ON CASCADE PRODUCTS OF ANSWER SET PROGRAMS

CHRISTIAN ANTIĆ

ABSTRACT. Describing complex objects by elementary ones is a common strategy in mathematics and science in general. In their seminal 1965 paper, Kenneth Krohn and John Rhodes showed that every finite deterministic automaton can be represented (or "emulated") by a cascade product of very simple automata. This led to an elegant algebraic theory of automata based on finite semigroups (Krohn-Rhodes Theory). Surprisingly, by relating logic programs and automata, we can show in this paper that the Krohn-Rhodes Theory is applicable in Answer Set Programming (ASP). More precisely, we recast the concept of a cascade product to ASP, and prove that every program can be represented by a product of very simple programs, the reset and standard programs. Roughly, this implies that the reset and standard programs are the basic building blocks of ASP with respect to the cascade product. In a broader sense, this paper is a first step towards an algebraic theory of products and networks of nonmonotonic reasoning systems based on Krohn-Rhodes Theory, aiming at important open issues in ASP and AI in general.

1. Introduction

Describing complex objects by elementary ones is a common strategy in mathematics and science in general. For instance, the fundamental theorem of number theory states that every natural number can be (uniquely) represented by its prime factors. Similarly, in their seminal 1965 paper "Algebraic theory of machines, I. Prime decomposition theorem for finite semigroups and machines", Kenneth Krohn and John Rhodes showed that every finite deterministic automaton can be represented (or "emulated") by a cascade product of very simple automata. This led to an elegant algebraic theory of automata based on finite semigroups (Krohn-Rhodes Theory) and, more recently, to an algebraic theory of networks of automata (cf. Dömösi & Nehaniv, 2005).

Answer Set Programming (ASP) (Gelfond & Lifschitz, 1991), on the other hand, has become a prominent knowledge representation and reasoning (KR&R) formalism over the last two decades, with a wide range of applications in AI-related subfields such as, e.g., nonmonotonic reasoning, diagnosis, and planning (cf. Brewka, Eiter, & Truszczynski, 2011b).

In this paper, we aim at combining these two vivid areas of research and will show that, surprisingly, the Krohn-Rhodes Theory is applicable in ASP. More precisely, we recast the concept of a cascade product to ASP, and prove that every program can be represented by a product of reset programs $R = \{1 \leftarrow \text{not } 1\}$ and n-standard programs S_n consisting only of rules of the simple form $i \leftarrow j$, not k (cf. Theorem 11). Roughly, this implies that the reset and standard programs are the basic building blocks of ASP with respect to the cascade product and, strikingly, while the reset and standard programs do not possess any interesting declarative meaning (the reset program is inconsistent and the standard programs have only the empty answer set), their interaction can "emulate" any given program. In other words, the

1

product semantics *emerges* from the interplay of its (simple) factors and allows for arbitrary complex behavior.

To the best of our knowledge, this is the first paper applying the Krohn-Rhodes Theory to logic programming. In a broader sense, it is a first step towards an algebraic theory of products and networks of nonmonotonic reasoning systems based on Krohn-Rhodes Theory, with far-reaching potential application areas including some important open issues in ASP and AI in general (cf. the discussion in Section 6).

The rest of the paper is structured as follows. In Section 2, we present the basic definitions and results concerning ASP and automata. In Section 3, we introduce the concept of a programmable automaton, and show that the distinguished reset and standard automata are programmable in this sense. In Section 4, the main part of this paper, we recast the concept of a cascade product to ASP and prove that every program can be (homomorphically) represented by reset and standard programs. In Section 5, we study the more restricted type of isomorphic representation and provide a complete class of programs with respect to it; moreover, we show that positive tight programs are isomorphically representable by reset programs. Finally, in Section 6, we conclude with a discussion on interesting lines for future research.

2. Preliminaries

We assume that the reader is familiar with the concept of a partially ordered set and that of a (complete) lattice. Following (Gécseg, 1986), we denote by [n], $n \ge 0$, the set $\{1, \ldots, n\}$. We denote, for $k \ge 1$ and $i \ge 0$, the least residue of i modulo n by i mod n. For a set X, we denote by |X| the cardinality of X. Given a function $f: X \times Y \to Z$, we denote by $f(\cdot, y)$ the function from X into Z mapping each $x \in X$ to $f(x,y) \in Z$, and we denote by $f(\cdot,y)$ the least fixpoint of $f(\cdot,y)$. We denote the power set of X by $\mathfrak{P}(X)$.

2.1. **Answer Set Programs.** We briefly recall the syntax and answer set semantics (Gelfond & Lifschitz, 1991) of nonmonotonic logic programs in an operator-based setting (cf. Denecker, Marek, & Truszczyński, 2000).

Syntax. In the sequel, Γ will denote a finite nonempty set of propositional atoms. A (normal logic) program P over some Γ_P is a finite nonempty set of rules of the form

(1)
$$a \leftarrow b_1, \dots, b_k, \text{ not } b_{k+1}, \dots, \text{ not } b_m, \quad m \ge k \ge 0,$$

where $a, b_1, \ldots, b_m \in \Gamma_P$ and not denotes negation-as-failure. For convenience, we define for a rule r of the form ((1)), H(r) = a, $B^+(r) = \{b_1, \ldots, b_k\}$, $B^-(r) = \{b_{k+1}, \ldots, b_m\}$, and $B(r) = B^+(r) \cup B^-(r)$. We call r a fact, if $B(r) = \emptyset$; and we call r positive if $B^-(r) = \emptyset$. We say that P is positive if every rule $r \in P$ is positive, and we call P tight if there is a mapping ℓ from Γ_P into the nonnegative integers such that for each rule r in P, $\ell(H(r)) > \ell(b)$ for every $b \in B^+(r)$.

Semantics. An interpretation of P is any subset $I \subseteq \Gamma_P$ and we denote the set of all interpretations of P by $\mathcal{I}_P = \mathfrak{P}(\Gamma_P)$. Define the 4-valued immediate consequence operator $\Psi_P : \mathcal{I}_P \times \mathcal{I}_P \to \mathcal{I}_P$ by

$$\Psi_P(I, J) = \{ H(r) : r \in P, B^+(r) \subseteq I, B^-(r) \cap J = \emptyset \}.$$

Intuitively, $\Psi_P(I, J)$ contains the heads H(r) of all rules r in P where the positive part of the body evaluates to true in I, and the negative part evaluates to true in J. Given some $I \in \mathcal{I}_P$, it is well-known that $\Psi_P(., I)$ is monotone on the complete lattice \mathcal{I}_P ordered by \subseteq ,

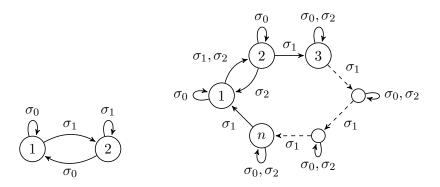


FIGURE 1. The (two-state) reset automaton \mathfrak{R} and the *n*-standard automaton \mathfrak{S}_n .

and hence has a *least fixpoint* denoted by lfp $\Psi_P(.,I)$. We say that $I \in \mathcal{I}_P$ is an answer set of P, or a Ψ_P -answer set, if $I = \text{lfp } \Psi_P(.,I)$.

2.2. **Krohn-Rhodes Theory.** In this section, we recall some basic definitions and results of Krohn-Rhodes Theory by mainly following the lines of (Gécseg, 1986, Chapters 1–3).

An automaton $\mathfrak{A} = (Q, \Sigma, \delta)$ consists of a finite set Q of states, a finite nonempty set Σ , called the *input alphabet*, and a mapping $\delta : Q \times \Sigma \to Q$ called the *transition function*.

Given two automata $\mathfrak{A} = (Q, \Sigma, \delta)$ and $\mathfrak{A}' = (Q', \Sigma', \delta')$, we say that \mathfrak{A}' is a subautomaton of \mathfrak{A} if $Q' \subseteq Q$, $\Sigma' \subseteq \Sigma$, and δ' is the restriction of δ to $Q' \times \Sigma'$. A pair $h = (h_1, h_2)$ of surjective mappings $h_1 : Q \to Q'$, $h_2 : \Sigma \to \Sigma'$ is a homomorphism of \mathfrak{A} onto \mathfrak{A}' if $h_1(\delta(q, x)) = \delta'(h_1(q), h_2(x))$, for every $q \in Q, x \in \Sigma$. The pair h is an isomorphism if h_1 and h_2 are bijective homomorphisms, and we say that \mathfrak{A} is isomorphic to \mathfrak{A}' if there exists an isomorphism h of \mathfrak{A} onto \mathfrak{A}' . If $\Sigma = \Sigma'$, then we omit h_2 and define $h = h_1$.

An equivalence relation \sim on Q is a congruence relation of \mathfrak{A} if $q \sim q'$ implies $\delta(q,x) \sim \delta(q',x)$, for all $q,q' \in Q$ and $x \in \Sigma$. We denote the congruence class of $q \in Q$ with respect to \sim by $q/_{\sim}$, and define the quotient automaton $\mathfrak{A}/_{\sim} = (Q/_{\sim}, \Sigma, \delta/_{\sim})$ by $\delta/_{\sim}(q/_{\sim}, x) = \delta(q, x)/_{\sim}$ for all $q \in Q$ and $x \in \Sigma$. Conversely, given a homomorphism $h = (h_1, h_2)$ of \mathfrak{A} onto \mathfrak{A}' , we mean by the congruence relation of \mathfrak{A} induced by h the binary relation \sim on Q given by $q \sim q'$ if $h_1(q) = h_1(q')$.

The following automata will play a central role throughout the rest of the paper (cf. Figure 1):

- (1) Define the (two-state) reset automaton $\mathfrak{R} = ([2], \{\sigma_0, \sigma_1\}, \delta_{\mathfrak{R}})$ by $\delta_{\mathfrak{R}}(i, \sigma_0) = 1$, and $\delta_{\mathfrak{R}}(i, \sigma_1) = 2$, for all $i \in [2]$.
- (2) We call an automaton $\mathfrak{S} = ([n], \{\sigma_0, \sigma_1, \sigma_2\}, \delta_{\mathfrak{S}}), n > 1$, standard if $\delta_{\mathfrak{S}}$ satisfies the following conditions, for all $i \in [n]$:
 - (a) $\delta_{\mathfrak{S}}(i, \sigma_0) = i$;
 - (b) $\delta_{\mathfrak{S}}(i, \sigma_1) = (i \mod n) + 1;$

(c)
$$\delta_{\mathfrak{S}}(i, \sigma_2) = \begin{cases} 2 & \text{if } i = 1, \\ 1 & \text{if } i = 2, \\ i & \text{otherwise.} \end{cases}$$

We denote the *n*-state standard automaton by \mathfrak{S}_n .

The following operators on arbitrary classes \mathcal{A} of automata will be useful:

- (1) S(A) denotes the set of subautomata of automata from A;
- (2) $\mathbf{H}(\mathcal{A})$ denotes the homomorphic images of automata from \mathcal{A} ;
- (3) $\mathbf{I}(\mathcal{A})$ denotes the isomorphic images of automata from \mathcal{A} .

We will write $\mathbf{XY}(\mathcal{A})$ for $\mathbf{X}(\mathbf{Y}(\mathcal{A}))$, where \mathbf{X} and \mathbf{Y} are operators from above.

We now define the cascade product for automata, which is also known as the wreath (Krohn & Rhodes, 1965) or α_0 -product (Gécseg, 1986) in the literature.

Definition 1 (Cascade Automata Product). For some k > 0, let $\mathfrak{A}_i = (Q_i, \Sigma_i, \delta_i)$, $i \in [k]$, be a family of automata, and let Σ be an alphabet. A feedforward function for $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ is a mapping $\psi : (Q_1 \times \ldots \times Q_k) \times \Sigma \to \Sigma_1 \times \ldots \times \Sigma_k$ with

$$\psi((q_1,\ldots,q_k),\sigma)=(\psi_1((q_1,\ldots,q_k),\sigma),\ldots,\psi_k((q_1,\ldots,q_k),\sigma))$$

where the component feedforward function ψ_i , $i \in [k]$, is a mapping from $(Q_1 \times \ldots \times Q_k) \times \Sigma$ into Σ_i . In the sequel, we omit those arguments q_j , $j \in [k]$, ψ_i does not depend on. The cascade (or loop-free) automata product of $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ with respect to $\Sigma_{\mathfrak{A}} = \Sigma$ and some feedforward function $\psi_{\mathfrak{A}}$

$$\mathfrak{A} = (Q_{\mathfrak{A}}, \Sigma_{\mathfrak{A}}, \delta_{\mathfrak{A}}) = \mathfrak{A}_1 \ltimes \ldots \ltimes \mathfrak{A}_k \left[\Sigma_{\mathfrak{A}}, \psi_{\mathfrak{A}} \right]$$

is given by $Q_{\mathfrak{A}} = Q_1 \times \ldots \times Q_k$ where ψ_i , $i \in [k]$, is independent of its j^{th} component, $j \in [k]$, whenever $j \geq i$. Finally, we define the transition function $\delta_{\mathfrak{A}} : Q_{\mathfrak{A}} \times \Sigma_{\mathfrak{A}} \to Q_{\mathfrak{A}}$ by

$$\delta_{\mathfrak{A}}((q_1,\ldots,q_k),\sigma)=(\delta_1(q_1,\psi_1(\sigma)),\ldots,\delta_k(q_k,\psi_k((q_1,\ldots,q_{k-1}),\sigma))).$$

Definition 2. We say that an automaton \mathfrak{A} homomorphically (resp., isomorphically) represents an automaton \mathfrak{A}' if $\mathfrak{A}' \in \mathbf{HS}(\{\mathfrak{A}\})$ (resp., $\mathfrak{A}' \in \mathbf{IS}(\{\mathfrak{A}\})$). Moreover, we say that a class \mathcal{A} of automata is homomorphically (resp., isomorphically) complete with respect to the cascade automata product if every automaton \mathfrak{A} can be homomorphically (resp., isomorphically) represented by a cascade automata product of automata from \mathcal{A} .

The following result is a consequence of the Krohn-Rhodes decomposition theorem (Krohn & Rhodes, 1965), and it will be of great importance for our main Theorem 11.

Theorem 3. [cf. (Gécseg, 1986), Theorem 2.1.5] Let $\mathfrak A$ be an automaton with n > 1 states. Then, $\mathfrak A$ can be homomorphically represented by a cascade automata product of reset and n-state standard automata over the same input alphabet as $\mathfrak A$.

We now turn to isomorphic completeness. Let $\mathfrak{T}_n = ([n], \Sigma_n, \delta_n), n \ge 1$, such that Σ_n is the set of all mappings $\sigma : [n] \to [n]$, and $\delta_n(j, \sigma) = \sigma(j)$, for all $j \in [n]$.

Theorem 4. [cf. (Gécseg, 1986), Theorem 3.2.1] A class A of automata is isomorphically complete with respect to the cascade automata product iff for every $n \ge 1$, there exists some $\mathfrak{A} \in A$ such that \mathfrak{T}_n can be embedded into a cascade automata product $\mathfrak{A}[\Sigma, \psi]$, consisting of a single factor.

3. Programmable Automata

In this section, we relate programs and automata and prove in Theorem 8 that the distinguished automata given in Section 2.2 can be "realized" by programs. This connection will serve as the basis for the rest of the paper, and for the main Theorem 11 in particular.

Given some program P, we define its characteristic automaton $\mathfrak{A}_P = (Q_P, \Sigma_P, \delta_P)$ by $Q_P = \Sigma_P = \mathcal{I}_P$ and $\delta_P = \Psi_P$. In the sequel, we will not distinguish between the operator Ψ_P and the characteristic automaton $\mathfrak{A}_P = (\mathcal{I}_P, \mathcal{I}_P, \Psi_P)$, i.e., we will refer to \mathfrak{A}_P simply by Ψ_P and will call Ψ_P the characteristic automaton of P (cf. Figure 2).

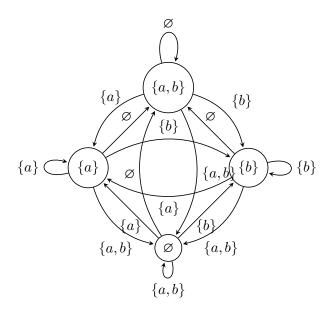


FIGURE 2. The characteristic automaton Ψ_B of the program $B = \{a \leftarrow \text{not } b; b \leftarrow \text{not } a\}$.

Definition 5. We say that an automaton \mathfrak{A} is homomorphically (resp., isomorphically) programmable if there exists some program P such that Ψ_P homomorphically (resp., isomorphically) represents \mathfrak{A} , that is, $\mathfrak{A} \in \mathbf{HS}(\{\Psi_P\})$ (resp., $\mathfrak{A} \in \mathbf{IS}(\{\Psi_P\})$). We then say that P homomorphically (resp., isomorphically) programs \mathfrak{A} .

We illustrate this concept with an example; in Theorem 8 we will see that the reset automaton \Re and the *n*-state standard automaton \mathfrak{S}_n , n > 1, are isomorphically programmable.

Example 6. Define the elevator automaton $\mathfrak{E} = ([2], \{\sigma_0, \sigma_1\}, \delta_{\mathfrak{E}})$ by $\delta_{\mathfrak{E}}(1, \sigma_0) = 1$, $\delta_{\mathfrak{E}}(1, \sigma_1) = 2$, and $\delta_{\mathfrak{E}}(2, \sigma_0) = \delta_{\mathfrak{E}}(2, \sigma_1) = 2$ (cf. Dömösi & Nehaniv, 2005, p.45). On the other hand, define the elevator program E by $E = \{e \leftarrow e; e \leftarrow \text{not } e\}$. Then, $h = (h_1, h_2)$ defined by $h_1(\emptyset) = 1$, $h_1(\{e\}) = 2$, $h_2(\{e\}) = \sigma_0$, and $h_2(\emptyset) = \sigma_1$ is an isomorphism of (the automaton) Ψ_E onto \mathfrak{E} ; hence, E isomorphically programs \mathfrak{E} .

For convenience, in the sequel we occasionally denote atoms by nonnegative integers.

Definition 7. The reset program R over $\Gamma_R = [1]$ consists of the following single rule:

$$1 \leftarrow \text{not } 1$$
.

The *n*-standard program (or *n*-program) S_n over $\Gamma_n = [n] \cup \{3\}$, n > 1, consists of the following rules, for all $i \in [n]$ and $j \in [n]$, j > 2:

$$i \leftarrow i, \text{ not } 1,$$
 $1 \leftarrow 2, \text{ not } 3,$ $(i \mod n) + 1 \leftarrow i, \text{ not } 2,$ $2 \leftarrow 1, \text{ not } 3,$ $j \leftarrow j, \text{ not } 3.$

Note that the reset program R is *inconsistent*, i.e., has no Ψ_R -answer sets, and for every n > 1, the n-program S_n has the Ψ_{S_n} -answer set \varnothing .

Theorem 8. The reset program R and the n-standard program S_n isomorphically program the reset automaton \mathfrak{R} and the n-state standard automaton \mathfrak{S}_n , n > 1, respectively.

Proof. Define $h_{R,1}: \mathcal{I}_R \to [2]$ and $h_{R,2}: \mathcal{I}_R \to \{\sigma_0, \sigma_1\}$ by $h_{R,1}(\emptyset) = 1$, $h_{R,1}(\{1\}) = 2$, $h_{R,2}(\emptyset) = \sigma_1$, and $h_{R,2}(\{1\}) = \sigma_0$. A straightforward computation shows that $h_R = (h_{R,1}, h_{R,2})$ is an isomorphism of Ψ_R onto \mathfrak{R} ; i.e., we have

$$h_{R,1}(\Psi_R(I,J)) = \delta_{\Re}(h_{R,1}(I), h_{R,2}(J)), \text{ for all } I, J \in \mathcal{I}_R.$$

Hence, $\mathfrak{R} \in \mathbf{IS}(\{\Psi_R\})$.

For the second part, let $\Psi'_{S_n} = (\mathcal{I}'_{S_n}, \mathcal{I}''_{S_n}, \Psi'_{S_n})$ be the subautomaton of Ψ_{S_n} given by $\mathcal{I}'_{S_n} = \{\{i\} : i \in [n]\} \subseteq \mathcal{I}_{S_n}, \mathcal{I}''_{S_n} = \{\{2,3\},\{1,3\},\{1,2\}\} \subseteq \mathcal{I}_{S_n}, \text{ and } \Psi'_{S_n} \text{ equals } \Psi_{S_n} \text{ restricted to } \mathcal{I}'_{S_n} \times \mathcal{I}''_{S_n}.$ Define $h_{S_n,1} : \mathcal{I}'_{S_n} \to [n]$ by $h_{S_n,1}(\{i\}) = i$, for all $i \in [n]$; and $h_{S_n,2} : \mathcal{I}''_{S_n} \to \{\sigma_0, \sigma_1, \sigma_2\}$ by $h_{S_n,2}(\{2,3\}) = \sigma_0, h_{S_n,2}(\{1,3\}) = \sigma_1, \text{ and } h_{S_n,2}(\{1,2\}) = \sigma_2.$ Then, $h = (h_{S_n,1}, h_{S_n,2})$ is an isomorphism of Ψ'_{S_n} onto \mathfrak{S}_n ; i.e., we have

$$h_{S_n,1}(\Psi_{S_n}'(\{i\},J)) = \delta_{\mathfrak{S}_n}(h_{S_n,1}(\{i\}),h_{S_n,2}(J)), \quad \text{for all } i \in [n] \text{ and } J \in \mathcal{I}_{S_n}''.$$
Hence, $\mathfrak{S}_n \in \mathbf{IS}(\{\Psi_{S_n}\}).$

4. Cascade Products and Homomorphic Representations

In this section, we recast the concept of a cascade automata product presented in Section 2.2 (cf. Definition 1) to the setting of ASP and study homomorphic representations.

Definition 9 (Cascade Program Product). Let $P_1, \ldots, P_k, k > 1$, be a family of programs over some alphabets $\Gamma_{P_1}, \ldots, \Gamma_{P_k}$, respectively, and let $\mathcal{I}_{\mathbf{P}}$ be some finite nonempty set. A feedforward function for P_1, \ldots, P_k is a mapping $\psi_{\mathbf{P}} : (\mathcal{I}_{P_1} \times \ldots \times \mathcal{I}_{P_k}) \times \mathcal{I}_{\mathbf{P}} \to \mathcal{I}_{P_1} \times \ldots \times \mathcal{I}_{P_k}$ with

$$\psi_{\mathbf{P}}((I_1,\ldots,I_k),\mathbf{J}) = (\psi_{\mathbf{P},1}((I_1,\ldots,I_k),\mathbf{J}),\ldots,\psi_{\mathbf{P},k}((I_1,\ldots,I_k),\mathbf{J}))$$

where the component feedforward function $\psi_{\mathbf{P},i}$, $i \in [k]$, is a mapping from $(\mathcal{I}_{P_1} \times \ldots \times \mathcal{I}_{P_k}) \times \mathcal{I}_{\mathbf{P}}$ into \mathcal{I}_{P_i} . In the sequel, we omit those arguments I_j , $j \in [k]$, $\psi_{\mathbf{P},i}$ does not depend on. The (cascade or loop-free program) product of P_1, \ldots, P_k with respect to $\mathcal{I}_{\mathbf{P}}$ and some feedforward function $\psi_{\mathbf{P}}$

$$\mathbf{P} = P_1 \ltimes \ldots \ltimes P_k \left[\mathcal{I}_{\mathbf{P}}, \psi_{\mathbf{P}} \right]$$

is given by its component feedforward functions $\psi_{\mathbf{P},i}$, $i \in [k]$, which are independent of their j^{th} component, $j \in [k]$, whenever $j \geq i$. Finally, we define the *characteristic automaton* $\Psi_{\mathbf{P}} = (Q_{\mathbf{P}}, \Sigma_{\mathbf{P}}, \Psi_{\mathbf{P}})$ of \mathbf{P} by $Q_{\mathbf{P}} = \mathcal{I}_{P_1} \times \ldots \times \mathcal{I}_{P_k}$, $\Sigma_{\mathbf{P}} = \mathcal{I}_{\mathbf{P}}$, and $\Psi_{\mathbf{P}} : (\mathcal{I}_{P_1} \times \ldots \times \mathcal{I}_{P_k}) \times \mathcal{I}_{\mathbf{P}} \to \mathcal{I}_{P_1} \times \ldots \times \mathcal{I}_{P_k}$ with

$$\Psi_{\mathbf{P}}((I_1,\ldots,I_k),\mathbf{J}) = (\Psi_{P_1}(I_1,\psi_{\mathbf{P},1}(\mathbf{J})),\ldots,\Psi_{P_k}(I_k,\psi_{\mathbf{P},k}((I_1,\ldots,I_{k-1}),\mathbf{J}))).$$

Intuitively, a cascade program product is a collection of programs which are connected to each other and exchange (local) information via a feedforward function, where each component program may depend only on the preceding components and on the global input; every state-transition of the characteristic automaton of the product is then the result of the *simultaneous local* state-transitions of the characteristic automata of its component programs.

Formally, a product is not a program according to the definition given in Section 2.1. However, we can relate products and programs as follows (cf. Definition 2).

Definition 10. We say that a cascade program product \mathbf{P} homomorphically (resp., isomorphically) represents a program P if $\Psi_{\mathbf{P}}$ homomorphically (resp., isomorphically) represents Ψ_P , that is, $\Psi_P \in \mathbf{HS}(\{\Psi_{\mathbf{P}}\})$ (resp., $\Psi_P \in \mathbf{IS}(\{\Psi_{\mathbf{P}}\})$). Moreover, we say that a class \mathcal{P} of programs is homomorphically (resp., isomorphically) complete with respect to the cascade program product if every program P can be homomorphically (resp., isomorphically) represented by a cascade program product of programs from \mathcal{P} .

We now make the relation between products and programs more explicit. In the context of logic programming, representation (or "emulation") means semantic equivalence (modulo some encoding). According to Definition 10, a product $\mathbf{P} = P_1 \ltimes \ldots \ltimes P_k [\mathcal{I}_{\mathbf{P}}, \psi_{\mathbf{P}}], k > 1$, represents a program P if the characteristic automaton $\Psi_{\mathbf{P}}$ represents the characteristic automaton Ψ_P (in the sense of Section 2.2); that is, if there exists a subautomaton $\Psi'_{\mathbf{P}} = (\mathcal{I}'_{P_1} \times \ldots \times \mathcal{I}'_{P_k}, \mathcal{I}'_{\mathbf{P}}, \Psi'_{\mathbf{P}})$ of $\Psi_{\mathbf{P}}$ and a congruence relation \sim on $\mathcal{I}'_{P_1} \times \ldots \times \mathcal{I}'_{P_k}$ such that $\Psi'_{\mathbf{P}}/_{\sim}$ is isomorphic to Ψ_P . Intuitively, every interpretation $I \in \mathcal{I}_P$ of P then corresponds to a congruence class of k-tuples from $\mathcal{I}'_{P_1} \times \ldots \times \mathcal{I}'_{P_k}$; if the representation is isomorphic, then I can be identified with a single k-tuple (I'_1, \ldots, I'_k) and in this case we can imagine (I'_1, \ldots, I'_k) to be an "encoding" of I.

Interestingly enough, by the forthcoming Theorem 11, we can assume that only reset and standard programs occur as factors in the product \mathbf{P} . That is, Theorem 11 roughly implies that by knowing the reset program R and all the n-programs S_n , n > 1, and by knowing how to form the cascade program product, we essentially know all programs; viz., the reset and standard programs are the basic building blocks of ASP with respect to the cascade program product.

We are now ready to state the main theorem of this paper.

Theorem 11. Every program P over some alphabet Γ_P , with $|\Gamma_P| = m$, can be homomorphically represented by a cascade program product \mathbf{P} of reset and 2^m -standard programs.

Proof. According to Definition 10, we have to show that there exists some product \mathbf{P} such that $\Psi_{\mathbf{P}}$ homomorphically represents Ψ_{P} . Since Ψ_{P} has 2^{m} states, Theorem 3 yields a cascade automata product $\mathfrak{A}_{P} = \mathfrak{A}_{1} \times \ldots \times \mathfrak{A}_{k} [\mathcal{I}_{P}, \psi_{P}]$, for some k > 0, consisting of reset and 2^{m} -standard automata homomorphically representing Ψ_{P} . Note that \mathfrak{A}_{P} has the same input alphabet \mathcal{I}_{P} as Ψ_{P} . Define the product $\mathbf{P} = P_{1} \times \ldots \times P_{k} [\mathcal{I}_{\mathbf{P}}, \psi_{\mathbf{P}}]$ as follows: (i) for every $i \in [k]$, if \mathfrak{A}_{i} is the reset automaton \mathfrak{R} (resp., 2^{m} -standard automaton $\mathfrak{S}_{2^{m}}$), then P_{i} is the reset program R (resp., 2^{m} -standard program $S_{2^{m}}$); (ii) $\mathcal{I}_{\mathbf{P}}$ is the input alphabet \mathcal{I}_{P} of \mathfrak{A}_{P} and Ψ_{P} ; (iii) $\psi_{\mathbf{P}}$ is a mapping from $(\mathcal{I}_{P_{1}} \times \ldots \times \mathcal{I}_{P_{k}}) \times \mathcal{I}_{P}$ into $\mathcal{I}_{P_{1}} \times \ldots \times \mathcal{I}_{P_{k}}$ where $\mathcal{I}_{P_{i}}$, $i \in [k]$, is \mathcal{I}_{R} (resp., $\mathcal{I}_{2^{m}}$) if P_{i} is the reset program R (resp., 2^{m} -standard program $S_{2^{m}}$), and $\psi_{\mathbf{P},i}$ coincides with $\psi_{P,i}$ on the appropriate subset of $\mathcal{I}_{P_{1}} \times \ldots \times \mathcal{I}_{P_{k}}$ modulo the isomorphisms defined in the proof of Theorem 8. Then, it follows from Theorem 8 that $\Psi_{\mathbf{P}}$ isomorphically represents \mathfrak{A}_{P} and, by transitivity of representation, it homomorphically represents Ψ_{P} , which proves our theorem.

It is worth noting that the proof of Theorem 11 yields a product \mathbf{P} whose characteristic automaton $\Psi_{\mathbf{P}}$ has the same input alphabet \mathcal{I}_P as the characteristic automaton Ψ_P of P. Therefore, we can characterize the answer sets of P by $\Psi_{\mathbf{P}}$ as follows. Roughly, the product semantics of \mathbf{P} emerges as an interaction of its (simple) factors P_1, \ldots, P_k with respect to P. More precisely, by the remarks given above, there exists a quotient subautomaton $\Psi'_{\mathbf{P}}/\sim$ of $\Psi_{\mathbf{P}}$ which is isomorphic to Ψ_P and which has the same input alphabet as Ψ_P . Let $h: \mathcal{I}'_{P_1} \times \ldots \times \mathcal{I}'_{P_k} \to \mathcal{I}_P$ be the corresponding homomorphism of $\Psi'_{\mathbf{P}}$ onto Ψ_P inducing \sim ; we order

 $(\mathcal{I}'_{P_1} \times \ldots \times \mathcal{I}'_{P_k})/_{\sim}$ by $(I_1, \ldots, I_k)/_{\sim} \subseteq_h (I'_1, \ldots, I'_k)/_{\sim}$ if $h(I_1, \ldots, I_k) \subseteq h(I'_1, \ldots, I'_k)$. Then, $((\mathcal{I}'_{P_1} \times \ldots \times \mathcal{I}'_{P_k})/_{\sim}, \subseteq_h)$ is isomorphic (as a lattice) to $(\mathcal{I}_P, \subseteq)$, and we say that $I \in \mathcal{I}_P$ is a $\Psi'_{\mathbf{P}}/_{\sim}$ -answer set if $I = h(\operatorname{lfp} \Psi'_{\mathbf{P}}/_{\sim}(\ldots, I))$. Then, we have the following correspondence:

(2) I is a
$$\Psi_P$$
-answer set $\Leftrightarrow I$ is a $\Psi'_P/_{\sim}$ -answer set.

By Theorem 11, we can assume that in the right hand side of (2), only reset and 2^m -standard programs occur.

We illustrate these concepts by giving some examples.

Example 12. Let $A = \{a \leftarrow\}$ be a program consisting of a single fact. We can interpret A as a database *storing* some information represented by a. Observe that neither the reset program R nor the 2-program S_2 contains a *fact*. However, we verify that

$$\mathbf{A} = R\left[\mathcal{I}_A, \psi_{\mathbf{A}}\right] = \left\{1 \leftarrow \text{not } 1\right\} \left[\mathcal{I}_A, \psi_{\mathbf{A}}\right]$$

defined by $\psi_{\mathbf{A}}(J) = \emptyset$, for all $J \in \mathcal{I}_A$, isomorphically represents A. Define $h : \mathcal{I}_R \to \mathcal{I}_A$ by $h(\emptyset) = \emptyset$ and $h(\{1\}) = \{a\}$. We check that h is an isomorphism:

$$h(\Psi_{\mathbf{A}}(I,J)) = h(\Psi_{R}(I,\psi_{\mathbf{A}}(J))$$

$$= h(\Psi_{R}(I,\varnothing))$$

$$= h(\{1\})$$

$$= \{a\}$$

$$= \Psi_{A}(h(I),J)$$

holds for all $I \in \mathcal{I}_R$ and $J \in \mathcal{I}_A$. Therefore, the congruence relation \sim induced by h is the trivial diagonal relation and $\Psi_{\mathbf{A}}/_{\sim}$ is isomorphic to $\Psi_{\mathbf{A}}$. Hence, $\Psi_A \in \mathbf{IS}(\{\Psi_{\mathbf{A}}\})$. The calculation above proves that $\{a\}$ is the only Ψ_A -answer set or, equivalently, the only Ψ_A -answer set. Intuitively, \mathbf{A} "emulates" the storage of the fact a by ignoring the input J appropriately. Generally, the program $A_m = \{a_1 \leftarrow; \ldots; a_m \leftarrow\}$, $m \geq 1$, is isomorphically represented by $\mathbf{A}_m = R \ltimes \ldots \ltimes R \left[\mathcal{I}_{A_m}, \psi_{\mathbf{A}_m}\right]$ (with m factors) where $\psi_{\mathbf{A}_m,i}((I_1,\ldots,I_{i-1}),J) = \emptyset$, for all $i \in [m]$, $I_1,\ldots,I_{i-1} \in \mathcal{I}_R$, and $J \in \mathcal{I}_{A_m}$. Here, an isomorphism is an arbitrary "binary encoding" h of \mathcal{I}_{A_m} ; e.g., $h(I_1,\ldots,I_m) = \{a_i \in \mathcal{I}_{A_m}: I_i = \{1\}, i \in [m]\}$.

Example 13. The program $B = \{a \leftarrow \text{not } b; b \leftarrow \text{not } a\}$ (cf. Figure 2) is isomorphically represented by the cascade program product

$$\mathbf{B} = R \ltimes R\left[\mathcal{I}_B, \psi_{\mathbf{B}}\right] = \left\{1 \leftarrow \text{not } 1\right\} \ltimes \left\{1 \leftarrow \text{not } 1\right\} \left[\mathcal{I}_B, \psi_{\mathbf{B}}\right]$$

defined by

$$\psi_{\mathbf{B},1}(\varnothing) = \psi_{\mathbf{B},1}(\{a\}) = \varnothing, \qquad \psi_{\mathbf{B},2}(I,\varnothing) = \psi_{\mathbf{B},2}(I,\{b\}) = \varnothing, \psi_{\mathbf{B},1}(\{b\}) = \psi_{\mathbf{B},1}(\{a,b\}) = \{1\}, \qquad \psi_{\mathbf{B},2}(I,\{a\}) = \psi_{\mathbf{B},2}(I,\{a,b\}) = \{1\},$$

for all $I \in \mathcal{I}_R$. Let $h : \mathcal{I}_R \times \mathcal{I}_R \to \mathcal{I}_B$ be the "binary encoding" of \mathcal{I}_B given by $h(\emptyset, \emptyset) = \emptyset$, $h(\{1\}, \emptyset) = \{a\}, h(\emptyset, \{1\}) = \{b\}, \text{ and } h(\{1\}, \{1\}) = \{a, b\}.$ It is straightforward to verify that

h is an isomorphism of $\Psi_{\mathbf{B}}$ onto $\Psi_{\mathbf{B}}$. For instance, we compute:

$$h(\Psi_{\mathbf{B}}((\varnothing,\varnothing),\{a\})) = h(\Psi_{R}(\varnothing,\psi_{\mathbf{B},1}(\{a\})), \Psi_{R}(\varnothing,\psi_{\mathbf{B},2}(\varnothing,\{a\})))$$

$$= h(\Psi_{R}(\varnothing,\varnothing), \Psi_{R}(\varnothing,\{1\}))$$

$$= h(\{1\},\varnothing)$$

$$= \{a\}$$

$$= \Psi_{B}(h(\varnothing,\varnothing),\{a\}).$$

Hence, $\Psi_B \in \mathbf{IS}(\{\Psi_{\mathbf{B}}\})$. By the remarks given above, I is a Ψ_B -answer set iff I is a $\Psi_{\mathbf{B}}$ -answer set and, clearly, $\{a\}$ and $\{b\}$ are the only ones.

5. Isomorphic Representations

In this section, we study the more restricted type of isomorphic representation and provide a complete class of programs with respect to it. Moreover, in Theorem 16 we show that every positive tight program can be isomorphically represented by a cascade program product of reset programs.

For some $n \ge 1$, let $\sigma_1, \ldots, \sigma_{n^n}$ be an enumeration of the set of all mappings from [n] into [n]. Define T_n over $\Gamma_{T_n} = [n^n]$ to be the program consisting of the rules, for all $j \in [n]$ and $k \in [n^n]$:

$$\sigma_k(j) \leftarrow j$$
, not k.

As a consequence of Theorem 4, we obtain the following completeness result.

Theorem 14. The class of programs consisting of all T_n , $n \ge 1$, is isomorphically complete with respect to the cascade program product.

Proof. According to Theorem 4 and Definition 10, we have to show that for every $n \geq 1$, the automaton $\mathfrak{T}_n = ([n], \Sigma_n, \delta_n)$ can be embedded into a cascade automata product of Ψ_{T_n} with a single factor. Define $\Psi_{\mathbf{T}_n} = \Psi_{T_n} [\mathcal{I}_{T_n}, \psi_{\mathbf{T}_n}]$ by $\psi_{\mathbf{T}_n}(J) = J$, for all $J \in \mathcal{I}_{T_n}$. Define the embedding $h = (h_1, h_2)$, with $h_1 : [n] \to \mathcal{I}_{T_n}$ and $h_2 : \Sigma_n \to \mathcal{I}_{T_n}$, by $h_1(j) = \{j\}$ and $h_2(\sigma_k) = \{1, \ldots, k-1, k+1, \ldots, n^n\}$, $k \in [n^n]$. Clearly, h_1 and h_2 are one-one, and the following computation proves that h is indeed an embedding:

$$\Psi_{\mathbf{T}_{n}}(h_{1}(j), h_{2}(\sigma_{k})) = \Psi_{T_{n}}(h_{1}(j), \psi_{\mathbf{T}_{n}}(h_{2}(\sigma_{k})))$$

$$= \Psi_{T_{n}}(h_{1}(j), h_{2}(\sigma_{k}))$$

$$= \Psi_{T_{n}}(\{j\}, \{1, \dots, k-1, k+1, \dots, n^{n}\})$$

$$= \{\sigma_{k}(j)\}$$

$$= h_{1}(\sigma_{k}(j))$$

$$= h_{1}(\delta_{n}(j, \sigma_{k}))$$

holds for all $j \in [n]$ and $k \in [n^n]$.

We now turn to the restricted class of positive (i.e., negation-free) tight programs.

Example 15. Consider the positive tight program $C = \{a \leftarrow ; b \leftarrow a; c \leftarrow a, b\}$. The product $\mathbf{C} = R \ltimes R \ltimes R [\mathcal{I}_C, \psi_{\mathbf{C}}]$ given by

$$\psi_{\mathbf{C},1}(J) = \varnothing \qquad \qquad \psi_{\mathbf{C},2}(I_1,J) = \{1\} - I_1 \qquad \qquad \psi_{\mathbf{C},3}((I_1,I_2),J) = \{1\} - (I_1 \cap I_2)$$

for all $I_1, I_2 \in \mathcal{I}_R$ and $J \in \mathcal{I}_C$, isomorphically represents C. Again, we define the isomorphism h to be a "binary encoding" of \mathcal{I}_C where, e.g., ($\{1\}, \emptyset, \emptyset$) is mapped to $\{a\}$, ($\{1\}, \emptyset, \{1\}$) is mapped to $\{a, c\}$ and so on. For instance, we can compute the least model $I = \{a, b, c\}$ of C as follows:

$$h(\Psi_{\mathbf{C}}((\varnothing,\varnothing,\varnothing),J)) = h(\Psi_{R}(\varnothing,\varnothing),\Psi_{R}(\varnothing,\{1\}),\Psi_{R}(\varnothing,\{1\})) = h(\{1\},\varnothing,\varnothing) = \{a\}$$

$$h(\Psi_{\mathbf{C}}((\{1\},\varnothing,\varnothing),J)) = h(\Psi_{R}(\{1\},\varnothing),\Psi_{R}(\varnothing,\varnothing),\Psi_{R}(\varnothing,\{1\})) = h(\{1\},\{1\},\varnothing) = \{a,b\}$$

$$h(\Psi_{\mathbf{C}}((\{1\},\{1\},\varnothing),J)) = h(\Psi_{R}(\{1\},\varnothing),\Psi_{R}(\{1\},\varnothing),\Psi_{R}(\varnothing,\varnothing)) = h(\{1\},\{1\},\{1\}) = I$$

$$h(\Psi_{\mathbf{C}}((\{1\},\{1\},\{1\},\{1\}),J)) = h(\Psi_{R}(\{1\},\varnothing),\Psi_{R}(\{1\},\varnothing),\Psi_{R}(\{1\},\varnothing)) = h(\{1\},\{1\},\{1\}) = I$$

where $J \in \mathcal{I}_C$ is arbitrary. The calculation shows that I is a Ψ_C -answer set or, equivalently, a Ψ_C -answer set and, clearly, it is the only one.

Now consider the slightly different program $C' = \{a \leftarrow ; b \leftarrow a; c \leftarrow a; c \leftarrow b\}$. Then, C' is isomorphically represented by the product $\mathbf{C}' = R \ltimes R \ltimes R[\mathcal{I}_{C'}, \psi_{\mathbf{C}'}]$ given by

$$\psi_{\mathbf{C}',1}(J) = \emptyset$$
 $\psi_{\mathbf{C}',2}(I_1,J) = \{1\} - I_1$ $\psi_{\mathbf{C}',3}((I_1,I_2),J) = \{1\} - (I_1 \cup I_2)$

for all $I_1, I_2 \in \mathcal{I}_R$ and $J \in \mathcal{I}_{C'}$. Let h be defined as before. Iterating $\Psi_{\mathbf{C}'}$ bottom-up as above yields, for all $J \in \mathcal{I}_{C'}$:

$$h(\Psi_{\mathbf{C}'}((\varnothing,\varnothing,\varnothing),J)) = h(\Psi_{R}(\varnothing,\varnothing),\Psi_{R}(\varnothing,\{1\}),\Psi_{R}(\varnothing,\{1\})) = h(\{1\},\varnothing,\varnothing) = \{a\}$$

$$h(\Psi_{\mathbf{C}'}((\{1\},\varnothing,\varnothing),J)) = h(\Psi_{R}(\{1\},\varnothing),\Psi_{R}(\varnothing,\varnothing),\Psi_{R}(\varnothing,\varnothing)) = h(\{1\},\{1\},\{1\}) = I$$

$$h(\Psi_{\mathbf{C}'}((\{1\},\{1\},\{1\}),J)) = h(\Psi_{R}(\{1\},\varnothing),\Psi_{R}(\{1\},\varnothing),\Psi_{R}(\{1\},\varnothing)) = h(\{1\},\{1\},\{1\}) = I$$
which shows that I is also a $\Psi_{C'}$ -answer set or, equivalently, a $\Psi_{\mathbf{C}'}$ -answer set.

It is straightforward to generalize Example 15 to the general case.

Theorem 16. Every positive tight program P can be isomorphically represented by a cascade program product of reset programs.

6. Discussion and Conclusion

In this paper, we applied the Krohn-Rhodes Theory (Krohn & Rhodes, 1965), presented here following (Gécseg, 1986), to Answer Set Programming (ASP) (Gelfond & Lifschitz, 1991). Particularly, we defined a cascade product for ASP and, by relating programs and automata, showed that every program can be represented (or "emulated") by a product of very simple programs. We thus obtained nice theoretical results regarding the structure of ASP programs, which can be straightforwardly generalized to wider classes of nonmonotonic reasoning formalisms. More precisely, as our concepts and results hinge on the operator Ψ_P , they can be directly reformulated in the algebraic framework of Approximation Fixpoint Theory (AFT) (Denecker et al., 2000), which captures, e.g., ordinary ASP, default and autoepistemic logic (Denecker, Marek, & Truszczyński, 2003), and ASP with external sources (Antić, Eiter, & Fink, 2013).

In a broader sense, this paper is a first step towards an algebraic theory of products and networks of nonmonotonic reasoning systems, including ASP and other formalisms. More precisely, we considered here only the very restricted (though powerful) kind of *cascade* product; it corresponds to the α_0 -product in (Gécseg, 1986), and to the wreath product in finite semigroup theory (Krohn & Rhodes, 1965). In the automata literature, however, many other important products have been studied (for an overview see Dömösi and Nehaniv (2005)). We believe that recasting these kinds of products to ASP will lead to interesting results.

References 11

Particularly, the notion of an asynchronous network (cf. Dömösi & Nehaniv, 2005, Chapter 7) seems very appealing from an ASP point of view, as current modular ASP formalisms (e.g., Dao-Tran, Eiter, Fink, and Krennwallner (2009) cannot cope with asynchronous module structures according to our knowledge. Moreover, as different formalisms can be unified in the AFT-setting, heterogeneous networks in the vein of multi-context systems (cf. Brewka, Eiter, & Fink, 2011a) arise naturally. Finally, our concept of a product semantics emerging from the interaction of its simple factors (cf. Section 4) seems interesting from a general AI perspective and we believe that it deserves a more intensive (and probably more intuitive) study in future work.

Although the Krohn-Rhodes decomposition theorem (Krohn & Rhodes, 1965) is now almost 50 years old, implementations and feasible applications of the Krohn-Rhodes Theory emerged only very recently (cf. Egri-Nagy & Nehaniv, 2005); reference our paper provides further evidence that it is a valuable tool for knowledge representation and reasoning in AI (e.g. Egri-Nagy, 2006), and implementations in the ASP-setting remain as future work.

References

- Antić, C., Eiter, T., & Fink, M. (2013). HEX semantics via approximation fixpoint theory. In Cabalar, P., & Son, T. C. (Eds.), *LPNMR 2013*, LNCS 8148, pp. 102–115.
- Brewka, G., Eiter, T., & Fink, M. (2011a). Nonmonotonic multi-context systems: a flexible approach for integrating heterogenous knowledge sources. In Balduccini, M., & Son, T. C. (Eds.), *Logic Programming, Knowledge Representation, and Nonmonotonic Reasoning*, pp. 233–258. Springer-Verlag, Berlin/Heidelberg.
- Brewka, G., Eiter, T., & Truszczynski, M. (2011b). Answer set programming at a glance. Communications of the ACM, 54(12), 92–103.
- Dao-Tran, M., Eiter, T., Fink, M., & Krennwallner, T. (2009). Modular nonmonotonic logic programming revisited. In *ICLP 2009*, LNCS 5649, pp. 145–159. Springer-Verlag, Berlin/Heidelberg.
- Denecker, M., Marek, V., & Truszczyński, M. (2003). Uniform semantic treatment of default and autoepistemic logics. *Artificial Intelligence*, 143(1), 79–122.
- Denecker, M., Marek, V., & Truszczyński, M. (2000). Approximations, stable operators, well-founded fixpoints and applications in nonmonotonic reasoning. In Minker, J. (Ed.), Logic-Based Artificial Intelligence, Vol. 597 of The Springer International Series in Engineering and Computer Science, pp. 127–144, Norwell, Massachusetts. Kluwer Academic Publishers.
- Dömösi, P., & Nehaniv, C. (2005). Algebraic Theory of Automata Networks: An Introduction. SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics, Philadelphia, PA.
- Egri-Nagy, A. (2006). Making sense of the sensory data—coordinate systems by hierarchical decomposition. In Gabrys, B., Howlett, R. J., & Jain, L. C. (Eds.), *KES 2006, Part III*, LNAI 4253, pp. 330–340. Springer-Verlag.
- Egri-Nagy, A., & Nehaniv, C. (2005). Algebraic hierarchical decomposition of finite state automata: comparison of implementations for Krohn-Rhodes theory. In Domaratzki, M., Okhotin, A., Salomaa, K., & Yu, S. (Eds.), CIAA 2004, LNCS 3317. Springer-Verlag, Berlin/Heidelberg.
- Gécseg, F. (1986). *Products of Automata*. EATCS Monographs on Theoretical Computer Science. Springer-Verlag, Berlin/Heidelberg.

12 References

- Gelfond, M., & Lifschitz, V. (1991). Classical negation in logic programs and disjunctive databases. New Generation Computing, 9(3-4), 365-385.
- Krohn, K., & Rhodes, J. (1965). Algebraic theory of machines. I. Prime decomposition theorem for finite semigroups and machines. *Transactions of the American Mathematical Society*, 116, 450–464.