CLUSTERED COLOURING OF GRAPH CLASSES WITH BOUNDED TREEDEPTH OR PATHWIDTH

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Abstract. The clustered chromatic number of a class of graphs is the minimum integer k such that for some integer c every graph in the class is k-colourable with monochromatic components of size at most c. We determine the clustered chromatic number of any minor-closed class with bounded treedepth, and prove a best possible upper bound on the clustered chromatic number of any minor-closed class with bounded pathwidth. As a consequence, we determine the fractional clustered chromatic number of every minor-closed class.

1 Introduction

This paper studies improper vertex colourings of graphs with bounded monochromatic degree or bounded monochromatic component size. This topic has been extensively studied recently [1–6, 8, 10, 12–21, 23–25]; see [26] for a survey.

A k-colouring of a graph G is a function that assigns one of k colours to each vertex of G. In a coloured graph, a *monochromatic component* is a connected component of the subgraph induced by all the vertices of one colour.

A colouring has *defect* d if each monochromatic component has maximum degree at most d. The *defective chromatic number* of a graph class \mathcal{G} , denoted by $\chi_{\Delta}(\mathcal{G})$, is the minimum integer k such that, for some integer d, every graph in \mathcal{G} is k-colourable with defect d.

A colouring has *clustering* c if each monochromatic component has at most c vertices. The *clustered chromatic number* of a graph class \mathcal{G} , denoted by $\chi_{\star}(\mathcal{G})$, is the minimum integer k such that, for some integer c, every graph in \mathcal{G} has a k-colouring with clustering c. We shall consider such colourings, where the goal is to minimise the number of colours, without optimising the clustering value.

Every colouring of a graph with clustering c has defect c-1. Thus $\chi_{\Delta}(\mathcal{G}) \leqslant \chi_{\star}(\mathcal{G})$ for every class \mathcal{G} .

The following is a well-known and important example in defective and clustered graph colouring. Let T be a rooted tree. The depth of T is the maximum number of vertices

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on a root-to-leaf path in T. The *closure* of T is obtained from T by adding an edge between every ancestor and descendant in T. For $h,k\geqslant 1$, let $C\langle h,k\rangle$ be the closure of the complete k-ary tree of depth h, as illustrated in Figure 1.

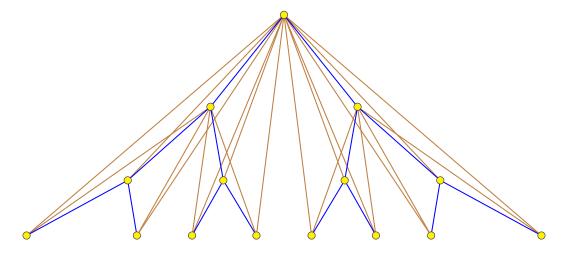


Figure 1: The standard example $C\langle 4,2\rangle$.

It is well known and easily proved (see [26]) that there is no (h-1)-colouring of $C\langle h,k\rangle$ with defect k-1, which implies there is no (h-1)-colouring of $C\langle h,k\rangle$ with clustering k. This says that if a graph class ${\mathscr G}$ includes $C\langle h,k\rangle$ for all k, then the defective chromatic number and the clustered chromatic number are at least k. Put another way, define the k-closure-number of a graph class ${\mathscr G}$ to be

$$tcn(\mathcal{G}) := \min\{h : \exists k \, C\langle h, k \rangle \not\in \mathcal{G}\} = \max\{h : \forall k \, C\langle h, k \rangle \in \mathcal{G}\} + 1;$$

then

$$\chi_{\star}(\mathcal{G}) \geqslant \chi_{\Delta}(\mathcal{G}) \geqslant \operatorname{tcn}(\mathcal{G}) - 1.$$

Our main result, Theorem 1 below, establishes a converse result for minor-closed classes with bounded treedepth. First we explain these terms. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from some subgraph of G by contracting edges. A class of graphs $\mathcal M$ is *minor-closed* if for every graph $G \in \mathcal M$ every minor of G is in $\mathcal M$, and $\mathcal M$ is *proper* minor-closed if, in addition, some graph is not in $\mathcal M$. The *connected treedepth* of a graph H, denoted by $\overline{\operatorname{td}}(H)$, is the minimum depth of a rooted tree G such that G is a subgraph of the closure of G. This definition is a variant of the more commonly used definition of the *treedepth* of G, denoted by G if G is a variant of the maximum connected treedepth of the connected components of G if G is a variant of the maximum connected treedepth of the connected components of G is a variant of the maximum connected treedepth of the connected components of G is a variant of the maximum connected treedepth of the connected components of G is a variant of the maximum connected treedepth. If G is connected, then G is an G is a variant of the maximum connected treedepth. If G is connected components of G is a variant of the maximum connected treedepth. If G is connected components of G is a variant of G is a variant of G is a variant of G in the class of graphs has bounded treedepth if there exists a constant G such that every graph in the class has treedepth at most G.

Theorem 1. For every minor-closed class \mathscr{G} with bounded treedepth,

$$\chi_{\Delta}(\mathcal{G}) = \chi_{\star}(\mathcal{G}) = \operatorname{tcn}(\mathcal{G}) - 1.$$

Our second result concerns pathwidth. A path-decomposition of a graph G consists of a sequence (B_1,\ldots,B_n) , where each B_i is a subset of V(G) called a bag, such that for every vertex $v\in V(G)$, the set $\{i\in [1,n]:v\in B_i\}$ is an interval, and for every edge $vw\in E(G)$ there is a bag B_i containing both v and w. Here $[a,b]:=\{a,a+1,\ldots,b\}$. The width of a path decomposition (B_1,\ldots,B_n) is $\max\{|B_i|:i\in [1,n]\}-1$. The pathwidth of a graph G is the minimum width of a path-decomposition of G. Note that paths (and more generally caterpillars) have pathwidth 1. A class of graphs has bounded pathwidth if there exists a constant c such that every graph in the class has pathwidth at most c.

Theorem 2. For every minor-closed class & with bounded pathwidth,

$$\chi_{\Delta}(\mathcal{G}) \leqslant \chi_{\star}(\mathcal{G}) \leqslant 2 \operatorname{tcn}(\mathcal{G}) - 2.$$

Theorems 1 and 2 are respectively proved in Sections 2 and 3. These results are best possible and partially resolve a number of conjectures from the literature, as we now explain.

Ossona de Mendez et al. [24] studied the defective chromatic number of minor-closed classes. For a graph H, let \mathcal{M}_H be the class of H-minor-free graphs (that is, not containing H as a minor). Ossona de Mendez et al. [24] proved the lower bound, $\chi_{\Delta}(\mathcal{M}_H) \geqslant \overline{\operatorname{td}}(H) - 1$ and conjectured that equality holds.

Conjecture 3 ([24]). For every graph H,

$$\chi_{\Delta}(\mathcal{M}_H) = \overline{\operatorname{td}}(H) - 1.$$

Conjecture 3 is known to hold in some special cases. Edwards et al. [10] proved it if $H=K_t$; that is, $\chi_{\Delta}(\mathcal{M}_{K_t})=t-1$, which can be thought of as a defective version of Hadwiger's Conjecture; see [25] for an improved bound on the defect in this case. Ossona de Mendez et al. [24] proved Conjecture 3 if $\overline{\operatorname{td}}(H) \leqslant 3$ or if H is a complete bipartite graph. In particular, $\chi_{\Delta}(\mathcal{M}_{K_{s,t}})=\min\{s,t\}$.

Norin et al. [23] studied the clustered chromatic number of minor-closed classes. They showed that for each $k \geqslant 2$, there is a graph H with treedepth k and connected treedepth k such that $\chi_{\star}(\mathcal{M}_H) \geqslant 2k-2$. Their proof in fact constructs a set \mathscr{X} of graphs in \mathcal{M}_H with bounded pathwidth (at most 2k-3 to be precise) such that $\chi_{\star}(\mathscr{X}) \geqslant 2k-2$. Thus the upper bound on $\chi_{\star}(\mathscr{G})$ in Theorem 2 is best possible.

Norin et al. [23] conjectured the following converse upper bound (analogous to Conjecture 3):

Conjecture 4 ([23]). For every graph H,

$$\chi_{\star}(\mathcal{M}_H) \leqslant 2 \, \overline{\operatorname{td}}(H) - 2.$$

While Conjectures 3 and 4 remain open, Norin et al. [23] showed in the following theorem that $\chi_{\Delta}(\mathcal{M}_H)$ and $\chi_{\star}(\mathcal{M}_H)$ are controlled by the treedepth of H:

Theorem 5 ([23]). For every graph H, $\chi_{\star}(\mathcal{M}_H)$ is tied to the (connected) treedepth of H. In particular,

$$\overline{\operatorname{td}}(H) - 1 \leqslant \chi_{\star}(\mathcal{M}_H) \leqslant 2^{\overline{\operatorname{td}}(H) + 1} - 4.$$

Theorem 1 gives a much more precise bound than Theorem 5 under the extra assumption of bounded treedepth.

Our third main result concerns fractional colourings. For real $t \ge 1$, a graph G is fractionally t-colourable with clustering c if there exist $Y_1, Y_2, \ldots, Y_s \subseteq V(G)$ and $\alpha_1, \ldots, \alpha_s \in [0, 1]$ such that 1:

- ullet Every component of $G[Y_i]$ has at most c vertices,
- $\sum_{i=1}^{s} \alpha_i \leqslant t$,
- $\sum_{i:v\in Y_i} \alpha_i \geqslant 1$ for every $v\in V(G)$.

The fractional clustered chromatic number $\chi_{\star}^f(\mathcal{G})$ of a graph class \mathcal{G} is the infimum of t>0 such that there exists $c=c(t,\mathcal{G})$ such that every $G\in\mathcal{G}$ is fractionally t-colourable with clustering c.

Fractionally t-colourable with defect d and fractional defective chromatic number $\chi_{\Delta}^f(\mathcal{G})$ are defined in exactly the same way, except the condition on the component size of $G[Y_i]$ is replaced by "the maximum degree of $G[Y_i]$ is at most d".

The following theorem determines the fractional clustered chromatic number and fractional defective chromatic number of any proper minor-closed class.

Theorem 6. For every proper minor-closed class \mathcal{G} ,

$$\chi^f_{\Delta}(\mathcal{G}) = \chi^f_{\star}(\mathcal{G}) = \mathrm{tcn}(\mathcal{G}) - 1.$$

This result is proved in Section 4.

We now give an interesting example of Theorem 6.

Corollary 7. For every surface Σ , if \mathscr{G}_{Σ} is the class of graphs embeddable in Σ , then

$$\chi_{\Delta}^{f}(\mathcal{G}_{\Sigma}) = \chi_{\star}^{f}(\mathcal{G}_{\Sigma}) = 3.$$

Proof. Note that $C\langle 3,k\rangle$ is planar for all k. Thus $\mathrm{tcn}(\mathscr{G}_{\Sigma})\geqslant 4$. Say Σ has Euler genus g. It follows from Euler's formula that $K_{3,2g+3}\not\in\mathscr{G}_{\Sigma}$. Since $K_{3,2g+3}\subseteq C\langle 4,2g+3\rangle$, we have $C\langle 4,2g+3\rangle\not\in\mathscr{G}_{\Sigma}$. Thus $\mathrm{tcn}(\mathscr{G}_{\Sigma})=4$. The result follows from Theorem 6.

In contrast to Corollary 7, Dvořák and Norin [8] proved that $\chi_{\star}(\mathcal{G}_{\Sigma}) = 4$. Note that Archdeacon [2] proved that $\chi_{\Delta}(\mathcal{G}_{\Sigma}) = 3$; see [5] for an improved bound on the defect.

¹ If c=1, then this corresponds to a (proper) fractional t-colouring, and if the α_i are integral, then this yields a t-colouring with clustering c.

2 Treedepth

Say G is a subgraph of the closure of some rooted tree T. For each vertex $v \in V(T)$, let T_v be the maximal subtree of T rooted at v (consisting of v and all its descendants), and let $G[T_v]$ be the subgraph of G induced by $V(T_v)$.

The *weak closure* of a rooted tree T is the graph G with vertex set V(T), where two vertices $v, w \in V(T)$ are adjacent in G whenever v is a leaf of T and w is an ancestor of v in T. As illustrated in Figure 2, let $W\langle h,k\rangle$ be the weak closure of the complete k-ary tree of height h.

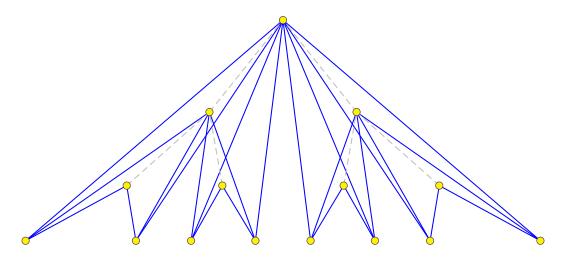


Figure 2: The weak closure $W\langle 4,2\rangle$.

Note that $W\langle h,k\rangle$ is a proper subgraph of $C\langle h,k\rangle$ for $h\geqslant 3$. On the other hand, Norin et al. [23] showed that $W\langle h,k\rangle$ contains $C\langle h,k-1\rangle$ as a minor for all $h,k\geqslant 2$. Therefore Theorem 1 is an immediate consequence of the following lemma.

Lemma 8. For all $d, k, h \in \mathbb{N}$ there exists $c = c(d, k, h) \in \mathbb{N}$ such that for every graph G with treedepth at most d, either G contains a $W\langle h, k \rangle$ -minor or G is (h-1)-colourable with clustering c.

Proof. Throughout this proof, d, k and h are fixed, and we make no attempt to optimise c

We may assume that G is connected. So G is a subgraph of the closure of some rooted tree of depth at most d. Choose a tree T of depth at most d rooted at some vertex r, such that G is a subgraph of the closure of T, and subject to this, $\sum_{v \in V(T)} \operatorname{dist}_T(v,r)$ is minimal. Suppose that $G[T_v]$ is disconnected for some vertex v in T. Choose such a vertex v at maximum distance from r. Since G is connected, $v \neq r$. By the choice of v, for each child w of v, the subgraph $G[T_w]$ is connected. Thus, for some child w of v, there is no edge in G joining v and $G[T_w]$. Let u be the parent of v. Let T' be obtained from T by deleting the edge vw and adding the edge uw, so that w is a child of u in T'. Note that G is a subgraph of the closure of T' (since v has no neighbour in $G[T_w]$). Moreover,

 $\operatorname{dist}_{T'}(x,r) = \operatorname{dist}_T(x,r) - 1$ for every vertex $x \in V(T_w)$, and $\operatorname{dist}_{T'}(y,r) = \operatorname{dist}_T(y,r)$ for every vertex $y \in V(T) \setminus V(T_w)$. Hence $\sum_{v \in V(T')} \operatorname{dist}_{T'}(v,r) < \sum_{v \in V(T)} \operatorname{dist}_T(v,r)$, which contradicts our choice of T. Therefore $G[T_v]$ is connected for every vertex v of T.

Consider each vertex $v \in V(T)$. Define the *level* $\ell(v) := \operatorname{dist}_T(r,v) \in [0,d-1]$. Let T_v^+ be the subtree of T consisting of T_v plus the vr-path in T, and let $G[T_v^+]$ be the subgraph of G induced by $V(T_v^+)$. For a subtree X of T rooted at vertex v, define the *level* $\ell(X) := \ell(v)$.

A *ranked graph* (for fixed d) is a triple (H, L, \preceq) where:

- H is a graph,
- ullet L:V(H)
 ightarrow [0,d-1] is a function,
- \leq is a partial order on V(H) such that L(v) < L(w) whenever $v \prec w$.

Here and throughout this proof, $v \prec w$ means that $v \preceq w$ and $v \neq w$. Up to isomorphism, the number of ranked graphs on n vertices is at most $2^{\binom{n}{2}} d^n \, 3^{\binom{n}{2}}$. For a vertex v of T, a ranked graph (H,L,\preceq) is said to be *contained in* $G[T_v^+]$ if there is an isomorphism ϕ from H to some subgraph of $G[T_v^+]$ such that:

- (A) for each vertex $v \in V(H)$ we have $L(v) = \ell(\phi(v))$, and
- (B) for all distinct vertices $v,w\in V(H)$ we have that $v\prec w$ if and only if $\phi(v)$ is an ancestor of $\phi(w)$ in T.

Say (H, L, \preceq) is a ranked graph and $i \in [0, d-1]$. Below we define the *i-splice* of (H, L, \preceq) to be a particular ranked graph (H', L', \preceq') , which (intuitively speaking) is obtained from (H, L, \preceq) by copying k times the subgraph of H induced by the vertices v with L(v) > i. Formally, let

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\begin{split} V(H') := & \{(v,0) : v \in V(H), L(v) \in [0,i]\} \cup \\ & \{(v,j) : v \in V(H), L(v) \in [i+1,d], j \in [1,k]\}. \\ E(H') := & \{(v,0)(w,0) : vw \in E(H), L(v) \in [0,i], L(w) \in [0,i]\} \cup \\ & \{(v,0)(w,j) : vw \in E(H), L(v) \in [0,i], L(w) \in [i+1,d], j \in [1,k]\} \cup \\ & \{(v,j)(w,j) : vw \in E(H), L(v) \in [i+1,d], L(w) \in [i+1,d], j \in [1,k]\}. \end{split}
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Define L'((v,j)) := L(v) for every vertex $(v,j) \in V(H')$. Now define the following partial order \leq' on V(H'):

- $(v,j) \leq' (v,j)$ for all $(v,j) \in V(H')$;
- if $v \prec w$ and $L(v), L(w) \in [0, i]$, then $(v, 0) \prec' (w, 0)$;
- if $v \prec w$ and $L(v) \in [0,i]$ and $L(w) \in [i+1,d]$, then $(v,0) \prec' (w,j)$ for all $j \in [1,k]$; and
- if $v \prec w$ and $L(v), L(w) \in [i+1,d]$, then $(v,j) \prec' (w,j)$ for all $j \in [1,k]$.

Note that if $(v,a) \prec' (w,b)$, then $a \leqslant b$ and $v \prec w$ (implying (L(v) < L(w))). It follows that \prec' is a partial order on V(H') such that L'((v,a)) < L'((w,b)) whenever $(v,a) \prec' (w,b)$. Thus (H',L',\preceq') is a ranked graph.

For
$$\ell \in [0, d-1]$$
, let

$$N_{\ell} := (d+1)(h-1)(k+1)^{d-1-\ell}$$
.

For each vertex v of T, define the *profile* of v to be the set of all ranked graphs (H, L, \preceq) contained in $G[T_v^+]$ such that $|V(H)| \leq N_{\ell(v)}$. Note that if v is a descendant of u, then the profile of v is a subset of the profile of u. For $\ell \in [0, d-1]$, if $N = N_\ell$ then let

$$M_{\ell} := 2^{2^{\binom{N}{2}} d^N 3^{\binom{N}{2}}}.$$

Then there are at most M_ℓ possible profiles of a vertex at level ℓ .

We now partition V(T) into subtrees. Each subtree is called a *group*. (At the end of the proof, vertices in a single group will be assigned the same colour.) We assign vertices to groups in non-increasing order of their distance from the root. Initialise this process by placing each leaf v of T into a singleton group. We now show how to determine the group of a non-leaf vertex. Let v be a vertex not assigned to a group at maximum distance from r. So each child of v is assigned to a group. Let Y_v be the set of children v of v, such that the number of children of v that have the same profile as v is in the range v. If v if v if then merge all the groups rooted at vertices in v into one group including v. This defines our partition of v into groups. Each group v is v is v in v closest to v in v and v is v in v in v in v is on the path in v from the root of v to v.

The next claim is the key to the remainder of the proof.

Claim 1. Let $uv \in E(T)$ where u is the parent of v, and u is in a different group to v. Then for every ranked graph (H, L, \preceq) in the profile of v, the $\ell(u)$ -splice of (H, L, \preceq) is in the profile of u.

Proof. Since (H,L,\preceq) is in the profile of v, there is an isomorphism ϕ from H to some subgraph of $G[T_v^+]$ such that for each vertex $x\in V(H)$ we have $L(x)=\ell(\phi(x))$, and for all distinct vertices $x,y\in V(H)$ we have that $x\prec y$ if and only if $\phi(x)$ is an ancestor of $\phi(y)$ in T.

Since u and v are in different groups, there are k children y_1,\ldots,y_k of u (one of which is v) such that the profiles of y_1,\ldots,y_k are equal. Thus (H,L,\preceq) is in the profile of each of y_1,\ldots,y_k . That is, for each $j\in[1,k]$, there is an isomorphism ϕ_j from H to some subgraph of $G[T_{y_j}^+]$ such that for each vertex $x\in V(H)$ we have $L(x)=\ell(\phi_j(x))$, and for all distinct vertices $x,y\in V(H)$ we have that $x\prec y$ if and only if $\phi_j(x)$ is an ancestor of $\phi_j(y)$ in T.

Let (H',L',\preceq') be the $\ell(u)$ -splice of (H,L,\preceq) . We now define a function ϕ' from V(H') to $V(G[T_u^+])$. For each vertex (x,0) of H' (thus with $x\in V(H)$ and $L(x)\in [0,\ell(u)]$), define $\phi'((x,0)):=\phi(x)$. For every other vertex (x,j) of H' (thus with $x\in V(H)$ and $L(x)\in [\ell(u)+1,d-1]$ and $j\in [1,k]$), define $\phi'((x,j)):=\phi_j(x)$.

We now show that ϕ' is an isomorphism from H' to a subgraph of $G[T_u^+]$. Consider an edge (x,a)(y,b) of H'. Thus $xy \in E(H)$. It suffices to show that $\phi'((x,a))\phi'((y,b)) \in$

 $E(G[T_u^+]). \text{ First suppose that } a=b=0. \text{ So } L(x) \in [0,\ell(u)] \text{ and } L(y) \in [0,\ell(u)].$ Thus $\phi'((x,a))=\phi(x)$ and $\phi'((y,b))=\phi(y).$ Since ϕ is an isomorphism to a subgraph of $G[T_v^+]$, we have $\phi(x)\phi(y)\in E(G[T_v^+])$, which is a subgraph of $G[T_u^+].$ Hence $\phi'((x,a))\phi'((y,b))\in E(G[T_u^+])$, as desired. Now suppose that a=0 and $b\in [1,k].$ Thus $\phi'((x,a))=\phi(x)$ and $\phi'((y,b))=\phi_b(y).$ Moreover, both $\ell(\phi(x))$ and $\ell(\phi_b(x))$ equal $L(x)\in [0,\ell(u)].$ There is only vertex z in T_v^+ with $\ell(z)$ equal to a specific number in $[0,\ell(u)].$ Thus $\phi'((x,a))=\phi(x)=\phi_b(x)$ (= z). Since ϕ_b is an isomorphism to a subgraph of $G[T_{y_b}^+]$, we have $\phi_b(x)\phi_b(y)\in E(G[T_{y_b}^+])$, which is a subgraph of $G[T_u^+].$ Hence $\phi'((x,a))\phi'((y,b))\in E(G[T_u^+])$, as desired. Finally, suppose that $a=b\in [1,k].$ Thus $\phi'((x,a))=\phi_a(x)$ and $\phi'((y,b))=\phi_b(y)=\phi_a(y).$ Since ϕ_a is an isomorphism to a subgraph of $G[T_{y_a}^+]$, we have $\phi_a(x)\phi_a(y)\in E(G[T_{y_a}^+])$, which is a subgraph of $G[T_u^+].$ Hence $\phi'((x,a))\phi'((y,b))\in E(G[T_u^+])$, as desired. This shows that ϕ' is an isomorphism from H' to a subgraph of $G[T_y^+].$

We now verify property (A) for (H',L',\preceq') . For each vertex (x,0) of H' (thus with $x\in V(H)$ and $L(x)\in [0,\ell(u)]$) we have $L'((x,0))=L(x)=\ell(\phi(x))=\ell(\phi'((x,0)))$, as desired. For every other vertex (x,j) of H' (thus with $x\in V(H)$ and $L(x)\in [\ell(u)+1,d-1]$ and $j\in [1,k]$) we have $L'((x,j))=L(x)=\ell(\phi_j(x))=\ell(\phi'((x,j)))$, as desired. Hence property (A) is satisfied for (H',L',\preceq') .

We now verify property (B) for (H',L',\preceq') . Consider distinct vertices $(x,a),(y,b)\in V(H')$. First suppose that a=0 and b=0. Then $(x,a)\prec'(y,b)$ if and only if $x\prec y$ if and only if $\phi(x)$ is an ancestor of $\phi(y)$ in T if and only if $\phi'((x,a))$ is an ancestor of $\phi'((y,b))$ in T, as desired. Now suppose that a=0 and $b\in [1,k]$. Then $(x,a)\prec'(y,b)$ if and only if $x\prec y$ if and only if $\phi(x)$ is an ancestor of $\phi_b(y)$ in T if and only if $\phi'((x,a))$ is an ancestor of $\phi'((y,b))$ in T, as desired. Now suppose that $a=b\in [1,k]$. Then $(x,a)\prec'(y,b)$ if and only if $x\prec y$ if and only if $\phi_a(x)$ is an ancestor of $\phi_b(y)$ in T if and only if $\phi'((x,a))$ is an ancestor of $\phi'((y,b))$ in T, as desired. Finally, suppose that $a,b\in [1,k]$ and $a\neq b$. Then (x,a) and (y,b) are incomparable under \prec' , and $\phi'((x,a))$ and $\phi'((y,b))$ in T are unrelated in T, as desired. Hence property (B) is satisfied for (H',L',\preceq') .

So ϕ' is an isomorphism from H' to a subgraph of $G[T_u^+]$ satisfying properties (A) and (B). Thus (H',L',\preceq') is contained in $G[T_u^+]$, as desired. Since (H,L,\preceq) is in the profile of v, we have $|V(H)| \leqslant (d+1)(h-1)(k+1)^{h-\ell(v)}$. Since $|V(H')| \leqslant (k+1)|V(H)|$ and $\ell(u) = \ell(v) - 1$, we have $|V(H')| \leqslant (d+1)(h-1)(k+1)^{h-\ell(v)} = (d+1)(h-1)(k+1)^{h-\ell(u)}$. Thus (H',L',\preceq') is in the profile of u.

The proof now divides into two cases. If some group X_0 is adjacent in G to at least h-1 other groups above X_0 , then we show that G contains $W\langle h,k\rangle$ as a minor. Otherwise, every group X is adjacent in G to at most h-2 other groups above X, in which case we show that G is (h-1)-colourable with bounded clustering.

Finding the Minor

Suppose that some group X_0 is adjacent in G to at least h-1 other groups X_1,\ldots,X_{h-1} above X_0 . We now show that G contains $W\langle h,k\rangle$ as a minor; refer to Figure 3. For $i\in[1,h-1]$, since X_i is above X_0 , the root v_i of X_i is on the v_0r -path in T. Without loss of generality, v_0,v_1,\ldots,v_{h-1} appear in this order on the v_0r -path in T. For $i\in[1,h-1]$, let w_i be a vertex in X_i adjacent to some vertex z_i in X_0 ; since G is a subgraph of the closure of T, w_i is on the v_0r -path in T. For $i\in[0,h-2]$, let u_i be the parent of v_i in T (which exists since $v_{h-2}\neq r$). So u_i is not in X_i (but may be in X_{i+1}). Note that $v_0,u_0,w_1,v_1,u_1,\ldots,w_{h-2},v_{h-2},u_{h-2},w_{h-1},v_{h-1}$ appear in this order on the v_0r -path in T, where v_0,v_1,\ldots,v_{h-1} are distinct (since they are in distinct groups).

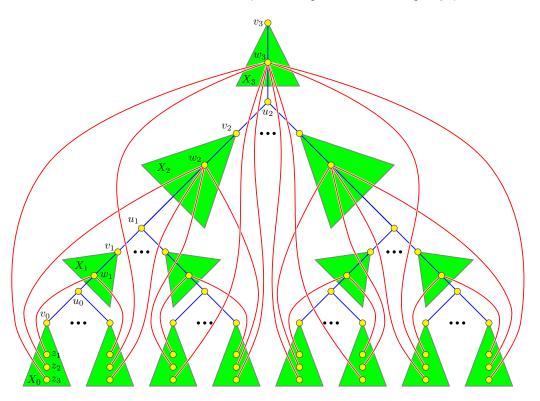


Figure 3: Construction of a W(4,k) minor (where u_i might be in X_{i+1}).

Let P_j be the z_jr -path in T for $j\in [1,h-1]$. Let H_0 be the graph with $V(H_0):=V(P_1\cup\cdots\cup P_{h-1})$ and $E(H_0):=\{z_jw_j:j\in [1,h-1]\}$. Define the function $L_0:V(H_0)\to [0,d-1]$ by $L_0(x):=\ell(x)$ for each $x\in V(H_0)$. Define the partial order \preceq_0 on $V(H_0)$, where $x\prec_0 y$ if and only if x is ancestor of y in T. Thus (H_0,L_0,\preceq_0) is a ranked graph. By construction, (H_0,L_0,\preceq_0) is contained in $G[T_{v_0}^+]$. Since H_0 has less than (d+1)(h-1) vertices, H_0 is in the profile of v_0 . For $i=0,1,\ldots,h-2$, let $(H_{i+1},L_{i+1},\prec_{i+1})$ be the $\ell(u_i)$ -splice of (H_i,L_i,\prec_i) .

By induction on i, using Claim 1 at each step and since $G[T^+_{u_i}] \subseteq G[T^+_{v_{i+1}}]$, we conclude that for each $i \in [0,h-1]$, the ranked graph (H_i,L_i,\preceq_i) is in the profile of v_i . In particular, $(H_{h-1},L_{h-1},\prec_{h-1})$ is in the profile of v_{h-1} , and H_{h-1} is isomorphic to a subgraph

of G. Note that each vertex of H_{h-1} is of the form $(((\ldots(x,d_1),d_2),\ldots),d_{h-1})$ for some $x\in V(H_0)$ and $d_1,\ldots,d_{h-1}\in [0,k]$. For brevity, call such a vertex $x\langle d_1,\ldots,d_{h-1}\rangle$. Note that if $x=w_j$ for some $j\in [1,h-1]$, then $d_1=\cdots=d_j=0$ (since w_j is above u_i whenever i< j, and $(H_{i+1},L_{i+1},\prec_{i+1})$ is the $\ell(u_i)$ -splice of (H_i,L_i,\preceq_i)).

For $x\in V(H_0)$, let Λ_x be the set of vertices $x\langle d_1,\dots,d_{h-1}\rangle$ in H_{h-1} . By construction, no two vertices in Λ_x are comparable under \leq_{h-1} . Therefore, by property (B), $V(T_a)\cap V(T_b)=\emptyset$ for all distinct $a,b\in\Lambda_x$. In particular, $V(T_a)\cap V(T_b)=\emptyset$ for all distinct $a,b\in\Lambda_{v_0}$. As proved above, $G[T_a]$ is connected for each $a\in V(T)$. Let G' be the graph obtained from G by contracting $G[T_a]$ into a single vertex $\alpha\langle d_1,\dots,d_{h-1}\rangle$, for each $a=v_0\langle d_1,\dots,d_{h-1}\rangle\in\Lambda_{v_0}$. So G' is a minor of G.

Let U be the tree with vertex set

$$\{\langle d_1,\ldots,d_{h-1}\rangle:\exists j\in[0,h-1]\ d_1=\cdots=d_i=0\ \text{and}\ d_{i+1},\ldots,d_{h-1}\in[1,k]\},\$$

where the parent of $(0,\ldots,0,d_{j+1},d_{j+2},\ldots,d_{h-1})$ is $(0,\ldots,0,d_{j+2},\ldots,d_{h-1})$. Then U is isomorphic to the complete k-tree of height h rooted at $\langle 0,\ldots,0\rangle$. We now show that the weak closure of U is a subgraph of G', where each vertex $\langle 0,\ldots,0,d_{j+1},\ldots,d_{h-1}\rangle$ of U with $j\in [1,h-1]$ is mapped to vertex $w_j\langle 0,\ldots,0,d_{j+1},\ldots,d_{h-1}\rangle$ of G', and each other vertex $\langle d_1,\ldots,d_{h-1}\rangle$ of U is mapped to $\alpha\langle d_1,\ldots,d_{h-1}\rangle$ of G'. For all $d_1,\ldots,d_{h-1}\in [1,k]$ and $j\in [1,h-1]$ the vertex $z_j\langle d_1,\ldots,d_{h-1}\rangle$ of G is contracted into the vertex $\alpha\langle d_1,\ldots,d_{h-1}\rangle$ of G'. By construction, $z_j\langle d_1,\ldots,d_{h-1}\rangle$ is adjacent to $w_j\langle 0,\ldots,0,d_{j+1},\ldots,d_{h-1}\rangle$ in G. So $\alpha\langle d_1,\ldots,d_{h-1}\rangle$ is adjacent to $w_j\langle 0,\ldots,0,d_{j+1},\ldots,d_{h-1}\rangle$ in G'. This implies that the weak closure of U (that is, $W\langle h,k\rangle$) is isomorphic to a subgraph of G', and is therefore a minor of G.

Finding the Colouring

Now assume that every group X is adjacent in G to at most h-2 other groups above X. Then (h-1)-colour the groups in order of distance from the root, such that every group X is assigned a colour different from the colours assigned to the neighbouring groups above X. Assign each vertex within a group the same colour as that assigned to the whole group. This defines an (h-1)-colouring of G.

Consider the function $s:[0,d-1]\to\mathbb{N}$ recursively defined by

$$s(\ell) := \begin{cases} 1 & \text{if } \ell = d-1 \\ (k-1) \cdot M_{\ell+1} \cdot s(\ell+1) & \text{if } \ell \in [0,d-2]. \end{cases}$$

Then every group at level ℓ has at most $s(\ell)$ vertices. By construction, our (h-1)-colouring of G has clustering s(0), which is bounded by a function of d, k and h, as desired.

3 Pathwidth

The following lemma of independent interest is the key to proving Theorem 2. Note that Eppstein [11] independently discovered the same result (with a slightly weaker bound

on the path length). The decomposition method in the proof has been previously used, for example, by Dujmović, Joret, Kozik, and Wood [7, Lemma 17].

Lemma 9. Every graph with pathwidth at most w has a vertex 2-colouring such that each monochromatic path has at most $(w+3)^w$ vertices.

Proof. We proceed by induction on $w \ge 1$. Every graph with pathwidth 1 is a caterpillar, and is thus properly 2-colourable. Now assume $w \ge 2$ and the result holds for graphs with pathwidth at most w-1. Let G be a graph with pathwidth at most w. Let (B_1,\ldots,B_n) be a path-decomposition of G with width at most w. Let t_1,t_2,\ldots,t_m be a maximal sequence such that $t_1 = 1$ and for each $i \ge 2$, t_i is the minimum integer such that $B_{t_i} \cap B_{t_{i-1}} = \emptyset$. For odd i, colour every vertex in B_{t_i} 'red'. For even i, colour every vertex in B_{t_i} 'blue'. Since $B_{t_i} \cap B_{t_{i-1}} = \emptyset$ for $i \ge 2$, no vertex is coloured twice. Let G'be the subgraph of G induced by the uncoloured vertices. By the choice of B_{t_i} , for $i \ge 2$ each bag B_j with $j \in [t_{i-1}+1, t_i-1]$ intersects $B_{t_{i-1}}$. Thus $(B_1 \cap V(G'), \dots, B_n \cap V(G'))$ is a path-decomposition of G^{\prime} of width at most w-1. By induction, G^{\prime} has a vertex 2-colouring such that each monochromatic path has at most $(w+3)^{w-1}$ vertices. Since $B_{t_i} \cup B_{t_{i+2}}$ separates $B_{t_{i+1}} \cup \cdots \cup B_{t_{i+2}-1}$ from the rest of G, each monochromatic component of G is contained in $B_{t_i+1}\cup\cdots\cup B_{t_{i+2}-1}$ for some $i\in[0,n-2].$ Consider a monochromatic path P in $G[B_{t_i+1} \cup \cdots \cup B_{t_{i+2}-1}]$. Then P has at most w+1 vertices in $B_{t_{i+1}}$. Note that $P-B_{t_{i+1}}$ is contained in G^\prime . Thus P consists of up to w+2monochromatic subpaths in G' plus w+1 vertices in $B_{t_{i+1}}$. Hence P has at most $(w+2)(w+3)^{w-1} + (w+1) < (w+3)^w$ vertices.

Nešetřil and Ossona de Mendez [22] showed that if a graph G contains no path on k vertices, then $\mathrm{td}(G) < k$ (since G is a subgraph of the closure of a DFS spanning tree with height at most k). Thus Lemma 9 implies:

Corollary 10. Every graph with pathwidth at most w has a vertex 2-colouring such that each monochromatic component has treedepth at most $(w+3)^w$.

Proof of Theorem 2. Let $\mathscr G$ be a minor-closed class of graphs, each with pathwidth at most w. Let h be the minimum integer such that $C\langle h,k\rangle \not\in \mathscr G$ for some $k\in \mathbb N$. Consider $G\in \mathscr G$. Thus $W\langle h,k+1\rangle$ is not a minor of G (since $C\langle h,k\rangle$ is a minor of $W\langle h,k+1\rangle$, as noted above). By Corollary 10, G has a vertex 2-colouring such that each monochromatic component H of G has treedepth at most $(w+3)^w$. Thus $W\langle h,k+1\rangle$ is not a minor of H. By Lemma 8, H is (h-1)-colourable with clustering $c((w+3)^w,k+1,h)$. Taking a product colouring, G is (2h-2)-colourable with clustering $c((w+3)^w,k+1,h)$. Hence $\chi_{\Delta}(\mathscr G)\leqslant \chi_{\star}(\mathscr G)\leqslant 2h-2$.

Note that Lemma 9 cannot be extended to the setting of bounded tree-width graphs: Esperet and Joret (see [17, Theorem 4.1]) proved that for all positive integers w and d there exists a graph G with tree-width at most w such that for every w-colouring of G there exists a monochromatic component of G with diameter greater than d (and thus with a monochromatic path on more than d vertices, and thus with treedepth at least $\log_2 d$).

4 Fractional Colouring

This section proves Theorem 6. The starting point is the following key result of Dvořák and Sereni [9].²

Theorem 11 ([9]). For every proper minor-closed class \mathcal{G} and every $\delta > 0$ there exists $d \in \mathbb{N}$ satisfying the following. For every $G \in \mathcal{G}$ there exist $s \in \mathbb{N}$ and $X_1, X_2, \ldots, X_s \subseteq V(G)$ such that:

- $\operatorname{td}(G[X_i]) \leqslant d$, and
- every $v \in V(G)$ belongs to at least $(1 \delta)s$ of these sets.

We now prove a lower bound on the fractional defective chromatic number of the closure of complete trees of given height.

Lemma 12. Let
$$\mathscr{C}_h := \{C\langle h, k \rangle\}_{k \in \mathbb{N}}$$
. Then $\chi_{\Delta}^f(\mathscr{C}_h) \geqslant h$.

Proof. We show by induction on h that if $C\langle h,k\rangle$ is fractionally t-colourable with defect d, then $t\geqslant h-(h-1)d/k$. This clearly implies the lemma. The base case h=1 is trivial.

For the induction step, suppose that $G := C\langle h, k \rangle$ is fractionally t-colourable with defect d. Thus there exist $Y_1, Y_2, \ldots, Y_s \subseteq V(G)$ and $\alpha_1, \ldots, \alpha_s \in [0, 1]$ such that:

- ullet every component of $G[Y_i]$ has maximum degree at most d,
- $\sum_{i=1}^{s} \alpha_i \leq t$, and
- $\sum_{i:v\in Y_i} \alpha_i \geqslant 1$ for every $v\in V(G)$.

Let r be the vertex of G corresponding to the root of the complete k-ary tree and let H_1,\ldots,H_k be the components of G-r. Then each H_i is isomorphic to $C\langle h-1,k\rangle$. Let $J_0:=\{j:r\in Y_j\}$, and let $J_i:=\{j:Y_j\cap V(H_i)\neq\emptyset\}$ for $i\in[1,k]$. Denote $\sum_{j\in J_i}\alpha_j$ by $\alpha(J_i)$ for brevity. Thus $\alpha(J_0)\geqslant 1$. For $i\in[1,k]$, the subgraph H_i is $\alpha(J_i)$ -colourable with defect d, and thus $\alpha(J_i)\geqslant h-1-(h-2)d/k$ by the induction hypothesis. Thus

$$(k-d)\alpha(J_0) + \sum_{i=1}^k \alpha(J_i) \geqslant (k-d) + k(h-1) - (h-2)d = kh - (h-1)d.$$

If $j \in J_0$ then Y_j intersects at most d of H_1, \ldots, H_k (since $G[Y_j]$ has maximum degree at most d). Thus every α_j appears with coefficient at most k in the left side of the above inequality, implying

$$(k-d)\alpha(J_0) + \sum_{i=1}^k \alpha(J_i) \leqslant k \sum_{i=1}^s \alpha_i \leqslant kt.$$

Combining the above inequalities yields the claimed bound on t.

² Dvořák and Sereni [9] expressed their result in the terms of "treedepth fragility". The sentence "proper minor-closed classes are fractionally treedepth-fragile" after Theorem 31 in [9] is equivalent to Theorem 11. Informally speaking, Theorem 11 shows that the fractional "treedepth" chromatic number of every minor-closed class equals 1.

Proof of Theorem 6. By Lemma 12,

$$\chi_{\star}^{f}(\mathcal{G}) \geqslant \chi_{\Delta}^{f}(\mathcal{G}) \geqslant \operatorname{tcn}(\mathcal{G}) - 1.$$

It remains to show that $\chi^f_\star(\mathcal{G}) \leqslant \mathrm{tcn}(\mathcal{G}) - 1$. Equivalently, we need to show that for all $h, k \in \mathbb{N}$ and $\varepsilon > 0$, if $C\langle h, k \rangle \not\in \mathcal{G}$ then there exists c such that every graph in \mathcal{G} is fractionally $(h-1+\varepsilon)$ -colourable with clustering c. This is trivial for h=1, and so we assume $h \geqslant 2$.

Let $d\in\mathbb{N}$ satisfy the conclusion of Theorem 11 for the class \mathscr{G} and $\delta=1-\frac{1}{1+\varepsilon/(h-1)}$. Choose c=c(d,k+1,h) to satisfy the conclusion of Lemma 8. We show that c is as desired.

Consider $G \in \mathcal{G}$. By the choice of d there exists $s \in \mathbb{N}$ and $X_1, X_2, \dots, X_s \subseteq V(G)$ such that:

- $\operatorname{td}(G[X_i]) \leqslant d$, and
- every $v \in V(G)$ belongs to at least $(1 \delta)s$ of these sets.

Since $C\langle h,k\rangle\not\in\mathcal{G}$, we have $W\langle h,k+1\rangle\not\in\mathcal{G}$, and by the choice of c, for each $i\in[1,s]$ there exists a partition $(Y_i^1,Y_i^2,\ldots,Y_i^{h-1})$ of X_i such that every component of $G[Y_i^j]$ has at most c vertices. Every vertex of G belongs to at least $(1-\delta)s$ sets Y_i^j where $i\in[1,s]$ and $j\in[1,h-1]$. Considering these sets with equal coefficients $\alpha_i^j:=\frac{1}{(1-\delta)s}$, we conclude that G is fractionally $\frac{h-1}{1-\delta}$ -colourable with clustering c, as desired (since $\frac{h-1}{1-\delta}=h-1+\varepsilon$).

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References

- [1] Noga Alon, Guoli Ding, Bogdan Oporowski, and Dirk Vertigan. Partitioning into graphs with only small components. J. Combin. Theory Ser. B, 87(2):231–243, 2003.
- [2] DAN ARCHDEACON. A note on defective colorings of graphs in surfaces. *J. Graph Theory*, 11(4):517–519, 1987.
- [3] NICOLAS BROUTIN AND ROSS J. KANG. Bounded monochromatic components for random graphs. *J. Comb.*, 9(3):411–446, 2018.
- [4] ILKYOO CHOI AND LOUIS ESPERET. Improper coloring of graphs on surfaces. *J. Graph Theory*, 91(1):16–34, 2019.
- [5] Lenore Cowen, Wayne Goddard, and C. Esther Jesurum. Defective coloring revisited. *J. Graph Theory*, 24(3):205–219, 1997.
- [6] VIDA DUJMOVIĆ, LOUIS ESPERET, PAT MORIN, BARTOSZ WALCZAK, AND DAVID R. WOOD. Clustered 3-colouring graphs of bounded degree. *Combin. Probab. Comput.*, 31(1):123–135, 2022.

- [7] VIDA DUJMOVIĆ, GWENAËL JORET, JAKUB KOZIK, AND DAVID R. WOOD. Nonrepetitive colouring via entropy compression. *Combinatorica*, 36(6):661–686, 2016.
- [8] ZDENĚK DVOŘÁK AND SERGEY NORIN. Islands in minor-closed classes. I. Bounded treewidth and separators. 2017, arXiv:1710.02727.
- [9] ZDENĚK DVOŘÁK AND JEAN-SÉBASTIEN SERENI. On fractional fragility rates of graph classes. *Electronic J. Combinatorics*, 27:P4.9, 2020.
- [10] Katherine Edwards, Dong Yeap Kang, Jaehoon Kim, Sang-il Oum, and Paul Seymour. A relative of Hadwiger's conjecture. *SIAM J. Discrete Math.*, 29(4):2385–2388, 2015.
- [11] DAVID EPPSTEIN. Pathbreaking for intervals. In 11011110. 2020.
- [12] Louis Esperet and Gwenaël Joret. Colouring planar graphs with three colours and no large monochromatic components. *Combin., Probab. Comput.*, 23(4):551–570, 2014.
- [13] Penny Haxell, Tibor Szabó, and Gábor Tardos. Bounded size components—partitions and transversals. *J. Combin. Theory Ser. B*, 88(2):281–297, 2003.
- [14] Dong Yeap Kang and Sang-il Oum. Improper coloring of graphs with no odd clique minor. *Combin. Probab. Comput.*, 28(5):740–754, 2019.
- [15] Ken-ichi Kawarabayashi. A weakening of the odd Hadwiger's conjecture. *Combin. Probab. Comput.*, 17(6):815–821, 2008.
- [16] Ken-ichi Kawarabayashi and Bojan Mohar. A relaxed Hadwiger's conjecture for list colorings. *J. Combin. Theory Ser. B*, 97(4):647–651, 2007.
- [17] Chun-Hung Liu and Sang-il Oum. Partitioning *H*-minor free graphs into three subgraphs with no large components. *J. Combin. Theory Ser. B*, 128:114–133, 2018.
- [18] Chun-Hung Liu and David R. Wood. Clustered coloring of graphs excluding a subgraph and a minor. 2019, arXiv:1905.09495.
- [19] Chun-Hung Liu and David R. Wood. Clustered graph coloring and layered treewidth. 2019, arXiv:1905.08969.
- [20] Chun-Hung Liu and David R. Wood. Clustered variants of Hajós' conjecture. *J. Combin. Theory, Ser. B,* 152:27–54, 2019.
- [21] Bojan Mohar, Bruce Reed, and David R. Wood. Colourings with bounded monochromatic components in graphs of given circumference. *Australas. J. Combin.*, 69(2):236–242, 2017.
- [22] JAROSLAV NEŠETŘIL AND PATRICE OSSONA DE MENDEZ. Sparsity, vol. 28 of *Algorithms and Combinatorics*. Springer, 2012.
- [23] SERGEY NORIN, ALEX SCOTT, PAUL SEYMOUR, AND DAVID R. WOOD. Clustered colouring in minor-closed classes. *Combinatorica*, 39(6):1387–1412, 2019.
- [24] Patrice Ossona de Mendez, Sang-il Oum, and David R. Wood. Defective colouring of graphs excluding a subgraph or minor. *Combinatorica*, 39(2):377–410, 2019.
- [25] JAN VAN DEN HEUVEL AND DAVID R. WOOD. Improper colourings inspired by Hadwiger's conjecture. J. London Math. Soc., 98:129–148, 2018. arXiv:1704.06536.
- [26] DAVID R. WOOD. Defective and clustered graph colouring. *Electron. J. Combin.*, DS23, 2018. Version 1.