

Relative Expressiveness of Defeasible Logics II

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Abstract

(Maher 2012) introduced an approach for relative expressiveness of defeasible logics, and two notions of relative expressiveness were investigated. Using the first of these definitions of relative expressiveness, we show that all the defeasible logics in the **DL** framework are equally expressive under this formulation of relative expressiveness. The second formulation of relative expressiveness is stronger than the first. However, we show that logics incorporating individual defeat are equally expressive as the corresponding logics with team defeat. Thus the only differences in expressiveness of logics in **DL** arise from differences in how ambiguity is handled. This completes the study of relative expressiveness in **DL** begun in (Maher 2012).

KEYWORDS: defeasible logic, non-monotonic reasoning, relative expressiveness

Introduction

Defeasible logics provide several linguistic features to support the expression of defeasible knowledge. There are also a variety of such logics, supporting different intuitions about reasoning in a defeasible setting. The **DL** framework (Antoniou et al. 2000; Billington et al. 2010) provides logics that allow ambiguity in the “truth” status of a literal to propagate, and logics that block ambiguity; it has logics that require an individual rule to defeat all competitors, and logics that allow a “team” of rules to defeat competitors. Given the different inferences supported by the different logics, it is interesting to determine whether these logics are equally powerful or whether, perhaps, some are more powerful than the others.

In terms of inference strength, (Billington et al. 2010) established the relationship between the different logics of **DL**. In terms of computational complexity, the logics of **DL** are equivalent: all have linear complexity (Maher 2001; Billington et al. 2010). Relative expressiveness of the different logics was first investigated in (Maher 2012), which developed a framework, based on simulation in the presence of additional elements. Two notions of relative expressiveness within this framework were investigated: polynomial simulation wrt the addition of facts, and simulation wrt the addition of rules.

In this paper we continue this investigation. We will see that all the logics of **DL** are equally expressive, using the first notion of relative expressiveness. Thus we cannot distinguish the logics based on this notion. We also establish that individual defeat has equal expressiveness to team defeat in the logics of **DL** wrt addition of rules. This is somewhat surprising, given the apparent greater sophistication of the team defeat inference rules. Given results in (Maher 2012), this completes the study of relative expressiveness for **DL**.

The next two sections summarize the **DL** framework of defeasible logics and the notions of relative expressiveness introduced in (Maher 2012). Then the following two sections together provide the proof that the logics of **DL** are of equal expressivity (in terms of simulation wrt addition of facts). The first shows the simulation of an ambiguity propagating logic by an ambiguity blocking logic, while the second shows a simulation in the reverse direction. Combined with results of (Maher 2012), this establishes that the logics in **DL** all have the same expressiveness in this formulation.

The following sections investigate relative expressiveness via the second, stronger formulation. Adapting simulations of (Maher 2012) to the stronger formulation, we establish that individual defeat has equal expressiveness to team defeat in the logics of **DL**. Proofs of the results in this paper are detailed and lengthy. They appear in an appendix.

Defeasible Logic

In this section we can only present an outline of the defeasible logics we investigate. Further details can be obtained from (Billington et al. 2010) and the references therein. We address propositional defeasible logics, but the results should extend to a first-order language.

A defeasible theory is built from a language Σ of literals (which we assume is closed under negation) and a language Λ of labels. A *defeasible theory* $D = (F, R, >)$ consists of a set of facts F , a finite set of rules R , each rule with a distinct label from Λ , and an acyclic relation $>$ on Λ called the *superiority relation*. This syntax is uniform for all the logics considered here. Facts are individual literals expressing indisputable truths. Rules relate a set of literals (the body), via an arrow, to a literal (the head), and are one of three types: a strict rule, with arrow \rightarrow ; a defeasible rule, with arrow \Rightarrow ; or a defeater, with arrow \rightsquigarrow . Strict rules represent inferences that are unequivocally sound if based on definite knowledge; defeasible rules represent inferences that are generally sound. Inferences suggested by a defeasible rule may fail, due to the presence in the theory of other rules. Defeaters do not support inferences, but may impede inferences suggested by other rules. The superiority relation provides a local priority on rules. Strict or defeasible rules whose bodies are established defeasibly represent claims for the head of the rule to be concluded. The superiority relation contributes to the adjudication of these claims by an inference rule, leading (possibly) to a conclusion. Given a theory D , the corresponding languages are expressed by $\Sigma(D)$ and $\Lambda(D)$.

Defeasible logics derive conclusions that are outside the syntax of the theories. Conclusions may have the form $+dq$, which denotes that under the inference rule d the literal q can be concluded, or $-dq$, which denotes that the logic can establish that under the inference rule d the literal q cannot be concluded. The syntactic element d is called a tag. In general, neither conclusion may be derivable: q cannot be concluded under d , but the logic is unable to establish that. Tags $+\Delta$ and $-\Delta$ represent monotonic provability (and unprovability) where inference is based on facts, strict rules, and modus ponens. We assume these tags and their inference rules are present in every defeasible logic. What distinguishes a logic is the inference rule for defeasible reasoning. The four logics discussed in the Introduction correspond to four different pairs of inference rules, labelled ∂ , δ , ∂^* , and δ^* ; they produce conclusions of the form (respectively) $+\partial q$, $-\partial q$, $+\delta q$, $-\delta q$, etc. The inference rules δ and δ^* require auxiliary tags and inference rules, denoted by σ and σ^* , respectively. For each of the four principal defeasible tags d , the corresponding logic is denoted by **DL**(d).

The four principal tags and corresponding inference rules represent different intuitions about

defeasible reasoning: in ∂ and ∂^* ambiguity is blocked, while in δ and δ^* ambiguity is propagated; in ∂ and δ rules for a literal act as a team to overcome competing rules, while in ∂^* and δ^* a single rule must overcome all competing rules. A more detailed discussion of ambiguity and team defeat in the **DL** framework is given in (Billington et al. 2010) and (Maher 2012).

The inference rules are presented in the appendix in the form of the definition of a function \mathcal{T}_D for a given theory D . Given a defeasible theory D , for any set of conclusions E , $\mathcal{T}_D(E)$ denotes the set of conclusions inferred from E using D and one application of an inference rule. \mathcal{T}_D is a monotonic function on the complete lattice of sets of conclusions ordered by containment. The least fixedpoint of \mathcal{T}_D is the set of all conclusions that can be drawn from D . We follow standard notation in that $\mathcal{T}_D \uparrow 0 = \emptyset$ and $\mathcal{T}_D \uparrow (n+1) = \mathcal{T}_D(\mathcal{T}_D \uparrow n)$.

The relative inference strength of the different logics in **DL** was established in the inclusion theorem of (Billington et al. 2010). For any tag d , $+d(D)$ denotes the set of conclusions of D of the form $+dq$ and similarly for $-d$.

Theorem 1 (Inclusion Theorem (Billington et al. 2010))

Let D be a defeasible theory.

- (a) $+\Delta(D) \subseteq +\delta^*(D) \subseteq +\delta(D) \subseteq +\partial(D) \subseteq +\sigma(D) \subseteq +\sigma^*(D)$.
- (b) $-\sigma^*(D) \subseteq -\sigma(D) \subseteq -\partial(D) \subseteq -\delta(D) \subseteq -\delta^*(D) \subseteq -\Delta(D)$.
- (c) $+\delta^*(D) \subseteq +\partial^*(D) \subseteq +\sigma^*(D)$
- (d) $-\sigma^*(D) \subseteq -\partial^*(D) \subseteq -\delta^*(D)$

Parts (a) and (b) are proved in (Billington et al. 2010). Parts (c) and (d) can be established by similar methods.

Simulating Defeasible Logics

(Maher 2012) introduced a framework for addressing the relative expressiveness of defeasible logics. The framework identifies the greater (or equal) expressiveness of L_2 compared to L_1 with the ability to simulate any theory D in a logic L_1 by a theory $T(D)$ in the logic L_2 . Simple simulation was shown not to be sufficiently discriminating, so simulation was required to hold in the presence of an addition to the theory.

The *addition* of a theory A to a theory D is denoted by $D + A$. Addition is essentially the union of the theories, but we require $\Lambda(D) \cap \Lambda(A) = \emptyset$, so that the addition of theories preserves the property that distinct rules have distinct labels. This requirement also has the effect that a superiority statement in D cannot affect a rule in A , and vice versa. Let $D = (F, R, >)$ and $A = (F', R', >')$. Then $D + A = (F \cup F', R \cup R', > \cup >')$. $\Lambda(D+A) = \Lambda(D) \cup \Lambda(A)$ and $\Sigma(D+A) = \Sigma(D) \cup \Sigma(A)$.

A simulating theory $T(D)$ in general will involve additional literals, rules and labels beyond those of D . If additions A were permitted to affect these, the notion of simulation would become trivial, so we restrict additions to have only an indirect effect on $T(D)$, via $\Sigma(D)$. Given a theory D and a possible simulating theory $T(D)$, we say an addition A is *modular* if $\Sigma(A) \cap \Sigma(T(D)) \subseteq \Sigma(D)$, $\Lambda(D) \cap \Lambda(A) = \emptyset$, and $\Lambda(T(D)) \cap \Lambda(A) = \emptyset$. In general, we will consider a class of additions but for any D and $T(D)$ only the modular additions in the class will be considered.

Since different logics involve different tags, conclusions from theories in different logics cannot be identical. For simulation it suffices that conclusions are equal modulo tags. Given logics L_1 and L_2 , with principal tags d_1 and d_2 , respectively, we say two conclusions α in L_1 and β in L_2 are *equal modulo tags* if α is $+d_1q$ and β is $+d_2q$ or α is $-d_1q$ and β is $-d_2q$.

Thus we have the following definition of simulation and relative expressiveness. For more discussion on the motivations for the definitions, see (Maher 2012).

Definition 2

Let C be a class of defeasible theories.

We say D_1 in logic L_1 is *simulated* by D_2 in L_2 with respect to a class C if, for every modular addition A in C , $D_1 + A$ and $D_2 + A$ have the same conclusions in $\Sigma(D_1 + A)$, modulo tags.

We say a logic L_1 can be simulated by a logic L_2 with respect to a class C if every theory in L_1 can be simulated by some theory in L_2 with respect to additions from C .

We say L_2 is *more (or equal) expressive than* L_1 if L_1 can be simulated by L_2 with respect C .

Different notions of relative expressiveness arise from different choices for C . There were two classes of additions investigated in (Maher 2012): the addition of *facts* (that is, A has the form $(F, \emptyset, \emptyset)$), and the addition of *rules* (that is, A has the form $(\emptyset, R, \emptyset)$). Simulation with respect to addition of rules is stronger than simulation with respect to addition of facts because any fact can equally be expressed as a strict rule with an empty body. We might also consider arbitrary additions, where A can be any defeasible theory.

The main results of (Maher 2012) are that:

- $DL(\partial)$ and $DL(\partial^*)$ have equal expressiveness, with respect to addition of *facts*, as do $DL(\delta)$ and $DL(\delta^*)$
- neither $DL(\partial)$ nor $DL(\partial^*)$ is more expressive than $DL(\delta)$ or $DL(\delta^*)$, and vice versa, with respect to addition of *rules*
- when arbitrary additions are permitted, of the four defeasible logics under consideration, none is more expressive than any other

Blocked Ambiguity Simulates Propagated Ambiguity

We now show that every theory over an ambiguity propagating logic can be simulated by a theory over the corresponding ambiguity blocking logic. To begin, we show that $\mathbf{DL}(\partial^*)$ can simulate $\mathbf{DL}(\delta^*)$. Any defeasible theory D is transformed into a new theory. The new theory employs new propositions $strict(q)$ and $supp(q)$, for each literal q , and $supp_body(r)$, $comp(r)$, and $o(r)$, for each rule r . The new theory also introduces labels $p_d(r)$, $n_d(r, s)$, $p_s(r)$, $n_s(r, s)$, for each pair r, s of opposing rules in D . These are families of propositions and labels, not predicates, despite the notation.

Definition 3

Let $D = (F, R, >)$ be a defeasible theory with language Σ . We define the transformation T of D to $T(D) = (F', R', >')$ as follows:

1. The facts of $T(D)$ are the facts of D . That is, $F' = F$.
2. Every strict rule of R is included in R' .
3. For every literal q , R' contains

$$\begin{aligned} str(q) : \quad q &\rightarrow str(q) \\ nstr(q) : \quad &\Rightarrow \neg str(q) \end{aligned}$$

and the superiority relation contains $nstr(q) >' str(q)$, for every q .

4. For each literal q in Σ , R' contains

$$q \Rightarrow \text{supp}(q)$$

5. For each strict or defeasible rule r of the form $b_1, \dots, b_n \hookrightarrow_r q$ in R , R' contains

$$\begin{aligned} \text{supp}(b_1), \dots, \text{supp}(b_n) &\Rightarrow \text{supp_body}(r) \\ \text{supp_body}(r), \neg o(r) &\Rightarrow \text{supp}(q) \end{aligned}$$

and, further, for each rule $s = B_s \hookrightarrow_s \sim q$ for $\sim q$ in R , where $s > r$, R' contains

$$\begin{aligned} n_s(r, s) : B_s &\Rightarrow o(r) \\ p_s(r) : &\Rightarrow \neg o(r) \end{aligned}$$

and the superiority relation contains $n_s(s, r) >' p_s(s)$.

6. For each strict or defeasible rule $r = B_r \hookrightarrow_r q$ in R , R' contains

$$\text{inf}(r) : B_r, \neg \text{comp}(r), \neg \text{strict}(\sim q) \Rightarrow q$$

and, further, for each rule $s = B_s \hookrightarrow_s \sim q$ for $\sim q$ in R , where $s \not\prec r$, R' contains

$$\begin{aligned} n_d(r, s) : \text{supp_body}(s) &\Rightarrow \text{comp}(r) \\ p_d(r) : &\Rightarrow \neg \text{comp}(r) \end{aligned}$$

and the superiority relation contains $n_d(r, s) >' p_d(r)$.

Parts 1 and 2 of the transformation preserve all the strict inferences from D . Part 3 allows us to distinguish strict conclusions from defeasible conclusions. The structure of these rules – where $\text{str}(q)$ is strict, $\text{nstr}(q)$ is defeasible, and $\text{nstr}(q) > \text{str}(q)$ – ensures that $\text{strict}(q)$ is inferred defeasibly iff q is inferred strictly, and $\text{strict}(q)$ fails iff strict inference of q fails. A similar structure of rules was previously used in (Maher 2012) in showing that $\mathbf{DL}(\partial^*)$ can simulate $\mathbf{DL}(\partial)$ wrt addition of facts.

We use the proposition $\text{supp}(q)$ to indicate that the literal q is supported (i.e. $+\sigma^*q$ can be inferred), while the literal q refers to defeasible provability (wrt δ^*). Part 4 ensures that every literal that holds defeasibly is also supported. This property is justified by the inclusion theorem of (Billington et al. 2010). Part 5 encodes the inference rules for support (i.e. σ^*). $\text{supp_body}(r)$ indicates that all literals in the body of rule r are supported. The head q of a rule r is supported if the body of r is supported and r is not overruled (i.e. all rules s that are superior to r fail). The overruling of r is indicated by $o(r)$. The rules $n_s(r, s)$ and $p_s(r)$ and the superiority relation ensure that $\neg o(r)$ is derived defeasibly iff there is no overruling rule s .

Rules $\text{inf}(r)$ in part 6 encode the inference rules for δ^* . q holds defeasibly iff the body of a rule r for q holds defeasibly and r has no competing rules (i.e. all rules for $\sim q$ not inferior to r have a body that fails wrt σ^*). The rules $n_d(r, s)$ and $p_d(r)$ and the superiority relation ensure that $\neg \text{comp}(r)$ is derived defeasibly iff there is no competing rule.

In this translation, the superiority relation in D is not directly represented by the superiority relation in $T(D)$. Instead, the superiority relation in D is used to restrict the instantiation of rules in the transformation, while the superiority relation in $T(D)$ is used to ensure that $o(r)$ and $\neg o(r)$ do not both fail, and similarly for $\text{comp}(r)$.

Example 4

To see the operation of this transformation, consider the following theory D , which demonstrates the difference between ambiguity propagation and blocking logics.

$$\begin{array}{ll}
r_1 : \Rightarrow p & r_3 : \neg p \Rightarrow \neg q \\
r_2 : \Rightarrow \neg p & r_4 : \Rightarrow q
\end{array}$$

In $\mathbf{DL}(\partial^*)$ from D we conclude $-\partial^*p$ and $-\partial^*\neg p$, $+\partial^*q$ and $-\partial^*\neg q$. In $\mathbf{DL}(\delta^*)$ from D we conclude $-\delta^*p$ and $-\delta^*\neg p$, $-\delta^*q$ and $-\delta^*\neg q$. We also conclude $+\sigma^*p$ and $+\sigma^*\neg p$, $+\sigma^*q$ and $+\sigma^*\neg q$.

$T(D)$ contains the following rules.

$$\begin{array}{ll}
\Rightarrow \text{supp_body}(r_1) & \text{supp_body}(r_1), \neg o(r_1) \Rightarrow \text{supp}(p) \\
\Rightarrow \text{supp_body}(r_2) & \text{supp_body}(r_2), \neg o(r_2) \Rightarrow \text{supp}(\neg p) \\
\text{supp}(\neg p) \Rightarrow \text{supp_body}(r_3) & \text{supp_body}(r_3), \neg o(r_3) \Rightarrow \text{supp}(\neg q) \\
\Rightarrow \text{supp_body}(r_4) & \text{supp_body}(r_4), \neg o(r_4) \Rightarrow \text{supp}(q)
\end{array}$$

$$\begin{array}{ll}
p \Rightarrow \text{supp}(p) & p_s(r_1) : \Rightarrow \neg o(r_1) \\
\neg p \Rightarrow \text{supp}(\neg p) & p_s(r_2) : \Rightarrow \neg o(r_2) \\
q \Rightarrow \text{supp}(q) & p_s(r_3) : \Rightarrow \neg o(r_3) \\
\neg q \Rightarrow \text{supp}(\neg q) & p_s(r_4) : \Rightarrow \neg o(r_4)
\end{array}$$

$$\begin{array}{ll}
\text{inf}(r_1) : \neg \text{comp}(r_1), \neg \text{strict}(\neg p) \Rightarrow p & \\
\text{inf}(r_2) : \neg \text{comp}(r_2), \neg \text{strict}(p) \Rightarrow \neg p & \\
\text{inf}(r_3) : \neg p, \neg \text{comp}(r_3), \neg \text{strict}(q) \Rightarrow \neg q & \\
\text{inf}(r_4) : \neg \text{comp}(r_4), \neg \text{strict}(\neg q) \Rightarrow q &
\end{array}$$

$$\begin{array}{ll}
p_d(r_1) : \Rightarrow \neg \text{comp}(r_1) & n_d(r_1, r_2) : \text{supp_body}(r_2) \Rightarrow \text{comp}(r_1) \\
p_d(r_2) : \Rightarrow \neg \text{comp}(r_2) & n_d(r_2, r_1) : \text{supp_body}(r_1) \Rightarrow \text{comp}(r_2) \\
p_d(r_3) : \Rightarrow \neg \text{comp}(r_3) & n_d(r_3, r_4) : \text{supp_body}(r_4) \Rightarrow \text{comp}(r_3) \\
p_d(r_4) : \Rightarrow \neg \text{comp}(r_4) & n_d(r_4, r_3) : \text{supp_body}(r_3) \Rightarrow \text{comp}(r_4)
\end{array}$$

$T(D)$ also contains the following superiority statements.

$$\begin{array}{ll}
nstr(p) >' str(p) & n_d(r_1, r_2) >' p_d(r_1) \\
nstr(\neg p) >' str(\neg p) & n_d(r_2, r_1) >' p_d(r_2) \\
nstr(q) >' str(q) & n_d(r_3, r_4) >' p_d(r_3) \\
nstr(\neg q) >' str(\neg q) & n_d(r_4, r_3) >' p_d(r_4)
\end{array}$$

There are some points to highlight in this example. Rules for *strict* and $\neg \text{strict}$ are omitted from the listing above because they are not of interest (D has no strict rules or facts); we will have conclusions $+\partial^*\neg \text{strict}(l)$ and $-\partial^*\text{strict}(l)$, for every literal l . There are no rules $n_s(r, s)$ in $T(D)$ because they only occur when $s > r$, and the superiority relation in D is empty. Consequently, there are no superiority statements of the form $n_s(r, s) >' p_s(r)$. It also follows that $+\partial^*\neg o(r)$ is concluded, for each rule r , and hence we can infer $+\partial^*\text{supp}(l)$, for each literal l except $\neg q$, reflecting the fact that these literals are supported in D , and $+\partial^*\text{supp_body}(r)$, for each rule r . We can then infer also $+\partial^*\text{supp}(\neg q)$. It then follows that $-\partial^*\neg \text{comp}(r)$ is concluded, for each r , using the superiority relation. Then, as a consequence of the rules $\text{inf}(r)$, we

find that all literals l fail to be inferred (i.e. we conclude $-\partial^*l$, for each literal l). This expresses the ambiguity propagating behaviour of $\mathbf{DL}(\delta^*)$ from within $\mathbf{DL}(\partial^*)$.

Theorem 5

The ambiguity blocking logics ($\mathbf{DL}(\partial)$ and $\mathbf{DL}(\partial^*)$) can simulate the ambiguity propagating logics ($\mathbf{DL}(\delta)$ and $\mathbf{DL}(\delta^*)$) with respect to addition of facts.

Propagated Ambiguity Simulates Blocked Ambiguity

We now show that every theory over an ambiguity blocking logic can be simulated by a theory over the corresponding ambiguity propagating logic. To begin, we simulate $\mathbf{DL}(\partial^*)$ by $\mathbf{DL}(\delta^*)$. Any defeasible theory D is transformed into a new theory $T(D)$. The new theory employs new propositions $strict(q)$ and $undefeated(q)$ for each literal q in Σ , and employs labels $str(q)$ and $nstr(q)$ for each literal q in Σ , and $n_d(r, s)$ and $p_d(r)$ for each pair of opposing rules r, s in R .

Definition 6

Let $D = (F, R, >)$ be a defeasible theory with language Σ . We define the transformation T of D to $T(D) = (F', R', >')$ as follows:

1. The facts of $T(D)$ are the facts of D . That is, $F' = F$.
2. Every strict rule of R is included in R' .
3. For every literal q , R' contains

$$\begin{aligned} str(q) : \quad q &\rightarrow strict(q) \\ nstr(q) : \quad &\Rightarrow \neg strict(q) \\ t(q) : \quad strict(q) &\Rightarrow true(q) \\ nt(q) : \quad &\Rightarrow \neg true(q) \end{aligned}$$

and the superiority relation contains $nstr(q) >' str(q)$ and $t(q) >' nt(q)$, for every q .

4. For each literal q , R' contains

$$undefeated(q) \Rightarrow q$$

For each strict or defeasible rule $r = B_r \hookrightarrow_r q$ for q in R , R' contains

$$p_d(r) : B_r, \neg true(\sim q) \Rightarrow undefeated(q)$$

and, further, for each rule $s = B_s \hookrightarrow_s \sim q$ for $\sim q$ in R , where $r \not\prec s$, R' contains

$$n_d(r, s) : B_s \Rightarrow \neg undefeated(q)$$

and the superiority relation contains $n_d(r, s) >' p_d(r)$.

Parts 1 and 2 preserve all the strict inferences from D . Part 3 allows us to distinguish strict conclusions from defeasible conclusions. For this transformation – compared to the transformation in the previous section – extra rules t and nt are needed. These rules ensure that δ^* and σ^* agree on the literals $true(q)$, that is, from $T(D)+A$ we conclude $+\delta^*true(q)$ iff we conclude $+\sigma^*true(q)$ iff $D+A \vdash +\Delta q$. (See Lemma 18 in the appendix.) In comparison, we never infer $-\sigma^*\neg strict(q)$ and always infer $+\sigma^*\neg strict(q)$, independent of D .¹

¹ This also demonstrates a flaw in (Maher 2012). In that paper, the transformation used to simulate $\mathbf{DL}(\delta)$ with $\mathbf{DL}(\delta^*)$ fails to use these extra rules, and thus is incorrect. Definition 13 has a corrected transformation.

Part 4 encodes the inference rules for ∂^* : q holds defeasibly if the body of a rule r for q holds defeasibly, $\sim q$ is not established strictly, and r is not defeated (i.e. all rules not inferior to r have a body that fails wrt ∂^*). The requirement that r is not defeated is expressed through the use of rules $n_d(r, s)$ opposing $p_d(r)$ for each rule s in D not inferior to r . The rules $n_d(r, s)$ are superior to $p_d(r)$ in $T(D)$, thus ensuring that $undefeated(q)$ is inferred iff r is not defeated.

Example 7

To see the operation of this transformation, consider (again) the following theory D , which demonstrates the difference between ambiguity propagation and blocking logics.

$$\begin{array}{ll} r_1 : \Rightarrow p & r_3 : \neg p \Rightarrow \neg q \\ r_2 : \Rightarrow \neg p & r_4 : \Rightarrow q \end{array}$$

In $\mathbf{DL}(\partial^*)$ from D we conclude $-\partial^*p$ and $-\partial^*\neg p$, $+\partial^*q$ and $-\partial^*\neg q$. In $\mathbf{DL}(\delta^*)$ from D we conclude $-\delta^*p$ and $-\delta^*\neg p$, $-\delta^*q$ and $-\delta^*\neg q$. We also conclude $+\sigma^*p$ and $+\sigma^*\neg p$, $+\sigma^*q$ and $+\sigma^*\neg q$.

$T(D)$ contains the following rules and superiority relation.

$$\begin{array}{ll} n_d(r_1, r_2) : & \Rightarrow \neg undefeated(p) & p_d(r_1) : & \neg true(\neg p) \Rightarrow undefeated(p) \\ n_d(r_2, r_1) : & \Rightarrow \neg undefeated(\neg p) & p_d(r_2) : & \neg true(p) \Rightarrow undefeated(\neg p) \\ n_d(r_3, r_4) : & \Rightarrow \neg undefeated(\neg q) & p_d(r_3) : & \neg p, \neg true(\neg q) \Rightarrow undefeated(\neg q) \\ n_d(r_4, r_3) : & \neg p \Rightarrow \neg undefeated(q) & p_d(r_4) : & \neg true(q) \Rightarrow undefeated(q) \end{array}$$

$$\begin{array}{l} undefeated(p) \Rightarrow p \\ undefeated(\neg p) \Rightarrow \neg p \\ undefeated(q) \Rightarrow q \\ undefeated(\neg q) \Rightarrow \neg q \end{array}$$

$$\begin{array}{ll} n_d(r_1, r_2) >' p_d(r_1) & nstr(p) >' str(p) \\ n_d(r_2, r_1) >' p_d(r_2) & nstr(\neg p) >' str(\neg p) \\ n_d(r_3, r_4) >' p_d(r_3) & nstr(q) >' str(q) \\ n_d(r_4, r_3) >' p_d(r_4) & nstr(\neg q) >' str(\neg q) \end{array}$$

The rules concerning *strict* and *true* have been omitted. Because there are no facts or strict rules in D we will infer $-\delta^*strict(s)$, and hence $+\delta^*\neg true(s)$ and $+\sigma^*\neg true(s)$ for each literal $s \in \Sigma$. However, because of the superiority of n_d over p_d , we infer $-\sigma^*undefeated(\neg p)$ and $-\delta^*undefeated(\neg p)$ and hence $-\sigma^*\neg p$ and $-\delta^*\neg p$ (and similarly for p). Hence, the body of $p_d(r_3)$ fails, and we infer $-\delta^*\neg q$. Similarly, the body of $n_d(r_4, r_3)$ fails, and hence we infer $+\delta^*q$. This reflects the ambiguity blocking behaviour of $\mathbf{DL}(\partial^*)$ from within the ambiguity propagating logic $\mathbf{DL}(\delta^*)$.

The proof of correctness of this simulation is complicated by the fact that inference rules for δ^* and σ^* are defined mutually recursively, while the inference rules for ∂^* are directly recursive. This difference in structure makes a direct inductive proof difficult. The problem is resolved by a “tight” simulating transformation that is able to simulate $\mathbf{DL}(\partial^*)$ (wrt addition of facts) in any of the \mathbf{DL} logics.

Theorem 8

For $d \in \{\delta, \delta^*, \partial\}$, $\mathbf{DL}(d)$ can simulate $\mathbf{DL}(\partial^*)$ with respect to addition of facts

Combining Theorems 5 and 8 with results from (Maher 2012), we see that all logics of the \mathbf{DL} framework are equally expressive in terms of simulation wrt addition of facts.

Simulation of Individual Defeat wrt Addition of Rules

The following definition defining $D' = (F', R', <')$ from D is repeated from (Maher 2012).

Definition 9

We add the following rules

1. The facts of D' are the facts of D . That is, $F' = F$.
2. For each rule $r = B \hookrightarrow_r q$ in R , R' contains

$$\begin{array}{lll} p(r) : & B & \hookrightarrow_r h(r) \\ s(r) : & h(r) & \rightarrow q \end{array}$$

and, further, for each rule $r' = B' \hookrightarrow_{r'} \sim q$ for $\sim q$ in R , R' contains

$$n(r, r') : B' \hookrightarrow_{r'} \neg h(r)$$

3. For every $r > r'$ in D , where r and r' are rules for opposite literals, D' contains $p(r) >' n(r, r')$ and $n(r', r) >' p(r')$.

It was shown in (Maher 2012) that, using this transformation, $\mathbf{DL}(\partial)$ simulates $\mathbf{DL}(\partial^*)$ and $\mathbf{DL}(\delta)$ simulates $\mathbf{DL}(\delta^*)$, wrt addition of facts.

On the surface, it might appear that this result extends readily to addition wrt rules: since the added rules do not participate in the superiority relation of the combined theory, it might be expected that the difference between team defeat and individual defeat is irrelevant. However, that expectation is misleading. The following example shows that this transformation *does not* provide a simulation of $\mathbf{DL}(\partial^*)$ by $\mathbf{DL}(\partial)$ wrt addition of *rules*.

Example 10

Let D consist of the rules

$$\begin{array}{ll} r_1 : & \Rightarrow p \\ r_2 : & \Rightarrow \neg p \end{array}$$

Then $T(D)$ consists of the following rules

$$\begin{array}{lll} p(r_1) : & & \Rightarrow h(r_1) \\ n(r_1, r_2) : & & \Rightarrow \neg h(r_1) \\ p(r_2) : & & \Rightarrow h(r_2) \\ n(r_2, r_1) : & & \Rightarrow \neg h(r_2) \\ s(r_1) : & h(r_1) & \Rightarrow p \\ s(r_2) : & h(r_2) & \Rightarrow \neg p \end{array}$$

Now, let A be the rule

$$\Rightarrow p$$

Clearly, $D + A \vdash -\partial^*p$ (and $D + A \vdash -\partial^*\neg p$), since r_2 cannot be overruled. However, $T(D) + A \vdash -\partial h(r_2)$, since $n(r_2, r_1)$ cannot be overruled, and hence $s(r_2)$ fails. This leaves the rule for p in A without competition, and so $T(D) + A \vdash +\partial p$.

A similar but more complex example (given in the appendix) shows the transformation also does not provide a simulation of $\mathbf{DL}(\delta^*)$ by $\mathbf{DL}(\delta)$ wrt addition of rules. These problems arise because if a rule body succeeds in D for each of q and $\sim q$, and the rules are not overruled, the simulation D' has all bodies for q and $\sim q$ failing. An applicable rule for q (or $\sim q$) in A thus has a competitor in D , but not in D' . In this way D' differs from D , and the examples show that addition of rules can make this difference observable.

To avoid these problems, we add extra rules to those in Definition 9.

Definition 11

We define $T(D)$ as the theory $(F', R', >')$ consisting of the facts, rules and superiority statements from D' in Definition 9, and the following.

4. For each literal q , R' contains

$$o(q) : \text{one}(q) \rightsquigarrow q$$

5. For each rule $r = B_r \hookrightarrow_r q$ in R , R' contains

$$B_r \Rightarrow \text{one}(q)$$

6. For every rule r for q , $>'$ contains $s(r) >' o(\sim q)$.

Parts 4 and 5 of this definition introduce an additional rule for each literal $\sim q$ which, however, is subordinate to the methods to derive q in the original transformation in the sense that a derivation of q in the original transformation will overrule (part 6) a derivation of $\sim q$ using part 4. The rules in part 4 are defeaters, so they cannot be used to derive any conclusions.

The effect of the extended definition on Example 10 is to add the following to the transformed theory:

$$\begin{array}{ll} \Rightarrow \text{one}(p) & o(p) : \text{one}(p) \rightsquigarrow p \\ \Rightarrow \text{one}(\neg p) & o(\neg p) : \text{one}(\neg p) \rightsquigarrow \neg p \end{array}$$

$$s(r_1) > o(\neg p) \qquad s(r_2) > o(p)$$

We now have $T(D) + A \vdash -\partial p$, since the rule $o(\neg p)$ provides a non-failed competitor to the rule in A . More generally, we find that, through the extended transformation, team defeat logics can simulate the corresponding individual defeat logics with respect to addition of rules.

Theorem 12

The logic $\mathbf{DL}(\partial)$ can simulate $\mathbf{DL}(\partial^*)$ with respect to addition of rules.

The logic $\mathbf{DL}(\delta)$ can simulate $\mathbf{DL}(\delta^*)$ with respect to addition of rules.

Simulation of Team Defeat wrt Addition of Rules

The same theory D and addition A as in Example 10 demonstrates that the simulation of $\mathbf{DL}(\partial)$ by $\mathbf{DL}(\partial^*)$ wrt addition of facts exhibited in (Maher 2012) does not extend to addition of rules.

The transformation below modifies the one of (Maher 2012) by treating strict rules differently

(following Definition 6), adding a competitor for each literal q (following Definition 11), and employing separate defeasible rules to accommodate differences between the δ and σ inference rules. We use a construction to restrict one class of defeasible rules to use only in simulating σ inference; it is not necessary to restrict the other class because $\delta \subseteq \sigma$, by the inclusion theorem.

Definition 13

We define the transformation T of D to $T(D) = (F', R', >')$ as follows:

1. The facts of $T(D)$ are the facts of D . That is, $F' = F$.
2. Every strict rule of R is included in R' .
3. For every literal q , R' contains

$$\begin{aligned} str(q) : & \quad q \rightarrow strict(q) \\ nstr(q) : & \quad \Rightarrow \neg strict(q) \\ t(q) : & \quad strict(q) \Rightarrow true(q) \\ nt(q) : & \quad \Rightarrow \neg true(q) \end{aligned}$$

and the superiority relation contains $nstr(q) >' str(q)$ and $t(q) >' nt(q)$, for every q .

4. For each ordered pair of opposing rules $r_i = (B_i \hookrightarrow_i \sim q)$ and $r_j = (B_j \hookrightarrow_j q)$ in R , where r_j is not a defeater, R' contains

$$\begin{aligned} R1_{ij} : & \quad B_i \hookrightarrow_i \neg d(r_i, r_j) \\ R2_{ij} : & \quad B_j \Rightarrow d(r_i, r_j) \\ R3_{ij} : & \quad true(q) \Rightarrow d(r_i, r_j) \\ & \quad d(r_i, r_j) \Rightarrow d(r_i) \\ & \quad fail(r_i) \Rightarrow d(r_i) \\ NF_i : & \quad B_i \Rightarrow \neg fail(r_i) \\ F_i : & \quad \Rightarrow fail(r_i) \end{aligned}$$

and $R2_{ij} >' R1_{ij}$ iff $r_j > r_i$, $R3_{ij} >' R1_{ij}$ for every i and j , and $NF_i > F_i$ for every i .

If there is no strict or defeasible rule r_j for q in D then only the last three rules appear in R' , for each i .

5. For each literal q , and each strict or defeasible rule $r = (B_r \hookrightarrow_r q)$ in R , R' contains

$$B_r \Rightarrow one(q)$$

6. For each literal q , R' contains

$$s(q) : one(q), \neg true(\sim q), d(s_1), \dots, d(s_k) \Rightarrow q$$

where s_1, \dots, s_k are the rules for $\sim q$

7. For each literal q and for each strict or defeasible rule r for q , R' contains

$$supp(q) : B_r, d_\sigma(s_1, r), \dots, d_\sigma(s_k, r), g, \neg g \Rightarrow q$$

where B_r is the body of r , s_1, \dots, s_k are the rules for $\sim q$, and for every strict or defeasible rule r and opposing rule s , R' contains

$$\begin{aligned} a(s, r) : & \quad B_s \Rightarrow \neg d_\sigma(s, r) \\ b(s, r) : & \quad B_r \Rightarrow d_\sigma(s, r) \end{aligned}$$

The superiority relation contains $a(s, r) > b(s, r)$ iff $s > r$. R' also contains the rules

$$\begin{aligned} & \Rightarrow g \\ & \Rightarrow \neg g \end{aligned}$$

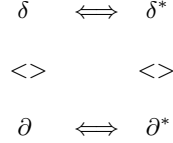


Fig. 1. Relative expressiveness of logics in **DL** using simulation wrt addition of rules

8. For each rule $r = B_r \hookrightarrow_r q$ in R , R' contains

$$B_r \Rightarrow o(q)$$

9. For each literal q , R' contains

$$o(q) : o(q) \rightsquigarrow q$$

and $>'$ contains $s(q) >' o(\sim q)$.

Parts 1–3 allow us to characterize strict conclusions. Part 4 expresses whether a rule is defeated or not, while part 6 expresses that q can be concluded if there is an applicable strict or defeasible rule for q , all attempts to strictly derive $\sim q$ fail finitely, and all opposing rules are defeated. While this expresses properly the inference rules for ∂ and δ , the inference rule for σ omits the condition on strict derivation of $\sim q$ and has a slightly different form of defeat. We need part 7 to express inference (and defeat) for σ . g and $\neg g$ are used to restrict the applicability of this rule to σ^* ; we have $T(D)+A \vdash +\sigma^*g$, but $T(D)+A \vdash -\partial^*g$ and $T(D)+A \vdash -\delta^*g$ (and the same for $\neg g$). Parts 8 and 9 redress the lack of a competitor in the same way as in Definition 11.

Theorem 14

The logic $\mathbf{DL}(\partial^*)$ can simulate $\mathbf{DL}(\partial)$ with respect to addition of rules.

The logic $\mathbf{DL}(\delta^*)$ can simulate $\mathbf{DL}(\delta)$ with respect to addition of rules.

Conclusions

We have shown that the logics of the **DL** framework are equally expressive when relative expressiveness is formulated as ability to simulate in the presence of additional facts. This involved the introduction of two new transformations simulating, respectively, a logic that blocks ambiguity and a logic that propagates ambiguity.

We also completed the study of relative expressiveness wrt addition of rules. Figure 1 shows this relation on the logics in **DL**, where an arrow from d_1 to d_2 expresses that $DL(d_1)$ can be simulated by $DL(d_2)$ with respect to the addition of rules. $<>$ between tags expresses that the two corresponding logics have incomparable expressiveness. It is clear that **DL** breaks into two classes of logics of different expressiveness.

While the issue of relative expressiveness within the framework **DL** is now largely resolved, this same approach can be applied to relate these logics to other logics. We can expect the same results for the **WFDL** logics (Maher and Governatori 1999; Maher et al. 2011), because of their similarity to **DL**, but their relation to the defeasible logics of Nute and Maier (Maier and Nute 2006; Maier and Nute 2010) will be of interest. Even more interesting will be to address other systems of defeasible reasoning, such as argumentation (Dung 1995; Rahwan and Simari 2009).

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Appendix

This appendix contains the inference rules for the logics in **DL**, proofs of results in the body of the paper, and some examples. Theorems, Lemmas, or Examples numbered 1–14 refer to items in the body of the paper. Larger numbers refer to items in this appendix.

Inference Rules for DL

For every inference rule $+d$ there is a closely related inference rule $-d$ allowing to infer that some literals q cannot be consequences of D via $+d$. The relationship between $+d$ and $-d$ is described as the Principle of Strong Negation (Antoniou et al. 2000). These inference rules are placed adjacently to emphasize this relationship.

Some notation in the inference rules requires explanation. Given a literal q , its complement $\sim q$ is defined as follows: if q is a proposition then $\sim q$ is $\neg q$; if q has form $\neg p$ then $\sim q$ is p . We say q and $\sim q$ (and the rules with these literals in the head) *oppose* each other. R_s (R_{sd}) denotes the set of strict rules (strict or defeasible rules) in R . $R[q]$ ($R_s[q]$, etc) denotes the set of rules (respectively, strict rules) of R with head q . Given a rule r , $A(r)$ denotes the set of literals in the body (or antecedent) of r .

$$\begin{array}{ll}
 +\Delta) +\Delta q \in \mathcal{T}_D(E) \text{ iff either} & -\Delta) -\Delta q \in \mathcal{T}_D(E) \text{ iff} \\
 \quad .1) q \in F; \text{ or} & \quad .1) q \notin F, \text{ and} \\
 \quad .2) \exists r \in R_s[q] \text{ such that} & \quad .2) \forall r \in R_s[q] \\
 \quad \quad .1) \forall a \in A(r), +\Delta a \in E & \quad .1) \exists a \in A(r), -\Delta a \in E
 \end{array}$$

$+\partial) +\partial q \in \mathcal{T}_D(E)$ iff either

- .1) $+\Delta q \in E$; or
- .2) The following three conditions all hold.
 - .1) $\exists r \in R_{sd}[q] \forall a \in A(r), +\partial a \in E$, and
 - .2) $-\Delta \sim q \in E$, and
 - .3) $\forall s \in R[\sim q]$ either
 - .1) $\exists a \in A(s), -\partial a \in E$; or
 - .2) $\exists t \in R_{sd}[q]$ such that
 - .1) $\forall a \in A(t), +\partial a \in E$, and
 - .2) $t > s$.

$-\partial) -\partial q \in \mathcal{T}_D(E)$ iff

- .1) $-\Delta q \in E$, and
- .2) either
 - .1) $\forall r \in R_{sd}[q] \exists a \in A(r), -\partial a \in E$; or
 - .2) $+\Delta \sim q \in E$; or
 - .3) $\exists s \in R[\sim q]$ such that
 - .1) $\forall a \in A(s), +\partial a \in E$, and
 - .2) $\forall t \in R_{sd}[q]$ either
 - .1) $\exists a \in A(t), -\partial a \in E$; or
 - .2) $\text{not}(t > s)$.

$+\partial^*) +\partial^* q \in \mathcal{T}_D(E)$ iff either

- .1) $+\Delta q \in E$; or
- .2) $\exists r \in R_{sd}[q]$ such that
 - .1) $\forall a \in A(r), +\partial^* a \in E$, and
 - .2) $-\Delta \sim q \in E$, and
 - .3) $\forall s \in R[\sim q]$ either
 - .1) $\exists a \in A(s), -\partial^* a \in E$; or
 - .2) $r > s$.

$-\partial^*) -\partial^* q \in \mathcal{T}_D(E)$ iff

- .1) $-\Delta q \in E$, and
- .2) $\forall r \in R_{sd}[q]$ either
 - .1) $\exists a \in A(r), -\partial^* a \in E$; or
 - .2) $+\Delta \sim q \in E$; or
 - .3) $\exists s \in R[\sim q]$ such that
 - .1) $\forall a \in A(s), +\partial^* a \in E$, and
 - .2) $\text{not}(r > s)$.

$+\delta) +\delta q \in \mathcal{T}_D(E)$ iff either

- .1) $+\Delta q \in E$; or
- .2) The following three conditions all hold.
 - .1) $\exists r \in R_{sd}[q] \forall a \in A(r), +\delta a \in E$, and
 - .2) $-\Delta \sim q \in E$, and
 - .3) $\forall s \in R[\sim q]$ either
 - .1) $\exists a \in A(s), -\sigma a \in E$; or
 - .2) $\exists t \in R_{sd}[q]$ such that
 - .1) $\forall a \in A(t), +\delta a \in E$, and
 - .2) $t > s$.

$-\delta) -\delta q \in \mathcal{T}_D(E)$ iff

- .1) $-\Delta q \in E$, and
- .2) either
 - .1) $\forall r \in R_{sd}[q] \exists a \in A(r), -\delta a \in E$; or
 - .2) $+\Delta \sim q \in E$; or
 - .3) $\exists s \in R[\sim q]$ such that
 - .1) $\forall a \in A(s), +\sigma a \in E$, and
 - .2) $\forall t \in R_{sd}[q]$ either
 - .1) $\exists a \in A(t), -\delta a \in E$; or
 - .2) $\text{not}(t > s)$.

$+\sigma) +\sigma q \in \mathcal{T}_D(E)$ iff either

- .1) $+\Delta q \in E$; or
- .2) $\exists r \in R_{sd}[q]$ such that
 - .1) $\forall a \in A(r), +\sigma a \in E$, and
 - .2) $\forall s \in R[\sim q]$ either
 - .1) $\exists a \in A(s), -\delta a \in E$; or
 - .2) $\text{not}(s > r)$.

$-\sigma) -\sigma q \in \mathcal{T}_D(E)$ iff

- .1) $-\Delta q \in E$, and
- .2) $\forall r \in R_{sd}[q]$ either
 - .1) $\exists a \in A(r), -\sigma a \in E$; or
 - .2) $\exists s \in R[\sim q]$ such that
 - .1) $\forall a \in A(s), +\delta a \in E$, and
 - .2) $s > r$.

$+\delta^*) +\delta^* q \in \mathcal{T}_D(E)$ iff either

- .1) $+\Delta q \in E$; or
- .2) $\exists r \in R_{sd}[q]$ such that
 - .1) $\forall a \in A(r), +\delta^* a \in E$, and
 - .2) $-\Delta \sim q \in E$, and
 - .3) $\forall s \in R[\sim q]$ either
 - .1) $\exists a \in A(s), -\sigma^* a \in E$; or
 - .2) $r > s$.

$-\delta^*) -\delta^* q \in \mathcal{T}_D(E)$ iff

- .1) $-\Delta q \in E$, and
- .2) $\forall r \in R_{sd}[q]$ either
 - .1) $\exists a \in A(r), -\delta^* a \in E$; or
 - .2) $+\Delta \sim q \in E$; or
 - .3) $\exists s \in R[\sim q]$ such that
 - .1) $\forall a \in A(s), +\sigma^* a \in E$, and
 - .2) $\text{not}(r > s)$.

$+\sigma^*) + \sigma^* q \in \mathcal{T}_D(E)$ iff either	$-\sigma^*) - \sigma^* q \in \mathcal{T}_D(E)$ iff
.1) $+\Delta q \in E$; or	.1) $-\Delta q \in E$, and
.2) $\exists r \in R_{sd}[q]$ such that	.2) $\forall r \in R_{sd}[q]$ either
.1) $\forall a \in A(r), +\sigma^* a \in E$, and	.1) $\exists a \in A(r), -\sigma^* a \in E$; or
.2) $\forall s \in R[\sim q]$ either	.2) $\exists s \in R[\sim q]$ such that
.1) $\exists a \in A(s), -\delta^* a \in E$; or	.1) $\forall a \in A(s), +\delta^* a \in E$, and
.2) $\text{not}(s > r)$.	.2) $s > r$.

Proofs of results

We now turn to proofs of results in the body of the paper, and some examples. This part of the appendix has the same structure as the paper itself, to make access easier.

All simulation proofs (of $\mathbf{DL}(d_1)$ by $\mathbf{DL}(d_2)$, say) have two parts: first we show every consequence of $D+A$ in $\mathbf{DL}(d_1)$ has a corresponding consequence of $T(D)+A$ in $\mathbf{DL}(d_2)$, and then we show that every consequence of $T(D)+A$ in $\mathbf{DL}(d_2)$ in the language of $D+A$ has a corresponding consequence of $D+A$ in $\mathbf{DL}(d_1)$. In both cases the proof is by induction on the level n of $\mathcal{T} \uparrow n$ where \mathcal{T} combines the functions in the inference rules for $\pm d_1$ and $\pm \Delta$ for $D+A$ in the first part, and combines the functions in the inference rules for $\pm d_2$ and $\pm \Delta$ for $T(D)+A$ in the second part. The induction hypothesis for the first part is: for $k \leq n$, if $\alpha \in \mathcal{T}_{D+A} \uparrow n$ then $T(D)+A \vdash \alpha'$, where α' is the counterpart, in $\mathbf{DL}(d_2)$, of α . For the second part it is: for $k \leq n$, if $\alpha \in \Sigma$ and $\alpha \in \mathcal{T}_{T(D)+A} \uparrow n$ then $D+A \vdash \alpha'$, where α' is the counterpart, in $\mathbf{DL}(d_1)$, of α . Since $\mathcal{T}_P \uparrow 0 = \emptyset$ the induction hypothesis is always valid for $n = 0$.

Throughout this appendix, if r is a rule then B_r refers to the body of that rule. For brevity, we write $+dB$, where B is a set of literals, to mean $\{+dq \mid q \in B\}$.

Blocked Ambiguity Simulates Propagated Ambiguity

The facts and strict rules of $D+A$ and $T(D)+A$ are the same, except for rules for $\text{strict}(q)$ in $T(D)+A$. However $\text{strict}(q)$ is not used in any other strict rule. Consequently, for any addition A , $D+A$ and $T(D)+A$ draw the same strict conclusions in $\Sigma(D+A)$. Furthermore, these conclusions are reflected in the defeasible conclusions of $\text{strict}(q)$.

Lemma 15

Let A be any defeasible theory, and let Σ be the language of $D+A$. Then, for every $q \in \Sigma$,

- $D+A \vdash +\Delta q$ iff $T(D)+A \vdash +\Delta q$
iff $T(D)+A \vdash +\partial^* \text{strict}(q)$ iff $T(D)+A \vdash -\partial^* \neg \text{strict}(q)$
- $D+A \vdash -\Delta q$ iff $T(D)+A \vdash -\Delta q$
iff $T(D)+A \vdash -\partial^* \text{strict}(q)$ iff $T(D)+A \vdash +\partial^* \neg \text{strict}(q)$

Proof

The proof of $D+A \vdash \pm \Delta q$ iff $T(D)+A \vdash \pm \Delta q$ is straightforward, by induction on length of proofs.

In the inference rule for $+\partial^* \text{strict}(q)$, clause .2.3 must be false, by the structure of the rules in part 3 of the transformation. Consequently, we infer $+\partial^* \text{strict}(q)$ iff we infer $+\Delta \text{strict}(q)$, which happens iff we infer $+\Delta q$ since there is only the one rule for $\text{strict}(q)$. Similarly, clause .2.3 of the inference rule for $-\partial^* \text{strict}(q)$ is true, so we infer $-\partial^* \text{strict}(q)$ iff we infer $-\Delta \text{strict}(q)$, which happens iff we infer $-\Delta q$ since there is only the one rule for $\text{strict}(q)$.

In the inference rule for $-\partial^* \neg \text{strict}(q)$, clause .2.1 is false because the body of $nstr(q)$ is empty, and clause .2.3 is false because $nstr(q) >' str(q)$. Thus we infer $-\partial^* \neg \text{strict}(q)$ iff we infer $+\Delta \text{strict}(q)$. Finally, in the inference rule for $+\partial^* \neg \text{strict}(q)$, clause .1 is false, because there is no fact or strict rule for $\neg \text{strict}(q)$. and clauses .2.1 and .2.3 are true (the latter because $nstr(q) >' str(q)$). Thus, we can infer $+\partial^* \neg \text{strict}(q)$ iff we can infer $-\Delta \text{strict}(q)$. \square

This lemma establishes that strict provability ($\pm\Delta$) from $D+A$ in $\mathbf{DL}(\delta^*)$ is captured in $\mathbf{DL}(\partial^*)$ by the transformation defined above, no matter what the addition A . We now show that $\mathbf{DL}(\partial^*)$ can simulate the behaviour of both δ^* and σ^* with respect to addition of facts.

Lemma 16

Let D be a defeasible theory, $T(D)$ be the transformed defeasible theory as described in Definition 3, and let A be a modular set of facts. Let Σ be the language of $D+A$ and let $q \in \Sigma$. Then

- $D+A \vdash +\sigma^* q$ iff $T(D)+A \vdash +\partial^* \text{supp}(q)$
- $D+A \vdash -\sigma^* q$ iff $T(D)+A \vdash -\partial^* \text{supp}(q)$
- $D+A \vdash +\delta^* q$ iff $T(D)+A \vdash +\partial^* q$
- $D+A \vdash -\delta^* q$ iff $T(D)+A \vdash -\partial^* q$

Proof

Suppose $+\sigma^* q \in \mathcal{T}_{D+A} \uparrow (n+1)$. Then, by the $+\sigma^*$ inference rule, there is a strict or defeasible rule r in D with head q and body B_r such that $+\sigma^* B_r \subseteq \mathcal{T}_{D+A} \uparrow n$, and for every rule s in D for $\sim q$ either there is a literal b in the body of s such that $-\delta^* b \in \mathcal{T}_{D+A} \uparrow n$ or $s \not\prec r$. Hence, by the induction hypothesis, there is a strict or defeasible rule r in D with head q and body B_r such that $T(D)+A \vdash +\partial^* \text{supp}(b)$ for each $b \in B_r$ and for every rule s in D for $\sim q$ either there is a literal b in the body of s such that $T(D)+A \vdash -\partial^* b$ or $s \not\prec r$. Then $T(D)+A \vdash +\partial^* \text{supp_body}(r)$ and for every rule s in D for $\sim q$ with $s > r$ $T(D)+A \vdash -\partial^* B_s$, and hence $T(D)+A \vdash +\partial^* \neg o(r)$. Combining these two conclusions, and given that there is no rule for $\neg \text{supp}(q)$, we have $T(D)+A \vdash +\partial^* \text{supp}(q)$.

Suppose $+\delta^* q \in \mathcal{T}_{D+A} \uparrow (n+1)$. Then, by the $+\delta^*$ inference rule, there is a strict or defeasible rule r in D with head q and body B_r such that $+\delta^* B_r \subseteq \mathcal{T}_{D+A} \uparrow n$, $-\Delta \sim q \in \mathcal{T}_{D+A} \uparrow n$, and for every rule s in D for $\sim q$ where $r \not\prec s$, there is a literal b in the body of s such that $-\sigma^* b \in \mathcal{T}_{D+A} \uparrow n$. Hence, by the induction hypothesis, there is a strict or defeasible rule r in D with head q and body B_r such that $T(D)+A \vdash +\partial^* B_r$, $T(D)+A \vdash -\Delta \sim q$, and for every rule s in D for $\sim q$ where $r \not\prec s$, there is a literal b in the body of s such that $T(D)+A \vdash -\partial^* \text{supp}(b)$. By Lemma 15, $T(D)+A \vdash -\partial^* \neg \text{strict}(\sim q)$. By repeated application of the $-\partial^*$ inference rule we have $T(D)+A \vdash -\partial^* \text{supp_body}(s)$ for each s , and then $T(D)+A \vdash +\partial^* \neg \text{comp}(r)$. Thus the body of the rule $\text{inf}(r)$ in $T(D)$ holds defeasibly. On the other hand, for every rule s for $\sim q$ in D where $r \not\prec s$ there is a literal b in the body of s such that $T(D)+A \vdash -\partial^* \text{supp}(b)$ so, using the inference rule for $-\partial^*$ and the rule from part 4 we must have $T(D)+A \vdash -\partial^* b$. $T(D)+A \vdash +\partial^* B_r$ so, using the rules in part 4 and part 5, $T(D)+A \vdash +\partial^* \text{supp_body}(r)$. Hence, for the rules for $\sim q$ where $r > s$, the rules $n_d(s, r)$ can be applied and $T(D)+A \vdash -\partial^* \neg \text{comp}(s)$. Consequently, all rules $\text{inf}(s)$ for $\sim q$ fail. From this fact and the fact that body of rule $\text{inf}(r)$ is proved defeasibly we conclude $T(D)+A \vdash +\partial^* q$.

Suppose $-\sigma^* q \in \mathcal{T}_{D+A} \uparrow (n+1)$. Then, by the $-\sigma^*$ inference rule, $-\Delta q \in \mathcal{T}_{D+A} \uparrow n$ and, for every strict or defeasible rule r in D with head q and body B_r , either $-\sigma^* b \in \mathcal{T}_{D+A} \uparrow n$ for

some $b \in B_r$, or there is a rule s in D for $\sim q$ with body B_s such that $+\delta^* B_s \subseteq \mathcal{T}_{D+A} \uparrow n$ and $s > r$. Hence, by the induction hypothesis, $T(D)+A \vdash -\Delta q$ and for every strict or defeasible rule r in D with head q either $T(D)+A \vdash -\partial^* \text{supp}(b)$ for some $b \in B_r$, or there is a rule s in D for $\sim q$ with $s > r$ and $T(D)+A \vdash +\partial^* B_s$. Hence, either $T(D)+A \vdash -\partial^* \text{supp_body}(r)$ or $T(D)+A \vdash -\partial^* \neg o(r)$. In either case, we have $T(D)+A \vdash -\partial^* \text{supp}(q)$.

Suppose $-\delta^* q \in \mathcal{T}_{D+A} \uparrow (n+1)$. Then, by the $-\delta^*$ inference rule, $-\Delta q \in \mathcal{T}_{D+A} \uparrow n$ or, for every strict or defeasible rule r in D with head q and body B_r , either $-\delta^* b \in \mathcal{T}_{D+A} \uparrow n$ for some $b \in B_r$, $+\Delta \sim q \in \mathcal{T}_{D+A} \uparrow n$, or there is a rule s in D for $\sim q$ with body B_s such that $+\sigma^* B_s \subseteq \mathcal{T}_{D+A} \uparrow n$ and $r \not\prec s$. Hence, by the induction hypothesis, $T(D)+A \vdash -\Delta q$ and for every strict or defeasible rule r in D with head q either (1) $T(D)+A \vdash -\partial^* b$ for some $b \in B_r$, (2) $T(D)+A \vdash +\Delta \sim q$, or (3) there is a rule s in D for $\sim q$ with $r \not\prec s$ and $T(D)+A \vdash +\partial^* \text{supp}(b')$ for every $b' \in B_s$. We consider these three cases in turn. In the first case, the rule $\text{inf}(r)$ fails. In the second case, using part 3, we can conclude $T(D)+A \vdash +\Delta \text{strict}(\sim q)$ and $T(D)+A \vdash -\partial^* \neg \text{strict}(\sim q)$, and hence the rule $\text{inf}(r)$ fails. In the third case, we can conclude $T(D)+A \vdash +\partial^* \text{supp_body}(s)$ and hence, using part 6, $T(D)+A \vdash -\partial^* \neg \text{comp}(r)$. Thus, the rule $\text{inf}(r)$ fails. In each case, the rule $\text{inf}(r)$ fails. Thus we can derive $T(D)+A \vdash -\partial^* q$.

Suppose $+\partial^* \text{supp}(q) \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$. Then, by the $+\partial^*$ inference rule, either $+\partial^* q \in \mathcal{T}_{T(D)+A} \uparrow n$, or there is a strict or defeasible rule r in D with head q and body B_r such that $+\partial^* \text{supp_body}(r) \in \mathcal{T}_{T(D)+A} \uparrow n$ and $+\partial^* \neg o(r) \in \mathcal{T}_{T(D)+A} \uparrow n$. Consequently, $+\partial^* \text{supp}(b) \in \mathcal{T}_{T(D)+A} \uparrow n$, for each $b \in B_r$ for every rule s in D for $\sim q$ where $s > r$, there is b in the body of s such that $-\partial^* b \in \mathcal{T}_{T(D)+A} \uparrow n$. In the first case, by the induction hypothesis, $D+A \vdash +\delta^* q$ and then, by the inclusion theorem, $D+A \vdash +\sigma^* q$. In the second case, by the induction hypothesis, $D+A \vdash +\sigma^* B_r$ for every rule s in D for $\sim q$ where $s > r$, there is b in the body of s such that $D+A \vdash -\partial^* b$. Applying the inference rule for $+\sigma^*$, $D+A \vdash +\sigma^* q$.

Suppose $-\partial^* \text{supp}(q) \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$. Then, by the $-\partial^*$ inference rule, $-\partial^* q \in \mathcal{T}_{T(D)+A} \uparrow n$, and for every strict or defeasible rule r in D for q either $-\partial^* \text{supp_body}(r) \in \mathcal{T}_{T(D)+A} \uparrow n$ or $-\partial^* \neg o(r) \in \mathcal{T}_{T(D)+A} \uparrow n$. In the former case we must have $-\partial^* \text{supp}(b) \in \mathcal{T}_{T(D)+A} \uparrow n$ for some b in the body B_r of r . In the latter case we must have that for some rule s in D with body B_s , $s > r$ and $+\partial^* B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$. By the induction hypothesis, we have $D+A \vdash -\delta^* q$ (and hence $D+A \vdash -\Delta q$) and, for each r either $D+A \vdash -\sigma^* b$ for some $b \in B_r$ or there is an opposing rule s with $s > r$ and $D+A \vdash +\delta^* B_s$. Applying the inference rule for $-\sigma^*$ we conclude $D+A \vdash -\sigma^* q$.

Suppose $+\partial^* q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$. Then, by the $+\partial^*$ inference rule, there is a strict or defeasible rule r in D with head q and body B_r such that $+\partial^* B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$, $+\partial^* \neg \text{strict}(\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$, and $+\partial^* \neg \text{comp}(r) \in \mathcal{T}_{T(D)+A} \uparrow n$. By Lemma 15, $D+A \vdash -\Delta \sim q$. Using the structure of $T(D)$ and the $+\partial^*$ inference rule, for every rule s in D for $\sim q$ where $r \not\prec s$ we must have $-\partial^* \text{supp_body}(s) \in \mathcal{T}_{T(D)+A} \uparrow n$, and hence $-\partial^* \text{supp}(b) \in \mathcal{T}_{T(D)+A} \uparrow n$, for some b in the body of s . By the induction hypothesis, $D+A \vdash +\delta^* B_r$ and for every rule s in D for $\sim q$ where $r \not\prec s$, there is b in the body of s such that $D+A \vdash -\sigma^* b$. Now, applying the $+\delta^*$ inference rule, we have $D+A \vdash +\delta^* q$.

Suppose $-\partial^* q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$. Then, by the $-\partial^*$ inference rule, $-\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ and, for every strict or defeasible rule r for q in D with body B_r , either (1) $-\partial^* b \in \mathcal{T}_{T(D)+A} \uparrow n$ for some $b \in B_r$, (2) $-\partial^* \neg \text{comp}(r) \in \mathcal{T}_{T(D)+A} \uparrow n$, (3) $-\partial^* \neg \text{strict}(\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$, or, (4) for some rule s for $\sim q$ in D , $+\partial^* B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$, $+\partial^* \neg \text{comp}(s) \in \mathcal{T}_{T(D)+A} \uparrow n$, and $+\partial^* \neg \text{strict}(q) \in \mathcal{T}_{T(D)+A} \uparrow n$.

Hence, using the structure of $T(D)$ and Lemma 15, $-\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ and, for every rule

r for q in D with body B_r , either (1) $-\partial^*b \in \mathcal{T}_{T(D)+A} \uparrow n$ for some $b \in B_r$, (2) for some rule s' for $\sim q$ in D we have $+\partial^*supp(b) \in \mathcal{T}_{T(D)+A} \uparrow n$ for each $b \in B_{s'}$, (3) $+\Delta\sim q \in \mathcal{T}_{T(D)+A} \uparrow n$, or, (4) for some rule s for $\sim q$ in D , $+\partial^*B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$, for every rule r' for q , there is b' in its body such that $-\partial^*supp(b') \in \mathcal{T}_{T(D)+A} \uparrow n$, and $-\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$.

By the induction hypothesis, $D+A \vdash -\Delta q$ and, for every strict or defeasible rule r for q in D with body B_r , either (1) $D+A \vdash -\delta b$ for some $b \in B_r$, (2) for some rule s for $\sim q$ in D we have $D+A \vdash +\sigma^*b$ for each $b \in B_s$, (3) $D+A \vdash +\Delta\sim q$, or (4) for some rule s for $\sim q$ in D , $D+A \vdash +\delta^*B_s$, for every rule r' for q , there is b' in its body such that $D+A \vdash -\sigma^*b'$, and $D+A \vdash -\Delta q$. For each disjunct, applying the inference rule for $-\delta^*$, we can conclude $D+A \vdash -\delta^*q$.

□

This result concerns only addition of facts. It was established in (Maher 2012) that it cannot be extended to addition of rules.

Given that the ambiguity blocking logics can simulate each other, as can the ambiguity propagating logics (see (Maher 2012)) we have

Theorem 17

The ambiguity blocking logics ($\mathbf{DL}(\partial)$ and $\mathbf{DL}(\partial^*)$) can simulate the ambiguity propagating logics ($\mathbf{DL}(\delta)$ and $\mathbf{DL}(\delta^*)$) with respect to addition of facts.

This is Theorem 5 from the body of the paper.

Propagated Ambiguity Simulates Blocked Ambiguity

As with the previous simulation, the facts and strict rules of D and $T(D)$ are the same, except for rules for $strict(q)$ in $T(D)$. Thus, again, for any addition A , $D+A$ and $T(D)+A$ draw the same strict conclusions in $\Sigma(D+A)$. Furthermore, these conclusions are reflected in the defeasible conclusions of $strict(q)$, $true(q)$ and $\neg true(q)$, and also in support conclusions.

Lemma 18

Let D be a defeasible theory, $T(D)$ be the transformed defeasible theory as described in Definition 6, and let A be a modular defeasible theory. Let Σ be the language of $D+A$ and let $q \in \Sigma$. Then

- $D+A \vdash +\Delta q$ iff $T(D)+A \vdash +\Delta q$ iff $T(D)+A \vdash +\delta^*strict(q)$
 iff $T(D)+A \vdash +\delta^*true(q)$ iff $T(D)+A \vdash +\sigma^*true(q)$
 iff $T(D)+A \vdash -\delta^*\neg true(q)$ iff $T(D)+A \vdash -\sigma^*\neg true(q)$
- $D+A \vdash -\Delta q$ iff $T(D)+A \vdash -\Delta q$ iff $T(D)+A \vdash -\delta^*strict(q)$
 iff $T(D)+A \vdash -\delta^*true(q)$ iff $T(D)+A \vdash -\sigma^*true(q)$
 iff $T(D)+A \vdash +\delta^*\neg true(q)$ iff $T(D)+A \vdash +\sigma^*\neg true(q)$

Proof

The proof of $D+A \vdash \pm\Delta q$ iff $T(D)+A \vdash \pm\Delta q$ is straightforward, by induction on length of proofs.

In the inference rule for $+\delta^*strict(q)$, clause .2.3 must be false, by the structure of the rules in part 3 of the transformation. Consequently, we infer $+\delta^*strict(q)$ iff we infer $+\Delta strict(q)$,

which happens iff we infer $+\Delta q$ since there is only the one rule for $strict(q)$. Similarly, clause .2.3 of the inference rule for $-\delta^*strict(q)$ is true, so we infer $-\delta^*strict(q)$ iff we infer $-\Delta strict(q)$, which happens iff we infer $-\Delta q$ since there is only the one rule for $strict(q)$.

Note that $-\Delta true(q)$ and $-\Delta \neg true(q)$ are consequences of $T(D)+A$ because there are no strict rules for such literals in $T(D)+A$. Using this fact, the two rules $t(q)$ and $nt(q)$ and the superiority $t(q) > nt(q)$, using the inference rule for $+\delta^*$, we can infer $+\delta^*\neg true(q)$ iff we can infer $-\sigma^*strict(q)$, because .1 of the inference rule is false, .2.1 and .2.2 are true, and .2.3.2 is false. Similarly, using the inference rule for $+\sigma^*$, we can infer $+\sigma^*\neg true(q)$ iff we can infer $-\delta^*strict(q)$. Using the inference rules for $-\delta^*$ and $-\sigma^*$, we can infer $-\delta^*\neg true(q)$ iff we can infer $+\sigma^*strict(q)$, and we can infer $-\sigma^*\neg true(q)$ iff we can infer $+\delta^*strict(q)$. \square

We need this more detailed characterization of strict consequence, compared to Lemma 15, because both δ^* and σ^* intermediate conclusions influence δ^* conclusions.

The next lemma is a key part of the proof. It shows that the structure of $T(D)+A$ tightly constrains the inferences that can be made in the sense that, for the literals of interest, the inference rules δ^* and σ^* draw the same conclusions.

Lemma 19

Let D be a defeasible theory, $T(D)$ be the transformed defeasible theory as described in Definition 6, and let A be a modular set of facts. Let Σ be the language of $D+A$ extended with literals of the forms $undefeated(p)$, $\neg undefeated(p)$ and $\neg true(p)$, for $p \in \Sigma(D)$.

Then, for any $q \in \Sigma$,

- $T(D)+A \vdash +\delta^*q$ iff $T(D)+A \vdash +\sigma^*q$
- $T(D)+A \vdash -\delta^*q$ iff $T(D)+A \vdash -\sigma^*q$

Proof

Two parts of the proof follow immediately from the inclusion theorem. These are the forward direction of the first statement and the backward direction of the second statement. Furthermore, it is immediate from Lemma 18 that the result holds for literals involving $true$ and for literals that are proved strictly. The remaining parts are proved by induction.

Recall that $T(D)+A \vdash s$ iff there is an integer n such that $s \in \mathcal{T}_{T(D)+A} \uparrow n$. Note that the result holds in $\mathcal{T}_{T(D)+A} \uparrow 0$, since it is empty. Suppose the result holds for conclusions s with $s \in \mathcal{T}_{T(D)+A} \uparrow n$. We show that it also holds for conclusions in $\mathcal{T}_{T(D)+A} \uparrow (n+1)$.

(1) If $+\sigma^*q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$ then $+\sigma^*undefeated(q) \in \mathcal{T}_{T(D)+A} \uparrow n$, because there is only one rule for q and it cannot be overruled. Further, if $+\sigma^*undefeated(q) \in \mathcal{T}_{T(D)+A} \uparrow n$ then for some rule r of D we must have $+\sigma^*B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and $+\sigma^*\neg true(\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$ and, for every rule s in D for $\sim q$ with $r \not\succ s$, there is $p \in B_s$ with $-\delta^*p \in \mathcal{T}_{T(D)+A} \uparrow n$, because clause .2.2.2 must be false, since $n_d(r, s) > p_d(r)$ for every such s . By the induction hypothesis, $+\delta^*B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$, and for each s there is $p \in B_s$ with $-\sigma^*p \in \mathcal{T}_{T(D)+A} \uparrow n$ and, by Lemma 18, $+\delta^*\neg true(\sim q)$ and $-\Delta \sim q$ are consequences of $T(D)+A$. Applying the $+\delta^*$ inference rule, $T(D)+A \vdash +\delta^*undefeated(q)$ and, applying the $-\sigma^*$ inference rule, $T(D)+A \vdash -\sigma^*undefeated(\sim q)$ since every rule $p_d(s)$ contains a p with $-\sigma^*p \in \mathcal{T}_{T(D)+A} \uparrow n$. Hence, applying the $+\delta^*$ inference rule, $T(D)+A \vdash +\delta^*q$.

If $+\sigma^*\neg undefeated(q) \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$ then, for some rule s for $\sim q$ in D , $+\sigma^*B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$. By the induction hypothesis, $+\delta^*B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$. Applying the $+\delta^*$ inference rule, noting that there is no fact or strict rule for $undefeated(q)$ and that $n_d(r, s) > p_d(r)$, we have $T(D)+A \vdash +\delta^*\neg undefeated(q)$.

(2) If $-\delta^*q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$ then, using the $-\delta^*$ inference rule and the structure of $T(D)+A$, $-\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ and either $-\delta^*\text{undefeated}(q) \in \mathcal{T}_{T(D)+A} \uparrow n$ or $+\Delta \sim q \in \mathcal{T}_{T(D)+A} \uparrow n$ or $+\sigma^*\text{undefeated}(\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$.

If $-\delta^*\text{undefeated}(q) \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$ then either for every rule $p_d(r)$, for some $p \in B_r$, $-\delta^*p \in \mathcal{T}_{T(D)+A} \uparrow n$ or $-\delta^*\neg\text{true}(\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$, or, for some rule $n_d(r, s)$, $+\sigma^*B_s$. By the induction hypothesis, either for each $p_d(r)$ there is a p in its body where $T(D)+A \vdash -\sigma^*p$, or $T(D)+A \vdash +\delta^*B_s$ for some s . Applying the $-\sigma^*$, we have $T(D)+A \vdash -\sigma^*\text{undefeated}(q)$. If $+\Delta \sim q \in \mathcal{T}_{T(D)+A} \uparrow n$ then, by Lemma 18, $T(D)+A \vdash -\sigma^*\neg\text{true}(\sim q)$. Hence we must have $T(D)+A \vdash -\sigma^*\text{undefeated}(q)$, since $\neg\text{true}(\sim q)$ appears in each rule for $\text{undefeated}(q)$.

If $+\sigma^*\text{undefeated}(\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$ then there is a rule $p_d(s)$ for $\text{undefeated}(\sim q)$ where $T(D)+A \vdash +\sigma^*B_s$ and $T(D)+A \vdash +\sigma^*\neg\text{true}(q)$ and, for every rule $n_d(s, r)$, $T(D)+A \vdash -\delta^*B_r$. By the induction hypothesis, $T(D)+A \vdash +\delta^*B_s$ and, for every rule $n_d(s, r)$ (where we must have $s \not\prec r$ in D), $T(D)+A \vdash -\delta^*B_r$. Hence, for every r for q in D where $r > s$ we have $T(D)+A \vdash -\sigma^*B_r$. For every other r for q in D there is $n_d(r, s)$ where $T(D)+A \vdash +\delta^*B_s$. Hence, applying the $-\sigma^*$ inference rule for $\text{undefeated}(q)$, we must have $T(D)+A \vdash -\sigma^*\text{undefeated}(q)$.

Thus, in every case we have $T(D)+A \vdash -\sigma^*\text{undefeated}(q)$ and consequently $T(D)+A \vdash -\sigma^*q$.

If $-\delta^*\neg\text{undefeated}(q) \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$ then for every rule $p_d(r)$, for some p in its body, $-\delta^*p \in \mathcal{T}_{T(D)+A} \uparrow n$. By the induction hypothesis, for every rule $p_d(r)$, for some p in its body, $T(D)+A \vdash -\sigma^*p$. Applying the $-\sigma^*$ inference rule, $T(D)+A \vdash -\sigma^*\neg\text{undefeated}(q)$.

□

As a consequence of the inclusion theorem and the previous lemma, any inference rule between σ^* and δ^* (that is, any inference rule except for Δ and δ) behaves the same way on Σ -literals in $T(D)+A$. In particular, it applies to ∂^* .

Corollary 20

Let Σ be the language of D , Σ' be as defined in the previous lemma. Let A be any set of facts. Then if $q \in \Sigma'$

- $T(D)+A \vdash +\delta^*q$ iff $T(D)+A \vdash +\partial^*q$
- $T(D)+A \vdash -\delta^*q$ iff $T(D)+A \vdash -\partial^*q$

Now we show that the transformation preserves the ∂^* consequences of $D+A$.

Theorem 21

Let D be a defeasible theory, $T(D)$ be the transformed defeasible theory as described in Definition 6, and let A be a modular set of facts. Let Σ be the language of $D+A$ and let $q \in \Sigma$. Then

- $D+A \vdash +\partial^*q$ iff $T(D)+A \vdash +\partial^*q$
- $D+A \vdash -\partial^*q$ iff $T(D)+A \vdash -\partial^*q$

Proof

Suppose $+\partial^*q \in \mathcal{T}_{D+A} \uparrow (n+1)$. Then, by the $+\partial^*$ inference rule, either $+\Delta q \in \mathcal{T}_{D+A} \uparrow n$ (in which case, we must have $T(D)+A \vdash +\delta^*$) or $+\Delta q \notin \mathcal{T}_{D+A} \uparrow n$ and there is a strict or defeasible rule r in D with head q and body B_r such that $+\partial^*B_r \subseteq \mathcal{T}_{D+A} \uparrow n$, $-\Delta \sim q \in \mathcal{T}_{D+A} \uparrow n$, and for every rule s in D for $\sim q$ either there is a literal b in the body of s such that

$-\partial^*b \in \mathcal{T}_{D+A} \uparrow n$ or $r > s$. Hence, in the latter case, by the induction hypothesis, there is a strict or defeasible rule r in $D+A$ with head q and body B_r such that $T(D)+A \vdash +\partial^*B_r$, $T(D)+A \vdash -\Delta\sim q$, and for every rule s in $D+A$ for $\sim q$ either $T(D)+A \vdash -\partial^*B_s$ or $r > s$.

From this statement we derive several facts. (1) By Lemma 18 and the inclusion theorem, $T(D)+A \vdash +\partial^*\neg true(\sim q)$. (2) Thus, $T(D)+A \vdash +\partial^*(B_r, \neg true(\sim q))$ and, for every rule $n_d(r, s)$ in $T(D)$, $T(D)+A \vdash -\partial^*B_s$ (since rules s where $r > s$ do not give rise to a rule $n_d(r, s)$). Hence, $T(D)+A \vdash +\partial^*undefeated(q)$. (3) Conversely, $T(D)+A \vdash -\partial^*undefeated(\sim q)$ because, for every rule $p_d(s)$ for $undefeated(\sim q)$, either $T(D)+A \vdash -\partial^*B_s$ or there is a rule $n_d(s, r)$ superior to $p_d(s)$ with $T(D)+A \vdash +\partial^*B_r$. Consequently, since the only rule in $T(D)$ for q has body $undefeated(q)$ (and similarly for $\sim q$), applying the $+\partial^*$ inference rule, we have $T(D)+A \vdash +\partial^*q$.

Suppose $-\partial^*q \in \mathcal{T}_{D+A} \uparrow (n+1)$. Then, by the $-\partial^*$ inference rule, $-\Delta q \in \mathcal{T}_{D+A} \uparrow n$ and, for every strict or defeasible rule r in D with head q and body B_r , either $-\partial^*b \subseteq \mathcal{T}_{D+A} \uparrow n$ for some $b \in B_r$, $+\Delta\sim q \in \mathcal{T}_{D+A} \uparrow n$, or there is a rule s in D for $\sim q$ with body B_s such that $+\partial B_s \subseteq \mathcal{T}_{D+A} \uparrow n$ and $r \not> s$. Hence, by the induction hypothesis, $T(D)+A \vdash -\Delta q$ and for every strict or defeasible rule r in D with head q either $T(D)+A \vdash -\partial^*b$ for some $b \in B_r$, $T(D)+A \vdash +\Delta\sim q$, or there is a rule s in D for $\sim q$ where $T(D)+A \vdash +\partial^*B_s$ and $r \not> s$. Hence, for every rule $p_d(r)$ in $T(D)$ for $undefeated(q)$ either $T(D)+A \vdash -\delta^*b$ for some $b \in B_r$, or $T(D)+A \vdash -\delta^*\neg strict(\sim q)$ (by Lemma 18), or there is a rule $n_d(r, s)$ where $T(D)+A \vdash +\partial^*B_s$. Applying the inference rule for $-\delta^*undefeated(q)$, we conclude $T(D)+A \vdash -\delta^*undefeated(q)$ and, hence, $T(D)+A \vdash -\delta^*q$.

Suppose $+\partial^*q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$. Then, by the $+\partial^*$ inference rule and using the structure of $T(D)$, either $+\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$, or $+\partial^*undefeated(q) \in \mathcal{T}_{T(D)+A} \uparrow n$, $-\Delta\sim q \in \mathcal{T}_{T(D)+A} \uparrow n$, and $-\partial^*undefeated(\sim q) \in \mathcal{T}_{D+A} \uparrow n$. In the first case we have $D+A \vdash +\Delta q$ and thus $D+A \vdash +\partial^*q$. Alternatively, there is a strict or defeasible rule r in D with head q and body B_r such that $+\partial^*B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$, $+\partial^*\neg true(\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$, and for every rule s in D for $\sim q$ where $r \not> s$ there is a literal b in the body B_s of s such that $-\partial^*b \in \mathcal{T}_{T(D)+A} \uparrow n$. By the induction hypothesis and Lemma 18, $D+A \vdash +\partial^*B_r$, $D+A \vdash -\Delta\sim q$, and for every rule s in D for $\sim q$ where $r \not> s$ there is b in the body of s such that $D+A \vdash -\partial b$. Applying the $+\partial$ inference rule, we conclude $D+A \vdash +\partial^*q$.

Suppose $-\partial^*q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$. Then, by the $-\partial^*$ inference rule and using the structure of $T(D)$, $-\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ and either (1) $-\partial^*undefeated(q) \in \mathcal{T}_{T(D)+A} \uparrow n$ or (2) $+\Delta\sim q \in \mathcal{T}_{T(D)+A} \uparrow n$, or (3) $+\partial^*undefeated(\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$. By Lemma 18 and Corollary 20 we have $D+A \vdash -\Delta q$ and $D+A \vdash +\partial^*\neg true(q)$.

In the first case, for each rule r for q in D either there is a literal $p \in B_r$ and $-\partial p \in \mathcal{T}_{T(D)+A} \uparrow n$ or $-\partial^*\neg true(\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$ or for some rule s for $\sim q$ in D where $r \not> s$, $+\partial^*B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$. By the induction hypothesis (and Lemma 18 and Corollary 20), for each rule r for q in D either there is a literal $p \in B_r$ and $D+A \vdash -\partial^*p$, or $D+A \vdash +\Delta\sim q$, or for some rule s for $\sim q$ in D where $r \not> s$, $D+A \vdash +\partial^*B_s$. Applying the $-\partial$ inference rule, $D+A \vdash -\partial^*q$.

In the second case, by Lemma 18 and Corollary 20, $D+A \vdash +\Delta\sim q$. Consequently, applying the $-\partial$ inference rule, $D+A \vdash -\partial^*q$. In the third case, for some rule s for $\sim q$ in D , $+\partial^*B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and, for all rules r for q in D where $s \not> r$, for some $p \in B_r$, $-\partial^*p \in \mathcal{T}_{T(D)+A} \uparrow n$. By the induction hypothesis, for every rule r for q in D where $r > s$, for some $p \in B_r$, $D+A \vdash -\partial^*p$, and $D+A \vdash +\partial^*B_s$. Applying the $-\partial$ inference rule, $D+A \vdash -\partial^*q$.

□

Combining Theorem 21 with Lemma 19 and the inclusion theorem, we see that $\mathbf{DL}(\partial^*)$ can be simulated by $\mathbf{DL}(\delta^*)$ and $\mathbf{DL}(\delta)$.

Theorem 22

For $d \in \{\delta, \delta^*, \partial\}$, $\mathbf{DL}(d)$ can simulate $\mathbf{DL}(\partial^*)$ with respect to addition of facts

Proof

$D+A \vdash +\partial^*q$ iff $T(D)+A \vdash +\partial^*q$ (by Theorem 21) iff $T(D)+A \vdash +\delta^*q$ (by Corollary 20) iff $T(D)+A \vdash +dq$ (by Lemma 19 and the inclusion theorem). The proof is similar for $-\partial^*q$.

□

This is Theorem 8 from the body of the paper.

Simulation of Individual Defeat wrt Addition of Rules

Example 10 does not apply to $\mathbf{DL}(\delta)$ and $\mathbf{DL}(\delta^*)$. We have $\mathcal{T}(D) + A \vdash +\sigma h(r_2)$ and, consequently, $\mathcal{T}(D) + A \vdash -\delta p$, in agreement with D under $\mathbf{DL}(\delta^*)$. The weaker inference strength of ambiguity propagation masks the distinction that is present for blocked ambiguity reasoning. However, the next example shows that the transformation does not provide a simulation wrt rules for the propagating ambiguity logics.

Example 23

Let D consist of the rules

$$\begin{array}{lll} r_1 : & \Rightarrow & p \\ r_2 : & \Rightarrow & \neg p \\ r_3 : & \Rightarrow & p \\ r_4 : & \Rightarrow & \neg p \end{array}$$

with $r_1 > r_2$ and $r_3 > r_4$.

Then $\mathcal{T}(D)$ consists of the following rules

$$\begin{array}{lll} p(r_1) : & \Rightarrow & h(r_1) \\ p(r_2) : & \Rightarrow & h(r_2) \\ p(r_3) : & \Rightarrow & h(r_1) \\ p(r_4) : & \Rightarrow & h(r_4) \end{array} \quad \begin{array}{lll} n(r_1, r_2) : & \Rightarrow & \neg h(r_1) \\ n(r_1, r_4) : & \Rightarrow & \neg h(r_1) \\ n(r_2, r_1) : & \Rightarrow & \neg h(r_2) \\ n(r_2, r_3) : & \Rightarrow & \neg h(r_2) \\ n(r_3, r_2) : & \Rightarrow & \neg h(r_3) \\ n(r_3, r_4) : & \Rightarrow & \neg h(r_3) \\ n(r_4, r_1) : & \Rightarrow & \neg h(r_4) \\ n(r_4, r_3) : & \Rightarrow & \neg h(r_4) \end{array}$$

$$\begin{array}{lll} s(r_1) : & h(r_1) & \Rightarrow p \\ s(r_2) : & h(r_2) & \Rightarrow \neg p \\ s(r_3) : & h(r_3) & \Rightarrow p \\ s(r_4) : & h(r_4) & \Rightarrow \neg p \end{array}$$

with $p(r_1) > n(r_1, r_2)$, $n(r_2, r_1) > p(r_2)$, $p(r_3) > n(r_3, r_4)$, and $n(r_4, r_3) > p(r_4)$.

Now, let A be the rule

$$\Rightarrow p$$

Then $D + A \vdash -\delta^*p$, because for every rule r for p , there is a rule for $\neg p$ that is not overruled by r (r_1 does not overrule r_4 , r_3 does not overrule r_2 and A overrules neither).

However, considering the transformed theory, $\mathcal{T}(D) + A \vdash -\sigma h(r_2)$, because $n(r_2, r_1) > p(r_2)$ and, similarly, $\mathcal{T}(D) + A \vdash -\sigma h(r_4)$. Consequently, both rules for $\neg p$ in $\mathcal{T}(D) + A$ fail. This leaves the rules for p without competition, and so $\mathcal{T}(D) + A \vdash +\delta p$, conflicting with the behaviour of $D+A$.

Following essentially the same argument, this example also applies to $\mathbf{DL}(\partial^*)$ and $\mathbf{DL}(\partial)$.

We show that the transformation defined in Definition 11 (and Definition 9) allows the team-defeat logics to simulate their individual-defeat counterparts. We treat the two cases separately, but first we address the effect of the transformation on strict inference.

Lemma 24

Consider the transformation T from Definition 11. For any D and A

- $D+A \vdash +\Delta q$ iff $T(D)+A \vdash +\Delta q$
- $D+A \vdash -\Delta q$ iff $T(D)+A \vdash -\Delta q$

The proof is a straightforward induction.

Theorem 25

The logic $\mathbf{DL}(\partial^*)$ can be simulated by $\mathbf{DL}(\partial)$ with respect to addition of rules.

Proof

We consider the transformation $T(D)$ of a defeasible theory D as defined in Definition 11 (and Definition 9) and show that this transformation provides a simulation of each defeasible theory D in $\mathbf{DL}(\partial^*)$ from within $\mathbf{DL}(\partial)$.

Fix any D and any A that satisfies the language separation condition. Let $\Sigma = \Sigma(D) \cup \Sigma(A)$.

[1] If $+\partial^*q \in \mathcal{T}_{D+A}\uparrow(n+1)$ then either $+\Delta q \in \mathcal{T}_{D+A}\uparrow n$ (in which case $T(D)+A \vdash +\partial q$) or $+\partial^*B_r \subseteq \mathcal{T}_{D+A}\uparrow n$, where B_r is the body of some strict or defeasible rule r in $D+A$. In the latter case, $\mathcal{T}_{D+A}\uparrow n$ also contains $-\Delta \sim q$ and for every rule s for $\sim q$ in $D+A$ either $r > s$ or $-\partial^*p \in \mathcal{T}_{D+A}\uparrow n$ for some literal p in the body of s . Then, by the induction hypothesis, $T(D)+A \vdash +\partial B_r$, $T(D)+A \vdash -\Delta \sim q$ and, if $r \in D$, for every rule $n(r, s)$ for $\neg h(r)$ in $T(D)$, either $p(r) >' n(r, s)$ or $T(D)+A \vdash -\partial p$ where p occurs in the body of $n(r, s)$. Thus, using the inference rule for $+\partial$, if $r \in D$ then $T(D)+A \vdash +\partial h(r)$. If $r \in A$ then $T(D)+A \vdash +\partial B_r$ so, whether $r \in D$ or $r \in A$, there is a rule for q in $T(D)+A$ with body B and $T(D)+A \vdash +\partial B$.

Applying the inference rule for $-\partial$ multiple times, for each strict or defeasible rule s for $\sim q$ in D we have $T(D)+A \vdash -\partial h(s)$. Furthermore, as noted above, for every rule $s \in A$ for $\sim q$, since $r \not> s$, $-\partial^*p \in \mathcal{T}_{D+A}\uparrow n$ for some literal p in the body of s . Thus, every rule for $\sim q$ in $T(D)+A$ fails. Now, again applying the inference rule for $+\partial$, we have $T(D)+A \vdash +\partial q$.

[2] If $-\partial^*q \in \mathcal{T}_{D+A}\uparrow(n+1)$ then $-\Delta q \in \mathcal{T}_{D+A}\uparrow n$ and either $+\Delta \sim q \in \mathcal{T}_{D+A}\uparrow n$ (in which case $T(D)+A \vdash -\partial q$) or, for every strict or defeasible rule r for q in $D+A$, either $-\partial^*p \in \mathcal{T}_{D+A}\uparrow n$ for some p in the body of r or there exists a rule s for $\sim q$ with body B , $+\partial^*B \subseteq \mathcal{T}_{D+A}\uparrow n$ and $r \not> s$. Then, for every strict or defeasible rule $p(r)$ in $T(D)$, either $-\partial^*p \in \mathcal{T}_{D+A}\uparrow n$ for some p in the body of $p(r)$ or there is a rule $n(r, s)$ with body B and $p(r) \not>' n(r, s)$, by the structure of $T(D)$. By the induction hypothesis, for every strict or defeasible rule $p(r)$ in $T(D)$, either $T(D)+A \vdash -\partial p$ for some p in the body of $p(r)$ or there is a rule $n(r, r')$ with body B where $T(D)+A \vdash +\partial B$ and $p(r) \not>' n(r, r')$. Since there is only one rule for $h(r)$,

application of the inference rule for $-\partial$ gives us $T(D)+A \vdash -\partial h(r)$ for each strict or defeasible rule $r \in D$ for q . Also by the induction hypothesis, for every strict or defeasible rule r for q in A , $T(D)+A \vdash -\partial p$ for some p in the body of r . Hence $T(D)+A \vdash -\partial q$.

[3] If $q \in \Sigma$ and $+\partial q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$ then either (1) $+\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D+A \vdash +\partial^* q$), or else (2) $+\partial h(r) \in \mathcal{T}_{T(D)+A} \uparrow n$ for some strict or defeasible rule r for q in D , or else (3) $+\partial B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ for some strict or defeasible rule r for q in A with body B_r . In case (3), by the induction hypothesis, $D+A \vdash +\partial B_r$. In case (2) we must also have that every rule for $\sim q$ in $T(D)+A$ fails (except for $o(\sim q)$, which is overruled); that is, for every rule s for $\sim q$ in D , $-\partial h(s) \in \mathcal{T}_{T(D)+A} \uparrow n$ and, for every rule for $\sim q$ in A with body B , for some literal p in B $-\partial p \in \mathcal{T}_{T(D)+A} \uparrow n$. In case (3) we must also have that $one(\sim q)$ fails, so that every rule for $\sim q$ in D with body B , for some literal p in B $-\partial p \in \mathcal{T}_{T(D)+A} \uparrow n$. Hence, by the induction hypothesis, in case (3), for every rule for $\sim q$ in $D+A$ with body B , for some literal p in B $D+A \vdash -\partial^* p$. In both cases (2) and (3), $-\Delta \sim q \in \mathcal{T}_{T(D)+A} \uparrow n$ and hence $D+A \vdash -\Delta \sim q$. Applying the $+\partial^*$ inference rule in case (3), $D+A \vdash +\partial^* q$.

In case (2), if $+\partial h(r) \in \mathcal{T}_{T(D)+A} \uparrow n$ then $p(r)$ is not a defeater, $+\partial B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ where B_r is the body of r and for every rule $n(r, s)$ with body B' either for some literal p in B' $-\partial p \in \mathcal{T}_{T(D)+A} \uparrow n$ or for some rule t for $h(r)$, its body is proved with respect to ∂ and $t > s$. There is only one rule for $h(r)$, so this last disjunct reduces to $p(r) > n(r, s)$. Using the construction of $T(D)$, r is not a defeater, $+\partial B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ where B_r is the body of r and for every rule s for $\sim q$ in D with body B' either for some literal p in B' , $-\partial p \in \mathcal{T}_{T(D)+A} \uparrow n$ or $r > s$. Furthermore, from the previous paragraph, for every rule for $\sim q$ in A with body B , for some literal p in B $-\partial p \in \mathcal{T}_{T(D)+A} \uparrow n$. Using the induction hypothesis, $D+A \vdash +\partial^* B_r$, and for every rule for $\sim q$ in $D+A$ either for some literal p in the body $D+A \vdash -\partial^* p$ or $r > s$. Applying the inference rule for $+\partial^*$, we obtain $D+A \vdash +\partial^* q$.

[4] If $q \in \Sigma$ and $-\partial q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$ then, using the $-\partial$ inference rule and the structure of $T(D)$, $-\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ and either (0) $+\Delta \sim q \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D+A \vdash -\partial^* q$), or else (1) $-\partial h(r) \in \mathcal{T}_{T(D)+A} \uparrow n$ for every rule r for q in D , while for every rule r in A there is a literal p in the body of r with $-\partial p \in \mathcal{T}_{T(D)+A} \uparrow n$, and $-\partial one(q) \in \mathcal{T}_{T(D)+A} \uparrow n$; or (2) $+\partial h(s) \in \mathcal{T}_{T(D)+A} \uparrow n$ for some rule s for $\sim q$ in D ; or (3) $+\partial one(\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$, in which case there is a rule s for $\sim q$ in D where $+\partial B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$, and $-\partial h(r) \in \mathcal{T}_{T(D)+A} \uparrow n$ for every rule r for q in D (so that $o(\sim q)$ is not overruled); or (4) there is a rule s for $\sim q$ in A where $+\partial B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$. In any case, using the induction hypothesis, we have $D+A \vdash -\Delta q$.

In case (1), since $-\partial one(q) \in \mathcal{T}_{T(D)+A} \uparrow n$, for every rule r in D for q there is a literal p in B_r with $-\partial p \in \mathcal{T}_{T(D)+A} \uparrow n$. Thus all rules for q in $D+A$ fail, and hence $D+A \vdash -\partial^* q$. In case (2), we must have, for every rule r in D for q , either $s > r$ (so that $p(s) > n(s, r)$) or there is a literal p in B_r with $-\partial p \in \mathcal{T}_{T(D)+A} \uparrow n$. By the induction hypothesis, we then have $D+A \vdash -\partial^* p$, for each such p . Furthermore, no rule in A can overrule s . Hence, applying the $-\partial^*$ inference rule, $D+A \vdash -\partial^* q$.

In case (3), since $-\partial h(r) \in \mathcal{T}_{T(D)+A} \uparrow n$, either there is a literal p in B_r with $-\partial p \in \mathcal{T}_{T(D)+A} \uparrow n$ or there is a rule s in D for $\sim q$ with $+\partial B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and $r \not> s$ (so that $p(r) \not> n(s, r)$). By the induction hypothesis, for every rule r for q in D either there is a literal p in B_r with $D+A \vdash -\partial^* p$ or there is a rule s in D for $\sim q$ with $D+A \vdash +\partial^* B_s$ and $r \not> s$. Furthermore, from $+\partial one(\sim q)$ we know there is an s in D with (using the induction hypothesis) $D+A \vdash +\partial^* B_s$, and this s cannot be overruled by any rule r in A . Consequently, applying the $-\partial^*$ inference rule, $D+A \vdash -\partial^* q$.

In case (4), by the induction hypothesis, we have there is a rule s for $\sim q$ in A where $D+A \vdash$

$+ \partial^* B_s$ and, since s cannot be inferior to any rule, applying the $-\partial^*$ inference rule we have $D+A \vdash -\partial^* q$. \square

This concludes the proof that $\mathbf{DL}(\partial)$ can simulate $\mathbf{DL}(\partial^*)$ with respect to addition of rules. We now turn to the corresponding proof for $\mathbf{DL}(\delta)$ and $\mathbf{DL}(\delta^*)$.

Theorem 26

The logic $\mathbf{DL}(\delta^*)$ can be simulated by $\mathbf{DL}(\delta)$ with respect to addition of rules.

Proof

Let A be any set of rules. Let Σ be the language of $D+A$ and let $q \in \Sigma$. Let $T(D)$ be the transformed defeasible theory as described in Definition 11. Then we claim

- $D+A \vdash +\sigma^* q$ iff $T(D)+A \vdash +\sigma q$
- $D+A \vdash -\sigma^* q$ iff $T(D)+A \vdash -\sigma q$
- $D+A \vdash +\delta^* q$ iff $T(D)+A \vdash +\delta q$
- $D+A \vdash -\delta^* q$ iff $T(D)+A \vdash -\delta q$

If $+\delta^* q \in \mathcal{T}_{D+A} \uparrow (n+1)$ then either $+\Delta q \in \mathcal{T}_{D+A} \uparrow n$ (in which case $T(D)+A \vdash +\delta q$) or else $+\delta^* B_r \subseteq \mathcal{T}_{D+A} \uparrow n$, where B_r is the body of some strict or defeasible rule r in $D+A$. In the latter case, $\mathcal{T}_{D+A} \uparrow n$ also contains $-\Delta \sim q$ and for every rule s for $\sim q$ in $D+A$ either $r > s$ or $-\sigma^* p \in \mathcal{T}_{D+A} \uparrow n$ for some literal p in the body of s . Then, by the induction hypothesis, $T(D)+A \vdash +\delta B_r$, $T(D)+A \vdash -\Delta \sim q$ and, if r and s are in D , for every rule $n(r, s)$ for $\neg h(r)$ in $T(D)$, either $p(r) >' n(r, s)$ or $T(D)+A \vdash -\sigma p$ where p occurs in the body of $n(r, s)$ and, similarly, the rule $p(s)$ for $h(s)$ in $T(D)$, either $p(s) <' n(s, r)$ or $T(D)+A \vdash -\sigma p$ where p occurs in the body of $p(s)$. If s is in A then $r > s$ cannot occur (since the rules of A do not participate in the superiority relation) and $T(D)+A \vdash -\sigma p$ where p occurs in the body of s . If r is in A and s is in D then, again, $r > s$ cannot occur and $T(D)+A \vdash -\sigma p$ where p occurs in the body of $p(s)$. Thus, using the inference rules for $+\delta$ and $-\sigma$, if r is in D then $T(D)+A \vdash +\delta h(r)$ and if s is in D then $T(D)+A \vdash -\sigma h(s)$. Now, applying the inference rule for $+\delta$, we conclude $T(D)+A \vdash +\delta q$.

If $-\delta^* q \in \mathcal{T}_{D+A} \uparrow (n+1)$ then $-\Delta q \in \mathcal{T}_{D+A} \uparrow n$ (and, hence, $T(D)+A \vdash -\Delta q$) and either $+\Delta \sim q \in \mathcal{T}_{D+A} \uparrow n$ (in which case $T(D)+A \vdash -\delta q$) or, for every strict or defeasible rule r for q in $D+A$, either $-\delta^* p \in \mathcal{T}_{D+A} \uparrow n$ for some p in the body of r or there exists a rule s for $\sim q$ with body B_s , where $+\sigma^* B_s \subseteq \mathcal{T}_{D+A} \uparrow n$ and $r \not> s$. Now, if, for some s for $\sim q$ in A , $+\sigma^* B_s \subseteq \mathcal{T}_{D+A} \uparrow n$ then, by the induction hypothesis, $T(D)+A \vdash +\sigma B_s$ and, applying the inference rule for $-\delta$ (and noting that no rule is superior to s), we have $T(D)+A \vdash -\delta q$.

Otherwise, for every strict or defeasible rule $p(r)$ in $T(D)$, either $-\delta^* p \in \mathcal{T}_{D+A} \uparrow n$ for some p in the body of $p(r)$ or there is a rule $n(r, s)$ with body B_s and $p(r) \not>' n(r, s)$, by the structure of $T(D)$. By the induction hypothesis, for every strict or defeasible rule $p(r)$ in $T(D)$, either $T(D)+A \vdash -\delta p$ for some p in the body of $p(r)$ or there is a rule $n(r, s)$ with body B_s where $T(D)+A \vdash +\sigma B_s$ and $p(r) \not>' n(r, s)$. In both cases, since there is only one rule for $h(r)$, application of the inference rule for $-\delta$ gives us $T(D)+A \vdash -\delta h(r)$ for each strict or defeasible rule r for q in D . Hence, no rule $s(r)$ can overrule $o(\sim q)$. Now, if every rule r for q in A has $p \in B_r$ with $-\delta^* p \in \mathcal{T}_{D+A} \uparrow n$ then, by the induction hypothesis, $T(D)+A \vdash -\delta p$ for every such rule and hence, by application of the $-\delta$ inference rule, $T(D)+A \vdash -\delta q$. Otherwise, there is s for $\sim q$ in D with $T(D)+A \vdash +\sigma B_s$. By the $+\sigma$ inference rule $T(D)+A \vdash +\sigma one(\sim q)$. Consequently, since $o(\sim q)$ cannot be overruled, $T(D)+A \vdash -\delta q$.

Hence, in every case, $T(D)+A \vdash -\delta q$.

If $q \in \Sigma$ and $+\delta q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$ then either $+\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D+A \vdash +\delta^* q$), or $+\delta h(r) \in \mathcal{T}_{T(D)+A} \uparrow n$ for some strict or defeasible rule r for q in D . or $+\delta B \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ for some strict or defeasible rule for q in A with body B . In the latter cases, $\mathcal{T}_{T(D)+A} \uparrow n$ also contains $-\Delta \sim q$; hence $D+A \vdash -\Delta \sim q$. In these cases we must also have, for each rule for $\sim q$ in A , for some p in its body $-\sigma p \in \mathcal{T}_{T(D)+A} \uparrow n$. Hence, by the induction hypothesis, for each rule for $\sim q$ in A , for some p in its body $D+A \vdash -\sigma^* p$. If $+\delta h(r) \in \mathcal{T}_{T(D)+A} \uparrow n$ then $p(r)$ is not a defeater, $+\delta B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ where B_r is the body of r and for every rule $n(r, s)$ with body B_s either for some literal p in B_s $-\sigma p \in \mathcal{T}_{T(D)+A} \uparrow n$ or for some rule t for $h(r)$, its body is proved with respect to δ and $t > s$. There is only one rule for $h(r)$, so this last disjunct reduces to $p(r) > n(r, s)$. Using the construction of $T(D)$, r is not a defeater, $+\delta B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and for every rule s for $\sim q$ in D either for some literal p in B_s , $-\sigma p \in \mathcal{T}_{T(D)+A} \uparrow n$ or $r > s$. Using the induction hypothesis, $D+A \vdash +\delta^* B_r$, and for every rule for $\sim q$ in D either for some literal p in the body $D+A \vdash -\sigma^* p$ or $r > s$. Applying the inference rule for $+\delta^*$ to this statement, and given we have shown that all rules for $\sim q$ in A fail, we obtain $D+A \vdash +\delta^* q$.

If $q \in \Sigma$ and $-\delta q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$ then $-\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ and either (1) $+\Delta \sim q \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D+A \vdash -\delta^* q$), or (2) $-\delta h(r) \in \mathcal{T}_{T(D)+A} \uparrow n$ for every rule r for q in D and every rule for q in A has a literal p in its body with $-\delta q \in \mathcal{T}_{T(D)+A} \uparrow n$, or (3) there is a rule s for $\sim q$ in D where $+\sigma h(s) \in \mathcal{T}_{T(D)+A} \uparrow n$, or (4) there is a rule for $\sim q$ in A with body B and $+\sigma B \subseteq \mathcal{T}_{T(D)+A} \uparrow n$. (Some conditions are simpler than the inference rule for $-\delta$ might suggest because the superiority relation in $T(D)+A$ does not involve the rules for q and $\sim q$.) Consequently, $D+A \vdash -\Delta q$. In the first case, using the induction hypothesis, we have $D+A \vdash -\Delta q$ and $D+A \vdash +\Delta \sim q$; hence, $D+A \vdash -\delta^* q$. In the second case, for each r , either r is a defeater, or there is a literal p in the body of r such that $-\delta p \in \mathcal{T}_{T(D)+A} \uparrow n$, or there is a rule s for $\sim q$ in D (corresponding to rule $n(r, s)$ in $T(D)$) with body B_s where $+\sigma B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and $r \not> s$. By the induction hypothesis, either r is a defeater, or there is a literal p in the body of r such that $D+A \vdash -\delta^* p$, or there is a rule s for $\sim q$ in D with body B_s where $D+A \vdash +\sigma^* B_s$ and $r \not> s$. Similarly, using the induction hypothesis, every rule for q in A has a literal p in its body with $D+A \vdash -\delta^* p$. Applying the inference rule for $-\delta^*$, we obtain $D+A \vdash -\delta^* q$.

In the third case, either $+\Delta h(s) \in \mathcal{T}_{T(D)+A} \uparrow n$, or $+\sigma B \subseteq \mathcal{T}_{T(D)+A} \uparrow n$, where B_s is the body of s , and, for every rule r for q in D , either $-\delta p \in \mathcal{T}_{T(D)+A} \uparrow n$ for some literal p in the body of r or $r \not> s$. If $+\Delta h(s) \in \mathcal{T}_{T(D)+A} \uparrow n$ then $+\Delta B \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and s is strict. Using the induction hypothesis, $D+A \vdash +\Delta B$ and, hence, $D+A \vdash +\Delta \sim q$ and, like case (1) above, $D+A \vdash -\delta^* q$. In the other case, by the induction hypothesis, $D+A \vdash +\sigma^* B_s$ and, for every rule r for q in D , either $D+A \vdash -\delta^* p$ for some literal p in the body of r or $r \not> s$. Applying the inference rule for $-\delta^*$ we conclude $D+A \vdash -\delta^* q$.

In the fourth case, using the induction hypothesis, there is a rule for $\sim q$ in A with body B and $D+A \vdash +\sigma^* B$. Applying the inference rule for $-\delta^*$ we conclude $D+A \vdash -\delta^* q$.

If $+\sigma^* q \in \mathcal{T}_{D+A} \uparrow (n+1)$ then either $+\Delta q \in \mathcal{T}_{D+A} \uparrow n$ (in which case $T(D)+A \vdash +\sigma q$) or $+\sigma^* B_r \subseteq \mathcal{T}_{D+A} \uparrow n$, where B_r is the body of some strict or defeasible rule r in $D+A$. In the latter case, for every rule s for $\sim q$ in $D+A$ either $s \not> r$ or $-\delta^* p \in \mathcal{T}_{D+A} \uparrow n$ for some literal p in the body of s . If $r \in A$ then r is not inferior to any rule. So, by the induction hypothesis, $T(D)+A \vdash +\sigma B_r$, and, by the $+\sigma$ inference rule $T(D)+A \vdash +\sigma q$. If $r \in D$, by the induction hypothesis, $T(D)+A \vdash +\sigma B_r$, for every rule $n(r, s)$ for $\neg h(r)$ in $T(D)$, either $n(r, s) \not> p(r)$ or $T(D)+A \vdash -\delta p$ where p occurs in the body of $n(r, s)$. Thus, using the inference rule for

$+\sigma, T(D)+A \vdash +\sigma h(r)$. Applying the inference rule for $-\delta$ multiple times, for each strict or defeasible rule s for $\sim q$ in D we have $T(D)+A \vdash -\delta h(s)$. Now, applying the inference rule for $+\sigma$, we have $T(D)+A \vdash +\sigma q$.

If $-\sigma^*q \in \mathcal{T}_{D+A}\uparrow(n+1)$ then $-\Delta q \in \mathcal{T}_{D+A}\uparrow n$ and for every strict or defeasible rule r for q in $D+A$, either $-\sigma^*p \in \mathcal{T}_{D+A}\uparrow n$ for some p in the body of r or there exists a rule s for $\sim q$ in $D+A$ with body B_s , $+\delta^*B_s \subseteq \mathcal{T}_{D+A}\uparrow n$ and $s > r$. If $r \in A$ then $-\sigma^*p \in \mathcal{T}_{D+A}\uparrow n$ for some p in the body of r and hence, by the induction hypothesis, $T(D)+A \vdash -\sigma p$. If $r \in D$ then, for every strict or defeasible rule $p(r)$ in $T(D)$, either $-\sigma^*p \in \mathcal{T}_{D+A}\uparrow n$ for some p in the body of $p(r)$ or there is a rule $n(r, s)$ with body B_s with $+\delta^*B_s \subseteq \mathcal{T}_{D+A}\uparrow n$ and $n(r, s) >' p(r)$, by the structure of $T(D)$. By the induction hypothesis, for every strict or defeasible rule $p(r)$ in $T(D)$, either $T(D)+A \vdash -\sigma p$ for some p in the body of $p(r)$ or there is a rule $n(r, s)$ with body B_s where $T(D)+A \vdash +\delta B$ and $n(r, s) >' p(r)$. Application of the inference rule for $-\sigma$ gives us $T(D)+A \vdash -\sigma h(r)$ for each strict or defeasible rule r for q in D . Rules in A for q also fail, as mentioned above. Hence $T(D)+A \vdash -\sigma q$.

If $q \in \Sigma$ and $+\sigma q \in \mathcal{T}_{T(D)+A}\uparrow(n+1)$ then either $+\Delta q \in \mathcal{T}_{T(D)+A}\uparrow n$ (in which case $D+A \vdash +\sigma^*q$), or $+\sigma B \subseteq \mathcal{T}_{T(D)+A}\uparrow n$ for some strict or defeasible rule for q in A with body B , or $+\sigma h(r) \in \mathcal{T}_{T(D)+A}\uparrow n$ for some strict or defeasible rule r for q in D . In the second case, by the induction hypothesis, $D+A \vdash +\sigma^*B$ and, applying the $+\sigma^*$ inference rule, $D+A \vdash +\sigma^*q$. In the third case, if $+\sigma h(r) \in \mathcal{T}_{T(D)+A}\uparrow n$ then $p(r)$ is not a defeater, $+\sigma B_r \subseteq \mathcal{T}_{T(D)+A}\uparrow n$ where B_r is the body of r and for every rule $n(r, s)$ with body B_s either for some literal p in B_s , $-\delta p \in \mathcal{T}_{T(D)+A}\uparrow n$ or $n(r, s) \not>' p(r)$. Using the construction of $T(D)$, r is not a defeater, $+\sigma B_r \subseteq \mathcal{T}_{T(D)+A}\uparrow n$ where B_r is the body of r and for every rule s for $\sim q$ with body B_s in D either for some literal p in B' , $-\delta p \in \mathcal{T}_{T(D)+A}\uparrow n$ or $s \not>' r$. Using the induction hypothesis, $D+A \vdash +\sigma^*B$, and for every rule for $\sim q$ in D either for some literal p in the body $D+A \vdash -\delta^*p$ or $s \not>' r$. Note also that no rule s for $\sim q$ in A can be superior to r . Applying the inference rule for $+\sigma^*$ to this statement, we obtain $D+A \vdash +\sigma^*q$.

If $q \in \Sigma$ and $-\sigma q \in \mathcal{T}_{T(D)+A}\uparrow(n+1)$ then $-\Delta q \in \mathcal{T}_{T(D)+A}\uparrow n$ (and, consequently, $D+A \vdash -\Delta q$) and for every strict or defeasible rule r for q in A with body B there is p in B with $-\sigma p \in \mathcal{T}_{T(D)+A}\uparrow n$, and, for every rule r for q in D , $-\sigma h(r) \in \mathcal{T}_{T(D)+A}\uparrow n$. From $-\sigma h(r)$ either r is a defeater, or there is a literal p in the body of r such that $-\sigma p \in \mathcal{T}_{T(D)+A}\uparrow n$, or there is a rule s for $\sim q$ in D (corresponding to rule $n(r, s)$ in $T(D)$) with body B_s where $+\delta B_s \subseteq \mathcal{T}_{T(D)+A}\uparrow n$ and $s > r$. By the induction hypothesis, either r is a defeater, or there is a literal p in the body of r such that $D+A \vdash -\sigma^*p$, or there is a rule s for $\sim q$ in D with body B_s where $D+A \vdash +\delta^*B_s$ and $s > r$. Applying the inference rule for $-\sigma^*$, we obtain $D+A \vdash -\sigma^*q$.

□

Combining Theorems 25 and 26 we have Theorem 12.

Simulation of Team Defeat wrt Addition of Rules

The same theory D and addition A as in Example 10 demonstrates that the simulation of $\mathbf{DL}(\partial)$ by $\mathbf{DL}(\partial^*)$ wrt addition of facts exhibited in (Maher 2012) does not extend to addition of rules.

Example 27

$$\begin{array}{ll} r_1 : & \Rightarrow p \\ r_2 : & \Rightarrow \neg p \end{array}$$
$$\Rightarrow p$$
$$\begin{array}{llll}
R1_{12} : & \Rightarrow & \neg d(r_1, r_2) & R1_{21} : & \Rightarrow & \neg d(r_2, r_1) \\
R2_{12} : & \Rightarrow & d(r_1, r_2) & R2_{21} : & \Rightarrow & d(r_2, r_1) \\
\\
NF_1 : & \Rightarrow & \neg fail(r_1) & NF_2 : & \Rightarrow & \neg fail(r_2) \\
F_1 : & \Rightarrow & fail(r_1) & F_2 : & \Rightarrow & fail(r_2) \\
\\
d(r_1, r_2) & \Rightarrow & d(r_1) & d(r_2, r_1) & \Rightarrow & d(r_2) \\
fail(r_1) & \Rightarrow & d(r_1) & fail(r_2) & \Rightarrow & d(r_2) \\
\\
& \Rightarrow & one(p) & one(p), d(r_2) & \Rightarrow & p \\
& \Rightarrow & one(\neg p) & one(\neg p), d(r_1) & \Rightarrow & \neg p
\end{array}$$

We say a rule r fails in D if, for some literal p in the body of r , $D \vdash \neg \partial^* p$. Similarly, r fails in $\mathcal{T} \upharpoonright n$ if $\neg \partial^* p \in \mathcal{T} \upharpoonright n$ for some literal p in the body of r .

Theorem 29

The logic $\mathbf{DL}(\partial)$ can be simulated by $\mathbf{DL}(\partial^*)$ with respect to addition of rules.

Proof

Let Σ be the language of $D+A$. Note, that, employing Lemma 18, $T(D)+A \vdash +\partial^*\neg\text{true}(q)$ iff $T(D)+A \vdash -\Delta q$ iff $D+A \vdash -\Delta q$. Because $T(D)+A \vdash -\partial^*q$, we can essentially ignore the rules $\text{supp}(q)$, which are only included for the simulation of $\mathbf{DL}(\delta)$ by $\mathbf{DL}(\delta^*)$.

Suppose $+\partial q \in \mathcal{T}_{D+A} \uparrow (n+1)$. Then either $+\Delta q \in \mathcal{T}_{D+A} \uparrow n$ (in which case $T(D) + A \vdash +\partial^*q$), or $-\Delta\sim q \in \mathcal{T}_{D+A} \uparrow n$ and there is a non-empty team of strict or defeasible rules for q such that $+\partial B_r \subseteq \mathcal{T}_{D+A} \uparrow n$ for each body B_r of each rule r and every rule s for $\sim q$ either has a body that fails in $\mathcal{T}_{D+A} \uparrow n$ or $s < t$ for some rule t in the team. $t \notin A$ because rules in A do not participate in the superiority relation. Then, by the induction hypothesis, $T(D) + A \vdash -\Delta\sim q$, $T(D) + A \vdash +\partial^*B_r$ for each rule r in the team, and for every rule s for $\sim q$ either its body fails in $T(D) + A$ or there is a rule t in the team and $t > s$. If $s \in A$ then its body B_s fails in $T(D) + A$. If $s \in D$ then either $T(D) + A \vdash +\partial^*\text{fail}(s)$ or $T(D) + A \vdash +\partial^*d(s, t)$; in either case, $T(D) + A \vdash +\partial^*d(s)$. Considering $T(D)$ and the inference rule for $+\partial^*$, we have $T(D) + A \vdash +\partial^*\text{one}(q)$. By Lemma 28, $T(D) + A \vdash +\partial^*\neg\text{true}(\sim q)$. Hence, the body of $s(q)$ is proved. Because $>$ is acyclic, there is a rule in the team for q that is not inferior to any rule in the team for $\sim q$. Hence this rule r' is not defeated, so $d(r')$ fails, and hence the rule $s(\sim q)$ in $T(D)$ (from point 6) fails. Hence all rules for $\sim q$ fail, with the possible exception of $o(\sim q)$. However $s(q) > o(\sim q)$ and hence, applying the $+\partial^*$ inference rule, $T(D) + A \vdash +\partial^*q$.

Suppose $-\partial q \in \mathcal{T}_{D+A} \uparrow (n+1)$. Then $-\Delta q \in \mathcal{T}_{D+A} \uparrow n$ (and hence $T(D) + A \vdash -\Delta q$) and either (1) $+\Delta\sim q \in \mathcal{T}_{D+A} \uparrow n$ (in which case $T(D) + A \vdash -\partial^*q$), or (2) every rule r for q fails, or (3) there is a rule s for $\sim q$ with body B_s such that $+\partial B_s \subseteq \mathcal{T}_{D+A} \uparrow n$ and, for every strict or defeasible rule t for q , either t fails in $\mathcal{T}_{D+A} \uparrow n$, or $t \not\prec s$. In case (2), the rules r in A for q fail and, by the induction hypothesis and the inference rule for $-\partial^*$, the rules r in A for q fail and, $T(D) + A \vdash -\partial^*\text{one}(q)$ and hence $T(D) + A \vdash -\partial^*q$. In case (3), by the induction hypothesis, there is a rule s for $\sim q$ with body B_s such that $T(D) + A \vdash +\partial^*B_s$ and for every strict or defeasible rule t for q , either t fails in $T(D) + A$, or $t \not\prec s$. If $s \in A$ then $t \not\prec s$, for every t , and hence $T(D) + A \vdash -\partial^*q$. If $s \in D$ then $T(D) + A \vdash -\partial^*d(s, t)$ (since, via Lemma 28, we also have $T(D) + A \vdash -\partial^*\text{true}(q)$). Using the $-\partial^*$ inference rule, $T(D) + A \vdash -\partial^*d(s)$ and hence $T(D) + A \vdash -\partial^*q$.

Suppose $q \in \Sigma$ and $+\partial^*q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$. Then either (1) $+\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D + A \vdash +\partial q$), or $-\Delta\sim q \in \mathcal{T}_{T(D)+A} \uparrow n$ (and hence $D + A \vdash -\Delta\sim q$) and either (2) for some r in A for q , $B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$, or (3) $+\partial^*\text{one}(q) \in \mathcal{T}_{T(D)+A} \uparrow n$, $+\partial^*\neg\text{true}(\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$, and $+\partial^*d(r)$ occurs in $\mathcal{T}_{T(D)+A} \uparrow n$, for each rule r for $\sim q$ in D . In both cases (2) and (3) we must have, for any rule s for $\sim q$ in A , for some p in the body B_s of s , $-\partial^*p \in \mathcal{T}_{T(D)+A} \uparrow n$. By the induction hypothesis, $D + A \vdash -\partial p$ for each such p .

In case (2), by the induction hypothesis, $D + A \vdash B_r$. Also, in case (2), the rule $o(\sim q)$ must fail. Consequently, every rule s for $\sim q$ in D fails in $\mathcal{T}_{T(D)+A} \uparrow n$. By the induction hypothesis, every rule s for $\sim q$ in D fails in $D + A$. Now, applying the inference rule for $+\partial$, $D + A \vdash +\partial q$.

In case (3) there must be a strict or defeasible rule r for q in D with body B_r such that $+\partial^*B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and, using the rules for $d(s)$ and $d(s, t)$, for every rule s for $\sim q$, either the body B_s of s fails or there is a strict or defeasible rule t for q with body B_t such that $+\partial^*B_t \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and $t > s$. By the induction hypothesis, $D + A \vdash +\partial B_r$, and, for every rule s for $\sim q$ in D , either $D + A \vdash -\partial B_s$ or there is a strict or defeasible rule t for q with body

B_t such that $D + A \vdash +\partial B_t$ and $t > s$. As noted above, for any rule s for $\sim q$ in A , B_s fails in $D + A$. Hence, by the inference rule for $+\partial$, $D + A \vdash +\partial q$.

If $q \in \Sigma$ and $-\partial^* q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$ then, using the inference rule for $-\partial^*$ and the structure of $T(D)$, $-\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ and either (a) $+\Delta \sim q \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D + A \vdash -\partial^* q$), or (b) $-\partial^* \text{one}(q) \in \mathcal{T}_{T(D)+A} \uparrow n$, or (c) $-\partial^* d(s) \in \mathcal{T}_{T(D)+A} \uparrow n$ for some rule s for $\sim q$ in D , or (d) $+\partial^* \text{one}(\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$ and $+\partial^* d(r) \in \mathcal{T}_{T(D)+A} \uparrow n$ for every rule r for q in D , or (e) for some s for $\sim q$ in A , $+\partial^* B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$.

If (b) $-\partial^* \text{one}(q) \in \mathcal{T}_{T(D)+A} \uparrow n$ then for every strict or defeasible rule for q in D fails. Applying the induction hypothesis and the inference rule for $-\partial$, we have $D + A \vdash -\partial q$. If (c) $-\partial^* d(s) \in \mathcal{T}_{T(D)+A} \uparrow n$ for some rule s for $\sim q$ in D , then there is no strict or defeasible rule r_2 for q that defeats s . If (d) then there is a rule s for $\sim q$ with body B_s such that $+\partial^* B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and every rule r for q in D is defeated by a strict or defeasible rule for $\sim q$. In both cases (c) and (d), applying the induction hypothesis and the inference rule for $-\partial$, we have $D + A \vdash -\partial q$. If (e) then, by the induction hypothesis, $D + A \vdash +\partial B_s$ and hence, applying the inference rule for $-\partial$, $D + A \vdash -\partial q$.

□

Theorem 30

The logic $\mathbf{DL}(\delta)$ can be simulated by $\mathbf{DL}(\delta^*)$ with respect to addition of rules.

Proof

Let Σ be the language of $D+A$. Note that, for any $q \in \Sigma(D)$, $T(D)+A \vdash +\sigma^* \neg \text{true}(q)$ and, employing Lemma 18, $T(D)+A \vdash +\delta^* \neg \text{true}(q)$ iff $T(D)+A \vdash -\Delta q$ iff $D+A \vdash -\Delta q$. Also note that $T(D)+A \vdash -\delta^* g$, $T(D)+A \vdash +\sigma^* g$, and $T(D)+A \vdash +\sigma^* \neg g$, where g is the proposition used in part 7 of Definition 13.

Suppose $+\delta q \in \mathcal{T}_{D+A} \uparrow (n+1)$. Then either $+\Delta q \in \mathcal{T}_{D+A} \uparrow n$, or $-\Delta \sim q \in \mathcal{T}_{D+A} \uparrow n$ and there is a non-empty team of strict or defeasible rules for q such that $+\delta B_r \subseteq \mathcal{T}_{D+A} \uparrow n$ for each body B_r of each rule r and every rule s for $\sim q$ either has a body that fails in $\mathcal{T}_{D+A} \uparrow n$ or $s < t$ for some rule t in the team. Then, by the induction hypothesis, either $T(D) + A \vdash +\Delta q$ (in which case $T(D) + A \vdash +\delta^* q$), or $T(D) + A \vdash -\Delta \sim q$, $T(D) + A \vdash +\delta^* B_r$ for each rule r in the team, and for every rule s for $\sim q$ with body B_s either $T(D) + A \vdash -\sigma B_s$ or there is a rule t in the team and $t > s$. If $s \in A$ then $T(D) + A \vdash -\sigma B_s$. If $s \in D$ then either $T(D) + A \vdash +\delta^* \text{fail}(s)$ or $T(D) + A \vdash +\delta^* d(s, t)$; in either case, $T(D) + A \vdash +\delta^* d(s)$. Considering $T(D)$, and the inference rule for $+\delta^*$ we have $T(D) + A \vdash +\delta^* \text{one}(q)$. By Lemma 18, $T(D) + A \vdash +$

$\delta^* \neg \text{true}(\sim q)$. Hence, the body of $s(q)$ is proved. Because $>$ is acyclic, there is a rule in the team for q that is not inferior to any rule in the team for $\sim q$. Hence this rule r' is not defeated, so $d(r')$ fails, and hence the rule for $\sim q$ in $T(D)$ from point 6 fails. Similarly, $d_\sigma(r', s)$ fails, and hence the rules for $\sim q$ in $T(D)$ from point 7 fail. Hence all rules for $\sim q$ fail, with the possible exception of $o(\sim q)$. However $s(q) > o(\sim q)$ and hence, applying the $+\delta^*$ inference rule, $T(D) + A \vdash +\delta^* q$.

Suppose $-\delta q \in \mathcal{T}_{D+A} \uparrow (n+1)$. Then $-\Delta q \in \mathcal{T}_{D+A} \uparrow n$ (and hence $T(D) + A \vdash -\Delta q$) and either (1) $+\Delta \sim q \in \mathcal{T}_{D+A} \uparrow n$ (in which case $T(D) + A \vdash -\delta^* q$), or (2) every rule r for q contains a body literal p with $-\delta p \in \mathcal{T}_{D+A} \uparrow n$, or (3) there is a rule s for $\sim q$ with body B_s such that $+\sigma B_s \subseteq \mathcal{T}_{D+A} \uparrow n$ and, for every strict or defeasible rule t for q , either t fails in $\mathcal{T}_{D+A} \uparrow n$, or $t \not> s$. In case (2), the rules r in A for q fail and, by the induction hypothesis and the inference

rule for $-\delta^*$, the rules r in A for q fail in $T(D) + A$, so $T(D) + A \vdash -\delta^* \text{one}(q)$ and hence $T(D) + A \vdash -\delta^* q$. In case (3), by the induction hypothesis, there is a rule s for $\sim q$ with body B_s such that $T(D) + A \vdash +\sigma^* B_s$ and for every strict or defeasible rule t for q , either t fails in $T(D) + A$, or $t \not\prec s$. If $s \in A$ then $t \not\prec s$, for every t , and hence $T(D) + A \vdash -\delta^* q$. If $s \in D$ then $T(D) + A \vdash -\delta^* d(s, t)$ (since, via Lemma 18, we also have $T(D) + A \vdash -\delta^* \text{true}(q)$). Using the $-\delta^*$ inference rule, $T(D) + A \vdash -\delta^* d(s)$. The bodies of rules from point 7 of the transformation also fail (wrt δ^*), because of the presence of g . Hence $T(D) + A \vdash -\delta^* q$.

Suppose $q \in \Sigma$ and $+\delta^* q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$. Then either (1) $+\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D + A \vdash +\delta q$), or else $-\Delta \sim q \in \mathcal{T}_{T(D)+A} \uparrow n$ and either (2) there is a strict or defeasible rule r for q in A where $+\delta^* B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and for all rules for $\sim q$ in $T(D)+A$, the body of the rule contains a literal p with $-\sigma^* p \in \mathcal{T}_{T(D)+A} \uparrow n$, or (3) each of $+\delta^* \text{one}(q)$, $+\delta^* \neg \text{true}(\sim q)$, and $+\delta^* d(s)$ occurs in $\mathcal{T}_{T(D)+A} \uparrow n$, for each rule s for $\sim q$ in D .

Hence, in case (3), there is a strict or defeasible rule r for q with body B_r such that $+\delta^* B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and, for every rule s for $\sim q$, either $-\sigma^* p \in \mathcal{T}_{T(D)+A} \uparrow n$, for some p in the body B_s of s , or there exists t in D for q with $+\delta^* B_t \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and $t > s$. By the induction hypothesis, $D + A \vdash -\Delta \sim q$, $D + A \vdash +\delta B_r$, and, for every rule s for $\sim q$, either $D + A \vdash -\sigma B_s$ or $D + A \vdash +\delta B_t$ and $t > s$. By the inference rule for $+\delta$, $D + A \vdash +\delta q$.

In case (2), using the structure of $T(D)$, for the rules $\text{supp}(\sim q)$, originating from some rule s for $\sim q$ in D , either for some p in B_s , $-\sigma^* p \in \mathcal{T}_{T(D)+A} \uparrow n$ or, for some t , $-\sigma^* d_\sigma(t, s) \in \mathcal{T}_{T(D)+A} \uparrow n$ (and, hence, $+\delta^* B_t \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and $t > s$). Now, by the induction hypothesis, $D + A \vdash +\delta B_r$; $D + A \vdash -\Delta \sim q$; for all rules for $\sim q$ in A , the body of the rule contains a literal p with $D + A \vdash -\sigma^* p$; and for all rules for $\sim q$ in D , either the body of the rule contains a literal p with $D + A \vdash -\sigma^* p$ or there is a rule t for q in D with $t > s$ and $D + A \vdash +\delta B_t$. Applying the inference rule for $+\delta$, $D + A \vdash +\delta q$.

If $q \in \Sigma$ and $-\delta^* q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$ then, using the inference rule for $-\delta^*$ and the structure of $T(D)+A$, $-\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ (and, hence, $D + A \vdash -\Delta q$) and either $+\Delta \sim q \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D + A \vdash -\delta q$), or else for every rule r for q in A , there is a literal p in B_r such that $-\delta^* p \in \mathcal{T}_{T(D)+A} \uparrow n$ and either (1) $-\delta^* \neg \text{true}(\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D + A \vdash +\Delta \sim q$ and hence $D + A \vdash -\delta q$), or (2) $-\delta^* \text{one}(q) \in \mathcal{T}_{T(D)+A} \uparrow n$, or (3) $-\delta^* d(s) \in \mathcal{T}_{T(D)+A} \uparrow n$ for some rule s for $\sim q$ in D . Or (4) $+\sigma^* \text{one}(\sim q) \in \mathcal{T}_{T(D)+A} \uparrow n$ and $+\sigma^* d(r) \in \mathcal{T}_{T(D)+A} \uparrow n$ for every rule r for q in D , or (5) there is a rule s for $\sim q$ in A and $+\sigma^* B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$. Or (6) the body of a rule $\text{supp}(\sim q, s)$ is supported for some rule s for $\sim q$ (that is, $+\sigma^* B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and, for each rule r for q , $+\sigma^* d_\sigma(r, s) \in \mathcal{T}_{T(D)+A} \uparrow n$).

For (2) and (3), by the induction hypothesis, for every rule r for q in A , there is a literal p in B_r such that $D + A \vdash -\delta p$. If (2) $-\delta^* \text{one}(q) \in \mathcal{T}_{T(D)+A} \uparrow n$ then every strict or defeasible rule for q in D fails. Applying the induction hypothesis and the inference rule for $-\delta$, we have $D + A \vdash -\delta q$. If (3) $-\delta^* d(s) \in \mathcal{T}_{T(D)+A} \uparrow n$ for some rule s for $\sim q$ in D , then $+\sigma^* B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$, where B_s is the body of s , and for every strict or defeasible rule r for q with body B either $-\delta^* B_r \in \mathcal{T}_{T(D)+A} \uparrow n$ or $+\sigma^* B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$, where B_s is the body of s , and $r \not\prec s$. Applying the induction hypothesis, $D + A \vdash +\sigma B_s$ and, for every r for q , $D + A \vdash -\delta B_r$ or $D + A \vdash +\sigma B_s$ and $r \not\prec s$. Hence, by the inference rule for $-\delta$, $D + A \vdash -\delta q$.

If (4) then there is a rule s for $\sim q$ with body B_s such that $+\sigma^* B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and for every rule r for q in D either there is a literal p in the body of r such that $-\delta^* p \in \mathcal{T}_{T(D)+A} \uparrow n$ or there is a rule s' for $\sim q$ with body B' such that $+\sigma^* B' \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and $s' > r$. Applying the induction hypothesis, for every rule r for q in D either there is a literal p in the body of r

such that $D + A \vdash -\delta p$ or there is a rule s' for $\sim q$ with body B' such that $D + A \vdash +\sigma B'$ and $s' > r$. It follows, by the inference rule for $-\delta$, that $D + A \vdash -\delta q$.

If (5) then, by the induction hypothesis, $D + A \vdash +\sigma B_s$ and, since s is not inferior to any rule, the inference rule for $-\delta$ gives us $D + A \vdash -\delta q$.

In case (6), since $+\sigma^* d_\sigma(r, s) \in \mathcal{T}_{T(D)+A} \uparrow n$, we must have $+\sigma^* B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and either there is a literal p in B_r such that $-\delta^* p \in \mathcal{T}_{T(D)+A} \uparrow n$ or $r \not\prec s$. By the induction hypothesis, $D + A \vdash +\sigma^* B_s$ and, for every rule r for q in D either there is a literal p in B_r such that $D + A \vdash -\delta^* p$ or $r \not\prec s$. By the $-\delta$ inference rule, $D + A \vdash -\delta q$.

Suppose $+\sigma q \in \mathcal{T}_{D+A} \uparrow (n+1)$. Then either $+\Delta q \in \mathcal{T}_{D+A} \uparrow n$, or there is a strict or defeasible rule r for q such that $+\sigma B_r \subseteq \mathcal{T}_{D+A} \uparrow n$ where B_r is the body of r and every rule s for $\sim q$ has a body with a literal p such that $-\delta p \in \mathcal{T}_{D+A} \uparrow n$ or $s \not\prec r$. Then, by the induction hypothesis, either $T(D) + A \vdash +\Delta q$ (in which case $T(D) + A \vdash +\sigma^* q$), or $T(D) + A \vdash +\sigma^* B_r$, and every rule s for $\sim q$ has a body with a literal p such that $-\delta^* p \in \mathcal{T}_{D+A} \uparrow n$ or $s \not\prec r$. If $r \in A$ then r is not inferior to any rule and, by the inference rule for $+\sigma^*$, $T(D) + A \vdash +\sigma^* q$. If $r \in D$ then, by the $+\sigma^*$ inference rule, $T(D) + A \vdash +\sigma^* B_r$. Furthermore, again by the $+\sigma^*$ inference rule, for every s for $\sim q$ in D , $T(D) + A \vdash +\sigma^* d_\sigma(s, r)$, since $a(s, r) \not\prec b(s, r)$ iff $s \not\prec r$. Note that there is no superiority relation between the rules in $T(D)$ for q and $\sim q$. Hence, applying the inference rule for $+\sigma^*$, $T(D) + A \vdash +\sigma^* q$.

Suppose $-\sigma q \in \mathcal{T}_{D+A} \uparrow (n+1)$. Then $-\Delta q \in \mathcal{T}_{D+A} \uparrow n$ and either every rule r for q contains a body literal p and $-\sigma p \in \mathcal{T}_{D+A} \uparrow n$, or there is a rule s for $\sim q$ with body B_s such that $+\delta B_s \subseteq \mathcal{T}_{D+A} \uparrow n$ and $s > r$. (Note that, for $r \in A$, only the first possibility can apply.) Then, by the induction hypothesis, $T(D) + A \vdash -\Delta q$ and either every rule r for q contains a body literal p such that $T(D) + A \vdash -\sigma^* p$, or there is a rule s for $\sim q$ with body B_s such that $T(D) + A \vdash +\delta^* B_s$ and $s > r$. (In particular, every rule for q in A contains a body literal p with $T(D) + A \vdash -\sigma^* p$.) If all rules for q in D fall in the former case, we have $-\sigma^* \text{one}(q)$, and all rules $\text{supp}(q)$ fail. Otherwise, there is an s that is not inferior to any rule for q and hence $T(D) + A \vdash -\sigma^* d(s, r)$ and $T(D) + A \vdash -\sigma^* d(s)$. Similarly, $T(D) + A \vdash -\sigma^* d_\sigma(s, r)$. In either case, all rules for q fail, and hence $T(D) + A \vdash -\sigma^* q$.

Suppose $q \in \Sigma$ and $+\sigma^* q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$. Then either (1) $+\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D + A \vdash +\sigma q$), or else either (2) for some rule r for q in A , $+\sigma^* B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$, or (3) $+\sigma^* \text{one}(q) \in \mathcal{T}_{T(D)+A} \uparrow n$, and $+\sigma^* d(s)$ occurs in $\mathcal{T}_{T(D)+A} \uparrow n$, for each rule s for $\sim q$ in D , or (4) for some strict or defeasible rule r for q in D , $+\sigma^* B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and $+\sigma^* d_\sigma(s, r)$ occurs in $\mathcal{T}_{T(D)+A} \uparrow n$, for each rule s for $\sim q$ in D .

In case (2), by the induction hypothesis, $D + A \vdash +\sigma B_r$ and hence, by the inference rule for σ , $D + A \vdash +\sigma q$.

In case (3), there is a strict or defeasible rule r for q with body B_r such that $+\sigma^* B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and, for every rule s for $\sim q$ in D , either $-\delta^* p \in \mathcal{T}_{T(D)+A} \uparrow n$, for some p in the body B_s of s or there is a rule t for q with body B_t such that $+\sigma^* B_t \in \mathcal{T}_{T(D)+A} \uparrow n$, and $s \not\prec t$. By the induction hypothesis, $D + A \vdash +\sigma B_r$, and, for every rule s for $\sim q$, either $D + A \vdash -\delta B_s$ or there is a rule t for q with body B_t such that $D + A \vdash +\sigma^* B_t$ and $s \not\prec t$. Because ζ is acyclic, there is a rule t for q such that $D + A \vdash +\sigma B_t$ and, for every rule s for $\sim q$ either $D + A \vdash -\delta B_s$ or $s \not\prec t$. By the inference rule for $+\sigma$, $D + A \vdash +\sigma q$.

In case (4), there is a strict or defeasible rule r for q with body B_r such that $+\sigma^* B_r \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and, for each rule s for $\sim q$ in D , either $-\delta^* p \in \mathcal{T}_{T(D)+A} \uparrow n$, for some p in the body B_s of s , or $s \not\prec r$. By the induction hypothesis, $D + A \vdash +\sigma B_r$, and, for each s , either $D + A \vdash -\delta p$ or $s \not\prec r$. By the $+\sigma$ inference rule, $D + A \vdash +\sigma q$.

If $q \in \Sigma$ and $-\sigma^*q \in \mathcal{T}_{T(D)+A} \uparrow (n+1)$ then, using the inference rule for $-\sigma^*$ and the structure of $T(D)$, $-\Delta q \in \mathcal{T}_{T(D)+A} \uparrow n$ (and hence $D + A \vdash -\Delta q$), and either (1) $+\Delta \sim q \in \mathcal{T}_{T(D)+A} \uparrow n$ (in which case $D + A \vdash -\sigma^*q$), or (2) for each strict or defeasible rule r for q in A , there is a literal p in B_r such that $-\sigma^*p \in \mathcal{T}_{T(D)+A} \uparrow n$ and for each strict or defeasible rule r for q in D , either there is a literal p in B_r such that $-\sigma^*p \in \mathcal{T}_{T(D)+A} \uparrow n$, or there is a rule s for $\sim q$ in D such that $+\delta^*B_s \subseteq \mathcal{T}_{T(D)+A} \uparrow n$ and $s > r$. By the induction hypothesis, in case (2), $D + A \vdash -\sigma B_r$ for the rules r in A and, for rules r in D , either $D + A \vdash -\sigma B_r$ or there is a rule s for $\sim q$ in D such that $D + A \vdash +\delta^*B_s$ and $s > r$. Applying the $-\sigma$ inference rule, $D + A \vdash -\sigma q$. \square

Combining Theorems 29 and 30, we have Theorem 14.