i$, colour every vertex in $B_{t_{i}}$ 'blue'. Since $B_{t_{i}} \cap B_{t_{i-1}}=\emptyset$ for $i \geqslant 2$, no vertex is coloured twice. Let $G^{\prime}$ be the subgraph of $G$ induced by the uncoloured vertices. By the choice of $B_{t_{i}}$, for $i \geqslant 2$ each bag $B_{j}$ with $j \in\left[t_{i-1}+1, t_{i}-1\right]$ intersects $B_{t_{i-1}}$. Thus $\left(B_{1} \cap V\left(G^{\prime}\right), \ldots, B_{n} \cap V\left(G^{\prime}\right)\right)$ is a path-decomposition of $G^{\prime}$ of width at most $w-1$. By induction, $G^{\prime}$ has a vertex 2-colouring such that each monochromatic path has at most $(w+3)^{w-1}$ vertices. Since $B_{t_{i}} \cup B_{t_{i+2}}$ separates $B_{t_{i}+1} \cup \cdots \cup B_{t_{i+2}-1}$ from the rest of $G$, each monochromatic component of $G$ is contained in $B_{t_{i}+1} \cup \cdots \cup B_{t_{i+2}-1}$ for some $i \in[0, n-2]$. Consider a monochromatic path $P$ in $G\left[B_{t_{i}+1} \cup \cdots \cup B_{t_{i+2}-1}\right]$. Then $P$ has at most $w+1$ vertices in $B_{t_{i+1}}$. Note that $P-B_{t_{i+1}}$ is contained in $G^{\prime}$. Thus $P$ consists of up to $w+2$ monochromatic subpaths in $G^{\prime}$ plus $w+1$ vertices in $B_{t_{i+1}}$. Hence $P$ has at most $(w+2)(w+3)^{w-1}+(w+1)<(w+3)^{w}$ vertices.

Nešetril and Ossona de Mendez [22] showed that if a graph $G$ contains no path on $k$ vertices, then $\operatorname{td}(G)<k$ (since $G$ is a subgraph of the closure of a DFS spanning tree with height at most $k$ ). Thus Lemma 9 implies:
Corollary 10. Every graph with pathwidth at most $w$ has a vertex 2-colouring such that each monochromatic component has treedepth at most $(w+3)^{w}$.

Proof of Theorem 2. Let $\mathscr{G}$ be a minor-closed class of graphs, each with pathwidth at most $w$. Let $h$ be the minimum integer such that $C\langle h, k\rangle \notin \mathscr{G}$ for some $k \in \mathbb{N}$. Consider $G \in \mathscr{G}$. Thus $W\langle h, k+1\rangle$ is not a minor of $G$ (since $C\langle h, k\rangle$ is a minor of $W\langle h, k+1\rangle$, as noted above). By Corollary 10, $G$ has a vertex 2-colouring such that each monochromatic component $H$ of $G$ has treedepth at most $(w+3)^{w}$. Thus $W\langle h, k+1\rangle$ is not a minor of $H$. By Lemma $8, H$ is $(h-1)$-colourable with clustering $c\left((w+3)^{w}, k+1, h\right)$. Taking a product colouring, $G$ is $(2 h-2)$-colourable with clustering $c\left((w+3)^{w}, k+1, h\right)$. Hence $\chi_{\Delta}\left(\varphi_{\varphi}\right) \leqslant \chi_{\star}(\varphi) \leqslant 2 h-2$.

Note that Lemma 9 cannot be extended to the setting of bounded tree-width graphs: Esperet and Joret (see [17, Theorem 4.1]) proved that for all positive integers $w$ and $d$ there exists a graph $G$ with tree-width at most $w$ such that for every $w$-colouring of $G$ there exists a monochromatic component of $G$ with diameter greater than $d$ (and thus with a monochromatic path on more than $d$ vertices, and thus with treedepth at least $\log _{2} d$ ).

## 4 Fractional Colouring

This section proves Theorem 6. The starting point is the following key result of Dvorák and Sereni $[9] .{ }^{2}$

Theorem 11 ([9]). For every proper minor-closed class $\mathscr{G}$ and every $\delta>0$ there exists $d \in \mathbb{N}$ satisfying the following. For every $G \in \mathscr{G}$ there exist $s \in \mathbb{N}$ and $X_{1}, X_{2}, \ldots, X_{s} \subseteq$ $V(G)$ such that:

- $\operatorname{td}\left(G\left[X_{i}\right]\right) \leqslant d$, and
- every $v \in V(G)$ belongs to at least $(1-\delta)$ s of these sets.

We now prove a lower bound on the fractional defective chromatic number of the closure of complete trees of given height.

Lemma 12. Let $\mathscr{C}_{h}:=\{C\langle h, k\rangle\}_{k \in \mathbb{N}}$. Then $\chi_{\Delta}^{f}\left(\mathscr{C}_{h}\right) \geqslant h$.
Proof. We show by induction on $h$ that if $C\langle h, k\rangle$ is fractionally $t$-colourable with defect $d$, then $t \geqslant h-(h-1) d / k$. This clearly implies the lemma. The base case $h=1$ is trivial.

For the induction step, suppose that $G:=C\langle h, k\rangle$ is fractionally $t$-colourable with defect $d$. Thus there exist $Y_{1}, Y_{2}, \ldots, Y_{s} \subseteq V(G)$ and $\alpha_{1}, \ldots, \alpha_{s} \in[0,1]$ such that:

- every component of $G\left[Y_{i}\right]$ has maximum degree at most $d$,
- $\sum_{i=1}^{s} \alpha_{i} \leqslant t$, and
- $\sum_{i: v \in Y_{i}} \alpha_{i} \geqslant 1$ for every $v \in V(G)$.

Let $r$ be the vertex of $G$ corresponding to the root of the complete $k$-ary tree and let $H_{1}, \ldots, H_{k}$ be the components of $G-r$. Then each $H_{i}$ is isomorphic to $C\langle h-1, k\rangle$. Let $J_{0}:=\left\{j: r \in Y_{j}\right\}$, and let $J_{i}:=\left\{j: Y_{j} \cap V\left(H_{i}\right) \neq \emptyset\right\}$ for $i \in[1, k]$. Denote $\sum_{j \in J_{i}} \alpha_{j}$ by $\alpha\left(J_{i}\right)$ for brevity. Thus $\alpha\left(J_{0}\right) \geqslant 1$. For $i \in[1, k]$, the subgraph $H_{i}$ is $\alpha\left(J_{i}\right)$-colourable with defect $d$, and thus $\alpha\left(J_{i}\right) \geqslant h-1-(h-2) d / k$ by the induction hypothesis. Thus

$$
(k-d) \alpha\left(J_{0}\right)+\sum_{i=1}^{k} \alpha\left(J_{i}\right) \geqslant(k-d)+k(h-1)-(h-2) d=k h-(h-1) d .
$$

If $j \in J_{0}$ then $Y_{j}$ intersects at most $d$ of $H_{1}, \ldots, H_{k}$ (since $G\left[Y_{j}\right]$ has maximum degree at most $d$ ). Thus every $\alpha_{j}$ appears with coefficient at most $k$ in the left side of the above inequality, implying

$$
(k-d) \alpha\left(J_{0}\right)+\sum_{i=1}^{k} \alpha\left(J_{i}\right) \leqslant k \sum_{i=1}^{s} \alpha_{i} \leqslant k t .
$$

Combining the above inequalities yields the claimed bound on $t$.

[^2]Proof of Theorem 6. By Lemma 12,

$$
\chi_{\star}^{f}(\mathscr{\varphi}) \geqslant \chi_{\Delta}^{f}(\varphi) \geqslant \operatorname{tcn}(\varphi)-1 .
$$

It remains to show that $\chi_{\star}^{f}(\mathscr{\varphi}) \leqslant \operatorname{tcn}(\mathscr{\varphi})-1$. Equivalently, we need to show that for all $h, k \in \mathbb{N}$ and $\varepsilon>0$, if $C\langle h, k\rangle \notin \mathscr{G}$ then there exists $c$ such that every graph in $\mathscr{G}$ is fractionally $(h-1+\varepsilon)$-colourable with clustering $c$. This is trivial for $h=1$, and so we assume $h \geqslant 2$.

Let $d \in \mathbb{N}$ satisfy the conclusion of Theorem 11 for the class $\mathscr{G}$ and $\delta=1-\frac{1}{1+\varepsilon /(h-1)}$. Choose $c=c(d, k+1, h)$ to satisfy the conclusion of Lemma 8 . We show that $c$ is as desired.

Consider $G \in \mathscr{G}$. By the choice of $d$ there exists $s \in \mathbb{N}$ and $X_{1}, X_{2}, \ldots, X_{s} \subseteq V(G)$ such that:

- $\operatorname{td}\left(G\left[X_{i}\right]\right) \leqslant d$, and
- every $v \in V(G)$ belongs to at least $(1-\delta) s$ of these sets.

Since $C\langle h, k\rangle \notin \mathscr{G}$, we have $W\langle h, k+1\rangle \notin \mathscr{G}$, and by the choice of $c$, for each $i \in[1, s]$ there exists a partition $\left(Y_{i}^{1}, Y_{i}^{2}, \ldots, Y_{i}^{h-1}\right)$ of $X_{i}$ such that every component of $G\left[Y_{i}^{j}\right]$ has at most $c$ vertices. Every vertex of $G$ belongs to at least $(1-\delta) s$ sets $Y_{i}^{j}$ where $i \in[1, s]$ and $j \in[1, h-1]$. Considering these sets with equal coefficients $\alpha_{i}^{j}:=\frac{1}{(1-\delta) s}$, we conclude that $G$ is fractionally $\frac{h-1}{1-\delta}$-colourable with clustering $c$, as desired (since $\left.\frac{h-1}{1-\delta}=h-1+\varepsilon\right)$.

## Acknowledgement

This work was partially completed while SN was visiting Monash University supported by a Robert Bartnik Visiting Fellowship. SN thanks the School of Mathematics at Monash University for its hospitality. Thanks to the referee for several helpful comments.

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[^0]:    January 24, 2022
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[^1]:    ${ }^{1}$ If $c=1$, then this corresponds to a (proper) fractional $t$-colouring, and if the $\alpha_{i}$ are integral, then this yields a $t$-colouring with clustering $c$.

[^2]:    ${ }^{2}$ Dvořák and Sereni [9] expressed their result in the terms of "treedepth fragility". The sentence "proper minor-closed classes are fractionally treedepth-fragile" after Theorem 31 in [9] is equivalent to Theorem 11. Informally speaking, Theorem 11 shows that the fractional "treedepth" chromatic number of every minor-closed class equals 1 .

