Demand Analysis with Partial Predicates*

Julio Mariño, Ángel Herranz and Juan José Moreno-Navarro Universidad Politécnica de Madrid

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Abstract

In order to alleviate the inefficiencies caused by the interaction of the logic and functional sides, integrated languages may take advantage of *demand* information — i.e. knowing in advance which computations are needed and, to which extent, in a particular context. This work studies *demand analysis* – which is closely related to *backwards strictness analysis* – in a semantic framework of *partial predicates*, which in turn are constructive realizations of ideals in a domain. This will allow us to give a concise, unified presentation of demand analysis, to relate it to other analyses based on abstract interpretation or strictness logics, some hints for the implementation, and, more important, to prove the soundness of our analysis based on *demand equations*. There are also some innovative results. One of them is that a set constraint-based analysis has been derived in a stepwise manner using ideas taken from the area of program transformation. The other one is the possibility of using program transformation itself to perform the analysis, specially in those domains of properties where algorithms based on constraint solving are too weak.

KEYWORDS: functional-logic programming, demand analysis, strictness analysis, program transformation, abstract interpretation, set-constraint analysis

1 Introduction

Although the main idea of declarative programming is to use mathematical elements for programming, the area is split in two main paradigms based on the subset of mathematics they are focused on: functional programming (functions: lambda calculus) and logic programming (predicate logic). However it is obvious that both paradigms have a common core and can be seen as different faces of a single idea.

Functional-logic languages aim at bringing together the advantages of functional programming and logic programming, see (Hanus 1994; Moreno-Navarro 1994), i.e. from functional programming they take higher-order features, polymorphic types, lazy evaluation, etc., while logic programming provides partial information, constraints, logical variables, search, etc. The language Curry (Hanus et al. 2003) is the de facto standard of functional-logic languages.

Probably, the combination of the last mentioned features of each paradigm (laziness, search) seems the more problematic to achieve in an efficient implementation. In other

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words, for executing a program it may be necessary to evaluate a functional-like expression containing uninstantiated (i.e. existentially quantified) logic variables.

The operational principle proposed for this situation is narrowing. Roughly speaking, narrowing guesses an instantiation for these variables. Functional nesting, nondeterminism, semantic unification, functional inversion, and lazy evaluation, which are key part for the expressiveness of functional-logic programs, are supported by this operational mechanism. In order to apply the general idea of narrowing to functional-logic languages, a functional-logic program is considered as a set of rewrite rules (plus some additional restrictions described later).

However, narrowing by itself is not enough: a brute-force approach to finding all the solutions would attempt to unify each rule with each nonvariable subterm of the given equation in every narrowing step. Even for small programs a huge search space would result. Therefore, an additional aspect to take into account in the implementation of functionallogic languages is the definition of an appropriate narrowing strategy. This strategy should be sound (i.e., only correct solutions are computed) and complete (i.e., all solutions or more general representatives of all solutions are computed).

Many narrowing strategies for limiting the size of the search space have been proposed, but we are interested on those with a lazy behaviour. To preserve completeness, see (Moreno-Navarro and Rodríguez-Artalejo 1992), a lazy narrowing step is applied at outermost positions with the exception that inner arguments of a function are evaluated, by narrowing them to their head normal forms, if their values are required for an outermost narrowing step. This property can only be ensured by looking-ahead on the rules tried in following steps. Unfortunately, a potentially infinite number of substitutions could arise but, in the case of inductively sequential programs it is possible to compute the property in an efficient way. This is the idea of needed narrowing introduced in (Antoy et al. 2000). The paper also proves completeness and optimality of the strategy.

But beyond this remarkable contribution to the implementation of functional-logic languages two problems remain: (i) the strategy is optimal with respect to the length of derivations but not in the size of the search space and additional improvements could be achieved, and (ii) it is defined only for a restricted class of programs (inductively sequential).

1.1 Demand analysis

Our proposal is to use a static analysis to improve the look-ahead approach. The analysis is able to extract demand information from a functional-logic program. This information can be used to guide and improve needed narrowing (thus cuting the search space and avoid-ing reevaluations). Additionally, we have some other advantages: transform nonsequential programs into sequential ones, following the ideas of (Mariño and Moreno-Navarro 2000) where strictness analysis is used; safe replacement of strict equality by unification; implementation of default rules (Moreno-Navarro 1996); improvement of the accuracy of groundness analysis; translation into Prolog (Mariño and Moreno-Navarro 1992; Mariño and Rey 1998), etc.

Demand analysis was introduced in (Mariño and Moreno-Navarro 1992) and then used in (Moreno-Navarro et al. 1993; Mariño and Herranz 1993; Mariño et al. 1993) as a way to improve the compilation of functional-logic programs. The essential idea was to perform backwards strictness analysis. The proposed solution consisted in generating, for a given program, a set of so called *demand equations* that were solved in a domain of regular trees (*demand patterns*). A strong point of demand patterns compared to existing strictness analyzers based on abstract interpretation was that they allowed, at least theoretically, inference in an infinite domain of properties.

The aim of this paper is to provide a semantic framework (*partial predicates*) for demand analysis, to prove the correctness of demand equations and to introduce a novel approach to the implementation of analyzers. In fact, the whole process is simplified and optimized: while it is easier to reason about demand analysis with partial predicates, the implementation is also simpler because we can reuse a lot of work already done in partial evaluation tools. Moreover, we claim that the method can be applied to different contexts with a similar success.

Partial predicates can be used to specify demand analysis as well as other classical analyses. One of the important point of this formalism is that it can be expressed by functionallogic programs. This has important consequences, allowing for reasoning about demand properties (for instance checking and inference of them) by using program transformation techniques.

Applications of demand analysis go beyond the usual applications of strictness analysis in functional programming, where strictness is used for trying to provide bigger computations that can be computed eagerly. Advances on this subject can have effects in getting a more efficient low level implementation, and in making easier and profitable the parallelization of programs. In functional-logic programming the gain in effiency means that some computations are not reevaluated or even that a wider class of programs can be used.

The Reevaluation Problem. There is an efficiency problem caused by the interaction of laziness and backtracking which are, in some sense, antagonistic in nature. The former tries to delay some computations while the second drives different computations through a tree of branching paths. The problem appears clear – if we place some computation beyond the branching point there exists the risk that the evaluation may be performed several times. This can be rather annoying, because laziness is intended to save work, not to waste it.

The toy example in Figure 1 shows how combining lazy evaluation and backtracking can lead to the repeated evaluation of delayed redexes. Observe that the redex (not True) is needed for the final result but, due to the outermost reduction strategy, its evaluation is delayed and evaluated twice, in different branches of the search tree. In this simple example, this is not serious, but in general, the redex reevaluated could have an expensive operation and the reevaluation can take place not only two, but an unbounded (even infinite) number of times.

Sequentiality Analysis. One of the stages in compiling the code for a function definition in a lazy language is to decide (i) which arguments need reduction to perform the matching against the patterns in the left hand sides of the rules, and (ii) in which order are these arguments to be reduced. For instance, the code for the *greater or equal* predicate (Figure 2) needs to obtain a topmost data constructor for the second argument in order to choose a rule. If this constructor is Zero, the only match is with the first rule and no evaluation is

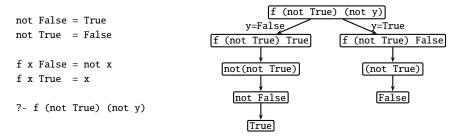


Fig. 1. Reevaluation example.

```
(>=) :: Nat -> Nat -> Bool
                                                          \underline{x} >= (Succ y)
                                x >= Zero
         >= Zero = True
m
                                   1
Zero
         >= (Succ n) = False
                                                                  (Succ x) >= (Succ y)
                                            Zero >= (Succ y)
                                  True
(Succ m) \ge (Succ n) = m \ge n
                                                    Т
                                                                            1
                                                  False
                                                                          x >= y
```

Fig. 2. Sample definition and its associated definitional tree.

needed on the first argument, but if it is Succ, the first argument needs to be examined in order to choose between the second and third rules.

Sloth¹ (Mariño and Rey 1998), our implementation of Curry, uses *definitional trees* (Antoy et al. 2000) as the intermediate structure to store these decisions. Figure 2 shows the definition for (>=) and the definitional tree obtained from it. Underlined positions are those where the branching of the decisions are done. Next section will provide a definition of this concept but the graphical notation could be enough at this point. Observe that arguments are not necessarily examined in a left to right order.

Sometimes, it is difficult – or impossible – to discover a definitional tree from the left hand sides alone. A typical example is the merging of two sorted lists:

```
merge :: [Nat] -> [Nat] -> [Nat]
merge [] ys = ys merge (x:xs) (y:ys) | x <= y = x:(merge xs (y:ys))
merge xs [] = xs | x >= y = y:(merge (x:xs) ys)
```

If we just look at the left hand sides there is no way of building a decision tree like the one for (>=), but looking at the right hand sides we immediately see that both arguments to *merge* are demanded. A programmer would write, in fact, a modified version where the second rule is rewritten as

merge (x:xs) [] = x:xs

instead. In our paper (Mariño and Moreno-Navarro 2000) we show that for the vast majority of programs, the information obtained from type inference and demand analysis can help a compiler to generate sequential definitional trees for programs even when they are not syntactically sequential.

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<sup>1</sup> http://babel.ls.fi.upm.es/software.
```

Avoiding Redundant Tests in Target Code. A substantial part of the overhead in implementing a lazy functional-logic language is precisely due to the code that implements the lazy evaluation of arguments (this was, in fact, what motivated research on strictness analysis of functional programs in the first place). This is particularly evident when looking at the translation scheme into Prolog of Sloth (Mariño and Rey 1998), but the problem appears also in abstract machine based implementations.

Demand information can help to generate, for most functions, eager code, with a drastic effect on efficiency. In the case of the translator to Prolog, this allows an almost verbatim translation with very little overhead.

Relation with Freeness and Sharing Analysis. Freeness and *sharing* (Jacobs and Langen 1992; Muthukumar and Hermenegildo 1991) are two operational properties of logic programs which have been extensively studied. Freeness analysis can tell whether a free variable present in a goal does not get bound during its evaluation. Sharing analysis can tell whether two variables present in a goal will not eventually share some structure. Freeness and sharing are important, for instance, for the parallel execution of logic programs. Moreover, freeness can be useful to detect deterministic computations in functional-logic programs.

Freeness and sharing are much more difficult to study in a lazy functional-logic language than in Prolog. From an operational point of view, the techniques used for Prolog can predict whether a given expression may bind some variables *provided that* this expression gets *actually evaluated*.

From a denotational point of view, this connection can be explained because the lazy semantics can be seen as a restricted form of free-variable semantics where all variables are collapsed into one symbol (\perp). In this setting, variable propagation (freeness) and \perp -propagation (strictness) share a common mechanism (Mariño 2002).

Transformation of Strict Equality into Unification. The behaviour of the equality operator is slightly different in logic programming and in functional-logic programming. While in the first case the expression x = t, where x is a free variable, can be dealt with by assigning t to x (provided that x does not occur in t), in functional-logic programming it is necessary to ensure that t can be evaluated to a *total* value, i.e. a term with no undefined subterms, otherwise the whole expression will be undefined. When logic programs are translated into functional-logic programs, this has two undesired effects: the expression cannot be resolved in constant time – which precludes the use of Prolog techniques such as difference lists – and the computed answers can be unnecessarily detailed.

The connection with strictness is twofold. On one hand, totality is a much stronger condition than not being undefined, which makes rules with equality expressions a source of useful information for a strictness analyzer. On the other hand, the same techniques employed to avoid redundant tests when applying strictness analysis can be used to avoid the test that *t* does not contain an occurrence of a defined function symbol, which is a sufficient condition to perform the assignment.

Some of these problems can be tackled by nonstandard implementation architectures like *memoization* or *bottom-up execution* (Mariño and Moreno-Navarro 1995) for the reevaluation problem, or *parallel definitional trees* (Antoy et al. 1997; Genius 1996) to cope with

the lack of sequentiality but, in practice, these methods introduce their own overheads and actual implementations are based either on extensions of abstract machines for the execution of functional languages or logic languages, or on the translation to another declarative language, like Prolog, see (Mariño and Moreno-Navarro 1992; Mariño and Rey 1998). This is why we chose to attack the problem at compile time or, in other words, how our research on demand analysis began.

1.2 Paper Organization

Section 2 introduces the subset of the Curry language we are going to use along the paper as well as the operational semantics used, namely narrowing. Section 3 introduces the formalism of partial predicates, their structure and how demand properties can be represented by using them. Section 4 discusses the problem of checking and inferring demand properties of a program. Different sublattices of demand properties, with increasing complexity, are introduced. Checking is demonstrated in an abstract way, by means of syntactic transformations (fold/unfold). Later, the harder problem of inferring demand properties is considered, firstly (Section 4.2), in an abstract way and then (Section 5) in connection with a particular analysis tool. Correctness of the aforementioned demand equations is shown there as well as the algorithms to compute approximate solutions to them. Code generation based on the information from the analyzer is discussed in Section 6 and a few experimental results showing the feasibility of the method are shown in Section 6. Some related work is discussed in Section 7 and open issues in Section 8. Finally, Section 9 concludes. In order to make this paper as self-contained as possible, Appendix A includes the reference denotational semantics for the kernel language. Appendix B contains some proofs that have been removed from the printed version due to lack of space.

2 Preliminaries

This section is devoted to fix the subset of Curry that will be used along the paper. The operational semantics assumed is discussed too.

2.1 Kernel Language

The language of choice is largely immaterial but a little syntax is needed in order to keep some coherency throughout the paper. We will use a simplified version of the functional-logic language Curry (Hanus et al. 2003), basically a language of recursion equations, with a Haskell-like syntax. In the sequel we assume some knowledge of functional-logic languages and Curry operational semantics.

We assume a ranked set $TC = \bigcup_n TC^n$ of *type constructors K* and a countably infinite set *TV* of *type variables* α . Any data type is uniquely denoted by an algebraic term $\tau \in \mathcal{T}(TC \cup TV)$ or a function type $(\tau_1 \to \tau_2)$. Next, we assume a set $DC = \bigcup_n DC^n$ of typed *data constructors C*, a countably infinite set *VS* of *variable symbols x*, and a set *FS* of *function symbols f* with declared principal type $f : \tau_1 \to \cdots \to \tau_n \to \tau$ where τ is not a function type. *TC*, *DC*, *VS* and *FS* are disjoint. The *arity* of a data constructor $C \in DC^n$ is *n* and is denoted *ar*(*C*). In practice, type and data constructors are both defined via *data* declarations of the form

data
$$K \alpha_1 \ldots \alpha_l = C_1 \tau_{11} \ldots \tau_{1m_1} | \ldots | C_n \tau_{n1} \ldots \tau_{nm_n}$$

A type constructor *Bool* with data constructors *True* and *False* is always assumed. Expressions are given by the grammar

$$e ::= C \mid x \mid f \mid e_1 \mid e_2$$
.

Expressions must be well typed. A program is a set of defining rules of the form

$$f e_1 \dots e_n = [b \rightarrow] e$$

The optional condition *b* of type *Bool* is called the *guard* of the rule. Several restrictions are imposed on the rules in a program in order to ensure confluence of reduction, and the following are used somewhere in the paper: (i) for every rule (l = r), *l* is a *pattern*, i.e. it has a single function symbol at its top and no variable occurs twice in *l*; (ii) rules must be well typed; (iii) for every pair of program rules $(l_1 = r_1)$, $(l_2 = r_2)$, if l_1 and l_2 have a unifier σ then $\sigma(r_1) = \sigma(r_2)$; (iv) free variables – i.e. those occurring in the right hand side but not in the left hand side – are allowed only if their rightmost occurrence is in the guard. Moreover, they must be of first order type. Observe that we are not forbidding overlaps. The set of rules defining function symbol *f* in program *P* is denoted *Rules*_{*P*}(*f*).

When looking at the syntactic shape of the left hand sides of defining rules, a total application $(f e_1 \dots e_n)$ is treated as the algebraic term $f(e_1, \dots, e_n)$ and then the standard notation for positions and substitutions is used. A position is a string of natural numbers that identifies a path to a subterm in a term. The expression $t|_p$ denotes the subterm of t at position p, i.e. $f(e_1, \dots, e_n)|_{i.p} = e_i|_p$ and $t|_{\epsilon} = t$, with ϵ the empty string. Replacement of $t|_p$ by t' is abbreviated $t[t']_p$. The topmost symbol of term t is denoted root(t).

A denotational semantics for the kernel language can be found in Appendix A. Although not strictly necessary to understand the techniques proposed here, this is the ultimate foundation for the validity of the equations and inequalities used and supports the validity of the fold/unfold transformations.

2.2 Narrowing

The fundamental computation mechanism of functional-logic languages is narrowing. Informally, to *narrow* an expression *e* means to apply a substitution that makes it reducible, and then reduce it. An expression *e* narrows to *e'* with substitution σ , if *p* is a nonvariable position of *e*, l = r is a variant of a program rule sharing no variables with *e*, and σ is a substitution such that $\sigma(l) = \sigma(e|_p)$, and $e' = \sigma(e[r]_p)$.

As we have mentioned, unrestricted application of the narrowing rule is too nondeterministic and many strategies have been proposed to improve this. From an expressiveness point of view, we prefer those with a lazy behaviour because functions can be defined more independently, without interaction among them, thus increasing modularity and reusability and allowing programming techniques like infinite objects. A general description of lazy narrowing for functional-logic languages can be found in (Moreno-Navarro and Rodríguez-Artalejo 1992).

The task of a narrowing strategy is the computation of the step, or steps, that must be applied to a term. A narrowing strategy suitable for functional-logic languages must be

sound, complete, and efficient. The intuition behind the soundness and the completeness of a strategy when the initial term of a derivation is an equation containing unknown variables is easy to describe: Soundness guarantees that any instantiation of the variables computed by the strategy is a solution of the equation. Completeness ensures that for any solution of the equation, the strategy computes another solution which is at least as general.

However, efficiency is more difficult to state. As usual, the goal is to minimize the overall time and memory consumed when finding one or all the values of an expression. In the narrowing context, it is related with the length of the derivations, and, specially, with the size of the search space. Basically, the two factors affecting the efficiency of a strategy are: (i) unnecessary steps should be avoided, and (ii) steps should be computed without consuming unnecessary resources. Lazy narrowing steps try to be applied at outermost positions but to preserve completeness (see (Moreno-Navarro and Rodríguez-Artalejo 1992)) inner arguments of a function are evaluated, by narrowing them to their head normal forms, if their values are required for an outermost narrowing step.

In general, a strategy cannot easily determine if a computation is unnecessary without look-ahead.

The strategy used in Curry is *needed narrowing* (Antoy et al. 2000), a lazy strategy where the program is translated into a set of definitional trees, one for every function symbol being defined. Definitional trees are given by the grammar

$$DT$$
 ::= branch (*Pattern*, Pos [, DT]⁺) | rule *Rule* | or (DT [, DT]⁺),

where *Pattern* stands for patterns made up of data constructors and different variables as in the left hand sides of program rules, *Pos* are positions defined in the standard way and *Rules* are program rules. Trees without *or* nodes are called (inductively) sequential, otherwise they are parallel definitional trees.

Given an expression e and a set of definitional trees for the defined symbols of the program, a position in e can be chosen to apply narrowing. This is done by first looking for an outermost application $fe_1 \dots e_n$ where f is a defined function symbol, and then descending some e_i according to f's definitional tree. Therefore, needed narrowing with sequential definitional trees establishes an efficient algorithm for implementing the lookahead for required evaluations. In (Antoy et al. 2000) the formal definition is given but also an interesting property is shown: the strategy is optimal with respect to the length of derivations.

To overcome the restriction to inductively sequential programs, other strategies have been proposed. For instance, the strategy of (Loogen et al. 1987) is based on some form of generalized definitional trees. The completeness of this strategy is unknown. These strategies are *demand driven*, which informally means the following: a subterm v of a term t is evaluated if there is a rule R potentially applicable to t that demands the evaluation of v.

The lack of well-defined strategies with provable properties motivated alternative efforts for computations in this class. In our case, we use the information provided by demand analysis to guide the computation. The analysis, its formal properties and the use of demand information for implementing efficiently functional-logic languages are the goal of the following sections.

3 Partial Predicates

This section is devoted to introduce the formalism of *partial predicates*. Informally speaking, they are logic predicates that represent the degree of evaluation of an expression (demand properties). The main feature is that they can be described using the language under analysis, so the language itself is used to abstract some properties of a given program. This fact is essential for their use as an analysis tool, as will be shown later.²

Definition 1 (Partial Predicate) Let *Two* be a two point domain; a *partial predicate* π defined on type τ is any continuous map $\pi \in \tau \rightarrow Two$ which can be defined in the kernel language. In the following, the type scheme *PP* α will be used as synonymous with $\alpha \rightarrow Two$.

A two point domain *Two* is isomorphic to the subset of the domain *Bool* = {*True*, *False*, \perp } after removing *False*³ and can be defined in the kernel language:

data Two = True

The definition of conjunction and disjunction functions in *Bool* can be restricted to this domain:

(&&), (||) :: Two -> Two -> Two True && True = True True || y = True; x || True = True

Observe that disjunction in *Two* is given by a parallel definition. While this could be problematic in case of trying to execute the program, it is not the case in our context as long as the definitions of the predicate transformers will mainly be used for program transformation.

Every partial predicate π of type *PP* τ represents subsets of the domain τ :

$$\pi^{-1}(True) = \{x \in \tau \mid \pi(x) = True\}.$$

Example 2 (Peano naturals) Some of the examples throughout the paper will make use of a data type for Peano naturals:

data Nat = Zero | Succ Nat

A pair of predicates *hnfNat* and *nfNat* can be introduced:

hnfNat, nfNat :: PP Nat	
hnfNat Zero = True;	<pre>hnfNat (Succ _) = True</pre>
nfNat Zero = True;	nfNat (Succ n) = nfNat n

The former yields *True* when its argument is evaluated enough to identify its topmost constructor. The latter yields *True* when its argument is evaluated to normal form. Observe that $hnfNat n = \bot \Leftrightarrow n = \bot$.

³ Hence the name of partial predicates.

² From this point on, notation based on *domain theory* is extensively used, and a denotational semantics for the kernel language is assumed. Readers less familiar with these topics are referred to (van Leeuwen 1990), Chapters 11 and 12.

Example 3 (Partial predicates *hnfList* **and** *spine*) Less trivial are those predicates that can be used to express properties of polymorphic types. Consider, for instance, *hnfList* or *spine* in the domain of polymorphic lists:

hnfList, spine :: PP [a]	
<pre>hnfList [] = True;</pre>	<pre>hnfList (h:ts) = True</pre>
<pre>spine [] = True;</pre>	<pre>spine (h:ts) = spine ts</pre>

Analogously to the example above, the former yields *True* when its list argument is evaluated enough to identify its topmost constructor. The latter yields *True* when the argument is evaluated enough to reach the end of the list. In both cases, the degree of definition of the elements in the list is immaterial. Observe again that *hnfList* $xs = \bot \Leftrightarrow xs = \bot$.

Example 4 (Partial predicates *any* **and** *nothing*) A pair of polymorphic partial predicates *any* and *nothing* can be defined for all types:

any, nothing :: PP a any x = True; nothing x = nothing x

3.1 Demand Typings

Partial predicates can be used to express a great number of program properties, including classic strictness. For instance, suppose we are interested in proving $f \in [\tau_1] \rightarrow Nat$ strict:⁴

$$\begin{array}{ll} f \text{ is strict} & \Leftrightarrow f \perp = \perp \Leftrightarrow \forall x. \ x = \perp \Rightarrow f \ x = \perp \\ & \Leftrightarrow \forall x. \ f \ x \sqsupset \perp \Rightarrow x \sqsupset \perp \\ & \Leftrightarrow \forall x. \ hnfNat(f \ x) = True \Rightarrow hnfList \ x = True \\ & \Leftrightarrow \forall x. \ hnfNat(f \ x) \sqsubseteq hnfList \ x \qquad \Leftrightarrow hnfNat \circ f \sqsubseteq hnfList \end{array}$$

So the property 'f is strict' is equivalent to $hnfNat \circ f \sqsubseteq hnfList$.

If we are interested in studying how much information is needed in a function's argument in order to obtain a certain amount in the result, this can be generalized to properties of the form $\pi_2 \circ f \sqsubseteq \pi_1$.

Definition 5 (Demand Properties, Demand Types and Demand Typings) A *demand property* is an inequality of the form $\pi_2 \circ f \sqsubseteq \pi_1$ where π_1 and π_2 are partial predicates defined on the domain and codomain types of f, respectively. It is denoted in the following way:

$$f: \pi_1 \Leftarrow \pi_2 \stackrel{\text{def}}{=} \pi_2 \circ f \sqsubseteq \pi_1.$$

 $\pi_1 \leftarrow \pi_2$ is a demand type and $f : \pi_1 \leftarrow \pi_2$ is a demand typing.

Demand typings $f : \pi_1 \leftarrow \pi_2$ are usually read 'f demands π_1 to its argument in order to give a result as evaluated as π_2 .' Our previous strictness example would be rewritten $f : hnfList \leftarrow hnfNat$. It is rather simple to show that for any partial predicate $\pi^{-1}(True)$ is an ideal – it is the inverse image of a closed set. In fact, 'partial predicate' can be taken

⁴ A formal definition of *strict* can be found in Definition 16.

as synonymous with 'computable ideal.' The lattice of partial predicates induces another on demand types $\pi_1 \leftarrow \pi_2$: covariantly on π_1 and contravariantly on π_2 .

3.2 Polymorphism

One advantage of partial predicates over other approaches to program analysis, such as abstract interpretation on finite domains, is a natural treatment of polymorphism. Being written in source code they have the same type constraints and expressiveness.

Some partial predicates presented so far are polymorphic and the same can be said of the predicate transformers to be introduced below.

Definition 6 (Predicate Transformers) *Predicate transformers* are higher order functions that take partial predicates as (part of) its argument and yield partial predicates as result.

Data constructors can be seen as predicate transformers, in particular, polymorphic constructors can be represented by polymorphic transformers.

Definition 7 (Constructor Predicate Transformer) Consider a type definition of the form

data $K \alpha_1 \ldots \alpha_l = \ldots |C_i \tau_1 \ldots \tau_{m_i}| \ldots$

The *constructor predicate transformer* c_i^5 associated with each data constructor is defined as follows:

Interesting partial predicates associated to data constructors and data types can be defined by using constructor predicate transformers (e.g. Definition 8 and Definition 10).

Definition 8 (Matching Predicates) For every data constructor *C* the partial predicate *isC* defined

isC :: PP ($K \alpha_1 \ldots \alpha_l$) isC = (c any ... any)

is called the *matching predicate for constructor C*.

Definition 9 (Meets and Joins) The greatest lower bound operator \sqcap (/\) and the least upper bound operator \sqcup (\/) can be defined:

 $(/\), (\/) :: PP a \rightarrow PP a \rightarrow PP a$ $(p /\ q) x = (p x) \&\& (q x); (p /\/ q) x = (p x) || (q x)$

Definition 10 (*hnf* **Predicates**) For every type constructor K with data constructors C_1 , ..., C_n , the partial predicate *hnfK* defined

⁵ We are not actually overloading data constructor names but just changing first letter of the name to lowercase.

 $hnfK :: PP (K \alpha_1 \dots \alpha_l)$ $hnfK = isC_1 \sqcup \dots \sqcup isC_n$

is the hnf predicate of K.

Example 11 Constructor predicate transformers and matching predicates associated with the list constructors [] :: $[\alpha]$ and (:) :: $\alpha \rightarrow [\alpha] \rightarrow [\alpha]^6$ are:

nil :: PP a	isNil :: PP [a]
nil [] = True	isNil = nil
cons :: PP a -> PP [a] -> PP [a]	isCons :: PP [a]
cons p q (x : xs) = (p x) && (q xs)	<pre>isCons = cons any any</pre>

The definition of *hnfList* in Example 3 can be rewriten

hnfList :: PP [a] hnfList = isNil \/ isCons

Example 12 The constructor predicate transformer and the matching predicate associated with the tuple constructor (,) :: $\alpha \rightarrow \beta \rightarrow (\alpha, \beta)^7$ are:

tup2 :: PP a -> PP b -> PP (a,b)	isTup2 :: PP (a,b)
tup2 p q (x,y) = (p x) && (q y)	isTup2 = tup2 any any
And the definition of <i>hnfTup2</i> is:	
hnfTup2 :: PP (a,b)	hnfTup2 = isTup2

Definition 13 (Cartesian Products) The *cartesian product* of two partial predicates is defined as the constructor predicate transformer *tup2*. In the following we will use the infix operator (×) :: $\alpha \rightarrow \beta \rightarrow (\alpha, \beta)$ as synonymous with *tup2*. The definition is generalised to arbitrary length tuples:

 $(p_1 \times \ldots \times p_n) (x_1, \ldots, x_n) = (p_1 x_1) \&\& \ldots \&\& (p_n x_n)$

Example 14 (Projections on Tuples) A cartesian product implies the existence of projections. We will show that there are actually two predicate transformers prj1 and prj2 with types

prj1 :: PP (a,b) -> PP a prj2 :: PP (a,b) -> PP b

although their definition is somewhat special. Mathematically, the following must hold:

$$(prj1 \ p)^{-1} = \{x \mid \exists y. \ p(x, y) = True\}$$

 $(prj2 \ p)^{-1} = \{y \mid \exists x. \ p(x, y) = True\}.$

As has been said in the introduction, the kernel language does not forbid free variables in the equations. In fact, the denotational semantics of rules treats them via a least upper bound quantified over all the possible values in their type. This means that an implementation of projections will be:

⁶ We are using *List*, *Nil* and *Cons* as names for ([]), [] and (:), respectively.

⁷ We are using Tup2 as the name for (,).

prj1 p x = p (x, y) -> True; prj2 p y = p (x, y) -> True

This can be surprising to the reader more biased towards functional programming but is by no means strange if we look at Prolog or functional-logic languages as Curry itself. In the special case that the variable being quantified is of a first order type, implementing such an equation is not a problem.

The extension of projections to arbitrary data types and data constructors is trivial. In particular projections on arbitrary length tuples will be used in the following section.

Definition 15 (Projections) Given the data declaration scheme

data $K \alpha_1 \ldots \alpha_l = \ldots |C_i \tau_1 \ldots \tau_{m_i}| \ldots$

for each data constructor C_i and for each $k \in \{1, ..., m_i\}$, the projection partial predicate $prjkC_i$ is defined as⁸

 $prjkC_i :: PP \ \tau_k \to PP \ (K \ \alpha_1 \ \dots \ \alpha_l)$ $prjkC_i \ p \ x = p \ (C_i \ x_1 \ \dots \ x \ \dots \ x_{m_i}) \to True .$

4 Checking and Inference of Demand Properties

This section studies some analyses and their domains of properties under the prism of partial predicates. The novelty of our approach is that, by expressing partial predicates in a subset of the programming language under analysis, a *program transformation approach* is feasible. All the examples below use the well known fold/unfold transformations and are, thus, trivially correct for a language with a lazy declarative semantics.

4.1 Checking

The problem of deciding if a given partial predicate fulfills the demand information of a given function is the *checking problem*: to prove that given f, π_1 and $\pi_2, f : \pi_1 \leftarrow \pi_2$ holds.

Concrete analyses fix a specific domain of checking properties, i.e. only a limited number of partial predicates are allowed. Let us show the translation of several domains of properties into the language of partial predicates and exemplify checking by means of equational reasoning.

Classic Strictness Analysis. The first attempt to mechanize strictness analysis is found in (Mycroft 1980). The aim is to detect when an argument can be safely reduced in advance without affecting the termination properties of the program.

Definition 16 A function *f* is said to be *strict* iff $f \perp = \perp$.

Due to evident practical reasons, this definition is relaxed to cope with the usual case of the argument belonging to a product type:

⁸ Observe that k actually expands: $prj1C_i, prj2C_i, \ldots$

Definition 17 A function f is said to be *strict in its i-th argument* iff

 $\forall x_1 \dots x_{i-1} x_{i+1} \dots x_n. f(x_1, \dots, x_{i-1}, \bot, x_{i+1}, \dots, x_n) = \bot.$

As we have seen in Section 3.1, the first case can be expressed in our setting by saying that the function demands an argument strictly more evaluated than \perp in order to produce a result strictly more evaluated than \perp . When both the argument and result types are constructed, values other than \perp can be finitely presented by enumeration of the different constructors in the type:

Lemma 18 If f is a function with type $f :: K\tau \to K'\tau'$ then it is strict iff $f : hnfK \leftarrow hnfK'$.

Proof

See proof in Section 3.1 and replace *Nat* and *List* with K' and K.

Let us see an example that will also show how to use equational reasoning in order to prove the demand typing correct.

Example 19 Function length

length :: [a] -> Nat length [] = Zero; length (h:ts) = Succ (length ts)

is strict. This will be expressed as *length* : $hnfList \leftarrow hnfNat$. Using the definitions of *hnfNat* and *hnfList* seen before, we have to prove that $hnfNat \circ length \sqsubseteq hnfList$. Then these equivalences follow from the semantics of the kernel language:⁹

Both rules coincide with the equations of hnfList.

Lemma 20 If f is a function with type $f :: (K_1\tau_1, ..., K_n\tau_n) \to K\tau'$ then it is strict in the *i*-th argument iff $f : any \times ... \times hnfK_i \times ... \times any \leftarrow hnfK$.

Proof

 $\begin{aligned} & hnf K \circ f \sqsubseteq any \times \ldots \times hnf K_i \times \ldots \times any \\ \Leftrightarrow & \forall x_1 \ldots x_n . hnf K(f(x_1, \ldots, x_n)) \sqsubseteq (any \times \ldots \times hnf K_i \times \ldots \times any)(x_1, \ldots, x_n) \\ \Leftrightarrow & \forall x_1 \ldots x_n . hnf K(f(x_1, \ldots, x_n)) \sqsubseteq hnf K_i(x_i) \\ \Leftrightarrow & \forall x_1 \ldots x_{i-1} x_{i+1} \ldots x_n . hnf K(f(x_1, \ldots, x_{i-1}, \bot, x_{i+1}, \ldots, x_n)) \sqsubseteq hnf K_i(\bot) \\ \Leftrightarrow & \forall x_1 \ldots x_{i-1} x_{i+1} \ldots x_n . hnf K(f(x_1, \ldots, x_{i-1}, \bot, x_{i+1}, \ldots, x_n)) \sqsubseteq \bot \\ \Leftrightarrow & \forall x_1 \ldots x_{i-1} x_{i+1} \ldots x_n . hnf K(f(x_1, \ldots, x_{i-1}, \bot, x_{i+1}, \ldots, x_n)) = \bot \\ \Leftrightarrow & \forall x_1 \ldots x_{i-1} x_{i+1} \ldots x_n . f(x_1, \ldots, x_{i-1}, \bot, x_{i+1}, \ldots, x_n)) = \bot . \end{aligned}$

⁹ Composition in Curry is denoted by with the symbol '.'.

Example 21 Function plus

is strict in its first argument: $plus : hnfNat \times any \leftarrow hnfNat$. Unfolding the definition of $hnfNat \times any$ we get

 $(\ln f \text{Nat} \times any) (x,y) = (\ln f \text{Nat} x) \& (any y) = \ln f \text{Nat} x$

Unfolding *hnfNat* o *plus*:

(hnfNat . plus) (Zero,m) = hnfNat m ⊑ hnfNat Zero (hnfNat . plus) (Succ n,m) = True = hnfNat (Succ n)

so $hnfNat \circ plus \sqsubseteq hnfNat \times any$ and, by Definition 5, $plus : hnfNat \times any \Leftarrow hnfNat$.

Wadler's Four Point Domain. Many interesting properties are related to the degree of evaluation required on recursive data structures, like lists. For instance, function *length* needs a nil-ending list in order to produce a definite result, but it is immaterial whether one or more of its elements is undefined.

In (Wadler 1987) a four point abstract domain of degrees of definiteness of monomorphic lists is introduced. The domain, in increasing order, can be given as:

 $4 = \{ \bot \sqsubset \infty \sqsubset \bot \in \sqsubset \top \in \}$

representing, respectively, the undefined list, any list with an undefined suffix, finite lists with some undefined elements and total lists. The original paper is not very formal and does not make clear that this semantics for the four elements does not provide conjunctive nor disjunctive closeness. This, of course, can be achieved if their semantics is changed into:

 $\begin{array}{ccc} \top \in & \text{any list} & \bot \in & \text{any nil ending list} \\ \infty & \text{any list in head normal form} & \bot & \text{an undefined list} . \end{array}$

These four levels of definiteness can be represented in our framework by the three partial predicates: *hnfList, spine*, and *nfListNat*:

```
nfListNat :: PP [Nat]
nfListNat (x:xs) = (nfNat x) && (nfListNat xs); nfListNat [] = True;
```

The following can help demonstrate the transformational approach. Let us prove *length* : *spine* \leftarrow *nfNat*. Let *R* denote *nfNat* \circ *length*. Applying standard *fusion* techniques (see Section 4.2) we successively obtain:

R [] = nfNat (length []) = nfNat Zero = True
R (x:xs) = nfNat (length x:xs) = nfNat (Succ (length xs))
= nfNat (length xs) = R xs

so we conclude that R = spine.

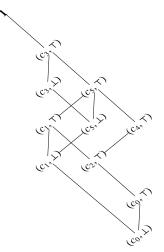


Fig. 3. The lattice of uniform properties on lists.

Uniform Properties. A number of proposals have been made to generalise Wadler's four point domain to any algebraic datatype. Intuitively, the *uniform properties* of a data structure are those invariant under any (type preserving) permutation of the elements of the structure. For instance, if p is an uniform property of lists of naturals, then $p(Zero : (Succ Zero) : \bot) \Leftrightarrow p((Succ Zero) : Zero : \bot)$.

In (Jensen 1994) a powerdomain construction for uniform properties over algebraic datatypes is given, using modalities, along with a strictness logic for reasoning about those properties. His domains are able to express certain properties that do not appear in Wadler's, like a list being empty or being finite with all their elements undefined, etc. The formalism is rather involved and the strictness logic does not lead very naturally to an implementation.

Trying to accommodate those ideas into our framework of partial predicates we immediately see that the domains that arise in Jensen's work correspond essentially to the *folds* on a given datatype. A fold on a data structure is a transformation that replaces every n-ary constructor by an n-ary function. Folds are generic programming constructs in the sense that folds can be defined for every algebraic datatype in an uniform way. For instance, folding lists is done using the following higher order operator:

foldl :: (a -> b -> b) -> b -> [a] -> b foldl f b [] = b; foldl f b (x:xs) = f x (foldl f b xs)

Due to type restrictions, the number of partial predicates on natural lists that can be defined as folds is limited. If only two degrees of definiteness are considered for the naturals (\perp and *True*) that leaves six possible combining functions of type *Two* \rightarrow *Two* \rightarrow *Two*:

 $(c_0) \lambda x. \lambda y. \perp (c_1) \lambda x. \lambda y. x (c_2) \lambda x. \lambda y. x \land y$ $(c_3) \lambda x. \lambda y. True (c_4) \lambda x. \lambda y. y (c_5) \lambda x. \lambda y. x \lor y$ times two values for the base case gives a lattice (Figure 3) of at most 13 abstract values (adding *any*).¹⁰ A similar domain appears in (Benton 1992).

The possibility of using catamorphisms (fold-like functions) on algebraic types as a way of automatically constructing domains for the analysis of programs has been studied and implemented in (Rey 2003).

4.2 Inference

Here we study the problem dual to checking, i.e. how to infer a partial predicate that describes (with reasonable accuracy) demand information for a given function. The concrete inference problem considered in this paper is the following: given f and π_2 , to find the *best* π_1 such that $f : \pi_1 \leftarrow \pi_2$ holds and, more important, to give a usable representation. The best π_1 such that $f : \pi_1 \leftarrow \pi_2$ is $\pi_1 = \pi_2 \circ f$ and an explicit (recursive) definition can be obtained using the transformational approach used for checking. However, giving a compact representation of π_1 suitable for code generation can be difficult and here is where a purely symbolic approach is better suited than the program transformation one.

In order to get an informal understanding of the connection between the program transformation and the symbolic approaches to inference, let us revisit Example 19 recast as an inference problem:

Example 22 The original program is

length [] = Zero; length (h:ts) = Succ(length ts)

We want to infer the degree of definiteness demanded π_1 on its argument by a result in normal form, i.e. $\pi_1 = nfNat \circ length$, so the following must hold:

$$\pi_1 ([]) = (nfNat \circ length) ([])$$

= $nfNat (Zero) = True$
$$\pi_1 (h:ts) = (nfNat \circ length) (h:ts)$$

= $(nfNat \circ Succ) (length (ts)).$

It is easy to see that *nfNat* o *Succ* simplifies to *nfNat* so

$$\pi_1 (h: ts) = nfNat (length (ts))$$

= (nfNat \circ length) (ts) = \pi_1 (ts),

resulting in a rather generative set of equations for π_1 (that coincide with equations for *spine*).

The following section shows a more systematic method to manipulate partial predicates properties in a fully symbolic way. From the program under analysis and the inference question, a set of inequalities among symbolic representations of partial predicates is generated, and is this set which is manipulated – although the meaning of these rewritings must mimic the original program transformations.

¹⁰ The notation (c, v) stands for *foldl* c v. The exact cardinality of the abstract domain is 11, as some of the combinations coincide: *foldl* $c_2 \perp = foldl c_4 \perp = foldl c_0 \perp$.

5 From Partial Predicates to Set Expressions

The most natural interpretation of a partial predicate $\pi \in \alpha \to Two$ is a subset of the domain D_{α} , a set of trees, more exactly an *ideal* set of *partial* trees. This section is devoted to show that set expressions and set constraint based analysis (Pacholski and Podelski 1997) can be used as a framework for the checking and inference of partial predicate typings.

5.1 Basic Notions

'Set constraints are first-order logic formulae interpreted over the domain of sets of trees' (Pacholski and Podelski 1997). Set expressions (*e*) are expressions built from variables interpreted over sets of trees, function symbols interpreted as functions over sets of trees and standard set operators (union, intersection, inclusion, complement, etc.). A system of set constraints is a conjunction of inclusions of the form $e_l \subseteq e_r$ with some restrictions on the set expressions that can appear in the left or right hand sides.

Co-definite set constraints (Charatonik and Podelski 1998) is the class of set constraints where constraints are inclusions between *positive* set expressions and where the set expression in the left hand side is restricted to contain variables, constants, unary function symbols and the union operator.¹¹ Sets of co-definite constraints, if satisfiable, always have a greatest solution. The satisfiability problem for co-definite set constraints is DEXPTIME-complete and an algorithm is given in (Charatonik and Podelski 1998).

The restriction to positive expressions - i.e. without the use of complementation - is essential for our purposes, as the complement of an ideal is not an ideal. The rest of operators are continuous, so we can translate existing results in set constraint theory to our domains.

The following definitions formalize these notions.

Definition 23 (Set Expressions) Given a typed alphabet Σ with constants (a, b, c, ...) and nonconstant function symbols (f, g, h, ...) and a typed set **VS** of variable symbols (u, v, x, y, ...), *set expressions* follow the syntax:¹²

$$e ::= x | a | f(u_1, ..., u_n) | f_{(k)}^{-1}(u) | e_1 \cup e_2 | \emptyset.$$

This syntax represents finite and infinite trees and we will use the notation T_{Σ} for the whole set of well-formed (w.r.t. types) trees.

Definition 24 (Valuation) A *valuation* (σ) is a function from variable symbols to proper subsets of T_{Σ} (σ : **VS** $\rightarrow 2^{T_{\Sigma}}$).

Definition 25 (Interpretation of Set Expressions) Given a valuation σ , the standard in-

¹¹ In Definition 28 a restricted but equivalent characterisation is proposed.

¹² The symbol originally used in set constraint theory is \perp .

terpretation I_{σ} of set expressions over Σ and **VS** is defined as¹³

$$I_{\sigma}(x) = \sigma(x)$$

$$I_{\sigma}(a) = \{a\}$$

$$I_{\sigma}(f(u_{1}, \dots, u_{n})) = \{f(t_{i}, \dots, t_{n}) \mid \forall i \in \{1, \dots, n\}. t_{i} \in I_{\sigma}(u_{i})\}$$

$$I_{\sigma}(f_{(k)}^{-1}(u)) = \{t \mid \exists t_{1}, \dots, t_{n}. t_{k} = t \land f(t_{1}, \dots, t_{n}) \in I_{\sigma}(u)\}$$

$$I_{\sigma}(e_{1} \cup e_{2}) = I_{\sigma}(e_{1}) \cup I_{\sigma}(e_{2})$$

$$I_{\sigma}(\emptyset) = \emptyset.$$

Definition 26 (Solution) A valuation σ is a *solution* of a set constraint $e_l \subseteq e_r$ iff

$$I_{\sigma}(e_l) \subseteq I_{\sigma}(e_r)$$
.

Definition 27 (Satisfaction) A system of set constraints *S* is *satisfiable* if there is some valuation σ that is a solution of every constraint in *S*.

Definition 28 (Co-definite Set Constraints) A constraint φ is a co-definite set constraint when it follows the syntax:

$$\tau ::= x | f(u_1, \dots, u_n) | \tau_1 \cup \tau_2 | \emptyset$$

$$\varphi ::= a \subseteq x | x \subseteq \tau | x \subseteq f_{(k)}^{-1}(u).$$

We will use the notation $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ to refer to the system $\varphi_1 \land \varphi_2 \land \dots \land \varphi_n$.

5.2 Co-definite Set Constraints and Partial Predicates

A partial predicate π is interpreted as the set $\pi^{-1}(True) = \{x \in T_{\Sigma} \mid \pi(x) = True\}$ that is in the codomain of interpretations of set expressions. Conversely, if *S* is an ideal, $\Pi(S)$ will denote its corresponding partial predicate, i.e. $\Pi(S)(x) = True$ if $x \in S$, otherwise $\Pi(S)(x) = \bot$. We will encode partial predicates as variables and the greatest solution of a system of co-definite set constraints.

Example 29 (Some basic partial predicates) The following table shows how some partial predicates can be encoded (z, s, n and c refer, respectively, to constructor predicate transformers *zero*, *succ*, *nil* and *cons* as described in Definition 7):

Partial predicate	System of set constraints	Variable
$hnfNat = z \sqcup s(any)$	${hnf \subseteq x \cup y, x \subseteq Zero, y \subseteq Succ(_)}$	hnf
$nfNat = z \sqcup s(nfNat)$	$\{nf \subseteq x \cup y, x \subseteq Zero, y \subseteq Succ(nf)\}$	nf
$nfListNat = n \sqcup c(nfNat, nfListNat)$	$\{nf \subseteq u \cup v, u \subseteq [], v \subseteq (nf' : nf),\$	nf
	$nf' \subseteq x \cup y, x \subseteq Zero, y \subseteq Succ(nf')$	
$spine = n \sqcup c(any, spine)$	$\{snf \subseteq u \cup v, u \subseteq [], v \subseteq (_: snf)\}$	snf

where _ represents fresh variables.

¹³ Where symbols \cup and \emptyset are overloaded.

Example 30 (Dependency) The intrinsic difficulty of working with dependency that was already patent in (Mariño et al. 1993) can be put in relation with the syntax of set constraints. A typical property that can be represented with dependent demand patterns is that a pair is made of lists of the same length. For instance, a demand typing for function *zip*

is zip : $spine \leftarrow same length$ where

samelength :: PP ([a],[b])
samelength ([],[]) = True; samelength (x:xs,y:ys) = samelength (xs,ys)

The greatest solution to the following set constraint system for the variable *sl* captures the dependent information of the partial predicate *samelength*:

 $\{sl \subseteq (l_1, l_2) \cup (l'_1, l'_2), l_1 \subseteq [], l_2 \subseteq [], l'_1 \subseteq x : xs, l'_2 \subseteq y : ys, (xs, ys) \subseteq sl\}.$

But this system is not co-definite (last constraint has a binary function symbol in the left hand side). In order to get a co-definite set constraint system, the last constraint is substituted by two constraints: $xs \subseteq (,)_{(1)}^{-1}(sl)$ and $ys \subseteq (,)_{(2)}^{-1}(sl)$. With the substitution we have lost the dependency information. Nevertheless, the solution is correct with respect to the interpretation of the partial predicate in the following formal sense:

samelength⁻¹(*True*) $\subseteq I_{\sigma}(sl)$

where σ is the greatest solution to the system of co-definite set constraints.

5.3 Generating Co-definite Set Constraints

DAC (*Demandedness Analysis for Curry*) is a tool that generates a system of co-definite set constraints from a given program. In this section we explain how DAC generates the system. Observe that the solving of a system of set constraints is completely independent of the application, i.e. the fact that we are encoding partial predicates is immaterial. Observe, as well, there could be other ways to generate correct systems of constraints.

In the first place, we need to introduce the logic that relates the variables in the system of set constraints with the meaning of the program. This connection relies on the fact the partial predicates the user is interested in are defined as functions in the kernel language.

Definition 31 (Variable Construction) Given $f \in FS$ and $p \in FS$, with types $\tau_1 \rightarrow \tau_2$ and $PP \tau_2$, respectively, the infix operator (•) is used to construct a new variable $p \bullet f \in VS$ that represents the degree of evaluation demanded by f in order to give a result as evaluated as p.

Auxiliary variables are introduced for different subterms in the program equations that define *f*:

- $-p \bullet f.i$ refers to the demandedness information introduced by the *i*-th equation defining f
- $p \bullet f.i.pos$ refers to the subterm $lhs|_{pos}$ if lhs is the left hand side of the *i*-th equation defining f.

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nf <i>•plus</i>	\subseteq	$nf \bullet plus.1 \cup nf \bullet plus.2$	(1)
nf•plus.1	\subseteq	(nf•plus.1.1, nf•plus.1.2)	(2)
nf•plus.1.1	\subseteq	Zero	(3)
nf•plus.1.2	\subseteq	nf	(4)
nf•plus.2	\subseteq	(Succ(nf•plus2.1.1), nf•plus.2.2)	(5)
Succ(nf_plus.2.1)	\subseteq	nf	(6)
nf•plus.2.1.1	\subseteq	nf_ <i>plus</i> .2.1.1.1	(7)
nf•plus.2.2	\subseteq	nf_ <i>plus</i> .2.1.1.2	(8)
(nf_plus.2.1.1.1, nf_plus.2.1.1.2))	\subseteq	nf e plus	(9)

Fig. 4. Set of constraints generated from program plus

 $-p_f.i.pos$ refers to the subterm $rhs|_{pos}$ if rhs is the right hand side of the *i*-th equation defining f.

Finally, the constraint generation algorithm generates variables of the form $(q \cdot g) \cdot f$. Although these can be given a neat interpretation, the constraint solver will treat them as indivisible, so they will have to be transformed in some way in order to be useful.

The intuitive meaning of $d \bullet f$ is to denote (an approximation of) $d \circ f$.¹⁴ This connection will be formalized below. The intuitive meaning of $p \bullet f.i.pos$ is the projection at position *pos* of the set $p \bullet f.i$. Variables generated from positions in the right hand sides have a less evident meaning or, perhaps, more operational but they provide valuable information for compilation.

Example 32 Figure 4 shows a system of constraints generated from function *plus* (Example 21) to give a result in normal form.

5.4 Generating Systems of Set Constraints

The generation scheme is presented here as a set of rules. These will be stated in a moderately informal way, in order to hide some of the details to the reader, especially those concerned with the handling of occurrence indices.

Rule 1 (Main Function Constraint) Given a partial predicate symbol *p* and function symbol *f*, with defining rules

$$f t_1 = b_1 \quad \dots \quad f t_n = b_n$$

the following constraint is generated:

$$p \bullet f \subseteq p \bullet f.1 \cup \ldots \cup p \bullet f.n.$$

¹⁴ Hence the choice of the symbol '•'.

Rule 2 (Main Rule Constraint) For every rule

$$f_i(t_1,\ldots,t_n) = b_i$$

the following constraint is added:

 $p \bullet f.i \subseteq (p \bullet f.i.1, \ldots, p \bullet f.i.n).$

Notice that this is a *lossy* step, i.e. possible dependencies among the arguments through the body of the rule (b_i) can be lost. This means that an analyzer based on program transformation techniques can, at least theoretically, achieve a better accuracy.

Rule 3 (Head Constraints) For every rule

 $f_i(t_1,\ldots,t_n) = b_i$

and for every $j \in \{1, ..., n\}$ the following constraint is added:

$$p \bullet f.i.j \subseteq \Delta(p, f, i.j, t_i),$$

where Δ is the function that constructs a set expression from a term by replacing every occurrence of a program variable with a demand variable decorated with its position, i.e.:

$$\Delta(p, f, i, c(t_1, \dots, t_m)) = c(\Delta(p, f, i.1, t_1), \dots, \Delta(p, f, i.m, t_m))$$

$$\Delta(p, f, i, x) = p \bullet f.i.$$

This step usually generates superfluous constraints of the form $v \subseteq v$ which can be discarded later.

Rule 4 (Body Constraints) We can distinguish several cases here:

1. (The body is a variable) If the rule is of the form

$$f_i(t_1,\ldots,t_n) = x$$

x being a program variable, the constraint

$$p \bullet f.i.1.pos \subseteq p$$

where *pos* is the position where *x* occurs in the left hand side, is added to the system.

2. (The body is a constant) If the rule is of the form

$$f_i(t_1,\ldots,t_n) = k$$

k being a constant, the constraint

$$k \subseteq p$$

is added to the system. This constraint will often be trivial.

3. (*The body is a function application*) This is the clumsiest case. To simplify the presentation, let us assume, without loss of generality, that the form of the rule is the following:

$$f_i(\bar{t}) = g(h_1(\bar{t}), \ldots, h_m(\bar{t})).$$

The constraints

$$p_f.i.1 \subseteq (,)^{-1}_{(1)}(p \bullet g)$$
$$\vdots$$
$$p_f.i.m \subseteq (,)^{-1}_{(m)}(p \bullet g)$$

are added to the system, and also the constraints :

$$p \bullet f.i \subseteq p_f.i.1 \bullet h_1$$

$$\vdots$$

$$p \bullet f.i \subseteq p_f.i.m \bullet h_m.$$

Notice that this step is responsible for the appearance of 'nested' demand variables.

Rule 5 (Simplification) In this step two kind of actions are performed: the shortcut of transitive chains and the simplification of nested demand variables.

If a variable of the form $(p \cdot g) \cdot f$ is found, and a concrete representation p' for $(p \cdot g)$ is known – which is usually the case when g is a data constructor – then it is replaced by $p' \cdot f$. This step is necessary when standard – i.e. *problem independent* – techniques for solving the constraint systems are going to be used.

Rule 6 (Weakening) Sometimes it is not easy to compute the $(p \cdot g)$ of the previous step, so some sort of approximation is necessary, i.e. using any p'' satisfying $p'' \supseteq p'$ instead of p'. This is often possible. In the worst case *any* can be used.

The following result states the soundness of the analysis based on the solution of this set of constraints.

Theorem 33 (Soundness of the Analysis) Let S denote the system of constraints generated from a given program applying the rules above. Let σ be a solution of S. For every variable $d \bullet f$ occurring in S the following must hold:

$$\Pi\llbracket \sigma(d \bullet f) \rrbracket \sqsupseteq d \circ f \, .$$

We will just sketch the proof here. A more detailed explanation can be found in Appendix B. The idea is to apply a set of program transformation rules to $d \circ f$, for every possible combination of *d* and *f*, so that the resulting program is structurally similar to *S*.

6 Application to Code Generation

For the sake of brevity, we will not develop the issues related to code generation in full here. A detailed discussion can be found in (Mariño 2002), first in an abstract fashion – by means of an operational semantics driven by degrees of definiteness – and then in the context of a stack-based machine. Anyway, code generation from the demand information (represented by partial predicates) is a challenging task on its own and some of the details of a full compiler are still open.

The basic idea is that different, specialized versions of a given function can be compiled

example	п	eager	naïve lazy	demand driven	
	10	0.47	1.00	0.62	
	11	0.91	1.98	1.24	
sublists	12	1.86	3.96	2.42	
	13	3.68	7.94	4.80	
	14	7.42	15.82	9.60	
	15	15.05	31.75	19.23	
	4	1.06	0.89	0.21	
n queens	5	14.50	10.55	2.60	
1	6	247.03	174.30	39.30	

Table 1. Runtimes of the example programs in seconds.

for different degrees of evaluation demanded on its result. For instance, if the main goal of a certain program is

?- mergeSort (f x)

the result must be shown in normal form, so a special version *mergeSort_nf* will be generated. In order to give a result in normal form, the argument to *mergeSort* must also be a total value, which implies that f can also be replaced by an specialized version $-f_nf$ – and so on, i.e. demand is back-propagated from the result to the argument expressions.

Some Experiments The following example programs were executed on a stack-based narrowing machine (Moreno-Navarro et al. 1990). The narrowing machine is an extension of a purely functional machine enriched by mechanisms for unification and backtracking, similar to the WAM.

Based on the implementation of the presented ideas on the stack-based narrowing machine, we have tried some example programs and measured their runtimes with the naive lazy approach and with our new approach. Additionally, we have measured the runtimes for eager narrowing.

We have investigated the following example programs: 1) the computation of all the sublists of *reverse* [1, ..., n], and 2) the *n*-queens problem (using a simple generate and test approach). Both examples have the property that a lot of reevaluations are needed since demanded arguments are not evaluated in advance. Due to space limitation we omit the code. The runtimes are depicted in Table 1. The examples show that the runtimes can be considerably improved, if the demanded arguments are evaluated in advance. Notice that in examples like *n*-queens, the lazy strategy is even better than the eager one. Moreover, bigger real examples have a lot of nested function calls, which implies a considerable risk of reevaluation.

Table 2 shows the results obtained with the *sublists* \circ *reverse* example using the translation into Prolog, with and without code optimization based on demand analysis. The measures have been taken in discrete resolution steps.

sublists \circ reverse $[1, \ldots, n]$	n	without demand anal.	with demand anal.	ratio
	3	41	36	1.13
	4	88	72	1.22
	5	183	141	1.29
	6	374	275	1.36
	7	757	538	1.40
	8	1524	1058	1.44

Table 2. Results in a lazy producer-consumer scheme.

7 Related Work

Our original work on demand analysis (Mariño and Moreno-Navarro 1992; Moreno-Navarro et al. 1993; Mariño and Herranz 1993; Mariño et al. 1993) was based on the generation and solution of a set of *demand equations* that were solved in a domain of regular trees (*demand patterns*). Similar, in spirit, to the techniques presented in Section 5, a semantic justification was missing and the solving method was ad-hoc. Partial predicates provide the necessary semantic ground and the advances in set constraint resolution makes unnecessary to reinvent the wheel.

With respect to partial predicates, the most striking similarity is with *projection analysis* (Wadler and Hughes 1993). However, the rationale and meaning for these two formalisms differ in some key aspects. A *projection*, in a domain-theoretic sense, is an idempotent approximation to the identity (in a given type), i.e. $\alpha :: \tau \to \tau$ is a projection (in τ) iff $\alpha \sqsubseteq id$ and $\alpha \circ \alpha = \alpha$.

While partial predicates try to be an extension of classic strictness analysis, projection analysis are designed to capture the property that a given function is invariant under certain program transformations. The typical example is *head-strictness*, the property that a function on lists gives the same results when the list constructor in its argument is replaced by a version strict in its first argument. Mathematically, this transformation is a projection $H :: [a] \rightarrow [a]$, so the property of f being head-strict is expressed as $f = f \circ H$. In general, projection analysis studies properties of the form $\alpha \circ f = \alpha \circ f \circ \beta$, where α and β are projections. These are abbreviated as $f : \alpha \Rightarrow \beta$.

Properties expressible in both formalisms are different. First of all, let us show that head strictness cannot be represented by a partial predicate typing.

Theorem 34 There is no pair of partial predicates π_1, π_2 such that the set of functions $\{f | f : \pi_1 \leftarrow \pi_2\}$ coincides with that of the head-strict ones.

Proof

Let us note that the property of being head-strict is just 'too polymorphic' as it does not take into account the type of the result, so equivalence just makes sense fixing a particular type, i.e. considering just the head-strict functions for a given type $[\sigma] \rightarrow \tau$. This makes the proof shorter, as we can restrict ourselves to the type $[Bool] \rightarrow Bool$. There will be just five possibilities for π_2 : nothing, true, false, hnf and any. The key to the proof is in

considering the functions *any*, *nothing* (which are head-strict) and *spine* (which is not). Any combination of partial predicates which would hold for both *any* and *nothing* would also hold for *spine*, contradiction.

Projections, on the other hand, are able to express partial predicate typings, but only if a tricky artifact is added to the formalism: assuming the existence in the domain of a new element $(\frac{1}{2})$ less defined than \perp . This had to be introduced by Wadler and Hughes in order to capture classic strictness with projections, but complicates the formalism in several ways. The following result holds assuming programs are $\frac{1}{2}$ -strict:

Theorem 35 For every pair of partial predicates π_1, π_2 there is a pair of projections α, β such that the set $\{f | f : \pi_1 \leftarrow \pi_2\}$ coincides with $\{f | f : \alpha \Rightarrow \beta\}$ — under reasonable type restrictions.

Proof

The proof is constructive. Define α and β in the following way:

$\alpha x = x$	if $\pi_2(f x) = true$	$\beta x = x$	if $\pi_1 x = true$
$\alpha x = 4$	otherwise	$\beta x = 4$	otherwise.

Let us examine both implications:

(i) $(f:\pi_1 \leftarrow \pi_2 \implies f:\alpha \Rightarrow \beta)$

There are two possibilities for any argument *x* to *f*:

a) $(x \in \pi_1)$ Trivial: $(\alpha \circ f \circ \beta) x = (\alpha \circ f)(\beta x) = (\alpha \circ f) x$.

b) $(x \notin \pi_1)$ In this case we know that $f x \notin \pi_2$. So, in one hand we have:

 $(\alpha \circ f \circ \beta) x = \alpha(f(\beta x)) = \alpha(f \not z) = \alpha \not z = \not z.$

On the other hand, using the fact that $f x \notin \pi_2$, $\alpha(f x) = \frac{1}{2}$.

(ii) $(f: \alpha \Rightarrow \beta \implies f: \pi_1 \leftarrow \pi_2)$

Using *reductio ad absurdum*: suppose there is some *z* s.t. $z \notin \pi_1$ and $f z \in \pi_2$. Then it is trivial to show that $(\alpha \circ f \circ \beta) z = \frac{1}{2}$ and $(\alpha \circ f) z = f z$, contradiction.

Projections in the lifted domain are no longer expressible in source code, precluding the possibility of using program transformation or the other techniques that are applicable to partial predicate typings.

The use of program transformation techniques for program analysis is used in other approaches, like *abstract compilation*, but to our best knowledge, the application of the fold/unfold method of program transformation for program analysis is a novel idea. The only similar approach appears in (Gallagher and Peralta 2000) to develop a type inference system for Prolog, and (Comini et al. 2000) to verify program properties.

There is also some existing work on using constraint generation for this kind of problems. In (Sekar and Ramakrishnan 1995), two degrees of definiteness are defined: normal form and head normal form, which leads to the notion of *e*-demand (normal form needed) and *d*-demand (head normal form needed). Recursive equations for computing how these degrees of demand are propagated are generated for a given program, based on an operational semantics. The authors also mention the possibility of using demand analysis for sequentiality recovery, although the idea is not developed there.

8 Open Issues and Future Work

The importance of demandedness analysis goes beyond functional-logic languages. In (?) dependent demand patterns are used for the analysis of concurrent (constraint) logic languages.

The question whether program transformation tools can be used for program analysis following the techniques presented here is still open, although several problems appear. In the first place, deciding on the equality of functions is harder, in general, than checking the equality of set expressions. Second, although the theory behind program transformation is well developed, practical implementations are scarce. However, this is an active area and we plan to study the possibility of adapting the tools by Vidal's group (Ramos et al. 2005) to serve this purpose.

Another possible extension of this work is to study the application of partial predicates to other analysis problems, like groundness, etc.

The extension of the analysis when higher order functions are used deserves an additional discussion. In fact, some complications do appear if higher order definitions are introduced. Let us consider an example involving curried definitions.

Take, for instance, the standard definition of the addition of Peano naturals of example 21. An interesting property that we would like to express is the fact that the first argument must be evaluated to head normal form in order to get a result in head normal form. This is very simple for the noncurried form: plus : $(hnfNat \times any) \leftarrow hnfNat$, but it is not clear at all how to express that for the curried version. In first place, (+) maps naturals to *a new function*, and it is not this function we are interested in, but the result of applying it to any other natural number. To grab the problem more formally, we will make use of the following lemma.

Lemma 36 (Currying lemma) Let $(f \circ)$ denote $\lambda x. f \circ x$. Then, the following holds:

$$\operatorname{curry}(f \circ g) = (f \circ) \circ (\operatorname{curry} g).$$

What we are looking for is a property of the form: (+) : $hnfNat \leftarrow \pi$ and what we actually have is *plus* : ($hnfNat \times any$) $\leftarrow hnfNat$. This is equivalent to:

 $hnfNat \circ plus \sqsubseteq (hnfNat \times any)$.

As *curry* is continuous, we can curry both sides of the inequality:

curry (*hnfNat* \circ *plus*) \sqsubseteq *curry* (*hnfNat* \times *any*)

and using the lemma above:

 $(hnfNat \circ) \circ (+) \sqsubseteq curry (hnfNat \times any)$

or equivalently

$$(+)$$
 : curry (hnfNat × any) \Leftarrow (hnfNat \circ).

Well, this gives *essentially* the same information as the demand typing for the noncurried version, but there is a problem: the functions in the typing are no longer partial predicates, i.e. they are not in the domain $\tau \rightarrow Two$.

There are essentially two ways of dealing with these problems in practice. One possibility is to avoid higher order definitions as much as possible, translating curried versions to noncurried ones – and vice-versa with the results of the analysis.

The other possibility is to generalize demand types to pairs of functions in the domain $\tau \rightarrow \tau'$. Although the theoretical interest of this lifting to higher order domains is apparent, and a higher order metalanguage seems feasible, the practical use will be very restricted, as the techniques in Section 4.2 will not be applicable.

9 Conclusion

We have presented a semantic framework for the denotation of demand properties, decoupling it from its abstraction or any implementation detail of the analysis. In spite of defining a new language and an associated demand logic (cf. the strictness logic in (Benton 1992)), properties are expressed in the language under study and problem independent techniques are used as proof methods: equational reasoning or set constraint solving.

The collection of analysis that can be modelled includes classic strictness analysis, Wadler's four point domain, etc. In particular it allows to describe demand analysis that is very important for the efficient implementation of several aspects of functional-logic languages: efficient implementation of lazy narrowing (Moreno-Navarro et al. 1993; Mariño et al. 1993), compilation of nonsequential programs (Mariño and Moreno-Navarro 2000) or the lazy management of default rules (Moreno-Navarro 1996). The formalism has been used to prove the correctness of a method based on set constraint solving, in a constructive way. It can also be used to generate domains suitable for abstract interpretation (uniform predicates).

Furthermore, the formalism is quite intuitive as it resembles the language to reason about and uses program transformation techniques. We have also shown how polymorphism and higher-order properties can be managed – at least theoretically – in this framework, although the extension for the analysis of higher-order program presents some challenges.

An original feature of this research is that it is completely based on a declarative (model theoretically) denotational semantics of the language, rather than on an operational one.¹⁵ From a practical point of view, performing the analysis on operational data is often easier, but considering that the domain of properties can be understood in a purely declarative setting, we wanted to explore the possibility of performing the analysis without using a particular operational semantics.

We also felt that the applications of set constraints to program analysis have been biased towards problems stated in an operational fashion, and that deriving set constraints from semantic equations was an original approach and a challenge worth taking up.

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¹⁵ We include here denotational presentations of essentially operational semantics like (Zartmann 1997).

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Appendix A Semantics of the Kernel Language

The following lines describe a denotational presentation of a declarative semantics for the language used in this paper. We start providing a declarative, logical semantics. Let us define the semantic domains first. H is the cpo completion of the Herbrand universe formed with all the (data) constructors in a program. The (higher order) domain of values D is given as the least solution to the equation

$$D \cong H + [D_{\perp} \to D_{\perp}] + \sum_{i} \{ (C \ d_1 \dots d_i) | C \in DC^i, \ \forall k.d_k \in D \}$$

Environments are type-preserving mappings from variable symbols to H, and interpretations (for a given program) map every function symbol to a value in D:

$$Env = VS \rightarrow H$$
$$Int = FS \rightarrow D$$

Environments can be lifted in the standard way to functions from terms (with variables) to H, with the usual overloading. We regard constructors as free, and thus their denotation is the usual, standard one.

Definition 37 (Models) An interpretation *I* is a model of *a ground instance* $l' = b' \rightarrow r'$ of a defining rule $l = b \rightarrow r$ iff

$$\mathbf{E}[[l']]I \supseteq \mathbf{E}[[b]]I \to \mathbf{E}[[r']]I$$

An interpretation is a model of a rule when it models all its ground instances:

$$\forall \sigma. \mathbf{E}[[\sigma l]] I \supseteq \mathbf{E}[[\sigma b]] I \to \mathbf{E}[[\sigma r]] I$$

being σ a well typed grounding substitution. An interpretation *I* is a model of a program *P* (in symbols $I \models P$) iff *I* is a model of every defining rule in *P*.

Next we define a denotational construction for such a semantics. We will make use of the following semantic functions¹⁶:

$$\mathbf{F} : Int$$
$$\mathbf{R} : Rule \to Int \to Int$$
$$\mathbf{E} : Exp \to Int \to D$$

E is just recursive evaluation of expressions according to the semantics of the program, and can be defined as the homomorphic extension of the semantics for function symbols (**F**). It is needed in order to evaluate the right hand sides of rules. **R** is the interpretation transformer associated with each rule of the program and represents the amount of information added by every possible application of that rule:

$$\mathbf{R}[[f \ t_1 \dots t_n = b \to r]]I = \lambda fs.(fs = f) \to \lambda x_1 \dots x_n. \bigsqcup_{\rho \in Env} (\rho t_1 \doteq x_1 \wedge \dots \wedge \rho t_n \doteq x_n \wedge \mathbf{E}[[\rho b]]I) \to \mathbf{E}[[\rho r]]I$$
$$\mathbf{F}_P = lfp(\bigsqcup_{rule \in P} (\mathbf{R}[[rule]]))$$

¹⁶ Properly speaking, they all depend on the program $-\mathbf{F}_P$, \mathbf{E}_P , etc. – but the subscript will be dropped when no confusion may arise.

The symbol (\doteq) denotes strict equality and ($\cdot \rightarrow \cdot$) is shorthand for ($\cdot \rightarrow \cdot | \perp$). For every equational program *P*, **F**_{*P*}, **R** and **E** are continuous¹⁷.

The following result, proved in (Mariño 2002), states the adequacy of both presentations:

Theorem 38 Let $Rules_P(f)$ be the set of rules defining function symbol f in propgram P. For every functional-logic program P, and function symbol $f \in FS_P$, $\mathbf{F}_P f$ is the minimal model for the rules in $Rules_P(f)$.

Appendix B Proof of Theorem 33

Theorem 33 states the soundness of the analysis based on the solution of a set of constraints. If S denotes the system of constraints generated from a given program applying the rules in Subsection 5.4 and σ is a solution of S, then for every variable $d \bullet f$ occurring in S the following must hold:

The following is still very sketchy — a full proof would be much longer. Some lemmata on valid program transformations are necessary in order to justify the different rules for constraint generation:

Lemma 39 The following program transformations are valid according to the semantics of the kernel language:

1. Any function symbol f defined by rules

$$f t_1 = b_1 \quad \cdots \quad f t_n = b_n$$

can be rewritten as

$$f = f.1 \quad \cdots \quad f = f.n$$

where

$$f.1 t_1 = b_1 \quad \cdots \quad f.n t_n = b_n$$

2. Any program rule

$$f(t_1,\ldots,t_n)=r$$

can be rewritten as

$$f(t) = b \rightarrow r^*$$

where *b* is a guard conveying all the matching information and r^* is obtained from *b* replacing every occurrence of a variable in the left hand side by an application of a selector function. In more detail, $f(t_1, ..., t_n) = r$ is rewritten as

$$f(t) = match(t_1, t) \land \cdots \land match(t_n, t) \rightarrow \delta(r, t)$$

¹⁷ This is due to the operators involved in their definition. Observe that the *lub* in the right hand side of the definition of **R** is not infinite because the conditional inside limits the possibilities to \perp or $\mathbf{E}[[\sigma r]]i$ – where σ is unique – and thus is well defined.

where

$$match(K,t) = isK(t)$$

$$match(C(w_1,...,w_m),t) = isC(t) \wedge match(w_1,t|_1) \wedge \cdots \wedge match(w_m,t|_m)$$

$$match(x,t) = True$$

and

$$\delta(f(e_1, \dots, e_j), t) = f(\delta(e_1, t), \dots, \delta(e_j, t))$$

$$\delta(x, t) = \operatorname{proj}(pos(x, t))(t)$$

provided that proj(p)(t) returns $t|_p$ and that pos(x, t) is the position where x occurs at t. 3. Given a rule

$$f(t_1,\ldots,t_n)=b\to r$$

either r is a variable, or a constant, or it can be rewritten as

$$f(\bar{t}) = b \rightarrow g(h_1(\bar{t}), \dots, h_m(\bar{t}))$$

Moreover, this is true for the whole set of rules in a program, i.e. the newly introduced function definitions – for h_1, \ldots, h_m – can again be normalized and the whole transformation is terminating.

Proof of Rule 1 (Main Function Constraint) From lemma 39.1, the definition of (\Box) and the semantics of the language,

$$p \circ f = p \circ f.1 \sqcup \cdots \sqcup p \circ f.n$$

for every partial predicate p and function symbol f.

Proof of Rule 2 (Main Rule Constraint) This is one of the lossy steps. This is justified by a generic property of projections: if

$$p \in PP(\tau_1 \times \cdots \times \tau_n)$$

then

$$p \sqsubset p_1 \times \cdots \times p_n$$

so if f.i is a function on tuples

$$p \circ f.i \sqsubset (p \circ f.i)_1 \times \cdots \times (p \circ f.i)_n$$

Proof of Rule 3 (Head Constraints) Given program rule

$$f.i(t_1,\ldots,t_n)=r_i$$

we have to prove the inequation

$$p \bullet f.i.k \subseteq \Delta(p, f, i.k, t_k)$$

and, in fact, we are going to prove the equality. Remember that $p \bullet f.pos$ is intended to represent the projection at position pos of $p \circ f$. From lemma 39.2, we have

$$f.i(t) = match(t_1, t) \land \cdots \land match(t_n, t) \rightarrow \delta(r_i, t)$$

so

$$p \circ f.i(t) = match(t_1, t) \land \dots \land match(t_n, t) \rightarrow p(\delta(r_i, t))$$

and from Def. 15

$$prjk(p \circ f.i) x_k = p(f.i(x_1, \dots, x_n)) \to True$$
$$= match(t_k, \bar{x}) \wedge \dots \wedge p(\delta(r_i, t))(\bar{x})$$

where only x_k and its subterms contribute to the result. From the definition of Δ and δ it can be proved that

$$\Delta(p, f, i.k, t_k) x_k = prjk(p \circ f.i) x_k$$

Proof of Rule 4 (Body Constraints) In Sec. 5.4 three cases were considered:

1. (The body is a variable) The inequality to prove is

$$p \bullet f.i.k \subseteq p$$

provided that the rule for f.i is of the form

 $f.i(t_1,\ldots,t_n)=x$

and that *x* occurs at position *k* at the head of the rule.

From lemma 39.2, we have that the definition of f.i can always be cast as:

$$f.i(t) = match(t_1, t) \land \dots \land match(t_n, t) \rightarrow proj(k)(t_n)$$

and then

$$p \circ f.i(t) = match(t_1, t) \land \dots \land match(t_n, t) \rightarrow p(proj(k)(t))$$

so

$$prjk(p \circ f.i) x_k = match(t_1, t) \land \cdots \land match(t_n, t) \land p(x_k)$$

which is clearly less defined or equal than *p*, hence the inequality.

- 2. (The body is a constant) The proof is very similar to that of the last case.
- 3. (*The body is a function application*) Without loss of generality (see lemma 39.3), we will restrict ourselves to rules of the form

$$f(t) = b \rightarrow g(h_1(t), \dots, h_m(t))$$

so

$$(p \circ f)(t) = b \rightarrow (p \circ g)(h_1(t), \dots, h_m(t))$$

To show the correctness of the inequalities

$$p_f.1 \subseteq (p \bullet g)_1$$

$$\vdots$$

$$p_f.m \subseteq (p \bullet g)_m$$

$$p \bullet f \subseteq p_f.1 \bullet h_1$$

$$\vdots$$

$$p \bullet f \subseteq p_f.m \bullet h_m$$

we can proceed by *reductio ad absurdum*. As the $p_f.i.j$ variables are only constrained by these inequalities, we can be sure that they will take the greatest values. The only possibility for the system to fail is that one inequality in the second set fails. Let z be an element such that $z \in (p \circ f)$ and $z \notin (p_f.k \bullet h_k)$. Introducing z in the equation above for $(p \circ f)$ leads to immediate contradiction.

Proof of Rule 5 (Simplification) Trivial, as this is essentially replacement of equals by equals.

Proof of Rule 6 (Weakening) Trivial as this is essentially replacement of a term by a greater one in the right hand side of "lesser than" inequation.