A Complete and Terminating Execution Model for Constraint Handling Rules

HARIOLF BETZ, FRANK RAISER

THOM FRÜHWIRTH

Faculty of Engineering and Computer Science, Ulm University, Germany
(*e-mail:* firstname.lastname@uni-ulm.de)

submitted January 2010*; revised* April 2010*; accepted* –

Note: This article has been published in *Theory and Practice of Logic Programming*, 10(4–6), 597–610, ©Cambridge University Press

Abstract

We observe that the various formulations of the operational semantics of Constraint Handling Rules proposed over the years fall into a spectrum ranging from the analytical to the pragmatic. While existing analytical formulations facilitate program analysis and formal proofs of program properties, they cannot be implemented as is. We propose a novel operational semantics ω_t, which has a strong analytical foundation, while featuring a terminating execution model. We prove its soundness and completeness with respect to existing analytical formulations and we provide an implementation in the form of a source-to-source transformation to CHR with rule priorities.

KEYWORDS: Constraint Handling Rules, Operational Semantics, Execution Model, Persistent Constraints

1 Introduction

Constraint Handling Rules (Frühwirth 2009) (CHR) is a declarative, multiset- and rule-based programming language suitable for concurrent execution and powerful program analysis. While it is known as a language that combines efficiency with declarativity, publications in the field display a tendency to favor one of these aspects over the other. We observe a spectrum of research directions ranging from the *analytical* to the *pragmatic*.

On the analytical end of the spectrum, emphasis is put on CHR as a mathematical formalism, declarativity, and the understanding of its logical foundations and theoretical properties. Several formalizations of the operational semantics, found in (Frühwirth and Hanschke 1993; Frühwirth 1998) and (Frühwirth and Abdennadher 2003), belong to this side of the spectrum. Notable results building on these analytical formalizations include decidable criteria for operational equivalence (Abdennadher and Frühwirth 1999) and confluence (Abdennadher et al. 1999), a strong foundation of CHR in linear logic (Betz and Frühwirth 2005), as well as weak and

strong parallelization, as presented in (Frühwirth 2005) and further developed toward concurrency in (Sulzmann and Lam 2007; Sulzmann and Lam 2008).

A recent analytical formalization is the operational semantics ω_e, given in (Raiser et al. 2009). It consists in a rewriting system of equivalence classes of states based on an axiomatic formulation of equivalence. It has been shown to coincide with the operational semantics ω_{va}, which has been introduced in (Frühwirth 2009) to set a standard for all other operational semantics to build upon.

On the downside, these operational semantics are detached from practical implementation in that they are oblivious to questions of efficiency and termination. Particularly, the class of rules called *propagation rules* causes trivial non-termination in both of them. Hence, it is safe to say that the existing analytical formalizations of the operational semantics lack a terminating execution model.

This contrasts with most work on the pragmatic side of the spectrum, which emphasizes practical implementation and efficiency over formal reasoning. It originates with (Abdennadher 1997), where a token-based approach is proposed in order to avoid trivial non-termination: Every propagation rule is applicable only once to a specific combination of constraints. This is realized by keeping a *propagation history* – sometimes called *token store* – in the CHR state. Thus, we gain a terminating execution model for the full segment of CHR.

Building upon (Abdennadher 1997), a plethora of operational semantics has been brought forth, such as the token-based operational semantics ω_t and its refinement ω_r (Duck et al. 2004). The latter reduces non-determinism for a gain in efficiency and sets the current standard for CHR implementations. Another notable exponent is the priority-based operational semantics ω_p (De Koninck et al. 2007).

On the downside, token stores break with declarativity: Two states that differ only in their token stores may exhibit different operational behavior while sharing the same logical reading. Therefore, we consider token stores as *non-declarative elements* in CHR states.

Recent work on linear logical algorithms (Simmons and Pfenning 2008) and the close relation of CHR to linear logic (Betz and Frühwirth 2005) suggest a novel approach that emphasizes aspects from both sides of the spectrum to a useful degree: In this work, we introduce the notion of *persistent constraints* to CHR, a concept reminiscent of unrestricted or "banged" propositions in linear logic. Persistent constraints provide a finite representation of the result of any number of propagation rule firings.

We furthermore introduce a state transition system based on persistent constraints, which is explicitly irreflexive. In combination, the two ideas solve the problem of trivial non-termination while retaining declarativity and preserving the potential for effective concurrent execution. This state transition system requires no more than two rules. As every transition step corresponds to a CHR rule application, it facilitates formal reasoning over programs.

In this work, we show that the resulting operational semantics $\omega_!$ is sound and complete with respect to ω_e. We show that $\omega_!$ can be faithfully embedded into the operational semantics ω_p, thus effectively providing an implementation in the form of a source-to-source transformation. All operational semantics developed with an

emphasis on pragmatic aspects lack this completeness property. Therefore, this work is the first to show that it is possible to implement CHR soundly and completely with respect to its abstract foundations, whilst featuring a terminating execution model.

Example 1.1
Consider the following straightforward CHR program for computing the transitive hull of a graph represented by edge constraints $e/2$:

$$t \quad @ \quad e(X,Y), e(Y,Z) \quad \Longrightarrow \quad e(X,Z)$$

This most intuitive formulation of a transitive hull is not a suitable implementation in most existing operational semantics. In fact, for goals containing cyclic graphs it is non-terminating in all aforementioned existing semantics. In this work we show that execution in our proposed semantics $\omega_!$ correctly computes the transitive hull whilst guaranteeing termination.

The remainder of this paper is structured as follows: We state the syntax of CHR and summarize the existing operational semantics ω_t and ω_e in Sect. 2. In Sect. 3, we present our semantics $\omega_!$, originally proposed in (Betz et al. 2009), and we state results concerning its soundness and completeness with respect to ω_e. In Sect. 4, we show how $\omega_!$ can be implemented by means of a faithful source-to-source transformation into ω_p. In Sect. 5, we discuss the termination behavior of $\omega_!$ as well as related work, before we conclude in Sect. 6. Proofs of the theorems presented in this work can be found in the accompanying technical report (Betz et al. 2010) [1], and will be omitted here.

2 Preliminaries

We first introduce the syntax of CHR and the equivalence-based operational semantics ω_e, which offers a foundation for all other semantics, although it lacks a terminating execution model. We furthermore present its refinements ω_t and ω_p.

2.1 The Syntax of CHR

Constraint Handling Rules distinguishes two kinds of constraints: *user-defined constraints* (or *CHR constraints*) and *built-in constraints*. Reasoning on built-in constraints is possible through a satisfaction-complete and decidable constraint theory $\mathcal{CT}$.

CHR is a programming language that offers advanced rule-based multiset rewriting. Its eponymous rules are of the form

$$r \ @ \ H_1 \backslash H_2 \Leftrightarrow G \mid B_c, B_b$$

where H_1 and H_2 are multisets of user-defined constraints, called the *kept head* and *removed head*, respectively. The *guard* G is a conjunction of built-in constraints and

[1] (Betz et al. 2010) is available from `http://vts.uni-ulm.de/doc.asp?id=7193`

the *body* consists of a conjunction of built-in constraints B_b and a multiset of user-defined constraints B_c. The *rule name* r is optional and may be omitted along with the @ symbol. Note that throughout this paper, we omit the curly braces around sets and multisets where there is no ambivalence. This applies especially to CHR rules and states.

In this work, we put special emphasis on the class of rules where $H_2 = \emptyset$, called *propagation rules*. Propagation rules can be written alternatively as

$$r \ @ \ H_1 \Longrightarrow G \mid B_c, B_b.$$

A *variant* of a rule $(r \ @ \ H_1 \backslash H_2 \Leftrightarrow G \mid B_c, B_b)$ with variables $\bar{x}$ is a rule of the form $(r \ @ \ H_1 \backslash H_2 \Leftrightarrow G \mid B_c, B_b)[\bar{x}/\bar{y}]$ for any sequence of pairwise distinct variables $\bar{y}$. For any rule $(r \ @ \ H_1 \backslash H_2 \Leftrightarrow G \mid B_c, B_b)$, the *local variables* $\bar{l}_r$ are defined as $\bar{l}_r ::= \mathrm{vars}(G, B_c, B_b) \setminus \mathrm{vars}(H_1, H_2)$. A rule where $\bar{l}_r = \emptyset$ is called *range-restricted*.

A CHR program $\mathcal{P}$ is a set of rules. A *range-restricted* CHR program is a set of range-restricted rules.

2.2 Equivalence-based Operational Semantics ω_e

In this section, we recall the *equivalence-based operational semantics* ω_e (Raiser et al. 2009). It is operationally close to the very abstract semantics ω_{va}, but we prefer it for its concise formulation and the explicit distinction of global variables, user-defined, and built-in constraints.

Definition 2.1 (ω_e *State*)
An ω_e *state* is a tuple $\langle \mathbb{G}; \mathbb{B}; \mathbb{V} \rangle$. The *user-defined (constraint) store* $\mathbb{G}$ is a multiset of CHR constraints. The *built-in (constraint) store* $\mathbb{B}$ is a conjunction of built-in constraints. $\mathbb{V}$ is a set of variables called the *global variables*. We use Σ_e to denote the set of all ω_e states. A variable $v \in \mathbb{B}$ is called a *strictly local variable* iff $v \notin (\mathbb{V} \cup \mathbb{G})$.

The operational semantics ω_e is founded on equivalence classes of states, based on the following definition of state equivalence.

Definition 2.2 (ω_e *State Equivalence*)
Equivalence between ω_e states is the smallest equivalence relation $\equiv_e$ over ω_e states that satisfies the following conditions:

1. $\langle \mathbb{G}; X \doteq t \wedge \mathbb{B}; \mathbb{V} \rangle \equiv_e \langle \mathbb{G}\,[X/t]\,; X \doteq t \wedge \mathbb{B}; \mathbb{V} \rangle$
2. If $\mathcal{CT} \models \exists \bar{s}.\mathbb{B} \leftrightarrow \exists \bar{s}'.\mathbb{B}'$ where $\bar{s}, \bar{s}'$ are the strictly local variables of $\mathbb{B}, \mathbb{B}'$, respectively, then $\langle \mathbb{G}; \mathbb{B}; \mathbb{V} \rangle \equiv_e \langle \mathbb{G}; \mathbb{B}'; \mathbb{V} \rangle$
3. If X is a variable that does not occur in $\mathbb{G}$ or $\mathbb{B}$ then $\langle \mathbb{G}; \mathbb{B}; \{X\} \cup \mathbb{V} \rangle \equiv_e \langle \mathbb{G}; \mathbb{B}; \mathbb{V} \rangle$
4. $\langle \mathbb{G}; \bot; \mathbb{V} \rangle \equiv_e \langle \mathbb{G}'; \bot; \mathbb{V} \rangle$

Definition 2.3 (ω_e *Transitions*)
For a CHR program $\mathcal{P}$, the state transition system $(\Sigma_e / \equiv_e, \rightarrowtail_e)$ is defined as follows. The transition is based on a variant of a rule r in $\mathcal{P}$ such that its local variables are disjoint from the variables occurring in the pre-transition state.

$$\frac{r \ @ \ H_1 \setminus H_2 \Leftrightarrow G \mid B_c \uplus B_b}{[\langle H_1 \uplus H_2 \uplus \mathbb{G}; G \wedge \mathbb{B}; \mathbb{V}\rangle] \rightarrowtail_e^r [\langle H_1 \uplus B_c \uplus \mathbb{G}; G \wedge B_b \wedge \mathbb{B}; \mathbb{V}\rangle]}$$

When the rule r is clear from the context or not important, we may write $\rightarrowtail_e$ rather than $\rightarrowtail_e^r$. By $\rightarrowtail_e^*$, we denote the reflexive-transitive closure of $\rightarrowtail_e$.

In the following, we freely mix equivalence classes and their representative, i.e. we often write $\sigma \rightarrowtail_e \tau$ instead of $[\sigma] \rightarrowtail_e [\tau]$.

An inherent problem of ω_e is its behavior with respect to propagation rules: If a state can fire a propagation rule once, it can do so again and again, ad infinitum. In the literature, this problem is referred to as *trivial non-termination* of propagation rules.

Example 2.1
Reconsider the transitivity rule from Example 1.1 and the following CHR state, which represents a cycle consisting of two edges:

$$\sigma = \langle e(A,B), e(B,A); \top; \emptyset\rangle$$

Let $t \ @ \ e(A',B'), e(B',C') \Longrightarrow e(A',C')$ be a variant of the transitivity rule, then it can be applied to σ, yielding an additional loop edge:

$$\begin{array}{ll} \sigma \equiv_e & \langle e(A',B'), e(B',C'); A = A' \wedge B = B' \wedge A = C'; \emptyset\rangle \\ \rightarrowtail^t & \langle e(A',B'), e(B',C'), e(A',C'); A = A' \wedge B = B' \wedge A = C'; \emptyset\rangle \\ \equiv_e & \langle e(A,B), e(B,A), e(A,A); \top; \emptyset\rangle \end{array}$$

It is easily verified, that the transitivity rule can be applied again to the same two constraints, yielding another $e(A,A)$ constraint, hence this program suffers from trivial non-termination in ω_e.

2.3 Operational Semantics with Rule Priorities

The extension of CHR with rule priorities was initially proposed in (De Koninck et al. 2007). It annotates rules with priorities and modifies the operational semantics such that among the applicable rules, we always select one of highest priority for execution. The operational semantics of this extension is denoted as ω_p and the formulation we use in work was given in (De Koninck et al. 2008).

The operational semantics ω_p uses a so-called *token store* to avoid trivial non-termination. A propagation rule can only be applied once to each combination of constraints matching the head. Hence, the token store keeps a history of fired propagation rules based on constraint identifiers, as defined below.

Definition 2.4 (*Identified CHR Constraints*)
An *identified CHR constraint* $c\#i$ is a CHR constraint c associated with a unique integer i, the *constraint identifier*. We introduce the functions $\mathrm{chr}(c\#i) = c$ and $\mathrm{id}(c\#i) = i$, and extend them to sequences and sets of identified CHR constraints in the obvious manner.

The definition of an ω_p state is more complicated, because identified constraints are distinguished from unidentified constraints and the token store is added.

Definition 2.5 (ω_p *State*)
An ω_p *state* is a tuple of the form $\langle \mathbb{G}; \mathbb{S}; \mathbb{B}; \mathbb{T} \rangle_n^{\mathbb{V}}$ where the *goal (store)* $\mathbb{G}$ is a multiset of constraints, the *CHR (constraint) store* $\mathbb{S}$ is a set of identified CHR constraints, the *built-in (constraint) store* $\mathbb{B}$ is a conjunction of built-in constraints. The *token store* (or *propagation history*) $\mathbb{T}$ is a set of tuples (r, I), where r is the name of a propagation rule and I is an ordered sequence of constraint identifiers. $\mathbb{V}$ is a set of variables called the *global variables*. We use Σ_p to denote the set of all ω_p states.

The corresponding transition system consists of the following three types of transitions.

Definition 2.6 (ω_p *Transitions*)
For a CHR program $\mathcal{P}$ with rule priorities, the state transition system $(\Sigma_p, \rightarrowtail_p)$ is defined as follows.

1. Solve. $\langle \{c\} \uplus \mathbb{G}; \mathbb{S}; \mathbb{B}; \mathbb{T} \rangle_n^{\mathbb{V}} \rightarrowtail_p \langle \mathbb{G}; \mathbb{S}; \mathbb{B}'; \mathbb{T} \rangle_n^{\mathbb{V}}$
where c is a built-in constraint and $\mathcal{CT} \models \forall((c \wedge \mathbb{B}) \leftrightarrow \mathbb{B}')$.

2. Introduce. $\langle \{c\} \uplus \mathbb{G}; \mathbb{S}; \mathbb{B}; \mathbb{T} \rangle_n^{\mathbb{V}} \rightarrowtail_p \langle \mathbb{G}; \{c\#n\} \cup \mathbb{S}; \mathbb{B}; \mathbb{T} \rangle_{n+1}^{\mathbb{V}}$
where c is a CHR constraint.

3. Apply. $\langle \emptyset; H_1 \cup H_2 \cup \mathbb{S}; \mathbb{B}; \mathbb{T} \rangle_n^{\mathbb{V}} \rightarrowtail_p \langle B; H_1 \cup \mathbb{S}; \Theta \wedge \mathbb{B}; \mathbb{T} \cup t \rangle_n^{\mathbb{V}}$ where $\mathcal{P}$ contains a rule of priority p with fresh variables of the form

$$p :: r \ @ \ H_1' \setminus H_2' \Leftrightarrow G \mid B$$

and a matching substitution Θ such that $\mathrm{chr}(H_1) = \Theta(H_1'), \mathrm{chr}(H_2) = \Theta(H_2')$, $\mathcal{CT} \models \exists(\mathbb{B}) \wedge \forall(\mathbb{B} \rightarrow \bar{\exists}_{\mathbb{B}}(\Theta \wedge G))$, $\Theta(p)$ is a ground arithmetic expression and $t = (r, \mathrm{id}(H_1) + \mathrm{id}(H_2)) \notin \mathbb{T}$. Furthermore, no rule of priority p' and substitution Θ' exists with $\Theta'(p') < \Theta(p)$ for which the above conditions hold.

When the rule r is clear from the context or not important, we may write $\rightarrowtail_p$ rather than $\rightarrowtail_p^r$. By $\rightarrowtail_p^*$, we denote the reflexive-transitive closure of $\rightarrowtail_p$.

3 Operational Semantics with Persistent Constraints $\omega_!$

In this section, we present the operational semantics with persistent constraints $\omega_!$, proposed in (Betz et al. 2009). Our semantics is built on the following basic ideas:

1. In ω_e, the body of a propagation rule can be generated any number of times, provided that the corresponding head constraints are present in the store. In order to give consideration to this theoretical behavior, we introduce those body constraints as so-called *persistent constraints*. A persistent constraint is a finite representation of a large, though unspecified number of identical constraints. For a proper distinction, constraints that are not persistent constraints are henceforth called *linear* constraints.
2. As a secondary consequence, arbitrary generation of rule bodies in ω_e affects other types of CHR rules as well. Consider the following program:

$$\begin{array}{lcll} \mathrm{r1} & @ & a & \Longrightarrow \ b \\ \mathrm{r2} & @ & b & \Leftrightarrow \ c \end{array}$$

If executed with a goal a, this program can generate an arbitrary number of constraints of the form b. As a consequence of this, it can also generate arbitrarily many constraints c. To take these indirect consequences of propagation rules into account, we introduce a rule's body constraints as persistent whenever its removed head can be matched completely with persistent constraints.

3. As a persistent constraint represents an arbitrary number of identical constraints, we consider multiple occurrences of a persistent constraint as idempotent. Thus, we implicitly apply a set semantics to persistent constraints.
4. We adapt the execution model such that a transition takes place only if the post-transition state is not equivalent to the pre-transition state. This entails two beneficial consequences: Firstly, in combination with the set semantics on persistent constraints, it avoids trivial non-termination of propagation rules. Secondly, as failed states are equivalent, it enforces termination upon failure.

The formal definition of $\omega_!$ is given in Sect. 3.1. In Sect. 3.2, we state results concerning its soundness and completeness with respect to ω_e.

3.1 Definition

In this section, we give a formal definition of our operational semantics $\omega_!$. We present our adapted notions of state and state equivalence and a transition system which consists of two distinct transition rules.

Definition 3.1 defines $\omega_!$ states. With respect to ω_e, the goal store $\mathbb{G}$ is split up into a store $\mathbb{L}$ of linear constraints and a store $\mathbb{P}$ of persistent constraints:

Definition 3.1 ($\omega_!$ *State*)
A $\omega_!$ *state* is a tuple of the form $\langle \mathbb{L}; \mathbb{P}; \mathbb{B}; \mathbb{V} \rangle$, where $\mathbb{L}$ and $\mathbb{P}$ are multisets of CHR constraints called the *linear (CHR) store* and *persistent (CHR) store*, respectively. $\mathbb{B}$ is a conjunction of built-in constraints and $\mathbb{V}$ is a set of variables called the *global variables*. We use $\Sigma_!$ to denote the set of all $\omega_!$ states.

Definition 3.2 is analogous to ω_e, though adapted to comply with Definition 3.1.

Definition 3.2 (*Variable Types*)
For the variables occurring in a $\omega_!$ state $\sigma = \langle \mathbb{L}; \mathbb{P}; \mathbb{B}; \mathbb{V} \rangle$ we distinguish three different types:

1. a variable $v \in \mathbb{V}$ is called a *global* variable
2. a variable $v \notin \mathbb{V}$ is called a *local* variable
3. a variable $v \notin (\mathbb{V} \cup \mathbb{L} \cup \mathbb{P})$ is called a *strictly local* variable

The following definition of state equivalence is adapted to comply with Definition 3.1 and extended to handle idempotence of persistent constraints.

Definition 3.3 (*Equivalence of* $\omega_!$ *States*)
Equivalence between $\omega_!$ states is the smallest equivalence relation $\equiv_!$ over $\omega_!$ states that satisfies the following conditions:

1. *(Equality as Substitution)* Let X be a variable, t be a term and $\doteq$ the syntactical equality relation.
$$\langle \mathbb{L}; \mathbb{P}; X \doteq t \wedge \mathbb{B}; \mathbb{V} \rangle \equiv_! \langle \mathbb{L}\,[X/t]\,; \mathbb{P}\,[X/t]\,; X \doteq t \wedge \mathbb{B}; \mathbb{V} \rangle$$
2. *(Transformation of the Constraint Store)* If $\mathcal{CT} \models \exists \bar{s}.\mathbb{B} \leftrightarrow \exists \bar{s}'.\mathbb{B}'$ where $\bar{s}, \bar{s}'$ are the strictly local variables of $\mathbb{B}, \mathbb{B}'$, respectively, then:
$$\langle \mathbb{L}; \mathbb{P}; \mathbb{B}; \mathbb{V} \rangle \equiv_! \langle \mathbb{L}; \mathbb{P}; \mathbb{B}'; \mathbb{V} \rangle$$
3. *(Omission of Non-Occurring Global Variables)* If X is a variable that does not occur in $\mathbb{L}$, $\mathbb{P}$, or $\mathbb{B}$ then:
$$\langle \mathbb{L}; \mathbb{P}; \mathbb{B}; \{X\} \cup \mathbb{V} \rangle \equiv_! \langle \mathbb{L}; \mathbb{P}; \mathbb{B}; \mathbb{V} \rangle$$
4. *(Equivalence of Failed States)*
$$\langle \mathbb{L}; \mathbb{P}; \bot; \mathbb{V} \rangle \equiv_! \langle \mathbb{L}'; \mathbb{P}'; \bot; \mathbb{V}' \rangle$$
5. *(Contraction)*
$$\langle \mathbb{L}; P \uplus P \uplus \mathbb{P}; \mathbb{B}; \mathbb{V} \rangle \equiv_! \langle \mathbb{L}; P \uplus \mathbb{P}; \mathbb{B}; \mathbb{V} \rangle$$

Based on the definition of $\equiv_e$, we define the operational semantics $\omega_!$ below. Since body constraints may be introduced either as linear or as persistent constraints, uniform rule application is replaced by two distinct application modes. Note that $\omega_!$ is only defined for *range-restricted* programs. In (Betz et al. 2010) it is shown that $\omega_!$ is no longer compliant with ω_e for non-range-restricted programs.

Definition 3.4 ($\omega_!$ *Transitions*)
For a range-restricted CHR program $\mathcal{P}$, the state transition system $(\Sigma_!/\equiv_!, \rightarrowtail_!)$ is defined as follows.

ApplyLinear:
$$\frac{r \ @ \ (H_1^l \uplus H_1^p) \backslash (H_2^l \uplus H_2^p) \Leftrightarrow G \mid B_c, B_b \quad H_2^l \neq \emptyset \quad \sigma \neq \tau}{\begin{array}{c}\sigma = [\langle H_1^l \uplus H_2^l \uplus \mathbb{L}; H_1^p \uplus H_2^p \uplus \mathbb{P}; G \wedge \mathbb{B}; \mathbb{V} \rangle] \\ \rightarrowtail_!^r [\langle H_1^l \uplus B_c \uplus \mathbb{L}; H_1^p \uplus H_2^p \uplus \mathbb{P}; G \wedge \mathbb{B} \wedge B_b; \mathbb{V} \rangle] = \tau\end{array}}$$

ApplyPersistent:
$$\frac{r \ @ \ (H_1^l \uplus H_1^p) \backslash H_2^p \Leftrightarrow G \mid B_c, B_b \quad \sigma \neq \tau}{\begin{array}{c}\sigma = [\langle H_1^l \uplus \mathbb{L}; H_1^p \uplus H_2^p \uplus \mathbb{P}; G \wedge \mathbb{B}; \mathbb{V} \rangle] \\ \rightarrowtail_!^r [\langle H_1^l \uplus \mathbb{L}; H_1^p \uplus H_2^p \uplus B_c \uplus \mathbb{P}; G \wedge \mathbb{B} \wedge B_b; \mathbb{V} \rangle] = \tau\end{array}}$$

When the rule r is clear from the context or not important, we may write $\rightarrowtail_!$ rather than $\rightarrowtail_!^r$. By $\rightarrowtail_!^*$, we denote the reflexive-transitive closure of $\rightarrowtail_!$.

Example 3.1
Again consider the transitive edge program from Example 1.1 and an analogous computation to that given in Example 2.1, using an **ApplyPersistent** transition:

$$\begin{array}{ll}\sigma \equiv_! & \langle e(A', B'), e(B', C'); \emptyset; A = A' \wedge B = B' \wedge A = C'; \emptyset \rangle \\ \rightarrowtail^t & \langle e(A', B'), e(B', C'); e(A', C'); A = A' \wedge B = B' \wedge A = C'; \emptyset \rangle \\ \equiv_! & \langle e(A, B), e(B, A); e(A, A); \top; \emptyset \rangle = \sigma'\end{array}$$

The operational semantics $\omega_!$ solves the trivial non-termination problem through the combination of persistent constraints and its irreflexive transition system, as the following observation shows:

$$\begin{array}{ll} \sigma' \equiv_! & \langle e(A',B'), e(B',C'); e(A,A); A = A' \wedge B = B' \wedge A = C'; \emptyset \rangle \\ \not\rightarrowtail^t & \langle e(A',B'), e(B',C'); e(A,A), e(A',C'); A = A' \wedge B = B' \wedge A = C'; \emptyset \rangle \\ \equiv_! & \langle e(A,B), e(B,A); e(A,A); \top; \emptyset \rangle = \sigma' \end{array}$$

3.2 Soundness and Completeness

The following two theorems state the soundness and completeness of $\omega_!$ with respect to ω_e.

Theorem 1 states that for every given state that can be derived in $\omega_!$, we can derive a corresponding state in ω_e which contains the linear constraints of the former state in equal multiplicities, but its persistent constraints in arbitrarily high multiplicities.

Theorem 1 (*Soundness*)
Let $\langle \mathbb{G}; \emptyset; \mathbb{B}; \mathbb{V} \rangle, \langle \mathbb{L}; \mathbb{P}; \mathbb{B}'; \mathbb{V} \rangle \in \Sigma_!$. If $\langle \mathbb{G}; \emptyset; \mathbb{B}; \mathbb{V} \rangle \rightarrowtail_!^* \langle \mathbb{L}; \mathbb{P}; \mathbb{B}'; \mathbb{V} \rangle$ then for every $N \in \mathbb{N}$ there exists a state $\langle \mathbb{G}'; \mathbb{B}'; \mathbb{V} \rangle \in \Sigma_e$ such that $\langle \mathbb{G}; \mathbb{B}; \mathbb{V} \rangle \rightarrowtail_e^* \langle \mathbb{G}'; \mathbb{B}'; \mathbb{V} \rangle$ and $\mathbb{L} \uplus N \cdot \mathbb{P} \subseteq \mathbb{G}'$.

Theorem 2 states that for every given state that can be derived in ω_e, we can derive a corresponding state in $\omega_!$, such that its linear store and some subset of its persistent store add up exactly to the user-defined store of the former state.

Theorem 2 (*Completeness*)
Let $\langle \mathbb{G}; \mathbb{B}; \mathbb{V} \rangle, \langle \mathbb{G}'; \mathbb{B}'; \mathbb{V} \rangle \in \Sigma_e$. If $\langle \mathbb{G}; \mathbb{B}; \mathbb{V} \rangle \rightarrowtail_e^* \langle \mathbb{G}'; \mathbb{B}'; \mathbb{V} \rangle$, then there exists a state $\langle \mathbb{L}; \mathbb{P}; \mathbb{B}'; \mathbb{V} \rangle \in \Sigma_!$ such that $\langle \mathbb{G}; \emptyset; \mathbb{B}; \mathbb{V} \rangle \rightarrowtail_!^* \langle \mathbb{L}; \mathbb{P}; \mathbb{B}'; \mathbb{V} \rangle$ and $\mathbb{L} \subseteq \mathbb{G}' \subseteq \mathbb{P} \uplus \mathbb{L}$.

4 Implementation via Source-To-Source Transformation

In this section we provide an implementation of the operational semantics $\omega_!$ in the form of a source-to-source transformation. A CHR program $\mathcal{P}$ is transformed into a program $[\![\mathcal{P}]\!]$ such that $[\![\mathcal{P}]\!]$'s execution in ω_p is sound and complete with respect to the execution of $\mathcal{P}$ in $\omega_!$.

The following definition of pathological rules is chosen such as to coincide with those rules that cause redundant rule applications – modulo state equivalence – in ω_e, i.e. in a non-pathological program every rule applied to a state σ results in a state $\tau \not\equiv_e \sigma$ (cf. (Betz et al. 2010)). This ensures that **ApplyLinear** transitions never fail due to irreflexivity, and hence, the resulting ω_p programs do not need to perform an explicit equivalence check.

Definition 4.1 (*Pathological Rules*)
A CHR rule $r \mathbin{@} H_1 \backslash H_2 \Leftrightarrow G \mid B_c, B_b$ is called *pathological* if and only if $\exists \mathbb{B}.\langle H_2; \mathbb{B} \wedge G; \emptyset \rangle \equiv_e \langle B_c; B_b; \emptyset \rangle$. It is called *trivially pathological* iff $\mathbb{B} = \top$. A CHR program $\mathcal{P}$ is called pathological if it contains at least one pathological rule.

Assuming a CHR program $\mathcal{P}$ without pathological rules, we now show how to encode it as $[\![\mathcal{P}]\!]$ for execution in ω_p.

For every n-ary constraint c/n in $\mathcal{P}$, there exists a constraint $c/(n+1)$ in $[\![\mathcal{P}]\!]$. In the following, for a multiset of user-defined $\omega_!$-constraints $M = \{c_1(\bar{t}_1), \ldots, c_n(\bar{t}_n)\}$ let $l(M) = \{c_1(l, \bar{t}_1), \ldots, c_n(l, \bar{t}_n)\}$, $p(M) = \{c_1(p, \bar{t}_1), \ldots, c_n(p, \bar{t}_n)\}$, and $c(M) = \{c_1(c, \bar{t}_1), \ldots, c_n(c, \bar{t}_n)\}$.

The rules of $[\![\mathcal{P}]\!]$ are constructed via the following source-to-source transformation.

1. For every rule $r \ @ \ H_1 \setminus H_2 \Leftrightarrow G \mid B$ in $\mathcal{P}$, and all multisets $H_1^l, H_1^p, H_2^l, H_2^p$ s.t. $H_1^l \uplus H_1^p = H_1$ and $H_2^l \uplus H_2^p = H_2$ and $H_2^l \neq \emptyset$, the following rule is in $[\![\mathcal{P}]\!]$:
$$3 :: l(H_1^l) \uplus p(H_1^p) \uplus p(H_2^p) \setminus l(H_2^l) \Leftrightarrow G \mid l(B_c), B_b$$
2. For every rule $r \ @ \ H_1 \setminus H_2 \Leftrightarrow G \mid B_c, B_b$ in $\mathcal{P}$, and all multisets H_1^l, H_1^p s.t. $H_1^l \uplus H_1^p = H_1$, the following rule is in $[\![\mathcal{P}]\!]$:
$$3 :: l(H_1^l) \uplus p(H_1^p) \uplus p(H_2) \Longrightarrow G \mid c(B_c), B_b$$
3. For every rule $\{c(p, \bar{t}), c(p, \bar{t}')\} \uplus H_1 \setminus H_2 \Leftrightarrow G \mid B$ in $[\![\mathcal{P}]\!]$, add also the following rule:
$$3 :: \{c(p, \bar{t})\} \uplus H_1 \setminus H_2 \Leftrightarrow \bar{t} = \bar{t}' \wedge G \mid B$$
4. For every user-defined constraint c/n in $\mathcal{P}$, add the following rules, where $\bar{t}$ is a sequence of n different variables:
$$1 :: c(p, \bar{t}) \backslash c(c, \bar{t}) \Leftrightarrow \top$$
$$2 :: c(c, \bar{t}) \Leftrightarrow c(p, \bar{t})$$

Example 4.1 (*Encoding of Transitive Hull*)
We consider the transitive hull program from Example 1.1:

$$t \quad @ \quad e(X,Y), e(Y,Z) \quad \Longrightarrow \quad e(X,Z)$$

According to the encoding given above, the program is transformed as follows:

$$\begin{array}{lll}
3 :: & e(l,X,Y), e(l,Y,Z) & \Longrightarrow \ e(c,X,Z) \\
3 :: & e(l,X,Y), e(p,Y,Z) & \Longrightarrow \ e(c,X,Z) \\
3 :: & e(p,X,Y), e(l,Y,Z) & \Longrightarrow \ e(c,X,Z) \\
3 :: & e(p,X,Y), e(p,Y,Z) & \Longrightarrow \ e(c,X,Z) \\
\\
3 :: & e(p,X,Y) & \Longrightarrow \ X = Y \wedge Y = Z \mid e(c,X,Z) \\
\\
1 :: & e(p,X,Y) \backslash e(c,X,Y) & \Leftrightarrow \ \top \\
2 :: & e(c,X,Y) & \Leftrightarrow \ e(p,X,Y)
\end{array}$$

The grouping of the rules above reflects the transformation steps 2, 3, and 4. Transformation step 1 is not productive in this example. The fifth rule above is operationally equivalent to $3 :: e(p,X,X) \Longrightarrow e(c,X,X)$, and hence, is redundant, as the resulting constraint will immediately be removed again by the rule with priority 1.

Furthermore, transformation step 3 also adds an additional symmetric version of the fifth rule, which was omitted here, as it is operationally equivalent as well.

Execution of a transformed program in ω_p is equivalent to execution of the original program in $\omega_!$, as the following theorem shows.

Theorem 3 (*Soundness and Completeness of Encoding*)
Let $G, \mathbb{L}, \mathbb{P}$ be multisets of user-defined constraints, $B, \mathbb{B}$ conjunctions of built-in constraints, and $\mathbb{V} = \text{vars}(G \wedge B)$. If $\mathcal{P}$ is a non-pathologic CHR program, then

$$\langle G; \emptyset; B; \mathbb{V}\rangle \rightarrowtail_!^* \langle \mathbb{L}; \mathbb{P}; \mathbb{B}; \mathbb{V}\rangle \not\rightarrowtail_! \text{ in } \mathcal{P}$$

iff

$$\exists \mathbb{T}, n. \langle l(G), B; \emptyset; \top; \emptyset\rangle_0^{\mathbb{V}} \rightarrowtail_p^* \langle \emptyset; l(\mathbb{L}) \uplus p(\mathbb{P}); \mathbb{B}; \mathbb{T}\rangle_n^{\mathbb{V}} \not\rightarrowtail_p \text{ in } [\![\mathcal{P}]\!]$$

Example 4.2 (*Example Runs of ω_p and $\omega_!$ Programs*)
The following example derivation shows how the translated program terminates with a state that corresponds with the result of an execution of the original program in $\omega_!$. For clarity's and brevity's sake, we do not show all intermediate states and we do not give the states' respective token stores explicitly.

$$\begin{array}{ll}
 & \langle e(l,A,B), e(l,B,A); \emptyset; \top; \emptyset\rangle_0^{\{A,B\}} \\
\rightarrowtail_p^* & \langle \emptyset; e(l,A,B)\#0, e(l,B,A)\#1; \top; \emptyset\rangle_2^{\{A,B\}} \\
\rightarrowtail_p^* & \langle \emptyset; e(l,A,B)\#0, e(l,B,A)\#1, e(c,A,A)\#2; \top; \ldots\rangle_3^{\{A,B\}} \\
\rightarrowtail_p^* & \langle \emptyset; e(l,A,B)\#0, e(l,B,A)\#1, e(p,A,A)\#3; \top; \ldots)\rangle_4^{\{A,B\}} \\
\rightarrowtail_p^* & \langle \emptyset; e(l,A,B)\#0, e(l,B,A)\#1, e(p,A,A)\#3, e(c,B,B)\#4; \top; \ldots\rangle_5^{\{A,B\}} \\
\rightarrowtail_p^* & \langle \emptyset; e(l,A,B)\#0, e(l,B,A)\#1, e(p,A,A)\#3, e(p,B,B)\#5; \top; \ldots)\rangle_6^{\{A,B\}} \\
\rightarrowtail_p^* & \langle \emptyset; e(l,A,B)\#0, e(l,B,A)\#1, e(p,A,A)\#3, e(p,B,B)\#5, e(c,A,B)\#6; \top; \ldots\rangle_7^{\{A,B\}} \\
\rightarrowtail_p^* & \langle \emptyset; e(l,A,B)\#0, e(l,B,A)\#1, e(p,A,A)\#3, e(p,B,B)\#5, e(p,A,B)\#7; \top; \ldots)\rangle_8^{\{A,B\}} \\
\rightarrowtail_p^* & \langle \emptyset; e(l,A,B)\#0, e(l,B,A)\#1, e(p,A,A)\#3, e(p,B,B)\#5, e(p,A,B)\#7, e(c,B,A)\#8; \top; \ldots\rangle_9^{\{A,B\}} \\
\rightarrowtail_p^* & \langle \emptyset; e(l,A,B)\#0, e(l,B,A)\#1, e(p,A,A)\#3, e(p,B,B)\#5, e(p,A,B)\#7, e(p,B,A)\#9; \top; \ldots)\rangle_{10}^{\{A,B\}} \\
\rightarrowtail_p^* & \langle \emptyset; e(l,A,B)\#0, e(l,B,A)\#1, e(p,A,A)\#3, e(p,B,B)\#5, e(p,A,B)\#7, e(p,B,A)\#9; \top; \ldots)\rangle_{24}^{\{A,B\}} \\
\not\rightarrowtail_p &
\end{array}$$

The above computation corresponds to the following execution in $\omega_!$:

$$\begin{array}{ll}
\sigma \equiv_! & \langle e(A,B), e(B,A); \emptyset; \top; \{A,B\}\rangle \\
\rightarrowtail_!^t & \langle e(A,B), e(B,A); e(A,A); \top; \{A,B\}\rangle \\
\rightarrowtail_!^t & \langle e(A,B), e(B,A); e(A,A), e(B,B); \top; \{A,B\}\rangle \\
\rightarrowtail_!^t & \langle e(A,B), e(B,A); e(A,A), e(B,B), e(A,B); \top; \{A,B\}\rangle \\
\rightarrowtail_!^t & \langle e(A,B), e(B,A); e(A,A), e(B,B), e(A,B), e(B,A); \top; \{A,B\}\rangle \\
\not\rightarrowtail_! &
\end{array}$$

This example also demonstrates how $\omega_!$ streamlines execution which in turn facilitates formal reasoning over derivations: the whole computation consists of 4 state transitions in $\omega_!$, whereas the corresponding computation in ω_p requires 60 state transitions.

The presented source-to-source transformation satisfies conditions for an *acceptable encoding* according to (Gabbrielli et al. 2009), modulo the necessary distinction between linear and persistent constraints in the translation.

5 Discussion

In this section, we discuss our insights on the behavior of $\omega_!$ in comparison with existing operational semantics.

5.1 Termination Behavior

Our proposed operational semantics $\omega_!$ exhibits a termination behavior different from ω_t, ω_p, and ω_e. Compared to ω_e, we have solved the problem of trivial non-termination of propagation rules, whereas any program terminating in ω_e also terminates in $\omega_!$. With respect to ω_t and ω_p, we found programs that terminate in $\omega_!$ but not in ω_t and ω_p, and vice versa.

We have seen in Example 2.1 and Example 3.1 that the transitivity rule displays different behavior in ω_e and $\omega_!$. The program's termination behavior in ω_t and ω_p has been investigated in (Pilozzi and De Schreye 2009), where it is shown to terminate for acyclic graphs. However, states containing cyclic graphs entail non-terminating behavior (cf. (Betz et al. 2010)). Contrarily, we show in the accompanying technical report (Betz et al. 2010) that in the operational semantics $\omega_!$, the computation of the transitive hull terminates for every possible input. At the same place, we present a CHR program that terminates in ω_t and ω_p, but not in $\omega_!$.

5.2 Related Work

In (Sarna-Starosta and Ramakrishnan 2007) the set-based semantics ω_{set} has been introduced. Its development was, among other considerations, driven by the intention to eliminate the propagation history. Besides addressing the problem of trivial non-termination in a novel manner, it reduces non-determinism similarly to the refined operational semantics ω_r (Duck et al. 2004). In ω_{set}, a propagation rule cannot be fired infinitely often for a possible matching. However, multiple firings are possible, the exact number depending on the built-in store.

The authors of (Sarna-Starosta and Ramakrishnan 2007) justify their set-based approach by the following statement:

> "When working with a multi-set-based constraint store, it appears that propagation history is essential to provide a reasonable semantics."

Our approach can be understood as a compromise since we avoid a propagation history by imposing an implicit set semantics on persistent constraints. The distinction between linear and persistent constraints, however, allows us to restrict the set behavior to those constraints, whereas the multiset semantics is preserved for linear constraints.

Linear logical algorithms (Simmons and Pfenning 2008) (LLA) is a programming language based on bottom-up reasoning in linear logic, inspired by logical algorithms (Ganzinger and McAllester 2002). The first implementation of logical algorithms was realized in CHR with rule priorities (De Koninck 2009).

Our proposed operational semantics $\omega_!$ is related to LLA (Simmons and Pfenning

2008), but displays significant differences: Firstly, the notion of a constraint theory with built-in constraints is absent in LLA. Secondly, LLA rules are restricted such that persistent propositions cannot be derived multiple times, whereas $\omega_!$ makes no such restriction and solves this problem via the irreflexive transition system. Thirdly, LLA requires a strict separation of propositions into linear and persistent ones. In $\omega_!$ a CHR constraint can occur in the linear store, in the persistent store, or both.

On the other hand, the separation of propositions in LLA allows the corresponding rules to freely mix linear and persistent propositions in bodies. This is not directly possible with our approach, as CHR constraints in a body are either added as linear or persistent constraints.

6 Conclusion and Future Work

The main motivation of this work was the observation that CHR research spans a spectrum ranging from an analytical to a pragmatic end: on the analytical side of the spectrum, emphasis is put on the formal aspects and properties of the language while on the pragmatic side, it is put on implementation and efficiency. A variety of operational semantics has been brought forth in the past, each aligning with one side of the spectrum. In this work we proposed the novel operational semantics $\omega_!$, heeding both analytical and pragmatic aspects.

Unlike other operational semantics with a strong analytical foundation, $\omega_!$ thus provides a terminating execution model and may be implemented as is. We provided evidence to this claim by presenting a sound and complete encoding of $\omega_!$ into ω_p, which can be used to implement $\omega_!$ by source-to-source transformation.

Our operational semantics $\omega_!$ is based on the concept of persistent constraints. These are finite representations of an arbitrarily large number of syntactically equivalent constraints. They enable us to subsume trivially non-terminating computations in a single derivation step.

We proved soundness and completeness of our operational semantics $\omega_!$ with respect to ω_e. The latter stands exemplarily for analytical formalizations of the operational semantics, thus providing a strong analytical foundation for $\omega_!$. This facilitates program analysis and formal proofs of program properties.

In its current formulation, $\omega_!$ is only applicable to range-restricted CHR programs – a limitation we plan to address in the future. Furthermore, similar to ω_t being the basis for numerous extensions to CHR (Sneyers et al. 2010), we plan to investigate the effect of building these extensions on $\omega_!$.

In a concurrent environment, some kind of conflict resolution is required for the case that multiple rules try to remove the same constraint. For example, in (Sulzmann and Lam 2008) a transaction-based approach is used, leading to a rollback, if the first evaluated rule application removed the constraint. The formulation of the **ApplyPersistent** transition reveals that for persistent constraints, no such conflicts have to be taken into account. A closer investigation of potential benefits of the persistent constraint approach in concurrent settings remains to be conducted.

References

Abdennadher, S. 1997. Operational semantics and confluence of constraint propagation rules. In *Principles and Practice of Constraint Programming*. 252–266.

Abdennadher, S. and Frühwirth, T. 1999. Operational equivalence of CHR programs and constraints. In *Principles and Practice of Constraint Programming, CP 1999*, J. Jaffar, Ed. Lecture Notes in Computer Science, vol. 1713. Springer-Verlag, 43–57.

Abdennadher, S., Frühwirth, T., and Meuss, H. 1999. Confluence and semantics of constraint simplification rules. *Constraints 4,* 2, 133–165.

Betz, H. and Frühwirth, T. 2005. A linear-logic semantics for constraint handling rules. In *Principles and Practice of Constraint Programming, 11th International Conference, CP 2005*, P. van Beek, Ed. Lecture Notes in Computer Science, vol. 3709. Springer-Verlag, Sitges, Spain, 137–151.

Betz, H., Raiser, F., and Frühwirth, T. 2009. Persistent constraint in constraint handling rules. In *Proceedings of 23rd Workshop on (Constraint) Logic Programming, WLP 2009*. to appear.

Betz, H., Raiser, F., and Frühwirth, T. 2010. A complete and terminating execution model for constraint handling rules. Tech. Rep. 01, Ulm University. January.

De Koninck, L. 2009. Logical Algorithms meets CHR: A meta-complexity result for Constraint Handling Rules with rule priorities. *Theory and Practice of Logic Programming 9,* 2 (March), 165–212.

De Koninck, L., Schrijvers, T., and Demoen, B. 2007. User-definable rule priorities for CHR. In *PPDP '07: Proceedings of the 9th ACM SIGPLAN international conference on Principles and practice of declarative programming*. ACM, New York, NY, USA, 25–36.

De Koninck, L., Stuckey, P. J., and Duck, G. J. 2008. Optimizing compilation of CHR with rule priorities. In *Functional and Logic Programming, 9th International Symposium (FLOPS)*, J. Garrigue and M. V. Hermenegildo, Eds. Lecture Notes in Computer Science, vol. 4989. Springer-Verlag, 32–47.

Duck, G. J., Stuckey, P. J., García de la Banda, M., and Holzbaur, C. 2004. The refined operational semantics of Constraint Handling Rules. In *Logic Programming, 20th International Conference, ICLP 2004*, B. Demoen and V. Lifschitz, Eds. Lecture Notes in Computer Science, vol. 3132. Springer-Verlag, Saint-Malo, France, 90–104.

Frühwirth, T. 1998. Theory and practice of constraint handling rules. *Journal of Logic Programming, Special Issue on Constraint Logic Programming 37,* 1-3 (October), 95–138.

Frühwirth, T. 2005. Parallelizing union-find in constraint handling rules using confluence analysis. In *Logic Programming, 21st International Conference, ICLP 2005*, M. Gabbrielli and G. Gupta, Eds. Lecture Notes in Computer Science, vol. 3668. Springer-Verlag, Sitges, Spain, 113–127.

Frühwirth, T. 2009. *Constraint Handling Rules*. Cambridge University Press.

Frühwirth, T. and Abdennadher, S. 2003. *Essentials of Constraint Programming*. Springer-Verlag.

Frühwirth, T. and Hanschke, P. 1993. Terminological reasoning with Constraint Handling Rules. In *Principles and Practice of Constraint Programming*. MIT Press, 80–89.

Gabbrielli, M., Mauro, J., and Meo, M. C. 2009. On the expressive power of priorities in CHR. In *Proceedings of the 11th International ACM SIGPLAN Conference on Principles and Practice of Declarative Programming*, A. Porto and F. J. López-Fraguas, Eds. ACM, Coimbra, Portugal, 267–276.

Ganzinger, H. and McAllester, D. A. 2002. Logical algorithms. In *Logic Programming, 18th International Conference, ICLP 2002*, P. J. Stuckey, Ed. Lecture Notes in Computer Science, vol. 2401. Springer-Verlag, 209–223.

Pilozzi, P. and De Schreye, D. 2009. Proving termination by invariance relations. In *25th International Conference Logic Programming, ICLP*, P. M. Hill and D. S. Warren, Eds. Lecture Notes in Computer Science, vol. 5649. Springer-Verlag, Pasadena, CA, USA, 499–503.

Raiser, F., Betz, H., and Frühwirth, T. 2009. Equivalence of CHR states revisited. In *6th International Workshop on Constraint Handling Rules (CHR)*, F. Raiser and J. Sneyers, Eds. 34–48.

Sarna-Starosta, B. and Ramakrishnan, C. 2007. Compiling Constraint Handling Rules for efficient tabled evaluation. In *9th Intl. Symp. Practical Aspects of Declarative Languages, PADL*, M. Hanus, Ed. Lecture Notes in Computer Science, vol. 4354. Springer-Verlag, Nice, France, 170–184.

Simmons, R. J. and Pfenning, F. 2008. Linear logical algorithms. In *Automata, Languages and Programming, 35th International Colloquium, ICALP 2008*, L. Aceto, I. Damgård, L. A. Goldberg, M. M. Halldórsson, A. Ingólfsdóttir, and I. Walukiewicz, Eds. Lecture Notes in Computer Science, vol. 5126. Springer-Verlag, 336–347.

Sneyers, J., Van Weert, P., Schrijvers, T., and De Koninck, L. 2010. As time goes by: Constraint Handling Rules – A survey of CHR research between 1998 and 2007. *Theory and Practice of Logic Programming 10,* 1, 1–47.

Sulzmann, M. and Lam, E. S. L. 2007. A concurrent constraint handling rules semantics and its implementation with software transactional memory. In *Proceedings of the POPL 2007 Workshop on Declarative Aspects of Multicore Programming*, N. Glew and G. E. Blelloch, Eds. ACM, 19–24.

Sulzmann, M. and Lam, E. S. L. 2008. Parallel execution of multi-set constraint rewrite rules. In *Proceedings of the 10th International ACM SIGPLAN Conference on Principles and Practice of Declarative Programming (PPDP)*, S. Antoy and E. Albert, Eds. ACM, Valencia, Spain, 20–31.