A purely syntactic and cut-free sequent calculus for the modal logic of provability

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Abstract

In this paper we present a sequent calculus for the modal propositional logic GL (the logic of provability) obtained by means of the tree-hypersequent method, a method in which the metalinguistic strength of hypersequents is improved, so that we can simulate trees-shapes. We prove that this sequent calculus is sound and complete with respect to the Hilbert-style system GL, that it is contraction-free and cut-free and that its logical and modal rules are invertible. No explicit semantic element is used in the sequent calculus and all the results are proved in a purely syntactic way.

1 Introduction

GL, from the initials of Gödel and Löb, or the logic of provability, is a system created at the end of the 70's in order to 'capture', in the simple framework of modal logic, the properties of the provability predicate of the formal system `PA` (Peano Arithmetic). Syntactically **GL** is usually presented by means of the Hilbert system composed by the modal system **K4**, plus the Löb's axiom: $\Box(\Box\alpha \to \alpha) \to \Box\alpha$; semantically it describes the class of transitive frames without infinite ascending $\mathcal{R}$-chains. **GL** is different from the other well-known normal systems of modal logic (e.g. **K**, **KT**, **S4**, **S5**) in at least two ways (that are probably not unrelated): (i) the property of not having infinite ascending $\mathcal{R}$-chains is a property that needs second-order logic to be described (while, for example, the well-known properties of reflexivity or symmetry can be described by first-order logic formulas); (ii) there exist only two attempts at finding a sequent calculus for this system (while we can count several attempts since the 50's until today at finding sequent calculi for the other well-known systems of modal logic). The first attempt, GLS, is quite easy to be presented: indeed it is the sequent calculus that results by adding to the sequent calculus for classical propositional logic the following rule:

$$\frac{M, \Box M, \Box\alpha \Rightarrow \alpha}{\Box M \Rightarrow \Box\alpha}\, gl$$

Several authors have tried to give a syntactic proof of cut-elimination for this sequent calculus. The first proof was proposed by Leivant (1981), but unfortunately it contains a gap, as it has been pointed out by Valentini (1983). A second attempt has been made by Valentini himself (see again Valentini (1983)), but, as Moen (2003) noted, in Valentini's proof one makes an essential use of the notion of set in a sequent, i.e. if one dealt with multisets, instead of sets, the proof would not work. Finally a third proof has been offered by Sasaki (2001). This proof, even if quite complicated and not developed by means of Gentzen-style methods, seems to work.

However, the sequent calculus GLS not only seems a difficult calculus to prove the cut-elimination theorem for, but it also presents other disadvantages: the structural rules are not admissible, and the rules are neither explicit (the logical rules of a sequent calculus are said to be explicit if they exhibit the constant they introduce only in the conclusion and not in the premises) nor symmetric (if for each constant of the language of a sequent calculus, there is at least one rule which introduces it on the left side of the sequent arrow, and at least one rule which introduces it on the right side of the sequent arrow, then the logical rules of the sequent calculus are said to be symmetric).[1] Some of these defects have been surmounted by the second attempted calculus for **GL**, $G3GL$, which has been proposed by Negri (2005). Indeed, even if her rules are not explicit, they are symmetric, and the cut-elimination proof does not seem to create any problem. On the other hand, Negri introduces and uses in her calculus explicit elements of Kripke semantics, such as labelled formulas of the form $x : \alpha$ and relational atoms of the form xRy, and this fact prevents us from considering $G3GL$ as a purely proof-theoretic instrument. Moreover it has as unpleasant consequence that the subformula property is not fully satisfied (on this point see also Negri (2005), p. 508)

Given this situation, it then seems worth developing a new sequent calculus for **GL** obtained by means of the tree-hypersequent method, a method introduced in Poggiolesi (2009a), Poggiolesi (2008) and Poggiolesi (2009b) whose basic idea, that we are going to explain presently, has already been used in the literature on the generalisations of the sequent calculus, e.g. in Kashima (1994). As we will see in the next sections, this sequent calculus has all the advantages of Negri's calculus (and also the disadvantage of not having explicit logical rules) since it is mainly inspired by it; moreover it is syntactically pure, that is to say it does not make any use of semantic elements, and it enjoys the subformula property.

2 Overview of the tree-hypersequents method

Since the 80's, several theorists have enriched the classical sequent calculus in order to extend it to non-classical logics. Some examples of methods by means of which we can improve the expressive power of the Gentzen's system are the well-known display method (e.g. see Wansing (1994)) or the hypersequent method (e.g. see Avron (1996)). The first one is based on the idea of adding several new metalinguistic symbols, the second one, by contrast, is based on the idea of dealing with many sequents at a time.

Another method by means of which we can generate extensions of the sequent

[1] For a full description of these properties and of their importance, see Wansing (2002) and Poggiolesi (2008).

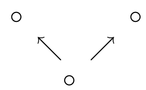
calculus, is the tree-hypersequent method: this method is based on the idea of working with n sequents a time, as in the hypersequent method. However it differs from the hypersequent method in that the order of the sequents is important - it yields a structure that can be read in terms of *trees*. Let us explain this idea by constructing a tree-hypersequent. Let us start by considering empty hypersequents, i.e. objects of the form:

$$\overbrace{-/-/-}^{n}$$

which is to say: n slashes separating $n+1$ dashes. By taking into account the order of the dashes, as it is not standardly done, we can look to the empty hypersequent as a linear and vertical tree-frame in Kripke semantics, where the dashes are the worlds of this tree-frame and the slashes the relations between worlds in the tree-frame. Following this analogy, the dash that is at the extreme left of the empty hypersequent corresponds to a world at distance one in a tree-frame, which is to say, to its root; the dash immediately after it, corresponds to a world a distance two, the next dash to a world a distance three, and so on.

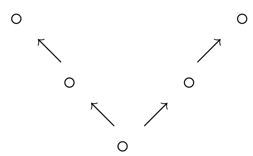
Given this situation a question seems to naturally arise: is it possible to reflect the full structure of the tree-frames in our syntactic object, i.e. the fact that in a tree-frame, at every distance, we can have n possible worlds? The answer is not only affirmative, but also quite simple: we separate the dashes that happen to be at the same distance with a semi-colon and we obtain in this way the notion of *empty tree-hypersequent*. So an example of an empty tree-hypersequent is an object of the following form (see the figure on the left):

$$- \,/\, - \,;\, - \qquad\qquad \rightsquigarrow$$



and it corresponds to a tree-frame (see the figure on the right) with a world at distance one related with two different worlds at distance two. Another example of an empty tree-hypersequent is an object of the following form (see the figure on the left):

$$-/(-/-);(-/-) \qquad\qquad \rightsquigarrow$$



and it corresponds to a tree-frame (see the figure on the right) with a world at distance one related with two different worlds at distance two, each of which is, in its turn, related with another world at distance three. Finally, in order to obtain a *tree-hypersequent*, and thus achieve our goal, we simply substitute the dashes with sequents. Therefore the following object is an example of a tree-hypersequent:

$$\Delta_1/(\Delta_2/\Delta_3);(\Delta_4/(\Delta_5/\Delta_6);\Delta_7)$$

where Δ_i, $\leq i \leq 7$ is a classical sequent.

Having clarified the basic intuition of the tree-hypersequents method, we can show how to exploit it in order to get a calculus for the system **GL**.

3 The calculus CSGL

We define the modal propositional language $\mathcal{L}^{\Box}$ in the following way:

- atoms: p^0, p^1, ...
- logical constant: $\Box$,
- connectives: $\neg$, $\vee$.

The other connectives can be defined as usual, as well as the constant $\diamond$ and the formulas of the modal language $\mathcal{L}^{\Box}$.

`Syntactic Conventions`:

α, β, ...: formulas,

M, N, ...: finite multisets of formulas,

Γ, Δ, ...: sequents (SEQ). The empty sequent ($\Rightarrow$) is included.

G, H, ...: tree-hypersequents (THS).

$\underline{X}$, $\underline{Y}$, ...: finite multisets of tree-hypersequents (MTHS), $\emptyset$ included.

For the sake of brevity we will use the following notation: given $\Gamma \equiv M \Rightarrow N$ and $\Pi \equiv P \Rightarrow Q$, we will write:

- β, Γ, α instead of $\beta, M \Rightarrow N, \alpha$,
- $\Gamma \centerdot \Pi$ instead of $M, P \Rightarrow N, Q$,

Note that from these abbreviations, it follows that $\beta, \Gamma \centerdot \Pi, \alpha$ stands for $\beta, M, P \Rightarrow N, Q, \alpha$.

As we will deal with tree-hypersequents, we remind the reader what a tree-hypersequent is.

Definition 3.1. The notion of *tree-hypersequent* is inductively defined in the following way:

- if $\Gamma \in$ SEQ, then $\Gamma \in$ THS,
- if $\Gamma \in$ SEQ and $\underline{X} \in$ MTHS, then $\Gamma/\underline{X} \in$ THS.

Definition 3.2. The intended interpretation of a tree-hypersequent is:

- $(M \Rightarrow N)^{\tau}\colon = \bigwedge M \to \bigvee N$
- $(\Gamma/G_1; ...; G_n)^{\tau}\colon = \Gamma^{\tau} \vee \Box G_1^{\tau} \vee ... \vee \Box G_n^{\tau}$

Therefore a tree-hypersequent is just a hypersequent to which one adjoins the meta-lingustic symbol ;. This addition allows to arrange the sequents that compose the tree-hyersequent in accordance with a tree-shape.

In order to display the rules of the calculi, we will use the notation $G[*]$ defined as follows:

Definition 3.3. The notion of zoom tree-hypersequent (ZTHS) is inductively defined in the following way:

- $[*] \in$ ZTHS,
- if $G_1, ..., G_n \in$ THS, then $[*]/G_1; ...; G_n \in$ ZTHS,
- if $G_1[*] \in$ ZTHS, $G_2, ..., G_n \in$ THS, then $[*]/G_1[*]; ...; G_n \in$ ZTHS,
- if $\Gamma \in$ SEQ, $G_1[*] \in$ ZTHS and $G_2, ..., G_n \in$ THS, then $\Gamma/G_1[*]; ...; G_n \in$ ZTHS.
- if $\Gamma \in$ SEQ, $G_1[*][*] \in$ ZTHS, $G_2, ..., G_n \in$ THS, then $\Gamma/G_1[*][*]; ...; G_n$ $\in$ ZTHS.

Definition 3.4. For any zoom tree-hypersequent $G[*]$, or $G[*][*]$, and tree-hypersequents H and I, we define $G[H]$ and $G[H][I]$, the result of substituting H into $G[*]$, and the result of substituting H and I in $G[*][*]$, respectively, as follows:

- if $G[*] = [*]$, then $G[H] = H$
- if $G[*] = [*]/G_1; ...; G_n$ and $H = \Delta/J_1; ...; J_m$, then $G[H] = \Delta/G_1; ...; G_n; J_1; ...; J_m$
- if $G[*][*] = [*]/G_1[*]; ...; G_n$ and $H = \Delta/J_1; ...; J_m$, then $G[H][I] = \Delta/G_1[I]; ...; G_n; J_1; ...; J_m$
- if $G[*] = \Gamma/G_1[*], ..., G_n$, then $G[H] = \Gamma/G_1[H], ..., G_n$
- if $G[*][*] = \Gamma/G_1[*][*], ..., G_n$, then $G[H][I] = \Gamma/G_1[H][I], ..., G_n$

Note that a sequent is a tree-hypersequent so that Definition 3.4 also applies to the case of substituting a sequent into a zoom tree-hypersequent.

What is the intuitive meaning of the last two definitions? Well, $G[*]$ can be thought of as a tree-hypersequent G together with one hole $[*]$, where the hole should be understood, metaphorically, as a zoom by means of which we can focus attention on a particular part, $*$, of G. The operation of substitution fills the hole with a sequent or a tree-hypersequent, and therefore allows us to make explicit the particular part in the tree-hypersequent that we want to concentrate our attention on. Similarly for $G[*][*]$.

The postulates of the sequent calculus $CSGL$ are:

Initial Tree-hypersequents

$G\ [p,\ \Gamma,\ p] \qquad\qquad G\ [\Box\alpha,\ \Gamma,\ \Box\alpha]$

Propositional Rules

$$\frac{G[\Gamma, \alpha]}{G[\neg\alpha, \Gamma]}\ \neg A \qquad\qquad \frac{G[\alpha, \Gamma]}{G[\Gamma, \neg\alpha]}\ \neg K$$

$$\frac{G[\alpha, \beta, \Gamma]}{G[\alpha \wedge \beta, \Gamma]}\ \wedge A \qquad \frac{G[\Gamma, \alpha] \qquad G[\Gamma, \beta]}{G[\Gamma, \alpha \wedge \beta]}\ \wedge K$$

Modal Rules

$$\frac{G[\Box\alpha, \Gamma/(\Sigma, \Box\alpha/\underline{X})] \qquad\qquad G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\ \Box A_{gl} \qquad \frac{G[\Gamma/\Box\alpha \Rightarrow \alpha]}{G[\Gamma, \Box\alpha]}\ \Box K_{gl}$$

Special Logical Rule

$$\frac{G[\Box\alpha, \Gamma/(\Box\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\ 4$$

In order to introduce the cut-rule, we firstly need the following two notions.

Definition 3.5. Given two tree-hypersequents $G[\Gamma]$ and $G^{'}[\Gamma^{'}]$ together with an occurrence of a sequent in each, the relation of *equivalent position* between two of their sequents, in this case Γ and $\Gamma^{'}$, $G[\Gamma] \sim G^{'}[\Gamma^{'}]$, is defined inductively in the following way:

- $\Gamma \sim \Gamma^{'}$
- $\Gamma/\underline{X} \sim \Gamma^{'}/\underline{X}^{'}$
- If $H[\Gamma] \sim H^{'}[\Gamma^{'}]$, then $\Delta/H[\Gamma]; \underline{X} \sim \Delta^{'}/H^{'}[\Gamma^{'}]; \underline{X}^{'}$

Definition 3.6. Given two tree-hypersequents $G[\Gamma]$ and $G^{'}[\Gamma^{'}]$ together with an occurrence of a sequent in each, such that $G[\Gamma] \sim G^{'}[\Gamma^{'}]$, the operation of *product*, $G[\Gamma] \otimes G^{'}[\Gamma^{'}]$, is defined inductively in the following way:

- $\Gamma \otimes \Gamma^{'} := \Gamma \centerdot \Gamma^{'}$

- $(\Gamma/\underline{X}) \otimes (\Gamma^{'}/\underline{X}^{'}) := \Gamma \centerdot \Gamma^{'}/\underline{X}; \underline{X}^{'}$

- $(\Delta/H[\Gamma]; \underline{X}) \otimes (\Delta^{'}/H^{'}[\Gamma^{'}]; \underline{X}^{'}) :=$

  $\Delta \centerdot \Delta^{'}/(H[\Gamma] \otimes H^{'}[\Gamma^{'}]); \underline{X}; \underline{X}^{'}$

Cut-Rule

Given two tree-hypersequents $G[\Gamma, \alpha]$ and $G^{'}[\alpha, \Pi]$ together with an occurrence of a sequent in each, such that $G[\Gamma, \alpha] \sim G^{'}[\alpha, \Pi]$, the cut-rule is:

$$\frac{G[\Gamma, \alpha] \quad G^{'}[\alpha, \Pi]}{G \otimes G'[\Gamma \centerdot \Pi]} \, cut_{\alpha}$$

As the reader can easily see from the above definition, the cut rule should respect two important criteria. The first one says that, given two tree-hypersequents, we can cut on any two sequents belonging to them provided that they are in an equivalent position, i.e. the number of the worlds that precede them on their branch is the same. The second one says that after the cut the two tree-hypersequents should not be randomly mixed but according to the inductive definition of product, i.e. the sequents that precede Γ and Π respectively, and are on their same branch, get merged; the rest of the two trees coexist separately. We underline that these two criteria are fundamental because they ensure that the result of a cut between two tree-hypersequents is still a *tree*-hypersequent, which is to say that the tree shape is kept.

4 Admissibility of the structural rules

In this section we will show which structural rules are admissible in the calculus $CSGL$. Moreover, in order to show that the two rules of contraction are admissible, we will show that all the logical and modal rules are invertible. The proof of the eliminability of the cut-rule will be given in the last section.

Definition 4.1. We associate to each proof d in $CSGL$ a natural number $h(d)$ (where h stand for the *height* of d). Intuitively, the height corresponds to the length of the longest branch in a tree-proof d, minus one. However we omit the standard inductive definition.

Definition 4.2. $d \vdash^n G$ means that d is a proof of G in $CSGL$, with $h(d) \leq n$. We write $^{\langle n \rangle}G$ for: G is the conclusion of a proof d with height $\leq n$.

Definition 4.3. Let G be a tree-hypersequent and $G^{'}$ be the result of the application of a certain rule $\mathcal{R}$ on G. We say that this rule $\mathcal{R}$ is *height-preserving admissible* when:

$$d \vdash^n G \quad \Rightarrow \quad \exists d^{'}(d^{'} \vdash^n G^{'})$$

We call a rule, $\mathcal{R}$, which transforms a tree-hypersequent G into a tree-hypersequent $G^{'}$, *admissible* when:

$$d \vdash^n G \quad \Rightarrow \quad \exists d^{'}(d^{'} \vdash G^{'})$$

Observation 4.4. In the sequent calculus for classical logic, we usually say that a (some) formula(s) is (are) auxiliary in the premise(s) of a rule when the rule operates on that (those) formula(s). In a similar way, we will say that a (some) sequent(s) is (are) auxiliary in the premise(s) of a rule, when the rule concerns that (those) sequent(s). More precisely we will consider as auxiliary those sequents that are displayed in the premise(s) of the rules of the tree-hypersequent calculi.
In the following proofs of the (height-preserving) admissibility of the structural rules and invertibility of the logical and modal rules, we will only take into account those cases in which the last applied rule operates on the auxiliary sequent(s) of the rule that we want to show to be admissible or invertible. All the other cases are dealt with easily, as shown in Lemmas 4.13 and 4.14, which are proved at the end of the current section.

Lemma 4.5. *The tree-hypersequents of the form $G[\alpha, \Gamma, \alpha]$, with α arbitrary formula, are derivable in $CSGL$.*

Proof. By induction on α. If $\alpha = p$ or $\alpha = \Box\beta$, then we have $G[p, \Gamma, p]$ and $G[\Box\beta, \Gamma, \Box\beta]$, respectively, that are initial tree-hypersequents. Let us then suppose that $\alpha = \beta \wedge \gamma$ (for $\alpha = \neg\beta$ the procedure is similar), and that $G[\beta, \Gamma, \beta]$ and $G[\gamma, \Gamma, \gamma]$ are derivable in $CSGL$. The we have:

$$\cfrac{\cfrac{\cfrac{G[\beta, \Gamma, \beta]}{G[\beta, \gamma, \Gamma, \beta]} \qquad \cfrac{G[\gamma, \Gamma, \gamma]}{G[\beta, \gamma, \Gamma, \gamma]}}{G[\beta, \gamma, \Gamma, \beta \wedge \gamma]}}{G[\beta \wedge \gamma, \Gamma, \beta \wedge \gamma]}$$

⊠

Lemma 4.6. *The rule:*

$$\frac{G}{\Rightarrow /G}\, rn$$

is height-preserving admissible in CSGL.

Proof. By induction on the height of the derivation of the premise.

If G is an initial tree-hypersequent, then so is the conclusion. If G is inferred by a logical rule or by the special logical rule, then the inference is clearly preserved. If G is inferred by the modal rule $\Box K_{gl}$, then the inference is preserved. We have:

$$\dfrac{{}^{\langle n-1\rangle} G[\Gamma/\Box\alpha \Rightarrow \alpha]}{{}^{\langle n\rangle} G[\Gamma, \Box\alpha]}\ {\scriptstyle \Box K_{gl}} \qquad \rightsquigarrow^{2}$$

$$\dfrac{{}^{\langle n-1\rangle} \Rightarrow /G[\Gamma/\Box\alpha \Rightarrow \alpha]}{{}^{\langle n\rangle} \Rightarrow /G[\Gamma, \Box\alpha]}\ {\scriptstyle \Box K_{gl}}$$

If G is inferred by the modal rule $\Box A_{gl}$, then the inference is preserved. We have:

$$\dfrac{{}^{\langle n-1\rangle} G[\Box\alpha, \Gamma/(\Sigma, \Box\alpha/\underline{X})] \qquad {}^{\langle n-1\rangle} G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{{}^{\langle n\rangle} G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\ {\scriptstyle \Box A_{gl}}$$

$$\rightsquigarrow$$

$$\dfrac{{}^{\langle n-1\rangle} \Rightarrow /G[\Box\alpha, \Gamma/(\Sigma, \Box\alpha/\underline{X})] \quad {}^{\langle n-1\rangle} \Rightarrow /G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{{}^{\langle n\rangle} \Rightarrow /G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\ {\scriptstyle \Box A_{gl}}$$

⊠

Lemma 4.7. *The rules of weakening:*

$$\dfrac{G[\Gamma]}{G[\alpha, \Gamma]}\ {\scriptstyle WA} \qquad \dfrac{G[\Gamma]}{G[\Gamma, \alpha]}\ {\scriptstyle WK}$$

are height-preserving admissible in CSGL.

Proof. By straightforward induction on the height of the derivation of the premise. ⊠

Lemma 4.8. *The rule of external weakening:*

$$\dfrac{G[\Gamma]}{G[\Gamma/\Sigma]}\ {\scriptstyle EA}$$

[2] The symbol $\rightsquigarrow$ means: the premise of the right side is obtained by induction hypothesis on the premise of the left side.

is height-preserving admissible in CSGL.

Proof. By straightforward induction on the height of the derivation of the premise. $\boxtimes$

Lemma 4.9. *The rule of merge:*

$$\frac{G[\Delta/(\Gamma/\underline{X});(\Pi/\underline{X}')]}{G[\Delta/(\Gamma\centerdot\Pi/\underline{X};\underline{X}')]}\;{\scriptstyle merge}$$

is height-preserving admissible in CSGL.

Proof. By induction on the height of the derivation of the premise. As the rule of merge has three auxiliary sequents, Δ, Γ and Π, we should, for each rule $\mathcal{R}$ applied to the premise, distinguish three subcases: one in which the rule $\mathcal{R}$ has been applied to the sequent Δ, one in which the rule $\mathcal{R}$ has been applied to the sequent Γ, one in which the rule $\mathcal{R}$ has been applied to the sequent Π. On the other hand these subcases are similar and, therefore, we do not need to analyse all of them; on the contrary, we will develop the proof by choosing the most significant one each time.

If G is an initial tree-hypersequent, then so is the conclusion. If G is inferred by a logical rule or by the special logical rule, then the inference is clearly preserved. If G is inferred by the modal rule $\Box K_{gl}$, then the inference is preserved. We have:

$$\frac{{}^{\langle n-1\rangle}G[\Delta/(\Gamma/\Box\alpha\Rightarrow\alpha;\underline{X});(\Pi/\underline{X}')]}{{}^{\langle n\rangle}G[\Delta/(\Gamma,\Box\alpha/\underline{X});(\Pi/\underline{X}')]}\;{\scriptstyle\Box K_{gl}}\qquad\rightsquigarrow$$

$$\frac{{}^{\langle n-1\rangle}G[\Delta/(\Gamma\centerdot\Pi/\Box\alpha\Rightarrow\alpha;\underline{X};\underline{X}')]}{{}^{\langle n\rangle}G[\Delta/(\Gamma\centerdot\Pi,\Box\alpha/\underline{X};\underline{X}')]}\;{\scriptstyle\Box K_{gl}}$$

If the premise is inferred by the rule $\Box A_{gl}$, then the inference is preserved. We have:

$$\frac{{}^{\langle n-1\rangle}G[\Box\alpha,\Delta/(\Gamma,\Box\alpha/\underline{X});(\Pi/\underline{X}')]\quad{}^{\langle n-1\rangle}G[\Box\alpha,\Delta/(\alpha,\Gamma/\underline{X});(\Pi/\underline{X}')]}{{}^{\langle n\rangle}G[\Box\alpha,\Delta/(\Gamma/\underline{X});(\Pi/\underline{X}')]}\;{\scriptstyle\Box A_{gl}}$$

$$\rightsquigarrow$$

$$\frac{{}^{\langle n-1\rangle}G[\Box\alpha,\Delta/(\Gamma\centerdot\Pi,\Box\alpha/\underline{X};\underline{X}')]\quad{}^{\langle n-1\rangle}G[\Box\alpha,\Delta/(\alpha,\Gamma\centerdot\Pi/\underline{X};\underline{X}')]}{{}^{\langle n\rangle}G[\Box\alpha,\Delta/(\Gamma\centerdot\Pi/\underline{X};\underline{X}')]}\;{\scriptstyle\Box A_{gl}}$$

$\boxtimes$

Lemma 4.10. *The rule* $\tilde{4}$*:*

$$\frac{G[\Gamma/(\Sigma/\underline{X})]}{G[\Gamma/(\Rightarrow/\Sigma/\underline{X})]}\ \tilde{4}$$

is admissible in $CSGL$*.*

Proof. By induction on the height of the derivation of the premise. As the rule $\tilde{4}$ has two auxiliary sequents, Γ and Σ, we should, for each rule $\mathcal{R}$ applied to the premise, distinguish two subcases: one in which the rule $\mathcal{R}$ has been applied to the sequent Γ, and one in which the rule $\mathcal{R}$ has been applied to the sequent Σ. On the other hand these subcases are similar and, therefore, we do not need to analyse both of them; on the contrary, we will develop the proof by choosing the most significant one each time.

If G is an initial tree-hypersequent, then so is the conclusion. If G is inferred by a logical rule, then the inference is clearly preserved. If G is inferred by the modal rule $\Box K_{gl}$, then the inference is preserved. We have:

$$\frac{{}^{\langle n-1\rangle}G[\Gamma/\Box\alpha\Rightarrow\alpha;(\Sigma/\underline{X})]}{{}^{\langle n\rangle}G[\Gamma,\Box\alpha/(\Sigma/\underline{X})]}\ \Box K_{gl} \qquad \rightsquigarrow$$

$$\frac{G[\Gamma/\Box\alpha\Rightarrow\alpha;(\Rightarrow/\Sigma/\underline{X})]}{G[\Gamma,\Box\alpha/(\Rightarrow/\Sigma/\underline{X})]}\ \Box K_{gl}$$

If G is inferred by the modal rule $\Box A_{gl}$ (the case of the special logical rule is analogous), then the inference is preserved. We have:

$$\frac{{}^{\langle n-1\rangle}G[\Box\alpha,\Gamma/(\Sigma,\Box\alpha/\underline{X})] \quad {}^{\langle n-1\rangle}G[\Box\alpha,\Gamma/(\alpha,\Sigma/\underline{X})]}{{}^{\langle n\rangle}G[\Box\alpha,\Gamma/(\Sigma/\underline{X})]}\ \Box A_{gl}$$

$\rightsquigarrow$

$$\frac{\dfrac{\dfrac{G[\Box\alpha,\Gamma/(\Rightarrow/\Sigma,\Box\alpha/\underline{X})]}{G[\Box\alpha,\Gamma/(\Box\alpha\Rightarrow/\Sigma,\Box\alpha/\underline{X})]}\ WA \quad \dfrac{G[\Box\alpha,\Gamma/(\Rightarrow/\alpha,\Sigma/\underline{X})]}{G[\Box\alpha,\Gamma/(\Box\alpha\Rightarrow/\alpha,\Sigma/\underline{X})]}\ WA}{G[\Box\alpha,\Gamma/(\Box\alpha\Rightarrow/\Sigma/\underline{X})]}\ \Box A_{gl}}{G[\Box\alpha,\Gamma/(\Rightarrow/\Sigma/\underline{X})]}\ 4$$

$\boxtimes$

Lemma 4.11. *The logical rules, the modal rules and the special logical rule of $CSGL$ are invertible.*

Proof. The proof is by induction on the height of the derivation of the premise of the rule considered. The cases of the logical rules are dealt with in the classical way. The only differences - the fact that we are dealing with tree-hypersequents and the cases where the last applied rule is one of the modal rules or the special logical rule - are dealt with easily.

The rules $\Box A_{gl}$ and 4 are (height-preserving) invertible by the (height-preserving) admissibility of internal weakening.

Let us then consider the invertibility of the $\Box K_{gl}$ rule. If $G[\Gamma, \Box\alpha]$ is an initial tree-hypersequent and $\Box\alpha$ is not principal in it, then $G[\Gamma/\Box\alpha \Rightarrow \alpha]$ is also an initial tree-hypersequent. On the other hand, if $G[\Gamma, \Box\alpha]$ is an initial tree-hypersequent and $\Box\alpha$ is principal in it, then Γ will be of the form $\Box\alpha, M^{'} \Rightarrow N$ and we need to prove that $G[\Box\alpha, M^{'} \Rightarrow N/\Box\alpha \Rightarrow \alpha]$ is derivable. This follows by $\Box A_{gl}$ from the initial tree-hypersequent $G[\Box\alpha, M^{'} \Rightarrow N/\Box\alpha \Rightarrow \Box\alpha, \alpha]$ and the derivable tree-hypersequent $G[\Box\alpha, M^{'} \Rightarrow N/\Box\alpha, \alpha \Rightarrow \alpha]$. The rest of the proof continues in the standard way. ⊠

Lemma 4.12. *The rules of contraction:*

$$\frac{G[\alpha, \alpha, \Gamma]}{G[\alpha, \Gamma]}\,CA \qquad \frac{G[\Gamma, \alpha, \alpha]}{G[\Gamma, \alpha]}\,CK$$

are admissible in $CSGL$.

Proof. By induction on the complexity of the formula α, $cmp(\alpha)$ (which is the number (≥ 0) of occurrences of logical symbols in α) with subinduction on the height of proofs of the premises. Let $CA_{<n}$ and CA_n mean that CA is admissible for $cmp(\alpha) < n$ and for $cmp(\alpha) = n$, respectively. Analogously for $CK_{<n}$ and CK_n. We prove, successively:

(i) For every k: if $CA_{<k}$ and $CK_{<k}$, then CA_k

(ii) For every k: if CA_k and $CK_{<k}$, then CK_k

Thus, if $CA_{<k}$ and $CK_{<k}$, then CA_k and CK_k and the conclusion follows by complete induction on k.

$k = 0$ is trivial. So suppose $k = n$, for n non zero. We treat this case in detail. As the reader will see, we are going to use the two (inductive) hypothesis $CA_{<n}$ and $\mathrm{CK}_{<n}$. We will indicate their use by IH.

(i) We only analyse those cases in which $G[\alpha, \alpha, \Gamma]$ is obtained by a rule $\mathcal{R}$ that has one of the two occurrences of the formula α as principal. The others can be dealt with easily by subinduction on the height of proofs.

- $\alpha \equiv \neg\beta$ and has been obtained by the rule $\neg A$. We solve this case by: exploiting Lemma 4.11, using IH (contracting on the right side of the sequent), and finally applying the rule $\neg A$. We have:

$$\frac{^{\langle n-1\rangle}G[\neg\beta,\Gamma,\beta]}{^{\langle n\rangle}G[\neg\beta,\neg\beta,\Gamma]}\,{\scriptstyle \neg A} \quad \dashrightarrow^{3} \quad \cfrac{\cfrac{G[\Gamma,\beta,\beta]}{G[\Gamma,\beta]}\,{\scriptstyle IH}}{G[\neg\beta,\Gamma]}\,{\scriptstyle \neg A}$$

- $\alpha \equiv \beta \wedge \gamma$ and has been obtained by the rule $\wedge A$. We solve this case by: exploiting Lemma 4.11, using IH and finally applying the rule $\wedge A$. Indeed we have:

$$\frac{^{\langle n-1\rangle}G[\beta\wedge\gamma,\beta,\gamma,\Gamma]}{^{\langle n\rangle}G[\beta\wedge\gamma,\beta\wedge\gamma,\Gamma]}\,{\scriptstyle \wedge A} \quad \dashrightarrow \quad \cfrac{\cfrac{\cfrac{G[\beta,\beta,\gamma,\gamma,\Gamma]}{G[\beta,\gamma,\gamma,\Gamma]}\,{\scriptstyle IH}}{G[\beta,\gamma,\Gamma]}\,{\scriptstyle IH}}{G[\beta\wedge\gamma,\Gamma]}\,{\scriptstyle \wedge A}$$

- $\alpha \equiv \Box\beta$ and has been obtained by the rule $\Box A_{gl}$. We easily solve this case by applying the subinduction on the height, as follows:

$$\frac{^{\langle n-1\rangle}G[\Box\beta,\Box\beta,\Gamma/(\Sigma,\Box\beta/\underline{X})] \quad ^{\langle n-1\rangle}G[\Box\beta,\Box\beta,\Gamma/(\beta,\Sigma/\underline{X})]}{^{\langle n\rangle}G[\Box\beta,\Box\beta,\Gamma/(\Sigma/\underline{X})]}\,{\scriptstyle \Box A_{gl}} \quad \rightsquigarrow$$

$$\frac{G[\Box\beta,\Gamma/(\Sigma,\Box\beta/\underline{X})] \quad G[\Box\beta,\Gamma/(\beta,\Sigma/\underline{X})]}{G[\Box\beta,\Gamma/(\Sigma/\underline{X})]}\,{\scriptstyle \Box A_{gl}}$$

- $\alpha \equiv \Box\beta$ and has been obtained by the rule 4. We easily solve this case by applying the subinduction on the height, as follows:

$$\frac{^{\langle n-1\rangle}G[\Box\beta,\Box\beta,\Gamma/(\Box\beta,\Sigma/\underline{X})]}{^{\langle n\rangle}G[\Box\beta,\Box\beta,\Gamma/(\Sigma/\underline{X})]}\,{\scriptstyle 4} \quad \rightsquigarrow \quad \frac{G[\Box\beta,\Gamma/(\Box\beta,\Sigma/\underline{X})]}{G[\Box\beta,\Gamma/(\Sigma/\underline{X})]}\,{\scriptstyle 4}$$

We have thus established (i) for $k = n$. Let us now establish (ii) for $k = n$.

(ii) We only analyse those cases in which $G[\Gamma,\alpha,\alpha]$ is obtained by a rule $\mathcal{R}$ that has one of the two occurrences of the formula α as principal. The others can be dealt with easily by subinduction on the height of proofs.

- $\alpha \equiv \neg\beta$ and has been obtained by the rule $\neg K$. This case can be dealt with analogously to the case (i) - $\neg A$ above.

- $\alpha \equiv \alpha \wedge \beta$ and has been obtained by the rule $\wedge K$. This case can be dealt with analogously to the case (i) - $\wedge A$ above.

- $\alpha \equiv \Box\beta$ and has been obtained by the rule $\Box K$. We solve this case by: exploiting Lemma 4.11, using the rule of *merge*, using the fact that CA_n (to contract the formula $\Box\beta$ on the left side), using IH (to contract the formula β on the right side), and finally by applying the rule $\Box K_{gl}$. We have:

[3] This symbol means: the premise of the right side is concluded by application of the Lemma 4.11 on the premise of the left side.

$$\dfrac{{}^{\langle n-1\rangle}G[\Gamma, \Box\beta/\Box\beta \Rightarrow \beta]}{{}^{\langle n\rangle}G[\Gamma, \Box\beta, \Box\beta]}\ \Box K_{gl} \quad \dashrightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{G[\Gamma/\Box\beta \Rightarrow \beta; \Box\beta \Rightarrow \beta]}{G[\Gamma/\Box\beta, \Box\beta, \Rightarrow \beta, \beta]}\ merge}{G[\Gamma/\Box\beta \Rightarrow \beta, \beta]}\ (i)}{G[\Gamma/\Box\beta \Rightarrow \beta]}\ IH}{G[\Gamma, \Box\beta]}\ \Box K_{gl}$$

⊠

Lemma 4.13. *Let $G[H]$ be any tree-hypersequent of the calculus $CSGL$ together with an occurrence of a tree-hypersequent in it, and $G^*[H]$ the result of the application of one of the (height-preserving admissible) rules - rn, WA, WK, EW, $merge$ - or of one of the (admissible) rules - CA, CK, $\tilde{4}$ - on $G[H]$. If for a rule $\mathcal{R}$ we have:*

$$\frac{G[H^{'}]}{G[H]}\ \mathcal{R}$$

then it holds that:

$$\frac{G^*[H^{'}]}{G^*[H]}\ \mathcal{R}$$

Proof. By induction on the form of the tree-hypersequent $G[H]$. ⊠

Lemma 4.14. *Let $G[H]$ be any tree-hypersequent of the calculus $CSGL$ together with an occurrence of a tree-hypersequent in it, and $G[H^{'}]$ the result of the application of one of the logical rules or of the rule $\Box K_{gl}$ on $G[H]$. If for a rule $\mathcal{R}$ we have:*

$$\frac{G^*[H^{'}]}{G[H']}\ \mathcal{R}$$

then it holds that:

$$\frac{G^*[H]}{G[H]}\ \mathcal{R}$$

Proof. By induction on the form of the tree-hypersequent $G[H^{'}]$. ⊠

4.1 Adequacy of CSGL

In this section we prove that the sequent calculus $CSGL$ proves exactly the same formulas as the corresponding Hilbert system **GL**. In order to get such result, we firstly need to introduce some preliminary notions and lemmas.

Definition 4.15. Let $\mathfrak{M} = \langle W, R, v\rangle$ be a model of Kripke semantics, $i \in W$, $G \in$ THS,

$$i \models G$$

is inductively defined in the following way:

[-] $i \models M \Rightarrow N$ iff $\exists\beta \in M(i \not\models \beta)$ or $\exists\gamma \in N(i \models \gamma)$

By adopting the convention, for P multiset of formulas:

$$i \models P := \exists\alpha \in P(i \models \alpha)$$

we can more succinctly write: $i \models M \Rightarrow N$ iff $i \models \neg M, N$, where $\neg M = \{\neg\beta \mid \beta \in M\}$.

[-] $i \models \Gamma/\underline{X}$ iff $i \models \Gamma$ or $\exists G \in \underline{X}\ \forall j(iRj \to j \models G)$

By adopting the convention:

$$i \models^* ... := \forall j(iRj \to j \models ...)$$

we can more succinctly write: $i \models \Gamma/\underline{X}$ iff $i \models \Gamma$ or $\exists G \in \underline{X}$, $i \models^* G$

Lemma 4.16. *(a) Let* Γ, $\Delta \in$ *SEQ*, $G \in$ *THS,*

if $\forall i\ (i \models \Gamma \to i \models \Delta)$

then, $\forall i(i \models G[\Gamma] \to i \models G[\Gamma/\Delta])$

(b) Let Γ_1, Γ_2, $\Delta \in$ *SEQ*, $G \in$ *THS,*

if $\forall i\ (i \models \Gamma_1$ *and* $i \models \Gamma_2 \to i \models \Delta)$

then, $\forall i(i \models G[\Gamma_1]$ *and* $i \models G[\Gamma_2] \to i \models [\Gamma/\Delta])$

(c) Let J, H, *and* $G \in$ *THS* $\setminus$ *SEQ,*

if $\forall i\ (i \models J \to i \models H)$

then, $\forall i(i \models G[J] \to i \models G[J/H])$

Proof. By induction on G. $\boxtimes$

The following important lemma has been established by Negri (2005).

Lemma 4.17. *For every interpretation in transitive frames without infinite ascending* $\mathcal{R}$*-chains, we have that, for all* i *and all* α*:*

$$i \models_{\mathfrak{cfgl}} \Box\alpha \quad iff \quad \forall j\ (iRj \text{ and } j \models_{\mathfrak{cfgl}} \Box\alpha \ \to\ j \models_{\mathfrak{cfgl}} \alpha)$$

where $i \models_{\mathfrak{cfgl}} \alpha$ stands for: the formula α is satisfied at the world i in the class of transitive frames without infinite ascending $\mathcal{R}$-chains.

Proof. See Negri (2005), pp. 524-525. $\boxtimes$

Lemma 4.18. *If $\vdash G$ in CSGL, then $\models_{\mathfrak{cfgl}} G$, where $\models_{\mathfrak{cfgl}} G$ stands for: G is valid in the class of transitive frames without infinite ascending R-chains.*

Proof. By induction on the height of the derivation of the premise, with the help of Lemmas 4.16 and 4.17. As an example we show the validity of the rule $\Box K_{gl}$. Let us consider the rule in the following form:

$$\frac{\Gamma/\Box\alpha \Rightarrow \alpha}{\Gamma, \Box\alpha}$$

By the inductive hypothesis we have: $\forall i\ (i \models_{\mathfrak{cfgl}} \Gamma$ or $i \models^*_{\mathfrak{cfgl}} \neg\Box\alpha, \alpha)$, i.e. $\forall i\ (i \models_{\mathfrak{cfgl}} \Gamma$ or $\forall j\ (iRj \to j \not\models_{\mathfrak{cfgl}} \Box\alpha$ or $j \models_{\mathfrak{cfgl}} \alpha))$. By logic we get: $\forall i$ $(i \models_{\mathfrak{cfgl}} \Gamma$ or $\forall j\ (iRj \to (j \models_{\mathfrak{cfgl}} \Box\alpha \to j \models_{\mathfrak{cfgl}} \alpha)))$, and hence, again by logic, we get: $\forall i\ (i \models_{\mathfrak{cfgl}} \Gamma$ or $\forall j\ (iRj$ and $j \models_{\mathfrak{cfgl}} \Box\alpha \to j \models_{\mathfrak{cfgl}} \alpha))$. By definition of the forcing relation in transitive frames without infinite ascending $\mathcal{R}$ - chains (see Lemma 4.17 above), we have $\forall i\ (i \models_{\mathfrak{cfgl}} \Gamma$ or $i \models_{\mathfrak{cfgl}} \Box\alpha)$ which is nothing else than the conclusion of the rule. Finally, by Lemma 4.16, we have that the rule $\Box K_{gl}$ is valid in the class of transitive frames without infinite ascending $\mathcal{R}$-chains. $\boxtimes$

Corollary 4.19. *If $\vdash G$ in CSGL, then $\vdash (G)^\tau$ in* **GL**.

Proof. By Lemma 4.18 and the completeness theorem between the class of transitive frames without infinite ascending R-chains and the Hilbert-style system **GL**. $\boxtimes$

In order to prove the completeness of the calculus $CSGL$, we firstly show the following Lemma and its Corollary.

Lemma 4.20. *All the tree-hypersequents of the form $G[\Box\alpha, \Gamma/\Sigma, \Box\alpha]$ are derivable in CSGL.*

Proof. Root-first, by steps of $\Box K_{gl}$, 4 and $\Box A_{gl}$. $\boxtimes$

Corollary 4.21. *The rule:*

$$\frac{G[\Box\alpha, \Gamma/(\alpha, \Sigma/\underline{X})]}{G[\Box\alpha, \Gamma/(\Sigma/\underline{X})]}\ \Box A$$

is derivable in CSGL.

Proof. By Lemma 4.20, the left premise of $\Box A_{gl}$ is derivable in $CSGL$. $\boxtimes$

Although the two rules $\Box A$ and $\Box A_{gl}$ are interderivable, the use of $\Box A_{gl}$ seems essential, as we will see in a moment, in the proof of cut-elimination (notice that Negri makes an analogous remark for her rules). However, we can think of a tree-hypersequent calculus $CSGL^*$ obtained by substituting in $CSGL$ the rule $\Box A_{gl}$ with the rule $\Box A$. The two tree-hypersequent calculi $CSGL$ and $CSGL^*$ are equivalent. The structural properties of the latter are the same as the ones the sequent calculus $G3KGL$, exposed on page 529 in Negri (2005).

Lemma 4.22. *If* $\vdash \alpha$ *in* **GL**, *then* $\vdash\Rightarrow \alpha$ *in* $CSGL$.

Proof. By induction on the height of proofs in **GL**. In order to further acquaint the reader with the calculus $CSGL$, we present the proof of the Löb's axiom. We have:

$CSGL \vdash\Rightarrow \Box(\Box\alpha \to \alpha) \to \Box\alpha$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box(\Box\alpha \to \alpha) \Rightarrow /\Box\alpha \Rightarrow \alpha, \Box\alpha \qquad \Box(\Box\alpha \to \alpha) \Rightarrow /\alpha, \Box\alpha \Rightarrow \alpha}{\Box(\Box\alpha \to \alpha) \Rightarrow /\Box\alpha \to \alpha, \Box\alpha \Rightarrow \alpha}\to A}{\Box(\Box\alpha \to \alpha) \Rightarrow /\Box\alpha \Rightarrow \alpha}\Box A}{\Box(\Box\alpha \to \alpha) \Rightarrow \Box\alpha}\Box K_{gl}}{\Rightarrow \Box(\Box\alpha \to \alpha) \to \Box\alpha}\to K$$

⊠

Theorem 4.23. *The calculus* $CSGL$ *is sound and complete with respect to the system* **GL**.

Proof. By Corollary 4.19 and Lemma 4.22. ⊠

4.2 Cut-elimination Theorem for CSGL

In this section we show that the cut-rule is admissible in the calculus $CSGL$. In order to prove this theorem we firstly need to introduce the following Lemma and Definition.

Lemma 4.24. *Given three zoom tree-hypersequents* $I[*]$, $J[*]$ *and* $H[*]$, *such that* $I[*] \sim J[*] \sim H[*]$, *if there is a rule* $\mathcal{R}$ *of* $CSGL$ *and a sequent* Γ *such that:*

$$\dfrac{J[\Gamma]}{I[\Gamma]}\mathcal{R}$$

then, for any Δ *we have:*

$$\frac{J \otimes H[\Delta]}{I \otimes H[\Delta]}\ \mathcal{R}$$

Proof. By induction on the form of the tree-hypersequent $J[*]$, noticing that the rule $\mathcal{R}$ does not change the structure of the tree-hypersequent. $\boxtimes$

Definition 4.25. Semantically when we consider a finite tree-frame, we can talk about its longest branch, i.e. a branch such that no other branch of the tree contains more worlds. Analogously, we can talk about the longest branch of a tree-hypersequent as the branch such that no other branch of the tree-hypersequent contains more sequents.

Let us call the *length* of a tree-hypersequent the number of sequents contained in its longest branch. The *position* of a sequent Γ in a tree-hypersequent G in a proof d is defined as the difference between the length of the longest tree-hypersequent occurring in the proof and the number of sequents that preceed Γ.

Lemma 4.26. *Let $G[\Gamma, \alpha]$ and $G^{'}[\alpha, \Pi]$ be such that $G[\Gamma, \alpha] \sim G^{'}[\alpha, \Pi]$. If:*

$$\frac{\begin{array}{c}\vdots\ {\scriptstyle d_1} \\ G[\Gamma, \alpha]\end{array} \quad \begin{array}{c}\vdots\ {\scriptstyle d_2} \\ G[\alpha, \Pi]\end{array}}{G \otimes G^{'}[\Gamma \centerdot \Pi]}\ cut_{\alpha}$$

and d_1 and d_2 do not contain any other application of the cut-rule, then we can construct a proof of $G \otimes G^{'}[\Gamma \centerdot \Pi]$ without any application of the cut- rule.

Proof. The proof is developed by induction on the complexity of the cut-formula (i.e. the number (≥ 0) of the occurrences of logical symbols in the cut-formula α), with subinduction on the position of the two sequents on which we apply the cut (see Definition 4.25) and with a third subinduction on the sum of the heights of the derivations of the premises of the cut-rule. We distinguish cases by the last rule applied on the left premise.

`Case 1.` $G[\Gamma, \alpha]$ is an initial tree-hypersequent. Then either the conclusion is also an initial tree-hypersequent or the cut can be replaced by various applications of the internal and external weakening rules on $G^{'}[\alpha, \Pi]$.

`Case 2.` $G[\Gamma, \alpha]$ is inferred by a rule $\mathcal{R}$ in which α is not principal. This case can be standardly solved by induction on the sum of the heights of the derivations of the premises of the cut-rule, with the help of Lemma 4.24. Indeed there is no rule that is able to change the position of the sequent where we cut, and, on the other hand, the definition of product assures that the structure of the tree-hypersequent stays unchanged, therefore no problem arises. However, for the sake of clarity, let us consider the example of the rule $\Box K_{gl}$. We have:

$$\dfrac{\dfrac{G[\Gamma, \alpha / \Box\beta \Rightarrow \beta]}{G[\Gamma, \alpha, \Box\beta]}\,\Box K_{gl} \qquad \begin{matrix}\vdots \\ G'[\alpha, \Pi]\end{matrix}}{G \otimes G'[\Gamma \centerdot \Pi, \Box\beta]}\, cut_{\alpha}$$

We reduce to:

$$\dfrac{\dfrac{G[\Gamma, \alpha / \Box\beta \Rightarrow \beta] \qquad G^{'}[\alpha, \Pi]}{G \otimes G^{'}[\Gamma \centerdot \Pi / \Box\beta \Rightarrow \beta]}\, cut_{\alpha}}{G \otimes G'[\Gamma \centerdot \Pi, \Box\beta]}\,\Box K_{gl}$$

`Case 3.` $G[\Gamma, \alpha]$ is inferred by a rule $\mathcal{R}$ in which α is principal. We distinguish two subcases: in one subcase $\mathcal{R}$ is a propositional rule, in the other $\mathcal{R}$ is a modal rule.

`Case 3.1.` We consider, for illustration, the case where rule that introduces $G[\Gamma, \alpha]$ is $\neg K$ and $\alpha \equiv \neg\beta$, we have:

$$\dfrac{\dfrac{G[\beta, \Gamma]}{G[\Gamma, \neg\beta]}\,\neg K \qquad \begin{matrix}\vdots \\ G'[\neg\beta, \Pi]\end{matrix}}{G \otimes G'[\Gamma \centerdot \Pi]}\, cut_{\neg\beta}$$

By applying Lemma 4.11 on $G^{'}[\neg\beta, \Pi]$, we obtain $G^{'}[\Pi, \beta]$. We replace the previous cut with the following one which is eliminable by induction on the complexity[4] of the cut-formula:

$$\dfrac{G^{'}[\Pi, \beta] \qquad G[\beta, \Gamma]}{G \otimes G'[\Gamma \centerdot \Pi]}\, cut_{\beta}$$

`Case 3.2.` $\mathcal{R}$ is $\Box K$ and $\alpha \equiv \Box\beta$. We have the following situation:

$$\dfrac{\dfrac{G[\Gamma / \Box\beta \Rightarrow \beta]}{G[\Gamma, \Box\beta]}\,\Box K_{gl} \qquad \begin{matrix}\vdots \\ G'[\Box\beta, \Pi]\end{matrix}}{G \otimes G'[\Gamma \centerdot \Pi]}\, cut_{\Box\beta}$$

We have to consider the last rule $\mathcal{R}^{'}$ of d_2. If there is no rule $\mathcal{R}^{'}$ that introduces $G^{'}[\Box\beta, \Pi]$ because $G^{'}[\Box\beta, \Pi]$ is an initial tree-hypersequent, then we can solve

[4]Note that, since we eliminate the cut by induction on the complexity of the cut-formula, the fact that the logical rules are invertible, and not height-preserving invertible, does not affect the course of the proof.

the case as in 1. If $\mathcal{R}^{'}$ is a rule in which $\Box\beta$ is not principal, we solve the case as in 2. The only problematic cases are those cases where $\mathcal{R}^{'}$ is the rule $\Box A_{gl}$ or the rule 4. We will analyse each of them.

$\Box A_{gl}$:[5]

$$\dfrac{\dfrac{G[\Gamma/\Box\beta \Rightarrow \beta]}{G[\Gamma, \Box\beta]}\,\Box K_{gl} \qquad \dfrac{G^{''}[\Box\beta, \Pi/(\Psi, \Box\beta/\underline{X})] \quad G^{''}[\Box\beta, \Pi/(\beta, \Psi/\underline{X})]}{G^{''}[\Box\beta, \Pi/(\Psi/\underline{X})]}\,\Box A_{gl}}{G \otimes G^{''}[\Gamma \centerdot \Pi/(\Psi/\underline{X})]}\,Cut_{\Box\beta}$$

The reduction is done in several passages. We expose them one-by-one.

First passage:

$$\dfrac{G[\Gamma, \Box\beta] \quad G^{''}[\Box\beta, \Pi/(\Psi, \Box\beta/\underline{X})]}{G \otimes G^{''}[\Gamma \centerdot \Pi/(\Psi, \Box\beta/\underline{X})]}\,Cut_{\Box\beta}$$

this cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule.

Second passage:

$$\dfrac{G[\Gamma, \Box\beta] \quad G^{''}[\Box\beta, \Pi/(\beta, \Psi/\underline{X})]}{G \otimes G^{''}[\Gamma \centerdot \Pi/(\beta, \Psi/\underline{X})]}\,Cut_{\Box\beta}$$

this cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule.

Third passage:

$$\dfrac{G \otimes G^{''}[\Gamma \centerdot \Pi/(\Psi, \Box\beta/\underline{X})] \quad G[\Gamma/\Box\beta \Rightarrow \beta]}{G \otimes G \otimes G^{''}[\Gamma \centerdot \Gamma \centerdot \Pi/(\Psi, \beta/\underline{X})]}\,Cut_{\Box\beta}$$

this cut is eliminable by induction on the position of the sequents that contain the cut-formula.

Fourth passage:

$$\dfrac{\dfrac{G \otimes G \otimes G^{''}[\Gamma \centerdot \Gamma \centerdot \Pi/(\Psi, \beta/\underline{X})] \quad G \otimes G^{''}[\Gamma \centerdot \Pi/(\beta, \Psi/\underline{X})]}{G \otimes G \otimes G^{''} \otimes G \otimes G^{''}[\Gamma \centerdot \Gamma \centerdot \Pi \centerdot \Gamma \centerdot \Pi/(\Psi \centerdot \Psi/\underline{X}; \underline{X})]}\,Cut_{\beta}}{G \otimes G^{''}[\Gamma \centerdot \Pi/(\Psi/\underline{X})]}\,merge^{*}+C^{*}$$

[5]Note that $G^{''}[\Box\beta, \Pi/(\Psi/\underline{X})]$ is just another way of writing $G^{'}[\Box\beta, \Pi]$

where the notation $merge^* + C^*$ stands for: iterated applications of the rules $merge$, CA and CK. This cut is eliminable by induction on the complexity of the cut-formula.

4:

$$\dfrac{\dfrac{G[\Gamma/\Box\beta \Rightarrow \beta]}{G[\Gamma, \Box\beta]}\,\Box K_{gl} \qquad \dfrac{G^{''}[\Box\beta, \Pi/(\Box\beta, \Psi/\underline{X})]}{G^{''}[\Box\beta, \Pi/(\Psi/\underline{X})]}\,4}{G \otimes G^{''}[\Gamma \centerdot \Pi/(\Psi/\underline{X})]}\,Cut_{\Box\beta}$$

In order to solve this case we must analyse each of the rules that may have introduced the tree-hypersequent $G^{''}[\Box\beta, \Pi/(\Box\beta, \Psi/\underline{X})]$. We go up the derivation until either a rule $\mathcal{R}^{''}$ applies to a formula different from the $\Box\beta$'s or a rule $\mathcal{R}^{''}$ different from 4 applies to some of the $\Box\beta$'s; this way we stop in front of the following situation (which is to say in front of the tree-hypersequent that is the conclusion of the rule $\mathcal{R}^{''}$):

$$\Diamond: \quad G^{'}[\Box\beta, \Pi]\ \{\Box\beta, \Upsilon\}$$

where $\{\Box\beta, \Upsilon\}$ represents all the sequents in G that are successive to the sequent Π and contain the formula $\Box\beta$ on the left side.

We analyse each of the rules that can have inferred this tree-hypersequent.

- $\Diamond$ is an initial tree-hypersequent. If $\Box\beta$ is not the principal formula of the axiom, then even the conclusion of the cut is an initial tree-hypersequent and the case is solved. Otherwise there is a sequence in $\Diamond$ that contains the formula $\Box\beta$ on both its right and left sides. This sequent can be the sequent $\Box\beta, \Pi$, and then we get the conclusion by several applications of the rules of external and internal weakening on the left premise of the cut. Otherwise this sequent may belong to $\{\Box\beta, \Upsilon\}$ and therefore be n steps after the sequent $\Box\beta, \Pi$. In this case, we apply the (admissible) rule $\tilde{4}$, $n+1$ times, on the left premise $G[\Gamma/\Box\beta \Rightarrow \beta]$, to obtain a tree-hypersequent where $\Box\beta \Rightarrow \beta$ is no longer after Γ, but n empty sequences after it. By applying, firstly, the rule $\Box K_{gl}$, then, repeatedly, the rules of external and internal weakening, on this tree-hypersequent, we obtain the conclusion.
- $\Diamond$ has been inferred by a rule $\mathcal{R}^{''}$ that does not have any $\Box\beta$ as principal formula. In this case the technique consists of: (i) applying the rule 4 n-times on the premise of the tree-hypersequent $\Diamond$; (ii) operating as in case 2.
- $\Diamond$ has been inferred by the modal rule $\Box A_{gl}$ that has $\Box\beta$ as principal formula. We shall first of all distinguish the following two subcases.

  (a) The rule $\Box A_{gl}$ has been applied on two sequents belonging to $\{\Box\beta, \Upsilon\}$, let us suppose the sequents $\Box\beta, \Xi/\Box\beta, \Omega$. Hence we have the following situation:

$$\dfrac{\dfrac{G[\Gamma/\Box\beta \Rightarrow \beta]}{G[\Gamma,\Box\beta]}\,{\scriptstyle \Box K_{gl}} \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{G^{'''}[\Box\beta,\Pi]\ [\Box\beta,\Xi/(\Box\beta,\Omega,\Box\beta/\underline{Y})] \qquad G^{'''}[\Box\beta,\Pi]\ [\Box\beta,\Xi/(\Box\beta,\beta,\Omega/\underline{Y})]}{G^{'''}[\Box\beta,\Pi]\ [\Box\beta,\Xi/(\Box\beta,\Omega/\underline{Y})]}\,{\scriptstyle \Box A_{gl}}}{\vdots}\,{\scriptstyle 4}\end{array}}{G^{'''}[\Box\beta,\Pi]\ [\Xi/(\Omega/\underline{Y})]}\,{\scriptstyle 4}}{G\otimes G^{'''}[\Gamma \centerdot \Pi]\ [\Xi/(\Omega/\underline{Y})]}\,{\scriptstyle cut_{\Box\beta}}$$

We proceed by the following three steps:

(a1) we apply on the premises of the rule $\Box A_{gl}$ the rule 4, n-times. This way we obtain the two tree-hypersequents:

Th1 : $\quad G^{'''}[\Box\beta,\Pi]\ [\Xi/(\Omega,\Box\beta/\underline{Y})]$

Th2 : $\quad G^{'''}[\Box\beta,\Pi]\ [\Xi/(\beta,\Omega/\underline{Y})]$

(a2) We apply the rule $\tilde{4}$ to the tree-hypersequent $G[\Gamma/\Box\beta \Rightarrow \beta]$ a number of times sufficient to get $\Box\beta \Rightarrow \beta$ in an equivalent position with the sequent β,Ω of the tree-hypersequent Th2 (and therefore also with the sequent $\Omega,\Box\beta$ of the tree-hypersequent Th1). This way we obtain a tree-hypersequent where $\Box\beta \Rightarrow \beta$ is no longer after Γ, but n empty sequences after. Let us note this as: $G[\Gamma]\ [\Box\beta \Rightarrow \beta]$.

(a3) We are now able to apply the following reductions:

First passage:

$$\dfrac{G[\Gamma,\Box\beta] \quad G^{'''}[\Box\beta,\Pi]\ [\Xi/(\Omega,\Box\beta/\underline{Y})]}{G\otimes G^{'''}[\Gamma \centerdot \Pi]\ [\Xi/(\Omega,\Box\beta/\underline{Y})]}\,{\scriptstyle Cut_{\Box\beta}}$$

this cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule.

Second passage:

$$\dfrac{G[\Gamma,\Box\beta] \quad G^{'''}[\Box\beta,\Pi]\ [\Xi/(\beta,\Omega/\underline{Y})]}{G\otimes G^{'''}[\Gamma \centerdot \Pi]\ [\Xi/(\beta,\Omega/\underline{Y})]}\,{\scriptstyle Cut_{\Box\beta}}$$

this cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule.

Third passage:

$$\frac{G \otimes G^{'''}[\Gamma \centerdot \Pi]\ [\Xi/(\Omega, \Box\beta/\underline{Y})] \quad G[\Gamma][\Box\beta \Rightarrow \beta]}{G \otimes G \otimes G'''[\Gamma \centerdot \Gamma \centerdot \Pi][\Xi/(\Omega, \beta/\underline{Y})]}\ Cut_{\Box\beta}$$

this cut is eliminable by induction on the position of the sequents that contain the cut-formula.

Fourth passage:

$$\cfrac{\cfrac{G \otimes G \otimes G^{'''}[\Gamma \centerdot \Gamma \centerdot \Pi][\Xi/(\Omega, \beta/\underline{Y})] \quad G \otimes G^{'''}[\Gamma \centerdot \Pi]\ [\Xi/(\beta, \Omega/\underline{Y})]}{G \otimes G \otimes G^{'''} \otimes G \otimes G^{'''}[\Gamma \centerdot \Gamma \centerdot \Pi \centerdot \Gamma \centerdot \Pi]\ [\Xi \centerdot \Xi/(\Omega \centerdot \Omega//\underline{Y}; \underline{Y})]}\ Cut_{\beta}}{G \otimes G'''[\Gamma \centerdot \Pi]\ [\Xi/(\Omega/\underline{Y})]}\ merge^{*}+C^{*}$$

this cut is eliminable by induction on the complexity of the cut-formula.

(b) The rule $\Box A_{gl}$, with $\Box\beta$ principal formula, has been applied on the sequents $\Box\beta, \Pi/\Box\beta, \Psi, \Box\beta$ and $\Box\beta, \Pi/\Box\beta, \beta, \Psi$. In this case we apply the rule 4, n-times (see (a1)), on the premises of the rule $\Box A_{gl}$ and then we simply proceed as in $\Box A_{gl}$.

Theorem 4.27. *Every proof d in $CSGL$ can be effectively transformed in a proof $d^{'}$ where there is no application of the cut-rule.*

Proof. It follows from the previous Lemma by induction on the number of cuts. ⊠

5 Conclusion and Further Work

In Poggiolesi (2009a) we suggested that the tree-hypersequent method could have been successfully applied to the system GL. In this paper we have shown that this intuition was correct: we have presented the tree-hypersequent calculus $CSGL$ which is contraction-free, cut-free, and satisfies the subformula property. It would now be interesting to analyse which other properties this calculus satisfies, such as the decidability property or the interpolation theorem.

This is a first topic for future research. A second one could be the following. As it has probably already emerged in the previous sections, $CSGL$ is quite similar to Negri's calculus $G3GL$: indeed, except for the rule 4 that only characterises $CSGL$, the propositional and modal rules of the two calculi seem to be based on a same intuition. Given this situation, a question naturally arises: what is the exact relation between the two calculi? Is it possible to find a translation from the tree-hypersequent calculi to the labelled calculi and vice-versa?

Finally a third question concerns the applicability of the tree-hypersequent method. The results obtained up to now are promising: we have tree-hypersequent calculi for the main systems of modal logic (namely, $K, KT, S4, S5$...), for GL and for propositional dynamic logic (see Hill and Poggiolesi (2009)). Are there other logics, such as temporal logics, that can be captured by this extension of the classical sequent calculus? The challenge is open.

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