On the complexity of identifying Head Elementary Set Free programs ∗

Fabio Fassetti[†]
ICAR/CNR
via P. Bucci, 41C
87036, Rende (CS), Italy
E-mail: f.fassetti@deis.unical.it

Luigi Palopoli
DEIS, University of Calabria
via P. Bucci, 41C
87036, Rende (CS), Italy
E-mail: palopoli@deis.unical.it

submitted 11 December 2008; revised 29 April 2009; accepted 22 May 2009

Abstract

Head-elementary-set-free programs were proposed in (Gebser et al. 2007) and shown to generalize over head-cycle-free programs while retaining their nice properties. It was left as an open problem in (Gebser et al. 2007) to establish the complexity of identifying head-elementary-set-free programs. This note solves the open problem, by showing that the problem is complete for coNP.

KEYWORDS: computational complexity, elementary set, disjunctive logic program, head-elementary-set-free program.

1 Introduction

Disjunctive Logic Programming (DLP) is a highly declarative yet powerful knowledge representation and problem solving formalism. However, the high expressive power of DLP corresponds to a high complexity of the associated entailment problems (Dantsin et al. 2001). Therefore, the task of defining easily recognizable fragments of DLP characterized by lower complexities than the general language has been looked at as a relevant problem in the literature, since general DLP resolution engines can speed up their computation by identifying subprograms matching those definitions. For instance, the DLV engine (Leone et al. 2006) takes advantage of identifying head-cycle-free (HCF) (sub)programs (Ben-Eliyahu and Dechter 1994;

∗ To appear in Theory and Practice of Logic Programming (TPLP).
† This work was partly done while the author was affiliated with *DEIS, University of Calabria*.

Ben-Eliyahu-Zohary and Palopoli 1997) in resolving disjunctive logic programs under the stable model semantics. Head-elementary-set-free (HEF) programs were recently introduced in (Gebser et al. 2007) as a strict generalization of HCF programs featuring the same nice properties of that smaller class. In detail, likewise HCF programs, HEF programs can be turned into equivalent nondisjunctive programs in polynomial time and space by shifting. As such, HEF programs can be regarded as "easy" disjunctive programs, since they actually denote syntactic variants of nondisjunctive coding. This fact has several formal consequences, which are precisely accounted for in (Gebser et al. 2007). Just for an example, while checking for a disjunctive program to have a stable model is Σ_2^P-complete in general, it is NP-complete for HEF programs.

It is therefore important to devise procedures to identify head-elementary-set-free programs. However, while checking for a program to be HCF can be done in linear time (Ben-Eliyahu and Dechter 1994), the complexity of identifying HEF programs is a problem left open in (Gebser et al. 2007), where it is read that: *It is an open question whether identifying HEF programs is tractable ...*. This note is intended to solve such an open problem, by showing that identifying HEF programs is, in fact, coNP-complete. Therefore, while HEF programs share several common properties with HCF programs, to identify them is much more difficult from the computational complexity standpoint.

The rest of the note is organized as follows. Preliminaries about DLP are illustrated in the next section. Section 3 recalls the definition of HEF programs and provides a couple of preliminary results. Section 4 and Section 5 settle the complexity of the problem accounting for the membership in coNP and its coNP-hardness, respectively.

2 Preliminaries

In this section we recall basic definitions about propositional disjunctive logic programming.

A *literal* is a propositional atom a or its negation $not\, a$. A rule is an expression of the form $B, F \rightarrow H$, where H, B and F are set of literals. In particular, sets H and B consist of positive atoms, whereas F consists of negated atoms. H and $B \cup F$ are referred to as, respectively, the head and body of the rule. If $|H| > 1$ then the rule is called *disjunctive*, otherwise it is called *non-disjunctive*.

A program $\mathcal{P}$ is a finite set of rules. If there is some disjunctive rule in $\mathcal{P}$ then $\mathcal{P}$ is called *disjunctive*, otherwise it is called *non-disjunctive*. A set S of atoms is called a *disjunctive set* for $\mathcal{P}$ if and only if there exists at least one rule $\delta : B, F \rightarrow H$ in $\mathcal{P}$ such that $|H \cap S| > 1$.

An interpretation I of $\mathcal{P}$ is a set of atoms from $\mathcal{P}$. An atom is *true* in the interpretation I if $a \in I$. A literal $not\, a$ is *true* in I if $a \notin I$. A conjunction C of literals is true in I if all the literals in C are true in I. A rule $B, F \rightarrow H$ is true in I if either H is true in I or $B \wedge F$ is false in I. An interpretation I is a model for a program $\mathcal{P}$ if all rules occurring in $\mathcal{P}$ are true in I. A model M for $\mathcal{P}$ is minimal if no proper subset of M is a model for $\mathcal{P}$. A model M of $\mathcal{P}$ is *stable* if M is a minimal model of the reduct of $\mathcal{P}$ w.r.t M, denoted by $\mathcal{P}^M$, that is the program built from $\mathcal{P}$ by (1) removing all rules that contain a negative literal $not\, a$ in the body with $a \in M$, and (2) removing all negative literals from the remaining rules (Gelfond and Lifschitz 1988).

Example 1
Consider for example the following program:

$$\begin{array}{rrl} \mathcal{P} = \{ & a & \rightarrow b, c \\ & not\, a, d & \rightarrow e \\ & c, not\, b, f & \rightarrow e \\ & not\, b & \rightarrow a \quad \} \end{array}$$

and the interpretation $M = \{a, c\}$. The ground positive program $\mathcal{P}^M$ is the following:

$$\begin{array}{rrl} \mathcal{P}^M = \{ & a & \rightarrow b, c \\ & c, f & \rightarrow e \\ & & \rightarrow a \quad \} \end{array}$$

Since M is a minimal model of $\mathcal{P}^M$, M is a stable model of $\mathcal{P}$.

3 Head-elementary-set-free programs

In this section, we recall the definition of HEF programs [Gebser et al. 2006] and provide a couple of preliminary results which will be useful in the following. We begin with introducing the concepts of outbound and elementary set.

Definition 1 (*Outbound Set[Gebser et al. 2006]*)
Let $\mathcal{P}$ be a disjunctive program. For any set Y of atoms occurring in $\mathcal{P}$, a subset Z of Y is *outbound* in Y for $\mathcal{P}$ if there is a rule $\delta : B, F \rightarrow H$ in $\mathcal{P}$ such that: (i) $H \cap Z \neq \emptyset$; (ii) $B \cap (Y \backslash Z) \neq \emptyset$; (iii) $B \cap Z = \emptyset$ and (iv) $H \cap (Y \backslash Z) = \emptyset$.

Intuitively, $Z \subseteq Y$ is outbound in Y for $\mathcal{P}$ if there exists a rule δ in $\mathcal{P}$ such that the partition of Y induced by Z $(that is, \langle Z; Y \setminus Z\rangle)$ separates head from body atoms of δ.

Example 2
Consider, for example, the program

$$\begin{array}{rrl} \mathcal{P}_{ex} = \{ & a & \rightarrow b, c \\ & c & \rightarrow b \\ & b & \rightarrow c \\ & b & \rightarrow a \\ & b, c & \rightarrow d \quad \} \end{array}$$

and the set $E_{ex} = \{a, b, c\}$. Consider, now, the subset $O = \{a, b\}$ of E_{ex}. O is outbound in E_{ex} for $\mathcal{P}_{ex}$ because of the rule $c \rightarrow b$, since $c \in E_{ex} \setminus O$, $c \notin O$, $b \in O$ and $b \notin E_{ex} \setminus O$.

Definition 2 (*Elementary Set[Gebser et al. 2006]*)
Let $\mathcal{P}$ be a disjunctive program. For any nonempty set Y of atoms occurring in $\mathcal{P}$, Y is *elementary* for $\mathcal{P}$ if all nonempty proper subsets of Y are outbound in Y for $\mathcal{P}$.

For example, the set E_{ex} of Example 2 is elementary for the program $\mathcal{P}_{ex}$, since each nonempty proper subset of E_{ex} is outbound in E_{ex} for $\mathcal{P}_{ex}$.

Definition 3 (*Head-Elementary-Set-Free Program[Gebser et al. 2007]*)
Let $\mathcal{P}$ be a disjunctive program. $\mathcal{P}$ is *Head Elementary Set Free (HEF)* if for each rule $B, F \to H$ in $\mathcal{P}$, there is no elementary set E for $\mathcal{P}$ such that $|E \cap H| > 1$.

So, a program $\mathcal{P}$ is HEF if there is no elementary set containing two or more atoms all appearing in the head of one rule of $\mathcal{P}$.

For example, the program $\mathcal{P}_{ex}$ of Example 2 is not HEF, because for the rule $\delta : a \to b, c$, and the elementary set E_{ex}: the intersection between the head of δ and $E_{ex} = \{a, b, c\}$ is $\{b, c\}$.

It follows from the definition that a program $\mathcal{P}$ is not HEF if and only if there exists a set X of atoms of $\mathcal{P}$ such that X is both a disjunctive set and an elementary set for $\mathcal{P}$.

Next, two theorems which are needed to prove our main results, given in the following sections, are proved. In particular, Theorem 1 tells about the connectedness of the subgraph an elementary set induces into a program positive dependency graph and actually immediately follows from (Gebser et al. 2006). Theorem 2, instead, tells that any atom that occurs in an elementary set must be "justified" by at least two rules, that atom being the only one in its elementary set occurring in the head of the first rule and in the body of the second rule, respectively. We begin by defining the concept of a positive dependency graph of a program.

A directed graph $\mathcal{G}$, called *positive dependency graph*, can be associated with a disjunctive program $\mathcal{P}$. Specifically, for each rule $B, F \to H$ of $\mathcal{P}$, each atom appearing in H or in B is associated with a node in $\mathcal{G}$, and there is a directed edge (m, n) from a node m to a node n if the atom associated with m is in B, and the atom associated with n is in H.

Theorem 1
Let E be an elementary set for a program $\mathcal{P}$ and let $\mathcal{G}$ be the positive dependency graph associated with $\mathcal{P}$. The subgraph induced by E is strongly connected.

Proof
The proof is given by contraposition. Specifically, it is supposed that the subgraph induced by E is not strongly connected and it is derived that E is not elementary.

If the subgraph induced by E is not strongly connected, then there exists some pair of node m and n such that n is not reachable from m. Then consider the set $E' \subset E$ of all the nodes reachable from m, and the set $E \setminus E'$. Since n is not reachable from m, $E \setminus E'$ is not empty, and then E' is a proper subset of E. Moreover, since reachability is a transitive relation, all the nodes in $E \setminus E'$ are not reachable from any node in E'. By definition of dependency graph, it follows that there is no rule $B, F \rightarrow H$ in $\mathcal{P}$ such that $B \cap E' \neq \emptyset$ and $H \cap (E \setminus E') \neq \emptyset$. Then $E \setminus E'$ is not outbound and, as a consequence, E is not elementary. $\square$

Theorem 2
Let $\mathcal{P}$ be a disjunctive program, let E be an elementary set for $\mathcal{P}$ such that $|E| > 1$ and let a be an atom belonging to E. Then: (i) there exists at least one rule $\delta_1 : B, F \rightarrow H$, such that $a \notin B$, $B \cap E \neq \emptyset$ and $H \cap E = \{a\}$, and (ii) there exists at least one rule $\delta_2 : B, F \rightarrow H$, such that $a \notin H$, $B \cap E = \{a\}$ and $H \cap E \neq \emptyset$.

Proof
(i) Consider the set $O = \{a\}$. If no rule $\delta_1 : B, F \rightarrow H$, such that $a \notin B$, $B \cap E \neq \emptyset$ and $H \cap E = \{a\}$, existed in $\mathcal{P}$, then O would not be outbound. Since $O \subset E$, E would not be elementary.
(ii) Consider the set $O = E \setminus \{a\}$. If no rule $\delta_2 : B, F \rightarrow H$, such that $a \notin H$, $B \cap E = \{a\}$ and $H \cap E \neq \emptyset$, existed in $\mathcal{P}$, then O would not be outbound in E and then E would not be elementary.
$\square$

Theorem 2 closes the preliminary part of this note. In the following Sections 4 and 5, the complexity of identifying HEF programs is analyzed.

4 Complexity Analysis: Membership

In this section, the membership of the problem in the class coNP is proved. To this end, some new properties of HEF programs are shown next.

Let X be a set of atoms of a disjunctive logic program $\mathcal{P}$. In the following, $\mathcal{P}_X$ will denote the disjunctive logic program built as follows: for each rule $\delta : B, F \rightarrow H$ of $\mathcal{P}$, add to $\mathcal{P}_X$ the rule $\delta' : B' \rightarrow H'$ obtained as the *projection* of δ on X, namely B' is $B \cap X$ and H' is $H \cap X$, if both B' and H' are not empty.

The following lemma is immediately proved.

Lemma 1
Let $\mathcal{P}$ be a logic program. E is an elementary set for $\mathcal{P}$ if and only if E is an elementary set for $\mathcal{P}_E$.

As a consequence of the above lemma, the definition of outbound set can be rewritten as follows: *let $\mathcal{P}$ be a disjunctive logic program, and let E be a set of atoms of $\mathcal{P}$. A subset O of E is outbound in E for $\mathcal{P}$ if and only if there is a rule $\delta : B' \rightarrow H'$ in $\mathcal{P}_E$ such that $\emptyset \subset H' \subseteq O$ and $\emptyset \subset B' \subseteq E \setminus O$.*

The following lemma states that elementary sets of a program $\mathcal{P}$ are preserved in supersets of $\mathcal{P}$.

Lemma 2
Let $\mathcal{P}$ be a logic program, and $\mathcal{P}^{red} \subseteq \mathcal{P}$ a logic program consisting of a subset of the rules of $\mathcal{P}$. If E is an elementary set for $\mathcal{P}^{red}$, then E is an elementary set for $\mathcal{P}$ as well.

Proof

If a set E is an elementary set in $\mathcal{P}^{red}$ then, by definition, each nonempty proper subset S of E is outbound in E for $\mathcal{P}^{red}$ and, therefore, there is a rule $\delta : B, F \rightarrow H$ in $\mathcal{P}^{red}$ such that $H \cap S \neq \emptyset$, $B \cap (E \setminus S) \neq \emptyset$, $B \cap S = \emptyset$ and $H \cap (E \setminus S) = \emptyset$.

Clear enough, if $\mathcal{P}^{red} \subseteq \mathcal{P}$ then δ is also in $\mathcal{P}$ and, as a consequence, each subset of E is outbound in E also for $\mathcal{P}$. $\square$

Let $\mathcal{P}$ be a logic program, and E an elementary set for $\mathcal{P}$. In the following, each program $\mathcal{P}_E^{red} \subseteq \mathcal{P}_E$ is called a *witness* of E if E is elementary in $\mathcal{P}_E^{red}$. Note, in particular, that $\mathcal{P}_E$ is a witness of E.

By Lemma 2, $\mathcal{P}_E^{red}$ shows that E is elementary for $\mathcal{P}_E$, and by Lemma 1 also for $\mathcal{P}$.

An important property of HEF programs is stated in the following theorem.

Theorem 3

Let $\mathcal{P}$ be a disjunctive logic program. $\mathcal{P}$ is not HEF if and only if there exists a pair $(E, \mathcal{P}_E^{red})$ such that E is a disjunctive set for $\mathcal{P}$ and $\mathcal{P}_E^{red}$ is both a non-disjunctive program and a witness of E.

Proof

For one direction, note that if such a pair exists, then E is a disjunctive set for $\mathcal{P}$ and, since it has a witness, it is also an elementary set for $\mathcal{P}$ and, therefore, $\mathcal{P}$ is not HEF.

Now, consider the case in which $\mathcal{P}$ is not HEF. In the following, it is proved that for each pair $(S, \mathcal{P}_S^{red})$ such that S is a disjunctive set, and $\mathcal{P}_S^{red}$ is a disjunctive witness of S, there exists a pair $(S', \mathcal{P}_{S'}^{red})$ such that S' is a disjunctive set and $\mathcal{P}_{S'}^{red}$ is a witness of S', such that the number of disjunctive rules in $\mathcal{P}_{S'}^{red}$ is strictly less than that of disjunctive rules occurring in $\mathcal{P}_S^{red}$.

Note that this would conclude the proof, since it would inductively imply the existence of a pair $(S^*, \mathcal{P}_{S^*}^{red})$ such that S^* is a disjunctive set, $\mathcal{P}_{S^*}^{red} \subseteq \mathcal{P}_{S^*}$ is a witness of S^* with no disjunctive rules.

Let $(S, \mathcal{P}_S^{red})$ be a pair such that S is a disjunctive set, and $\mathcal{P}_S^{red}$ is a witness of S. Note that at least one of these pairs exists since, by definition, for each non-HEF program, there exists an elementary set E and, by Lemma 1, a witness $\mathcal{P}_E$ of E therefore exists as well. Assume that $\mathcal{P}_S^{red}$ is a disjunctive program. Then, at least one rule $\delta^* : B \rightarrow H$, $|H| > 1$ belongs to $\mathcal{P}_S^{red}$. Two cases are possible: (i) S is not an elementary set for $\mathcal{P}_S^{red} \setminus \{\delta^*\}$; (ii) S is an elementary set for $\mathcal{P}_S^{red} \setminus \{\delta^*\}$.

(i) Since S is not elementary for $\mathcal{P}_S^{red} \setminus \{\delta^*\}$, then there exists at least one proper subset of S which is not outbound in S for $\mathcal{P}_S^{red} \setminus \{\delta^*\}$. In particular, let S' be a minimal subset of S which is not outbound in S for $\mathcal{P}_S^{red} \setminus \{\delta^*\}$. Since S' is outbound in $\mathcal{P}_S^{red}$, δ^* is such that $H \subseteq S'$ and $B \subseteq S \setminus S'$, namely, δ^* is needed to prove S' to be outbound. It is worth noting that, because of δ^*, S' is a disjunctive set for $\mathcal{P}$. Consider now each nonempty proper subset S'' of S'. Note that one of such subsets exists, since S' contains at least all of the atoms belonging to the head of δ^*, and then its cardinality is greater than 1.

Since S' is a mimimal subset of S which is not outbound in $\mathcal{P}_S^{red} \setminus \{\delta^*\}$, S'' is outbound in $\mathcal{P}_S^{red} \setminus \{\delta^*\}$. Therefore, there exists a rule $\delta' : B' \to H'$ in $\mathcal{P}_S^{red} \setminus \{\delta^*\}$, such that $\emptyset \subset H' \subseteq S''$ and $\emptyset \subset B' \subseteq S \setminus S''$.
Moreover, it must hold that $S' \cap B' \neq \emptyset$. Indeed, were $S' \cap B' = \emptyset$ then $\delta' : B' \to H'$ would be a rule such that $\emptyset \subset H' \subseteq S'' \subset S'$ and $\emptyset \subset B' \subseteq S \setminus S'$; hence, because of δ', S' would be outbound also in $\mathcal{P}_S^{red} \setminus \{\delta^*\}$, which does not hold by hypothesis.
Consider, now, the program $\mathcal{P}_{S'}^{red}$ consisting of the projections of the rules $\delta : B \to H$ of $\mathcal{P}_S^{red}$ such that $B \cap S' \neq \emptyset$ and $H \cap S' \neq \emptyset$. Note that, as the rule δ^* has the body contained in $S \setminus S'$, the projection of δ^* is not added to $\mathcal{P}_{S'}^{red}$.
Since, as stated above, the set S' is such that for each nonempty proper subset $S'' \subset S'$ there is a rule $\delta' : B' \to H'$ in $\mathcal{P}_S^{red}$ where $\emptyset \subset H' \subseteq S''$ and $\emptyset \subset B' \subseteq S' \setminus S''$, it follows that δ' is also in $\mathcal{P}_{S'}^{red}$ and, therefore, S'' is outbound in S'; this implies, in turn, that $\mathcal{P}_{S'}^{red}$ is a witness of S'.
Summarizing, for each pair $(S, \mathcal{P}_S^{red})$ such that S is an elementary set for $\mathcal{P}$ and $\mathcal{P}_S^{red}$ is a witness of S containing at least one disjunctive rule δ, there exist both a non-empty disjunctive set $S' \subset S$ such that S' is a disjunctive set for $\mathcal{P}$ and a witness $\mathcal{P}_{S'}^{red}$ of S', such that $\mathcal{P}_{S'}^{red}$ contains a number of disjunctive rules strictly less than the number of disjunctive rules occurring in $\mathcal{P}_S^{red}$ (as the former does not contain δ^*).

(ii) In this second case, consider the pair $(S', \mathcal{P}_{S'}^{red})$, where $S' = S$ and $\mathcal{P}_{S'}^{red} = \mathcal{P}_S^{red} \setminus \{\delta^*\}$. S' is a disjunctive set for $\mathcal{P}$ and $\mathcal{P}_{S'}^{red}$ is a witness of S' that does not contain the disjunctive rule δ^*.

□

Example 3
In order to clarify the proof of the Theorem 3, consider the following example. Let $\mathcal{P}$ be the following program

$$\mathcal{P} = \{ \begin{array}{rl} a & \to b, c \\ c & \to b \\ b & \to d \\ b & \to e \\ d, e & \to f \\ f & \to e \\ e & \to c \\ d & \to a \end{array} \} \qquad \mathcal{P}_{S'}^{red} = \{ \begin{array}{rl} c & \to b \\ b & \to e \\ e & \to f \\ f & \to e \\ e & \to c \end{array} \}$$

which is not HEF, since the set $E = \{a, b, c, d, e, f\}$ is elementary for $\mathcal{P}$. Furthermore, E is a disjunctive set, due to the rule $\delta^* : a \to b, c$ and $\mathcal{P}$ is a witness of E. E is not elementary for $\mathcal{P} \setminus \{\delta^*\}$ since $S' = \{b, c, e, f\}$ is not outbound in E for $\mathcal{P} \setminus \{\delta^*\}$ and, moreover, S' is a minimal non-outbound subset of E. Note that S' is outbound in $\mathcal{P}$ just for the presence of δ^*, and S' is a disjunctive set since it contains the whole head of δ^*. Consider the program $\mathcal{P}_{S'}^{red}$. Since S' is a minimal non-outbound subset of E, each nonempty subset of S' is outbound in $\mathcal{P}_{S'}^{red}$, and then S' is elementary for $\mathcal{P}_{S'}^{red}$. Summarizing, S' is a disjuctive set and is also an elemetary set for $\mathcal{P}_{S'}^{red}$ and then for $\mathcal{P}$. Thus, $\mathcal{P}_{S'}^{red}$ is a witness of S' and it is also non-disjunctive, since it does not contain δ^*.

Using the result stated in Theorem 3, it is possible to prove the coNP-membership theorem.

Theorem 4 (*HEF Problem-Membership*)
Let $\mathcal{P}$ be a disjunctive logic program. Deciding if $\mathcal{P}$ is HEF is in coNP.

Proof
By Therorem 3, a nondeterministic polynomial-time Turing machine can disqualify the HEF-Problem by first guessing a pair $(Y, \mathcal{P}_Y^{red})$ where Y is a set of atoms and $\mathcal{P}_Y^{red}$ is a non-disjunctive program. Next, the machine verifies in polynomial time that at least two atoms, belonging to the head of a rule in $\mathcal{P}$, are contained in Y (that is, that Y is a disjunctive set for $\mathcal{P}$) and, finally, checks that Y is an elementary set for $\mathcal{P}_Y^{red}$, by verifying that $\mathcal{P}_Y^{red}$ is a witness of Y. This last task can be accomplished in polynomial time as stated in (Gebser et al. 2006). If this holds, by Lemmata 1 and 2, it follows that Y is elementary for $\mathcal{P}$ and then $\mathcal{P}$ is not HEF. □

5 Complexity Analysis: Hardness

In this section the coNP-hardness of the problem is proved.

Let $\Phi = C_1 \wedge \ldots \wedge C_n, n \geq 1$ be a 3-CNF formula, namely a conjunctive Boolean formula where each clause C_i consists exactly of three literals. From Φ, a logic program $\mathcal{P}^\Phi$ is constructed as follows. Let $A_1, \ldots, A_m$ be the variables of Φ; and let $\mathcal{A}^\Phi$ be a set of atoms consisting of: an atom ϕ; an atom a_i and an atom na_i for each variable A_i; an atom c_i for each clause C_i; and, finally, two further atoms c_0 and c_{n+1}. Thus, note that $\mathcal{A}^\Phi$ is always non-empty. In the following, the atom na_i is referred to as the *opposite* of the atom a_i and vice versa. For each atom c_i, $V(c_i)$ denotes the set of atoms associated with the literals appearing in the clause C_i. In particular, an atom a_j belongs to $V(c_i)$ if A_j appears in C_i and na_j belongs to $V(c_i)$ if $\neg A_j$ appears in C_i. Moreover, for each atom c_i, $NV(c_i)$ denotes the set of the opposites of the atoms in $V(c_i)$, namely the atom a_j (resp. na_j) is in $NV(c_i)$ if na_j (resp. a_j) is in $V(c_i)$. $\mathcal{P}^\Phi$, the disjunctive program associated with Φ and built on $\mathcal{A}^\Phi$, consists in the following rules:

1. $\phi \rightarrow c_0 \vee c_{n+1}$
2. $c_0 \rightarrow c_1$
3. $c_i \wedge \alpha_j^i \rightarrow c_{i+1}$, for each $1 \leq i \leq n$ and for each $\alpha_j^i \in NV(c_i), 1 \leq j \leq 3$
4. $c_{n+1} \wedge na_1 \rightarrow a_1$
5. $c_{n+1} \wedge a_1 \rightarrow na_1$
6. $a_i \wedge na_{i+1} \rightarrow a_{i+1}$, $1 \leq i \leq m-1$;
7. $a_i \wedge a_{i+1} \rightarrow na_{i+1}$, $1 \leq i \leq m-1$;
8. $na_i \wedge na_{i+1} \rightarrow a_{i+1}$, $1 \leq i \leq m-1$;
9. $na_i \wedge a_{i+1} \rightarrow na_{i+1}$, $1 \leq i \leq m-1$;
10. $a_m \wedge na_m \rightarrow c_0$;

Theorem 5 (*HEF Problem-Hardness*)
Let $\mathcal{P}$ be a disjunctive logic program. Deciding if $\mathcal{P}$ is HEF is coNP-hard.

Proof
The proof is given by reduction of 3-SAT, which is well known to be NP-complete (Garey and Johnson 1979).

Let $\Phi = C_1 \wedge \ldots \wedge C_n$ be a 3-CNF and $\mathcal{P}^\Phi$ the disjunctive program associated with Φ. First, we note that the size of $\mathcal{P}^\Phi$ is polynomially bounded in the size of Φ. Next, it is proved that $\mathcal{P}^\Phi$ is not HEF if and only if Φ is satisfiable.

Since the only rule of $\mathcal{P}^\Phi$ containing more than one atom in the head is $\phi \rightarrow c_0 \vee c_{n+1}$, in order to prove that $\mathcal{P}^\Phi$ is not HEF, an elementary set E containing both c_0 and c_{n+1} must be found.

Before proceeding with the proof of the theorem, some claims are shown about this.

Claim 1
E does not contain both a_i and na_i for any $i \in [1, m]$.

Proof of Claim 1.
If there existed i such that both a_i and na_i are in E, then the set $\{a_i, na_i\} \subset E$ would not be outbound in E and E would not be elementary. $\square$

Claim 2
E contains c_j, for all $1 \leq j \leq n$.

Proof of Claim 2.
Because of Theorem 1, the subgraph induced by the atoms in E must be strongly connected; then, since E contains both c_0 and c_{n+1} and since the only path from c_0 to c_{n+1} passes through atoms $c_1, \ldots, c_n$, all these atoms must belong to E. $\square$

Claim 3
E contains at least one atom out of a_i and na_i, for each $i \in [1, m]$.

Proof of Claim 3.
Because of Theorem 1, the subgraph induced by the atoms in E must be strongly connected; then, since E contains both c_0 and c_{n+1} and since all the paths from c_{n+1} to c_0 pass through either the atom a_i or the atom na_i for each $i \in [1, m]$, either the atom a_i or the atom na_i must belong to E. $\square$

Summarizing the results of previous claims, a potential elementary set E for $\mathcal{P}^\Phi$ consists of:

- the atoms $c_0, c_1 \ldots, c_n, c_{n+1}$;
- either the atom a_i or the atom na_i (but not both of them), for each $i \in [1, m]$.

Claim 4
Let E be as described above. Then, for each clause C_i, at least one atom in $NV(c_i)$ is not in E.

Proof of Claim 4.
There are only three rules having c_i in their body, namely $c_i \wedge \alpha^i_j \rightarrow c_{i+1}$ for each $\alpha^i_j \in NV(c_i)$. Due to Theorem 2, in order for E to be elementary, at least one rule $B \rightarrow H$ such that $B \cap E = \{c_i\}$ must occur in $\mathcal{P}^\Phi$; then at least one atom $\alpha^i_j \in NV(c_i)$ has not to belong in E. $\square$

The above claim asserts that, in order for E to be elementary, for each clause C_i a *necessary* condition is that at least one atom in $NV(c_i)$ must be not in E. It can be shown that this is also a *sufficient* condition.

Claim 5
Let E be as described above. Then, if for each clause C_i at least one atom in $NV(c_i)$ is not in E, then E is an elementary set for $\mathcal{P}^\Phi$.

Proof of Claim 5.
The proof is given by picking a generic nonempty proper subset O of E and by showing that it is outbound in E for $\mathcal{P}^\Phi$.

Let $Q \subset E$ be the subset of E consisting of exactly one of the atoms a_i and na_i for each $i \in [1, m]$; and let Q_i be the atom a_i (resp., na_i), if a_i (resp., na_i) belongs to Q. Moreover, let $\mathcal{G}_E$ denote the subgraph induced by the atoms in E and consider the path π in $\mathcal{G}_E$ consisting of: (i) the directed edge from the c_i to c_{i+1} for each $0 \leq i \leq n$, (ii) the directed edge from c_{n+1} to Q_1, (iii) the directed edge from Q_i to Q_{i+1} for each $1 \leq i \leq m-1$ and, finally (iv) the directed edge from Q_m to c_0. Note that π is an Hamiltonian cycle. Since O is a nonempty proper subset of E then at least one node of E is not in O. Therefore, there exists a pair of nodes n_1 and n_2 in $\mathcal{G}_E$ such that the atom x_1 associated with n_1 is in $E \setminus O$, the atom x_2 associated with n_2 is in O and there exists a directed edge from n_1 to n_2 in π. Since there exists a directed edge from n_1 to n_2, then there is a rule $\delta : B \rightarrow H$ in $\mathcal{P}^\Phi$ such that $x_1 \in B \cap E$ and $x_2 \in H \cap E$. In particular, it will be shown next that there exists a rule $\delta' : B' \rightarrow H'$ such that $B' \cap E = \{x_1\}$ and $H' \cap E = \{x_2\}$. Note that this will conclude the proof, since O is outbound just by the virtue of δ'.

Since there exists a directed edge from n_1 to n_2, simply consider all the pairs of atoms associated with the directed edges in π; the following cases exhaust all possibilities: (i) $x_1 = c_i$ and $x_2 = c_{i+1}$ for some $0 \leq i \leq n$; (ii) $x_1 = c_{n+1}$ and $x_2 = Q_1$; (iii) $x_1 = Q_i$ and $x_2 = Q_{i+1}$ for some $1 \leq i \leq m-1$; (iv) $x_1 = Q_m$ and $x_2 = c_0$.

Consider case (i). Since for each clause C_i at least one atom in $NV(c_i)$ is not in E, there exists at least one rule $\delta' : c_i \wedge \alpha^i_j \rightarrow c_{i+1}$ in $\mathcal{P}^\Phi$ such that the intersection between E and the body of δ is $\{c_i\}$. As for case (ii), assume w.l.o.g. that $Q_1 = a_1$ and then that $na_1 \notin E$. Then, the rule $\delta' : c_{n+1} \wedge na_1 \rightarrow a_1$ is such that the intersection between E and the body of δ' is $\{c_{n+1}\}$. Consider case (iii), assume w.l.o.g., that $Q_i = a_i$ and $Q_{i+1} = a_{i+1}$. Then, the rule $\delta' : a_i \wedge na_{i+1} \rightarrow a_{i+1}$ is such that the intersection between E and the body of δ' is $\{a_i\}$. Finally, as for case (iv), assume w.l.o.g., that $Q_m = a_m$. The rule $\delta' : a_m \rightarrow c_0$ is such that the intersection between E and the body of δ' is $\{a_m\}$. □

Now, the proof of the theorem can be resumed.

Let X be a truth assignment to the variables in Φ. Let Q^X be the set of atoms associated with X. In particular, a_i (resp., na_i) is in Q^X, if A_i is true (resp., false) in X. It is proved that: X is satisfies Φ, if and only if the set $E = \{c_0, \ldots, c_{n+1}\} \cup Q^X$ is elementary for $\mathcal{P}^\Phi$. Note that this will conclude the theorem proof, since E contains both c_0 and c_{n+1}.

($\Rightarrow$) If X satisfies Φ then Q^X contains at least one atom $\alpha \in V(c_i)$ for each $c_i, i \in [1, n]$. Therefore, at least one atom, in particular the opposite of the atom α, that belongs to $NV(c_i)$ for each $c_i, i \in [1, n]$, is not in E. Thus, by Claim 5, E is elementary.

($\Leftarrow$) By Claim 4, if E is elementary then Q^X does not contain any $\alpha \in NV(c_i)$ for each $c_i, i \in [1, n]$. Then, for each clause C_i, Q^X contains one of the atoms associated with the literals satisfying C_i. Therefore, the truth assignment associated with Q^X satisfies Φ.

□

6 Conclusions

In this work the complexity of verifying if a disjunctive logic program is head-elementary-set-free is analyzed. We have proved here that the problem at hand is coNP-complete, hereby providing an answer to a question left open in (Gebser et al. 2007). This, basically negative, result leaves open the further problem of singling out a polynomial-time recognizable fragment of DLP, generalizing over HCF programs, while sharing their nice computational characteristics. In this respect, a direction to go is supposedly that of identifying some simple subclasses of programs for which checking for head-elementary-set-freeness is easier than for the general case[5].

References

Ben-Eliyahu, R. and Dechter, R. 1994. Propositional semantics for disjunctive logic programs. *Annals of Mathematics and Artificial Intelligence 12,* 1-2, 53–87.

Ben-Eliyahu-Zohary, R. and Palopoli, L. 1997. Reasoning with minimal models: Efficient algorithms and applications. *Artificial Intelligence 96,* 2, 421–449.

Dantsin, E., Eiter, T., Gottlob, G., and Voronkov, A. 2001. Complexity and expressive power of logic programming. *ACM Computuing Surveys 33,* 3, 374–425.

Garey, M. R. and Johnson, D. S. 1979. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman.

Gebser, M., Lee, J., and Lierler, Y. 2006. Elementary sets for logic programs. In *Proceedings of the 21st National Conference on Artificial Intelligence (AAAI).*

[5] Authors thank one of the anonymous referees for having pointed this out.

Gebser, M., Lee, J., and Lierler, Y. 2007. Head-elementary-set-free logic programs. In *Proceeding of 9th International conference on Logic Programming and Nonmonotonic Reasoning (LPNMR)*. 149–161.

Gelfond, M. and Lifschitz, V. 1988. The stable model semantics for logic programming. In *Proceedings of the Fifth International Conference and Symposium*. 1070–1080.

Leone, N., Pfeifer, G., Faber, W., Eiter, T., Gottlob, G., Perri, S., and Scarcello, F. 2006. The dlv system for knowledge representation and reasoning. *ACM Transactions on Computational Logic 7*, 3, 499–562.