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2 authors:

Eduardo Fermé
Universidade da Madeira

Maurício D. L. Reis
Universidade da Madeira

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EPISTEMIC ENTRENCHMENT-BASED MULTIPLE CONTRACTIONS

EDUARDO FERMÉ and MAURÍCIO D. L. REIS

Centro de Ciências Exactas e da Engenharia,
Universidade da Madeira and Centre for Artificial Intelligence (CENTRIA),
Universidade Nova de Lisboa

Abstract. In this article we present a new class of multiple contraction functions—the *epistemic entrenchment-based multiple contractions*—which are a generalization of the epistemic entrenchment-based contractions (Gärdenfors, 1988; Gärdenfors & Makinson, 1988) to the case of contractions by (possibly nonsingleton) sets of sentences and provide an axiomatic characterization for that class of functions. Moreover, we show that the class of *epistemic entrenchment-based multiple contractions* coincides with the class of *system of spheres-based multiple contractions* introduced in Fermé & Reis (2012).

§1. Introduction. The standard model of theory change was proposed by Alchourrón, Gärdenfors and Makinson in their seminal paper (Alchourrón *et al.*, 1985) and is, nowadays, known as the AGM model. One of the main issues that is addressed by this model is the modeling of how information is removed from the set of beliefs of an agent, that is, the characterization of contraction functions. In that regard, in the mentioned paper, the class of *partial meet contractions* was introduced and axiomatically characterized. Subsequently, several constructive models have been presented in the literature for the class of contraction functions proposed in the AGM framework, such as the *system of spheres-based contractions* (Grove, 1988), *safe/kernel contractions* (Alchourrón & Makinson, 1985; Hansson, 1994), and the *epistemic entrenchment-based contractions* (Gärdenfors, 1988; Gärdenfors & Makinson, 1988).

In a posterior stage of the development of the theory of belief contraction, several researchers (e.g., Niederée, 1991; Hansson, 1989; Fuhrmann, 1991; Fuhrmann & Hansson, 1994) pointed out the need for defining operations that could account for the removal of sets with more than one element from a theory. In particular it was remarked in Fuhrmann & Hansson (1994) that a simple evidence of the usefulness and the necessity of the study of package contractions is the fact that, in general, a set which is intuitively acceptable as possible result of the package contraction of a theory **K** by a set of sentences, say $\{\alpha, \beta\}$, is different from the set which results of:

1. contracting **K** by (the single sentence) $\alpha \wedge \beta$, because to remove a conjunction it suffices to remove one of the conjuncts.
2. contracting **K** by $\alpha \vee \beta$, since, however the removal of a disjunction from a theory implies the removal of both disjuncts, the converse does not hold, that is, in order to remove the set $\{\alpha, \beta\}$ from **K** it is not necessary to remove the sentence $\alpha \vee \beta$ from **K** (to see that this is so it is enough to consider the case when $\beta = \neg\alpha$).

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3. first contracting by α and then (contracting the result of such contraction) by β, given that, on the one hand, the result of first contracting by α and then by β is not, in general, identical to that of first contracting by β and then by α, and, on the other hand, it is implicit in the notion of multiple contraction that, in such a process, all the sentences to be contracted are treated equally.

In all that follows we will use the expression *multiple contraction* to refer to an operation of the above described kind. More precisely, given a belief set $\mathbf{K}$, by a multiple contraction on $\mathbf{K}$ we mean a function that receives any finite set of sentences B and returns (if possible) a new belief set that is contained in $\mathbf{K}$ and which does not contain any of the elements of B (note that in Fuhrmann & Hansson, 1994 a (multiple) contraction function of this kind is designated by *package contraction*). We will use the expression *singleton contraction* to designate contractions by a single sentence (like the ones mentioned in the first paragraph above).

That led, in particular, to the generalization of several of the existing models (of AGM contractions): The partial meet multiple contractions were presented in Hansson (1989) and Fuhrmann & Hansson (1994), the kernel multiple contractions were introduced in Fermé *et al.* (2003), Hansson (2010) exposed the class of multiple specified meet contraction, Spohn (2010) proposed a ranking-theoretic account of multiple package contraction and, in Reis (2011) and Fermé & Reis (2012), the class of *system of spheres-based multiple contractions* (which are a generalization of Grove's system of spheres-based (singleton) contractions) was introduced.

In the present article we will generalize the epistemic entrenchment-based contractions to the case of multiple contraction and obtain an axiomatic characterization for that class of multiple contraction functions. Additionally we will show that the class of such operations coincides with the class of system of spheres-based multiple contractions.

This article is organized as follows: In Section §2. we provide the notation and background needed for the rest of the paper. In Section §3. we propose a definition of *epistemic entrenchment-based multiple contraction* which fulfils the required properties. After that, in Section §4., we present an axiomatic characterization for that (newly defined) class of multiple contraction functions. Then, in Section §5. we show that the class of epistemic entrenchment-based multiple contractions coincides with the class of system of spheres-based multiple contractions. Finally, in Section §6. we summarize the main contributions of the paper and present a brief discussion regarding the postulates included in the axiomatic characterization presented.

§2. Background.

2.1. Formal Preliminaries. We will assume a language $\mathcal{L}$ that is closed under truth-functional operations. We shall make use of a consequence operation Cn that takes sets of sentences to sets of sentences and which satisfies the standard Tarskian properties, namely *inclusion, monotony*, and *iteration*. Furthermore we will assume that Cn satisfies *supraclassicality, compactness* and *deduction*. We will sometimes use $Cn(\alpha)$ for $Cn(\{\alpha\})$, $A \vdash \alpha$ for $\alpha \in Cn(A)$, $\vdash \alpha$ for $\alpha \in Cn(\emptyset)$, $A \nvdash \alpha$ for $\alpha \notin Cn(A)$, $\nvdash \alpha$ for $\alpha \notin Cn(\emptyset)$. The letters $\alpha, \alpha_i, \beta, \ldots$ (except for γ) will be used to denote sentences. $\top$ stands for an arbitrary tautology and $\perp$ for an arbitrary contradiction. $A, A_i, B, \ldots$ shall denote subsets of sentences of $\mathcal{L}$. $\mathbf{K}$ is reserved to represent a set of sentences that is closed under logical consequence (i.e., $\mathbf{K} = Cn(\mathbf{K})$) – such a set is called a *belief set* or *theory*. We shall denote the set of all maximal consistent subsets of $\mathcal{L}$ by $\mathcal{M}_{\mathcal{L}}$. We will use the expression *possible* *world* (or just *world*) to designate an element of $\mathcal{M}_{\mathcal{L}}$. $\mathcal{M}, \mathcal{N}_i, \mathcal{W}, \ldots$, (except for $\mathcal{L}$ and $\mathcal{P}$) shall be used to denote subsets of $\mathcal{M}_{\mathcal{L}}$. Such sets are called *propositions*. Given a set of sentences R, the set consisting of all the possible worlds that contain R is denoted by $\|R\|$. The elements of $\|R\|$ are the R-worlds. $\|\varphi\|$ is an abbreviation of $\|\{\varphi\}\|$ and the elements of $\|\varphi\|$ are the φ-worlds. To any set of possible worlds $\mathcal{V}$ we associate a belief set $Th(\mathcal{V})$ given by $Th(\mathcal{V}) = \bigcap \mathcal{V}$ – under the assumption that $\bigcap \emptyset = \mathcal{L}$.

2.2. Singleton Contraction. Singleton contraction is the contraction of a belief set $\mathbf{K}$ by a sentence α. We will present two different models of contraction and the interrelation among them.

2.2.1. System of Spheres-based Contractions. Now we recall the definitions of a *system of spheres* and of the *system of spheres-based contractions* presented in Grove (1988).

DEFINITION 2.1 (Grove, 1988). *Let $\mathcal{X}$ be a subset of $\mathcal{M}_{\mathcal{L}}$. A system of spheres, or spheres' system, centred on $\mathcal{X}$ is a collection $\mathbb{S}$ of subsets of $\mathcal{M}_{\mathcal{L}}$, that is, $\mathbb{S} \subseteq \mathcal{P}(\mathcal{M}_{\mathcal{L}})$, that satisfies the following conditions:*

($\mathbb{S}1$) $\mathbb{S}$ is totally ordered with respect to set inclusion, that is, if $\mathcal{U}, \mathcal{V} \in \mathbb{S}$, then $\mathcal{U} \subseteq \mathcal{V}$ or $\mathcal{V} \subseteq \mathcal{U}$.
($\mathbb{S}2$) $\mathcal{X} \in \mathbb{S}$, and if $\mathcal{U} \in \mathbb{S}$ then $\mathcal{X} \subseteq \mathcal{U}$.
($\mathbb{S}3$) $\mathcal{M}_{\mathcal{L}} \in \mathbb{S}$ (and so it is the largest element of $\mathbb{S}$).
($\mathbb{S}4$) For every $\varphi \in \mathcal{L}$, if there is any element in $\mathbb{S}$ intersecting $\|\varphi\|$ then there is also a smallest element in $\mathbb{S}$ intersecting $\|\varphi\|$.

The elements of $\mathbb{S}$ are called spheres. For any consistent sentence $\varphi \in \mathcal{L}$, the smallest sphere in $\mathbb{S}$ intersecting $\|\varphi\|$ is denoted by $\mathbb{S}_\varphi$ and $f_{\mathbb{S}}(\varphi)$ denotes the set consisting of the φ-worlds closest *to $\mathcal{X}$, i.e.,*

$$f_{\mathbb{S}}(\varphi) = \|\varphi\| \cap \mathbb{S}_\varphi.$$

DEFINITION 2.2 (Grove, 1988). *Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. The* $\mathbb{S}$-based contraction on $\mathbf{K}$ *is the contraction operation $-_{\mathbb{S}}$ defined, for any $\varphi \in \mathcal{L}$, by:*

$$\mathbf{K} -_{\mathbb{S}} \varphi = \begin{cases} Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\varphi)) & \text{, if } \nvdash \varphi \\ \mathbf{K} & \text{, if } \vdash \varphi. \end{cases}$$

An operation – on $\mathbf{K}$ is a system of spheres-based contraction on $\mathbf{K}$ *if and only if there is a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, such that, for all sentences $\varphi \in \mathcal{L}$, $\mathbf{K} - \varphi = \mathbf{K} -_{\mathbb{S}} \varphi$.*

2.2.2. Epistemic Entrenchment-based Contractions. We start by introducing the concept of *epistemic entrenchment relation* which is the fundamental concept underlying the definition of the above-mentioned class of contraction functions.

DEFINITION 2.3 (Gärdenfors, 1988, Gärdenfors & Makinson, 1988). *An ordering of* epistemic entrenchment *with respect to a belief set $\mathbf{K}$ is a binary relation $\leq$ on $\mathcal{L}$ which satisfies the following postulates:*

(EE1) For all $\alpha, \beta, \delta \in \mathcal{L}$, if $\alpha \leq \beta$ and $\beta \leq \delta$ then $\alpha \leq \delta$. (Transitivity)
(EE2) For all $\alpha, \beta \in \mathcal{L}$, if $\alpha \vdash \beta$ then $\alpha \leq \beta$. (Dominance)
(EE3) For all $\alpha, \beta \in \mathcal{L}$, $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$. (Conjunctiveness)
(EE4) When $\mathbf{K} \neq \mathcal{L}$, $\alpha \notin \mathbf{K}$ iff $\alpha \leq \beta$ for all $\beta \in \mathcal{L}$. (Minimality)
(EE5) If $\beta \leq \alpha$ for all $\beta \in \mathcal{L}$, then $\vdash \alpha$. (Maximality)

$\alpha \leq \beta$ is seen as meaning that "β is at least as epistemically entrenched as α". $<$ denotes the strict and $=_{\leq}$ the symmetric part of $\leq$. $\alpha \not\leq \beta$ denotes the negation of $\alpha \leq \beta$ and it follows from (EE1)-(EE3) that $\alpha \not\leq \beta$ iff $\beta < \alpha$.

Now we proceed to the presentation of the definition of the *epistemic entrenchment-based contractions* which has been introduced in Gärdenfors (1988) and Gärdenfors & Makinson (1988).

DEFINITION 2.4 (Gärdenfors, 1988, Gärdenfors & Makinson, 1988). *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$. *The* $\leq$-based contraction *on* $\mathbf{K}$ *is the contraction operation* $-_{\leq}$ *defined, for any* $\alpha \in \mathcal{L}$, *by:*

$$\mathbf{K}-_{\leq}\alpha = \begin{cases} \{\beta \in \mathbf{K} : \alpha < \alpha \vee \beta\} & \text{, if } \not\vdash \alpha \\ \mathbf{K} & \text{, if } \vdash \alpha. \end{cases} \qquad (C_{-_{\leq}})$$

An operation – *on* $\mathbf{K}$ *is an* epistemic entrenchment-based contraction *on* $\mathbf{K}$ *if and only if there is an epistemic entrenchment relation with respect to* $\mathbf{K}$ *such that, for all sentences* $\alpha \in \mathcal{L}$, $\mathbf{K}-\alpha = \mathbf{K}-_{\leq}\alpha$.

Apart from presenting a way of defining a contraction operation based on an epistemic entrenchment relation (by means of condition $(C_{-_{\leq}})$), Gärdenfors (1988) and Gärdenfors & Makinson (1988) have also exposed the following way of proceeding to the converse construction (namely, of an epistemic entrenchment relation $\leq$ by means of a contraction function $-$):

$$\forall \alpha, \beta \in \mathcal{L},\ \alpha \leq \beta \text{ iff } \alpha \notin \mathbf{K}-\alpha \wedge \beta \text{ or } \vdash \alpha \wedge \beta. \qquad (C_{\leq})$$

The following lemma presents some properties that are satisfied by any epistemic entrenchment relation which will be useful further ahead.

LEMMA 2.5. *Let* $\leq$ *be a relation that satisfies (EE1)-(EE4). Then the following statements hold:*

(i) $\alpha \notin \mathbf{K}$ *if and only if for all* $\beta \in \mathbf{K}$, $\alpha < \beta$.
(ii) *If* $\alpha < \beta \vee \delta$ *and* $\alpha < \varepsilon \vee \neg\beta$, *then* $\alpha < \varepsilon \vee \delta$.
(iii) *If* $\alpha < \alpha \vee \neg\beta$, *then* $\beta \leq \alpha$.
(iv) *If* $\delta \leq \alpha$, $\alpha < \alpha \vee \beta$ *and* $\delta < \delta \vee \neg\alpha$, *then* $\delta < \delta \vee \beta$.
(v) *If* $\vdash \alpha \leftrightarrow \alpha'$ *and* $\vdash \beta \leftrightarrow \beta'$, *then* $\alpha \leq \beta$ *iff* $\alpha' \leq \beta'$ *(intersubstitutivity (Gärdenfors & Rott, 1995, Lemma* 4.2.1-*(i))).*
(vi) *If* $\delta < \alpha$ *and* $\delta < \beta$, *then* $\delta < \alpha \wedge \beta$ *(conjunction up (Foo, 1990, pp.* 5, *Lemma (iv))).*

2.2.3. Interrelation Between System of Spheres-based Contractions and Epistemic Entrenchment-based Contractions. Given a belief set $\mathbf{K}$, it follows immediately from the axiomatic characterizations for each of such two classes of functions, presented in Grove (1988) and Gärdenfors & Makinson (1988), respectively, that the class of the system of spheres-based contractions on $\mathbf{K}$ coincides with the class of the epistemic entrenchment-based contractions on $\mathbf{K}$.

The following proposition presents a way of defining an epistemic entrenchment relation by means of a system of spheres:

PROPOSITION 2.6 (Grove, 1988, Gärdenfors, 1988). *Let* **K** *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$. *If* $\leq$ *is the binary relation on* $\mathcal{L}$ *defined in the following way:*

$$\forall \alpha, \beta \in \mathcal{L},\ \alpha \leq \beta \text{ iff either } \mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta} \text{ or } \vdash \beta, \qquad (\leq -\mathbb{S})$$

then $\leq$ *is an epistemic entrenchment relation with respect to* **K**, *that is,* $\leq$ *satisfies conditions (EE1)–(EE5).*

On the other hand, given an arbitrary epistemic entrenchment relation $\leq$, a system of spheres $\mathbb{S}$ such that condition $(\leq -\mathbb{S})$ holds, can be defined (by means of $\leq$) as exposed in the following proposition.

PROPOSITION 2.7 (Reis, 2011). *Let* **K** *be a belief set,* $\leq$ *be an epistemic entrenchment relation with respect to* **K**, *and* $\mathbb{S}'$ *be the class of subsets of* $\mathcal{M}_{\mathcal{L}}$ *defined by:*

$$\mathbb{S}' = \{\mathcal{W}_{\alpha_i} : \alpha_i \in \mathcal{L} \setminus Cn(\emptyset)\},$$

where, for any $\alpha_i \in \mathcal{L} \setminus Cn(\emptyset)$, $\mathcal{W}_{\alpha_i}$ *is the set defined as follows:*

$$\mathcal{W}_{\alpha_i} = \|\{\alpha \in \mathcal{L} : \alpha_i < \alpha\}\|,$$

then the following statements hold:

(i) *If* $\mathbf{K} \neq \mathcal{L}$ *(i.e.,* **K** *is a consistent belief set), then the set* $\mathbb{S} = \mathbb{S}' \cup \{\mathcal{M}_{\mathcal{L}}\}$ *is a system of spheres centred on* $\|\mathbf{K}\|$.

(ii) *If* $\mathbf{K} = \mathcal{L}$, *then the set* $\mathbb{S} = \{\emptyset\} \cup \mathbb{S}' \cup \{\mathcal{M}_{\mathcal{L}}\}$ *is a system of spheres centred on* $\|\mathbf{K}\|$.

Moreover, in both cases $\mathbf{K} \neq \mathcal{L}$ *and* $\mathbf{K} = \mathcal{L}$, *it holds that the (respective) thus constructed system of spheres* $\mathbb{S}$ *and the given epistemic entrenchment relation* $\leq$ *satisfy condition* $(\leq -\mathbb{S})$.

The next proposition exposes that condition $(\leq -\mathbb{S})$ is a necessary and sufficient condition for the $\mathbb{S}$-based contraction to coincide with the $\leq$-based contraction.

PROPOSITION 2.8 (Reis, 2011, Peppas & Williams, 1995, Hansson, 1999). *Let* **K** *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\leq$ *be an epistemic entrenchment relation with respect to* **K**. *Then the* $\leq$*-based contraction on* **K**, $-_{\leq}$ *(cf. condition* $(C_{-_{\leq}})$ *in Definition* 2.4*), and the* $\mathbb{S}$*-based contraction on* **K**, $-_{\mathbb{S}}$ *(cf. Definition* 2.2*), coincide, that is,*

$$\forall \varphi \in \mathcal{L},\ \mathbf{K}{-_{\leq}}\varphi = \mathbf{K}{-_{\mathbb{S}}}\varphi,$$

if and only if condition $(\leq -\mathbb{S})$ *is satisfied.*

2.3. Multiple Contraction. As already mentioned in the introduction, several models of AGM-related contraction operators have been generalized in order to account for the case of multiple contraction, that is, contraction by sets of sentences. In this subsection we will recall the models of partial meet multiple contraction and of system of spheres-based multiple contractions that we will use in the rest of the paper.

2.3.1. Partial Meet Multiple Contractions. The *partial meet multiple contractions* are a generalization of the partial meet contraction functions introduced in Alchourrón *et al.* (1985). We start by recalling the basic concepts necessary for the definition of the partial meet contractions (Alchourrón & Makinson, 1981; Alchourrón *et al.*, 1985).

Given a belief set **K** and a set of sentences B, the *remainder set* of **K** by B is the set of maximal subsets of **K** that do not imply any element of B and is denoted by $\mathbf{K} \perp B$. The elements of $\mathbf{K}\perp B$ are the *remainders* (of $\mathbf{K}$, by B) (Alchourrón & Makinson, 1981). Since Cn is compact, it holds that $\mathbf{K}\perp B \neq \emptyset$ if and only if $B \cap Cn(\emptyset) = \emptyset$ (Alchourrón & Makinson, 1981).

DEFINITION 2.9 (Hansson, 1989, Fuhrmann & Hansson, 1994). *Let* $\mathbf{K}$ *be a belief set. A* package selection function *for* $\mathbf{K}$ *is a function* γ *such that for all sets of sentences* B*: if* $\mathbf{K}\perp B \neq \emptyset$*, then* $\emptyset \neq \gamma(\mathbf{K}\perp B) \subseteq \mathbf{K}\perp B$*, and if* $\mathbf{K}\perp B = \emptyset$ *then* $\gamma(\mathbf{K}\perp B) = \{\mathbf{K}\}$.

An operation $\div$ *is a* partial meet multiple contraction *on* $\mathbf{K}$ *if and only if there is some package selection function* γ *for* $\mathbf{K}$*, such that for all sets of sentences* B*:* $\mathbf{K}\div B = \bigcap \gamma(\mathbf{K}\perp B)$.

Fuhrmann & Hansson (1994) provided the following axiomatic characterization for *partial meet multiple contraction*:

PROPOSITION 2.10 (Fuhrmann & Hansson, 1994). *Let* $\mathbf{K}$ *be a belief set. An operation* $\div$ *is a* partial meet multiple contraction *on* $\mathbf{K}$ *if and only if it satisfies the following postulates:*

Package success *If* $B \cap Cn(\emptyset) = \emptyset$*, then* $B \cap \mathbf{K}\div B = \emptyset$.

Package inclusion $\mathbf{K}\div B \subseteq \mathbf{K}$.

Package relevance *If* $\beta \in \mathbf{K}$ *and* $\beta \notin \mathbf{K}\div B$*, then there is a set* K' *such that* $\mathbf{K}\div B \subseteq K' \subseteq \mathbf{K}$ *and* $B \cap Cn(K') = \emptyset$ *but* $B \cap Cn(K' \cup \{\beta\}) \neq \emptyset$.

Package uniformity *If every subset* X *of* $\mathbf{K}$ *implies some element of* B *if and only if* X *implies some element of* C*, then* $\mathbf{K}\div B = \mathbf{K}\div C$.

2.3.2. System of Spheres-based Multiple Contractions. In this subsection we present the definition of *system of spheres-based multiple contraction* which was introduced in Reis (2011) and Fermé & Reis (2012) and is a generalization to the case of multiple contraction of Grove's definition of system of spheres-based (singleton) contraction.

Such definition makes use of the concept of $\mathbb{S}$-based filtration of a set of sentences B, where $\mathbb{S}$ is a system of spheres centred on $\|\mathbf{K}\|$, for some given belief set $\mathbf{K}$:

DEFINITION 2.11 (Fermé & Reis, 2012). *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$.

Consider a set of sentences $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \mathcal{L}$ *such that* $B \setminus Cn(\emptyset) \neq \emptyset$.

Denote by $C_1, \ldots, C_m$ *the (different) equivalence classes in the quotient set of* $(B \setminus Cn(\emptyset))$ *by* $\backsim$*, that is,* $\{C_1, \ldots, C_m\} = (B \setminus Cn(\emptyset))/_{\backsim}$*, where* $\backsim$ *is the equivalence relation on* $B \setminus Cn(\emptyset)$ *defined by:*

$$\forall \alpha, \beta \in B \setminus Cn(\emptyset),\ \alpha \backsim \beta \ \text{ iff } \ \mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}.$$

Moreover, assume that the equivalence classes $C_1, \ldots, C_m$ *are ordered according the following condition:*

$$\text{If } 1 \leq i < j \leq m, \text{ then } \forall \alpha_r \in C_i \forall \alpha_s \in C_j\ \mathbb{S}_{\neg\alpha_s} \subset \mathbb{S}_{\neg\alpha_r}.$$

Now consider the following list of subsets of B*:*

$$\begin{aligned} B_0 &= B \cap Cn(\emptyset) \\ C_1' &= C_1 \\ C_1'' &= \{\alpha_i \in C_1' : \forall \alpha_j \in C_1'\ f_{\mathbb{S}}(\neg\alpha_j) \not\subset f_{\mathbb{S}}(\neg\alpha_i)\} \\ B_1 &= C_1''. \end{aligned}$$

Moreover, if $m > 1$ *for all* $l \in \{2, \ldots, m\}$, *let* C_l', C_l'' *and* B_l *be the sets defined by:*

$$\begin{array}{rcl} C_l' & = & \{\alpha_i \in C_l : \forall \alpha_j \in B_{l-1}\ f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|\}; \\ C_l'' & = & \{\alpha_i \in C_l' : \forall \alpha_j \in C_l'\ f_{\mathbb{S}}(\neg\alpha_j) \not\subset f_{\mathbb{S}}(\neg\alpha_i)\}; \\ B_l & = & B_{l-1} \cup C_l''. \end{array}$$

The set $B_{\mathbb{S}} = B_m$ *is the* $\mathbb{S}$-based filtration of B.

If D *is a set of sentences such that* $D \subseteq Cn(\emptyset)$, *then the* $\mathbb{S}$-based filtration of D *is the empty set and is denoted by* $D_{\mathbb{S}}$, *that is,* $D_{\mathbb{S}} = \emptyset$.

DEFINITION 2.12 (Fermé & Reis, 2012). *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$. *The* $\mathbb{S}$-based multiple contraction on $\mathbf{K}$ *is the multiple contraction function* $\div_{\mathbb{S}}$ *defined by:*

$$\mathbf{K}\div_{\mathbb{S}} B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) & , \text{if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & , \text{if } B \cap Cn(\emptyset) \neq \emptyset \end{cases}$$

for any set of sentences B *and where* $B_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based filtration of* B. *An operator* $\div$ *on* $\mathbf{K}$ *is a* system of spheres-based multiple contraction on $\mathbf{K}$ *if and only if there is a system of spheres* $\mathbb{S}$ *centred on* $\|\mathbf{K}\|$, *such that* $\mathbf{K}\div B = \mathbf{K}\div_{\mathbb{S}} B$, *for any set of sentences* B.

The next proposition shows that the above-defined system of spheres-based multiple contractions are indeed a generalization of Grove's system of spheres-based (singleton) contractions.

PROPOSITION 2.13 (Fermé & Reis, 2012). *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$. *If* $\div_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based multiple contraction and* $-_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based (singleton) contraction, then:*

- *For any set of sentences* B *such that* $B \cap Cn(\emptyset) = \emptyset$ *it holds that:*

$$\mathbf{K}\div_{\mathbb{S}} B = \bigcap_{\alpha_i \in B_{\mathbb{S}}} \mathbf{K}\div_{\mathbb{S}}\{\alpha_i\} = \bigcap_{\alpha_i \in B_{\mathbb{S}}} \mathbf{K}-_{\mathbb{S}}\alpha_i,$$

  where $B_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based filtration of* B.
- *The identity* $\mathbf{K}\div_{\mathbb{S}}\{\alpha\} = \mathbf{K}-_{\mathbb{S}}\alpha$ *is satisfied for any sentence* $\alpha \in \mathcal{L}$.

§3. Epistemic Entrenchment-based Multiple Contraction Functions. In this section our main goal is to introduce a new class of multiple contraction functions, namely the *epistemic entrenchment-based multiple contractions* which generalize the epistemic entrenchment-based (singleton) contractions (Gärdenfors, 1988; Gärdenfors & Makinson, 1988) to the case of contractions by (possibly nonsingleton) sets of sentences rather than by a single sentence. Analogously to what is the case in what concerns the system of spheres-based multiple contractions, in the definition of the epistemic entrenchment-based multiple contractions we shall need to use a filtration of the input set of sentences B, which is defined as follows:

DEFINITION 3.1. *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$.

Consider a finite set of sentences $B = \{\alpha_1, \ldots, \alpha_n\}$ *such that* $B \setminus Cn(\emptyset) \neq \emptyset$.

Denote by $C_1, \ldots, C_m$ *the (different) equivalence classes in the quotient set of* $(B \setminus Cn(\emptyset))$ *by* $\backsim$, *that is,* $\{C_1, \ldots, C_m\} = (B \setminus Cn(\emptyset))/_{\backsim}$, *where* $\backsim$ *is the equivalence relation* *on $B \setminus Cn(\emptyset)$ defined by:*

$$\forall \alpha, \beta \in B \setminus Cn(\emptyset),\ \alpha \backsim \beta \ \textit{ iff } \ \alpha =_{\leq} \beta.$$

Moreover, assume that the equivalence classes $C_1, \ldots, C_m$ are ordered according to the following condition:

$$\textit{If } 1 \leq i < j \leq m,\ \textit{then } \forall \alpha_r \in C_i\ \forall \alpha_s \in C_j\ \ \alpha_s < \alpha_r.$$

Now consider the following list of subsets of B:

$$\begin{array}{rcl} B_0 & = & B \cap Cn(\emptyset) \\ C_1' & = & C_1 \\ C_1'' & = & \{\alpha_i \in C_1' : \forall \alpha_j \in C_1'\ \alpha_j \vee \neg\alpha_i \leq \alpha_j \textit{ or } \alpha_i < \alpha_i \vee \neg\alpha_j\} \\ B_1 & = & C_1''. \end{array}$$

Moreover, if $m > 1$ for all $l \in \{2, \ldots, m\}$, let C_l', C_l'' and B_l be the sets defined by:

$$\begin{array}{rcl} C_l' & = & \{\alpha_i \in C_l : \forall \alpha_j \in B_{l-1}\ \alpha_j \vee \neg\alpha_i \leq \alpha_j\}; \\ C_l'' & = & \{\alpha_i \in C_l' : \forall \alpha_j \in C_l'\ \alpha_j \vee \neg\alpha_i \leq \alpha_j \textit{ or } \alpha_i < \alpha_i \vee \neg\alpha_j\}; \\ B_l & = & B_{l-1} \cup C_l''. \end{array}$$

The $\leq$-based filtration of *B is the set $B_{\leq} = B_m$.*

If D is a set of sentences such that $D \subseteq Cn(\emptyset)$, then the $\leq$-based filtration of *D is the empty set, i.e. $D_{\leq} = \emptyset$.*

The following proposition shows an interesting property of the elements of B that are kept in the set $B_{\leq}$:

PROPOSITION 3.2. *Let $\mathbf{K}$ be a belief set, $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$ and B be a finite set of sentences. If $\alpha_i \in B_{\leq}$, then for any $\alpha_t \in B_{\leq}$, $\alpha_t \vee \neg\alpha_i \leq \alpha_t$ or $\alpha_i < \alpha_i \vee \neg\alpha_t$.*

Proof. If $B \subseteq Cn(\emptyset)$ then, according to Definition 3.1, $B_{\leq} = \emptyset$ and we are done.

Let $B \setminus Cn(\emptyset) \neq \emptyset$ and let $C_1, \ldots, C_m, C_1', \ldots, C_m', C_1'', \ldots, C_m'', B_1, \ldots, B_m$ be the subsets of B considered in the process of construction of the set $B_{\leq}$ described in Definition 3.1.

Now let α_i be an arbitrary element of $B_{\leq}$ and assume by *reductio* that there is some $\alpha_t \in B_{\leq}$ such that $\alpha_t < \alpha_t \vee \neg\alpha_i$ and $\alpha_i \not< \alpha_i \vee \neg\alpha_t$. It follows from Lemma 2.5-(ii) that $\alpha_i \leq \alpha_t$.

Case 1, $\alpha_i =_{\leq} \alpha_t$. Then, there is some class C_l such that $\alpha_i, \alpha_t \in C_l$. Since $\alpha_i, \alpha_t \in B_{\leq}$ we must have $\alpha_i, \alpha_t \in C_l'$. But, in that case, from $\alpha_t < \alpha_t \vee \neg\alpha_i$ and $\alpha_i \not< \alpha_i \vee \neg\alpha_t$ we conclude that $\alpha_i \notin C_l''$, which contradicts $\alpha_i \in B_{\leq}$.

Case 2, $\alpha_i < \alpha_t$. Then, there are two classes C_l and C_n, with $l < n$ such that $\alpha_t \in C_l$ and $\alpha_i \in C_n$. Since $\alpha_t \in B_{\leq}$ we have that $\alpha_t \in C_l''$. Hence, from $C_l'' \subseteq B_l \subseteq B_{n-1}$ we can conclude that $\alpha_t \in B_{n-1}$. But then, from $\alpha_t < \alpha_t \vee \neg\alpha_i$ it follows that $\alpha_i \notin C_n'$, which contradicts $\alpha_i \in B_{\leq}$. □

On the other hand, the following proposition presents a property satisfied by any element of a set B which is not included in its filtration $B_{\leq}$:

PROPOSITION 3.3. *Let $\mathbf{K}$ be a belief set, $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$ and B be a finite set of sentences such that $B \cap Cn(\emptyset) = \emptyset$. If $\alpha_j \in B \setminus B_{\leq}$, then there is some $\alpha_l \in B_{\leq}$ such that $\alpha_l < \alpha_l \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_l$.*

Proof. In order to prove this proposition, we start first by proving the following lemma:

LEMMA 3.4. *Let* $\mathbf{K}$ *be a belief set,* $\leq$ *be a relation on* $\mathbf{K}$ *that satisfies (EE1), (EE2), and (EE3). Assume that* $C = \{\alpha_1, \ldots, \alpha_n\}$, *with* $n \geq 1$ *is a nonempty finite set of sentences such that* $\alpha_1 =_\leq \cdots =_\leq \alpha_n$, *and, for each* $\alpha_j \in C$ *let* C_{α_j} *be the set defined by* $C_{\alpha_j} = \{\alpha_k \in C : \alpha_k < \alpha_k \vee \neg\alpha_j$ *and* $\alpha_j \not< \alpha_j \vee \neg\alpha_k\}$.

If $\alpha_j \in C$ *and* $C_{\alpha_j} \neq \emptyset$ *then there is some* $\alpha_l \in C_{\alpha_j}$ *such that* $C_{\alpha_l} = \emptyset$.

Proof. Let $\alpha_j \in C$ be such that $C_{\alpha_j} \neq \emptyset$. We must show that there is some $\alpha_l \in C_{\alpha_j}$ such that $C_{\alpha_l} = \emptyset$.

First notice that given $\alpha_s, \alpha_t \in C$ if $\alpha_s \in C_{\alpha_j}$ and $\alpha_t < \alpha_t \vee \neg\alpha_s$, then $\alpha_t \in C_{\alpha_j}$. In order to verify this we take α_s and α_t in the mentioned conditions. From $\alpha_s \in C_{\alpha_j}$ it follows that $\alpha_s < \alpha_s \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_s$. Now, on the one hand, from $\alpha_s < \alpha_s \vee \neg\alpha_j$ and $\alpha_t < \alpha_t \vee \neg\alpha_s$ it follows by Lemma 2.5-(iii) that $\alpha_t < \alpha_t \vee \neg\alpha_j$. On the other hand, by the same lemma, from $\alpha_t < \alpha_t \vee \neg\alpha_s$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_s$, we obtain that $\alpha_j \not< \alpha_j \vee \neg\alpha_t$. Hence $\alpha_t \in C_{\alpha_j}$.

From the above it follows immediately that for any $\alpha_r \in C$, if $\alpha_r \in C_{\alpha_j}$ then $C_{\alpha_r} \subset C_{\alpha_j}$. Indeed, let $\alpha_r, \alpha_p \in C$ be such that $\alpha_r \in C_{\alpha_j}$ and $\alpha_p \in C_{\alpha_r}$, we will start by showing that $\alpha_p \in C_{\alpha_j}$. From $\alpha_p \in C_{\alpha_r}$ it follows that $\alpha_p < \alpha_p \vee \neg\alpha_r$ and, as we have seen above, from this condition and the fact that $\alpha_r \in C_{\alpha_j}$ we can conclude that $\alpha_p \in C_{\alpha_j}$. Hence $C_{\alpha_r} \subseteq C_{\alpha_j}$. Finally, since $\alpha_r \in C_{\alpha_j}$ but $\alpha_r \notin C_{\alpha_r}$ we can conclude that $C_{\alpha_r} \subset C_{\alpha_j}$.

In what follows, given a finite set S, $\#S$ denotes the number of elements of S.

Now, in order to prove that there is some $\alpha_l \in C_{\alpha_j}$ such that $C_{\alpha_l} = \emptyset$ we start by noticing that, since $\emptyset \neq C_{\alpha_j} \subset C$ (note that $\alpha_j \in C \setminus C_{\alpha_j}$), we have that $1 \leq \#C_{\alpha_j} < n$. Now we proceed according to the following (finite) sequence of steps:

Step 1: Pick some $\alpha_{j1} \in C_{\alpha_j}$. As we have seen above we have that $C_{\alpha_{j1}} \subset C_{\alpha_j}$, hence $0 \leq \#C_{\alpha_{j1}} < n-1$.

Step 2: If $\#C_{\alpha_{j1}} = 0$ we have that $C_{\alpha_{j1}} = \emptyset$ and then we can take $\alpha_l = \alpha_{j1}$ and this finishes the proof. Otherwise, pick some $\alpha_{j2} \in C_{\alpha_{j1}}$. Then $C_{\alpha_{j2}} \subset C_{\alpha_{j1}}$, hence $0 \leq \#C_{\alpha_{j2}} < n-2$.

$$\vdots$$

Step i: If $\#C_{\alpha_{j(i-1)}} = 0$ we have that $C_{\alpha_{j(i-1)}} = \emptyset$ and then we can take $\alpha_l = \alpha_{j(i-1)}$ and this finishes the proof.
Otherwise, pick some $\alpha_{ji} \in C_{\alpha_{j(i-1)}}$. Then $C_{\alpha_{ji}} \subset C_{\alpha_{j(i-1)}}$, hence $0 \leq \#C_{\alpha_{ji}} < n-i$.

$$\vdots$$

Suppose after $n-2$ steps the above-described process has not finished yet. Furthermore, assume that $\#C_{\alpha_{j(n-2)}} \neq 0$. Since, by construction, $0 \leq \#C_{\alpha_{j(n-2)}} < n-(n-2) = 2$, we can conclude that $C_{\alpha_{j(n-2)}}$ is a singleton set. Then, at step $n-1$ we pick the only element of $C_{\alpha_{j(n-2)}}$, which we denote by $\alpha_{j(n-1)}$, and from the fact that $C_{\alpha_{j(n-1)}} \subset C_{\alpha_{j(n-2)}}$ it must be the case that $C_{\alpha_{j(n-1)}} = \emptyset$.

Hence, after at most n steps the process must have finished. That is, at most at step n we must find some $\alpha_l \in C_{\alpha_j}$ such that $C_{\alpha_l} = \emptyset$ as we wished to prove. □

Now we are in conditions to prove the proposition:

If $B = \emptyset$, then the proposition is trivially true. So, in what remains of this proof we assume that $B \neq \emptyset$ and let $C_1, \ldots, C_m, C'_1, \ldots, C'_m, C''_1, \ldots, C''_m, B_1, \ldots, B_m$ be the subsets of B considered in the process of construction of the set $B_{\leq}$ described in Definition 3.1 (notice that in the conditions of the proposition it follows from $B \neq \emptyset$ that $B \setminus Cn(\emptyset) \neq \emptyset$).

Now assume α_j is such that $\alpha_j \in B$ but $\alpha_j \notin B_{\leq}$. We must show that there is some $\alpha_l \in B_{\leq}$ such that $\alpha_l < \alpha_l \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_l$.

To do that, we consider separately the two possibilities $\alpha_j \in C_1$ or $\alpha_j \in C_n$, with $1 < n \leq m$.

Case 1, $\alpha_j \in C_1$. Then, since by construction $C_1 = C'_1$, it holds that $\alpha_j \in C'_1$. On the other hand, from $\alpha_j \notin B_{\leq}$ it follows that $\alpha_j \notin C''_1$. Therefore, there is some $\alpha_k \in C'_1$ such that $\alpha_k < \alpha_k \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_k$.

Now we notice that, since B is finite, by construction we have that C'_1 is finite and for any $\alpha_r, \alpha_s \in C'_1$ it holds that $\alpha_r =_{\leq} \alpha_s$. Then, according to Lemma 3.4, there is some $\alpha_l \in C'_1$ such that $\alpha_l < \alpha_l \vee \neg\alpha_j$, $\alpha_j \not< \alpha_j \vee \neg\alpha_l$ and, for any $\alpha_m \in C'_1$ it holds that $\alpha_m \not< \alpha_m \vee \neg\alpha_l$ or $\alpha_l < \alpha_l \vee \neg\alpha_m$.

It remains to remark that, according to the definition of C''_1, it follows from the conditions above that $\alpha_l \in C''_1$. Consequently $\alpha_l \in B_{\leq}$, and we are done.

Case 2, $\alpha_j \in C_n$, with $1 < n \leq m$. From $\alpha_j \notin B_{\leq}$ it follows that $\alpha_j \notin C''_n$, and we have to consider the two possibilities $\alpha_j \notin C'_n$ or $\alpha_j \in C'_n$.

Case 2.1, $\alpha_j \notin C'_n$. Then, there is some $\alpha_l \in B_{n-1}$ such that $\alpha_l < \alpha_l \vee \neg\alpha_j$. Now, since $\alpha_l \in B_{n-1}$, on the one hand we have that $\alpha_l \in B_{\leq}$ and, on the other hand, it follows from the construction of B_{n-1} and C_n that $\alpha_j < \alpha_l$. So, $\alpha_l \not\leq \alpha_j$ and it follows from Lemma 2.5-(ii) that $\alpha_j \not< \alpha_j \vee \neg\alpha_l$. Hence α_l is such that $\alpha_l \in B_{\leq}$ and $\alpha_l < \alpha_l \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_l$ as we wished to prove.

Case 2.2, $\alpha_j \in C'_n$. Then, since $\alpha_j \notin C''_n$, there is some $\alpha_k \in C'_n$ such that $\alpha_k < \alpha_k \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_k$. Reasoning as we did in Case 1 above we can conclude that there is some $\alpha_l \in B_{\leq}$ such that $\alpha_l < \alpha_l \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_l$, and this finishes the proof. □

At this point we notice that, taking the above results into account, the $\leq$-based filtration of a finite set of sentences can be described in one of the more concise forms presented in the following remark:

REMARK 3.5. *Let* **K** *be a belief set,* $\leq$ *be an epistemic entrenchment relation with respect to* **K**, B *be a finite set of sentences and* $B_{\leq}$ *be the* $\leq$*-based filtration of* B. *Then:*

1. $\alpha_i \in B_{\leq}$ *if and only if* $\alpha_i \in B$ *and for every* $\alpha_t \in B_{\leq}$ *the following conditions are satisfied:*

   (a) *If* $\alpha_i < \alpha_t$, *then* $\alpha_t \vee \neg\alpha_i \leq \alpha_t$

   (b) *If* $\alpha_i =_{\leq} \alpha_t$, *then* $\alpha_t \vee \neg\alpha_i \leq \alpha_t$ or $\alpha_i < \alpha_i \vee \neg\alpha_t$

2. $B_{\leq} = \{\alpha_i \in B : \forall\alpha_t \in B_{\leq}(\alpha_t \vee \neg\alpha_i \leq \alpha_t \text{ or } \alpha_i < \alpha_i \vee \neg\alpha_t)\}$.

By means of the $\leq$-based filtration of B we are now in conditions to define the class of epistemic entrenchment-based multiple contractions functions:

DEFINITION 3.6 (Epistemic entrenchment-based multiple contractions). *Let* **K** *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* **K**. *The* $\leq$-based multiple contraction on $\mathbf{K}$ *by finite sets is the multiple contraction function* $\div_{\leq}$ *defined by:*

$$\mathbf{K}\div_{\leq} B = \begin{cases} \{\beta \in \mathbf{K} : \forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \beta\} & \text{, if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{, if } B \cap Cn(\emptyset) \neq \emptyset \end{cases}, \qquad (CM_{\div_{\leq}})$$

for any finite set of sentences B and where $B_{\leq}$ is the $\leq$-based filtration of B.

An operator $\div$ on $\mathbf{K}$ *is an* epistemic entrenchment-based multiple contraction on $\mathbf{K}$ *by finite sets if and only if there is an epistemic entrenchment relation $\leq$ with respect to* $\mathbf{K}$*, such that* $\mathbf{K}\div B = \mathbf{K}\div_{\leq} B$*, for any finite set of sentences B.*

It is interesting to explore the relation between multiple and singleton epistemic entrenchment-based contraction. The next proposition, which is is an immediate consequence of Definitions 2.4 and 3.6, formally states that the $\leq$-based multiple contraction of a belief set $\mathbf{K}$ by a finite set of sentences B consists of the intersection of the results of the $\leq$-based singleton contractions of $\mathbf{K}$ by each of the sentences of $B_{\leq}$. Therefore, in particular, we can conclude that, as desired, the epistemic entrenchment-based multiple contractions are indeed a generalization of the epistemic entrenchment-based (singleton) contractions.

PROPOSITION 3.7. *Let* $\mathbf{K}$ *be a belief set and $\leq$ be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. If $\div_{\leq}$ is the $\leq$-based multiple contraction by finite sets and $-_{\leq}$ is the $\leq$-based (singleton) contraction, then:*

- *For any finite set of sentences B such that $B \cap Cn(\emptyset) = \emptyset$ it holds that:*

$$\mathbf{K}\div_{\leq} B = \bigcap_{\alpha_i \in B_{\leq}} \mathbf{K}\div_{\leq}\{\alpha_i\} = \bigcap_{\alpha_i \in B_{\leq}} \mathbf{K}-_{\leq}\alpha_i,$$

  where $B_{\leq}$ is the $\leq$-based filtration of B.
- *The identity $\mathbf{K}\div_{\leq}\{\alpha\} = \mathbf{K}-_{\leq}\alpha$ is satisfied for any sentence α.*

3.1. Normalization of a set** B **and its relation with** $B_{\leq}$**. Let $\alpha_j, \alpha_i \in \mathcal{L}$ be such that $\nvdash \alpha_i \rightarrow \alpha_j$ and $\vdash \alpha_j \rightarrow \alpha_i$. In order to contract a belief set $\mathbf{K}$ by the set $\{\alpha_j, \alpha_i\}$, each of those two sentences needs to be removed from $\mathbf{K}$. However, the removal of α_i imposes the removal of α_j. Thus, the result of contracting $\mathbf{K}$ by $\{\alpha_j, \alpha_i\}$ is identical to the outcome of the contraction of $\mathbf{K}$ by $\{\alpha_i\}$. Generalizing this thought we can conclude that the outcome of the contraction of a belief set $\mathbf{K}$ by a set of sentences B is the same of that of the contraction of $\mathbf{K}$ by the subset of B, which contains only those elements that do not imply any elements of B other than those to which they are equivalent. Such set is formally introduced in the following definition, after which we present some interrelations between that set and the filtration of B.

DEFINITION 3.8 (Reis & Fermé, 2012). *Let B be a set of sentences. The* normalization *of B is the subset $B_N \subseteq B$ defined in the following way*[1]

$$B_N = \{\alpha_i \in B : \forall \alpha_j \in B,\ \nvdash \alpha_i \rightarrow \alpha_j \text{ or } \vdash \alpha_j \rightarrow \alpha_i\}.$$

PROPOSITION 3.9. *Let* $\mathbf{K}$ *be a belief set, $\leq$ be an epistemic entrenchment relation with respect to* $\mathbf{K}$*, B be a finite set of sentences and β be a sentence. Then:*

(i) $B_N \cap \mathbf{K} = (B \cap \mathbf{K})_N$.

(ii) $B_{\leq} \cap \mathbf{K} = (B \cap \mathbf{K})_{\leq}$.

[1] In Hansson (2010) such set is designated by nimp(B).

(iii) $(B_N)_{\leq} = (B_{\leq})_N$.

(iv) $\forall \alpha_i \in B_{\leq}\ \alpha_i < \alpha_i \vee \beta$ *if and only if* $\forall \alpha_i \in (B_{\leq})_N\ \alpha_i < \alpha_i \vee \beta$.

Proof. In order to prove this proposition, we start first by proving the following lemmas:

LEMMA 3.10. *Let B be a finite set of sentences and $\alpha_l \in B$. If $\alpha_l \notin B_N$, then there is some $\alpha_k \in B_N$ such that $\vdash \alpha_l \rightarrow \alpha_k$ and $\nvdash \alpha_k \rightarrow \alpha_l$.*

Proof. Follows immediately from Reis & Fermé (2012, Lemma 4.3) and the fact that for any $\alpha_l, \alpha_k \in \mathcal{L}$, $\vdash \alpha_l \rightarrow \alpha_k$ if and only if $\|\neg\alpha_k\| \subseteq \|\neg\alpha_l\|$. □

LEMMA 3.11. *Let $\mathbf{K}$ be a belief set and G and H be two finite sets of sentences. If every subset X of $\mathbf{K}$ implies some element of G if and only if X implies some element of H, then for every element of $G_N \cap \mathbf{K}$ there is a logically equivalent element of $H_N \cap \mathbf{K}$.*

Proof. Follows immediately from Reis & Fermé (2012, Lemma 4.6) and the fact that for any $\beta_i, \varsigma_i \in \mathcal{L}$, $\vdash \beta_i \rightarrow \varsigma_i$ if and only if $\|\neg\varsigma_i\| \subseteq \|\neg\beta_i\|$. □

LEMMA 3.12. *Let $\mathbf{K}$ be a belief set, $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$ and G and H be two finite sets of sentences. If every subset X of $\mathbf{K}$ implies some element of G if and only if X implies some element of H, then for every element of $(G_N \cap \mathbf{K})_{\leq}$ there is a logically equivalent element of $(H_N \cap \mathbf{K})_{\leq}$.*

Proof. Assume every subset X of $\mathbf{K}$ implies some element of G if and only if X implies some element of H. According to Lemma 3.11 it holds that for all $\beta_i \in G_N \cap \mathbf{K}$ there is some $\varsigma_i \in H_N \cap \mathbf{K}$ such that $\vdash \beta_i \leftrightarrow \varsigma_i$. Hence, it follows trivially from the definition of $\leq$-based filtration of a set of sentences (Definition 3.1) and intersubstitutivity that for all $\beta_i \in (G_N \cap \mathbf{K})_{\leq}$ there is some $\varsigma_i \in (H_N \cap \mathbf{K})_{\leq}$ such that $\vdash \beta_i \leftrightarrow \varsigma_i$, as required. □

Now we are in conditions to prove the proposition

(i)–(iii) follow immediately from the definitions of normalization and of $\leq$-based filtration of a set of sentences.

(iv) We must show that $\forall \alpha_i \in B_{\leq}\ \alpha_i < \alpha_i \vee \beta$ if and only if $\forall \alpha_i \in (B_{\leq})_N\ \alpha_i < \alpha_i \vee \beta$. The *left-to-right* implication follows immediately from the fact that $(B_{\leq})_N \subseteq B_{\leq}$. Now we prove that the converse implication also holds. Assume $\forall \alpha_i \in (B_{\leq})_N\ \alpha_i < \alpha_i \vee \beta$ and let α_j be an arbitrary element of $B_{\leq}$. We only need to show that $\alpha_j < \alpha_j \vee \beta$. If $\alpha_j \in (B_{\leq})_N$, then it follows from our above assumption that $\alpha_j < \alpha_j \vee \beta$, and we are done. Now assume that $\alpha_j \notin (B_{\leq})_N$. Then, according to Lemma 3.10, there is some $\alpha_k \in (B_{\leq})_N$ such that $\vdash \alpha_j \rightarrow \alpha_k$ (and $\nvdash \alpha_k \rightarrow \alpha_j$). Then (EE2) yields $\alpha_j \leq \alpha_k$. Furthermore, according to the above assumptions it holds that $\alpha_k < \alpha_k \vee \beta$.

On the other hand, according to the definition of $B_{\leq}$, from $\alpha_k \in B_{\leq}$ we can conclude that $\nvdash \alpha_k$. Then, since $\vdash \alpha_k \vee \neg\alpha_j$, it follows (see Foo, 1990) that $\alpha_k < \alpha_k \vee \neg\alpha_j$. Now, since $\alpha_j, \alpha_k \in B_{\leq}$, it follows from Proposition 3.2 that $\alpha_j < \alpha_j \vee \neg\alpha_k$.

Finally, from $\alpha_j \leq \alpha_k$, $\alpha_k < \alpha_k \vee \beta$ and $\alpha_j < \alpha_j \vee \neg\alpha_k$ we can conclude by Lemma 2.5-(iii) that $\alpha_j < \alpha_j \vee \beta$, and (iv) is proved. □

§4. Axiomatic Characterization for Epistemic Entrenchment-based Multiple Contraction Functions. Our main goal in the present section is to obtain an axiomatic characterization for the epistemic entrenchment-based multiple contraction.

Having that in mind, we start by presenting a proposition which states that any multiple contraction function $\div$ that satisfies the postulates of *Package inclusion*, *Package* *uniformity*, and *Package relevance* (whose formulations can be found in the statement of Proposition 2.10), also satisfies some other postulates which have been proposed in the literature (e.g., Hansson, 1989, 1991, 1992; Fuhrmann & Hansson, 1994) as properties which are naturally expectable from a multiple contraction function, and which we will need to refer to in what follows.

PROPOSITION 4.1. *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be a multiple contraction function on* $\mathbf{K}$*. If* $\div$ *satisfies* Package inclusion*,* Package uniformity*, and* Package relevance *then it also satisfies*

Package closure $\mathbf{K}\div B = Cn(\mathbf{K}\div B)$.
Package vacuity *If* $B \cap \mathbf{K} = \emptyset$*, then* $\mathbf{K}\div B = \mathbf{K}$*.*
Package extensionality *If for every sentence* α *in* B *there is a sentence* β *in* C *such that* $\vdash \alpha \leftrightarrow \beta$*, and vice versa, then* $\mathbf{K}\div B = \mathbf{K}\div C$*.*
Package recovery $\mathbf{K} \subseteq Cn((\mathbf{K}\div B) \cup B)$.

Proof. Let $\mathbf{K}$ be a belief set and $\div$ be a multiple contraction function on $\mathbf{K}$ that satisfies *Package inclusion*, *Package uniformity*, and *Package relevance*. Then, it follows immediately from Fuhrmann & Hansson (1994, Lemma 3-1.) and Fuhrmann & Hansson (1994, Observation 10) that $\div$ satisfies *Package closure* and *Package recovery*.[2] On the other hand, it is easy to check that *Package vacuity* follows from *Package relevance* and that *Package uniformity* entails *Package extensionality*. Hence, we can conclude that $\div$ satisfies also *Package vacuity* and *Package extensionality*, and the proof is complete. □

Now we introduce the following couple of postulates:

Package conjunctive overlap $\mathbf{K}\div\{\alpha\} \cap \mathbf{K}\div\{\beta\} \subseteq \mathbf{K}\div\{\alpha \wedge \beta\}$.
Package conjunctive inclusion If $\alpha \notin \mathbf{K}\div\{\alpha \wedge \beta\}$, then $\mathbf{K}\div\{\alpha \wedge \beta\} \subseteq \mathbf{K}\div\{\alpha\}$.

Package conjunctive overlap and Package conjunctive inclusion are generalization of the singleton contraction postulates of conjunctive overlap and conjunctive inclusion, respectively, which were introduced in Alchourrón *et al.* (1985).[3]

Gärdenfors (1988) and Gärdenfors & Makinson (1988) proposed condition ($C_{\leq}$) to define an epistemic entrenchment relation by means of a given contraction function. In what follows we will need to make use of the following similar condition which defines a binary relation on $\mathcal{L}$ by means of a given multiple (rather than singleton) contraction function $\div$ on $\mathbf{K}$:

$$\forall \alpha, \beta \in \mathcal{L},\ \ \alpha \leq \beta \text{ iff } \alpha \notin \mathbf{K}\div\{\alpha \wedge \beta\} \text{ or } \vdash \alpha \wedge \beta. \tag{$CM_{\leq}$}$$

The following proposition, which can be straightforwardly proven, presents some results which materialize the intuition that has driven the formulations of the postulates of *Package conjunctive overlap* and *Package conjunctive inclusion* as well as of condition ($CM_{\leq}$).

PROPOSITION 4.2. *Let* $\mathbf{K}$ *be a belief set and* $-$ *and* $\div$ *be, respectively, a singleton contraction function on* $\mathbf{K}$ *and a multiple contraction function on* $\mathbf{K}$ *such that for all sentences* α *it holds that* $\mathbf{K}-\alpha = \mathbf{K}\div\{\alpha\}$*. Then:*

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[2] Notice that, since in this paper we are only considering multiple contraction functions by finite sets, here the expression *Package recovery* refers to the same postulate that was designated by *finite P-recovery* in Fuhrmann & Hansson (1994).

[3] **Conjunctive overlap** $\mathbf{K}-\alpha \cap \mathbf{K}-\beta \subseteq \mathbf{K}-\alpha \wedge \beta$.
Conjunctive inclusion If $\alpha \notin \mathbf{K}-\alpha \wedge \beta$, then $\mathbf{K}-\alpha \wedge \beta \subseteq \mathbf{K}-\alpha$.

(i) *Conditions* ($C_\leq$) *and* ($CM_\leq$) *are equivalent, that is, they define the same binary relation* ($\leq$) *on* $\mathcal{L}$.

(ii) $-$ *satisfies* Conjunctive overlap (respectively, *Conjunctive inclusion*) if and only if $\div$ satisfies *Package conjunctive overlap* (respectively, *Package conjunctive inclusion*).

(iii) If $\div$ satisfies *Package closure*, *Package inclusion*, *Package vacuity*, *Package success*, *Package recovery*, *Package extensionality*, then $-$ satisfies *closure*, *inclusion*, *vacuity*, *success*, *recovery*, *extensionality*, respectively.[4]

Now we present a result which asserts that, as long as the multiple contraction $\div$ satisfies certain properties the binary relation $\leq$ defined by condition ($CM_\leq$) is an epistemic entrenchment relation.

PROPOSITION 4.3. *Let* $\mathbf{K}$ *be a belief set. If* $\div$ *is a multiple contraction function on* $\mathbf{K}$ *that satisfies* Package inclusion*,* Package success*,* Package uniformity*,* Package relevance*,* Package conjunctive overlap*, and* Package conjunctive inclusion*, then the binary relation* $\leq$ *defined from* $\div$ *by means of condition* ($CM_\leq$) *is an epistemic entrenchment relation with respect to* $\mathbf{K}$.

Proof. Let $\mathbf{K}$ be a belief set, $\div$ be a multiple contraction function on $\mathbf{K}$ that satisfies *Package inclusion*, *Package success*, *Package uniformity*, *Package relevance*, *Package conjunctive overlap*, and *Package conjunctive inclusion* and $\leq$ be the binary relation defined by condition ($CM_\leq$).

We start by noticing that it follows from Proposition 4.1 that $\div$ also satisfies *Package closure*, *Package vacuity*, *Package extensionality*, and *Package recovery*.

Now we define a singleton contraction function on $\mathbf{K}$ in the following way:

$$\forall \alpha \in \mathcal{L},\ \mathbf{K}-\alpha = \mathbf{K}\div\{\alpha\}.$$

From the above we can conclude by Proposition 4.2-(ii) and (iii) that $-$ satisfies all the basic and supplementary AGM postulates for belief set contraction (namely, *closure*, *inclusion*, *vacuity*, *success*, *recovery*, *extensionality*, *Conjunctive overlap*, and *Conjunctive inclusion*). Therefore Gärdenfors & Makinson (1988, Theorem 5) allows us to conclude that the binary relation on $\mathcal{L}$ defined from $-$ by means of condition ($C_\leq$) is an epistemic entrenchment relation with respect to $\mathbf{K}$.

Finally, since, according to Proposition 4.2-(i) Conditions ($C_\leq$) and ($CM_\leq$) define the same binary relation $\leq$ on $\mathcal{L}$, the required conclusion follows immediately. □

We now proceed to formulate the last postulate which will be necessary for the axiomatic characterization of the epistemic entrenchment-based multiple contractions that we shall present further ahed.

Having in mind that $\mathbf{K}\div_\leq B = \bigcap_{\alpha_i \in B_\leq} \mathbf{K}\div_\leq\{\alpha_i\}$ (cf. Proposition 3.7), we realize that if a multiple contraction $\div$ is an epistemic entrenchment-based multiple contraction, then it must be such that, $\mathbf{K}\div B = \bigcap_{\alpha_i \in B'} \mathbf{K}\div\{\alpha_i\}$, where B' contains some of the elements of B whose *selection* is made by a process similar to the one presented in Definition 3.1, which is based on some epistemic entrenchment relation. On the other hand, Proposition 4.3 presentes a way of defining an epistemic entrenchment relation by means of a multiple

[4] The (singleton) contraction postulates of *closure*, *inclusion*, *vacuity*, *success*, *recovery*, and *extensionality* are known as the basic AGM postulates for contraction and their formulations can be found for example, in Alchourrón *et al.* (1985).

contraction function $\div$ (provided that such function satisfies certain postulates). Thus, combining these two remarks, we are led to expect that in order to assure that a multiple contraction function $\div$ is an epistemic entrenchment-based multiple contraction it may be necessary (or, at least, useful) to impose that it satisfies the following property:

- For any set B such that $B \cap Cn(\emptyset) = \emptyset$, if $\div$ satisfies *Package inclusion*, *Package success*, *Package uniformity*, *Package relevance*, *Package conjunctive overlap*, and *Package conjunctive inclusion*, then $\mathbf{K}\div B = \bigcap_{\alpha_i \in B_{\leq_\div}} \mathbf{K}\div\{\alpha_i\}$, where $\leq_\div$ is the epistemic entrenchment relation with respect to $\mathbf{K}$ defined by

  $$\alpha \leq_\div \beta \text{ if and only if } \alpha \notin \mathbf{K}\div\{\alpha \wedge \beta\} \text{ or } \vdash \alpha \wedge \beta,$$

  and $B_{\leq_\div}$ is the $\leq_\div$-based filtration of B.

Notice that the need to include in the formulation of the above property the condition that $\div$ satisfies certain multiple contraction postulates is due to the fact that, since we have only assured that the binary relation $\leq$ defined from $\div$ by means of condition $(CM_\leq)$ is an epistemic entrenchment relation as long as $\div$ satisfies those other postulates (cf. Proposition 4.3), it only makes sense to refer to the $\leq_\div$-based filtration of a finite set of sentences (cf. Definition 3.1), which is used in the formulation of such property, if the multiple contraction operation $\div$ satisfies those *extra* postulates.

However, since throughout the remainder of this text whenever we refer to (or make use of) the above property of a multiple contraction function $\div$ that shall be done in settings where the remaining above-mentioned postulates are (or are assumed to be) also satisfied by $\div$, to lighten the writing we can (abdicating of some rigour) use the following *lighter* formulation of that property:

Package singleton reduction For any set B such that $B \cap Cn(\emptyset) = \emptyset$, it holds that $\mathbf{K}\div B = \bigcap_{\alpha_i \in B_{\leq_\div}} \mathbf{K}\div\{\alpha_i\}$, where $\leq_\div$ is the epistemic entrenchment relation with respect to $\mathbf{K}$ defined by
$\alpha \leq_\div \beta$ if and only if $\alpha \notin \mathbf{K}\div\{\alpha \wedge \beta\}$ or $\vdash \alpha \wedge \beta$, and
$B_{\leq_\div}$ is the $\leq_\div$-based filtration of B.

Loosely speaking to say that a certain multiple contraction function $\div$ satisfies the postulate of *Package singleton reduction* is to say that the result of the (multiple) contraction of $\mathbf{K}$ by any finite set B can be obtained by intersecting the results of the singleton contractions (by means of that same multiple contraction function) of $\mathbf{K}$ by some appropriately chosen sentences of B (more precisely, the sentences in the $\leq_\div$-based filtration of B).

At this point we notice that if we compare the above postulate to the following simpler (and considerably more *intuitively appealing*) property of a multiple contraction function $\div$:

- For any set B such that $B \cap Cn(\emptyset) = \emptyset$, it holds that $\mathbf{K}\div B = \bigcap_{\alpha_i \in B} \mathbf{K}\div\{\alpha_i\}$,

we may be led to think that the latter is a more adequate multiple contraction postulate than the former. However, it has been shown (e.g., in Fuhrmann & Hansson (1994, p. 62) and in (Fermé & Reis, 2012, Counterexample 3.1)) that, in general, the partial meet multiple contractions do not satisfy such property. And, recalling that the epistemic entrenchment-based multiple contractions are a generalization of the epistemic entrenchment-based (singleton) contractions (cf. Proposition 3.7) and that the latter form a subclass of the class of partial meet (singleton) contractions, it is natural to expect that all epistemic entrenchment-based multiple contractions are partial meet multiple contractions. Thus the above property is presumably not adequate to be included in an axiomatic characterization of the epistemic entrenchment-based multiple contractions. Furthermore, we notice that this condition can, in some cases, lead to unintuitive results, as we illustrated by the following example:

> I bought three cats, Benny, Spook and Barny. Benny is my favourite, in second comes Spook and the last is Barny. I believed that α: "Benny can live at home", β: "Spook can live at home", γ: "Barny can live at home". However, I discover that Benny and Spook can't live together nor can Barny and Spook. Due to that I must contract my set of beliefs $\mathbf{K}$ by $B = \{\alpha \wedge \beta, \beta \wedge \gamma\}$. If I take the intersection of the outputs of the singleton contractions (having in mind my order of preference) by each one of the different sentences in B I am led to abandon my beliefs β and γ, and consequently I have to drive out of my house Barny and Spook. However it is more adequate in this scenario to search for a *global solution* which would naturally leed to the conclusion that it would be enough to drive out of my house Spook.

All these facts provide further support to the adequacy of the postulate of Package singleton reduction in the present context, since that postulate can be seen as a weaker version of the above property (in the sense that it states that the result of the multiple contraction by a set of sentences B is the intersection of the results of *some*—rather than of *all*—of the singleton contractions by one of the sentences in B).

We are now in a position to provide an axiomatic characterization for the epistemic entrenchment-based multiple contraction functions:

THEOREM 4.4. *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be a multiple contraction function on* $\mathbf{K}$. *Then the following statements are equivalent:*

1. $\div$ *is an epistemic entrenchment-based multiple contraction on* $\mathbf{K}$.
2. $\div$ *satisfies Package success, Package inclusion, Package relevance, Package uniformity, Package conjunctive overlap, Package conjunctive inclusion, and Package singleton reduction.*

Proof. The following lemmas are needed for the proof of the theorem.

LEMMA 4.5. *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$. *Then the* $\leq$*-based multiple contraction* $\div$ *on* $\mathbf{K}$ *satisfies* Package closure.

Proof. If $B \cap Cn(\emptyset) \neq \emptyset$ it follows from $(CM_{\div_{\leq}})$ that $\mathbf{K}\div B = \mathbf{K}$. Then it follows from the hypothesis that $\mathbf{K}\div B = Cn(\mathbf{K}\div B)$. Hence, for the rest of this proof we assume that $B \cap Cn(\emptyset) = \emptyset$. That $\mathbf{K}\div B \subseteq Cn(\mathbf{K}\div B)$ follows from the fact that the consequence operation Cn satisfies *inclusion* (i.e., Cn is such that $A \subseteq Cn(A)$, for any set of sentences A).

To show the converse inclusion we take $\delta \in Cn(\mathbf{K}\div B)$ and we prove that $\delta \in \mathbf{K}\div B$. In order to do that, according to $(CM_{\div_{\leq}})$, and since we are assuming that $B \cap Cn(\emptyset) = \emptyset$, we must show that $\delta \in \mathbf{K}$ and $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee \delta$.

Case 1, $\mathbf{K}\div B \neq \emptyset$. Since we are assuming the consequence operation Cn is compact, there is a finite set $S \subseteq \mathbf{K}\div B$ such that $S \vdash \delta$. Furthermore, given that $\mathbf{K}\div B \neq \emptyset$, we can assume that $S \neq \emptyset$. Now we consider separately two possibilities.

Case 1.1, S is a singleton set, that is, $S = \{\beta_1\}$ for some $\beta_1 \in \mathbf{K}\div B$. Then it follows from Definition 3.6 that $\beta_1 \in \mathbf{K}$ and, since $\beta_1 \vdash \delta$ and $\mathbf{K}$ is a belief set we can conclude that $\delta \in \mathbf{K}$. On the other hand, it follows from $\beta_1 \in \mathbf{K}\div B$ and $B \cap Cn(\emptyset) = \emptyset$ that $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee \beta_1$. Finally, observing that from $\beta_1 \vdash \delta$ it follows that, for any $\alpha_j \in B_{\leq}$, it holds that $\alpha_j \vee \beta_1 \vdash \alpha_j \vee \delta$, by (EE1)–(EE3) we obtain that $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee \delta$, as we wished to show.

Case 1.2, $S = \{\beta_1, \ldots, \beta_n\}$, with $n \geq 2$. Since $\beta_1, \ldots, \beta_n \in \mathbf{K} \div B \subseteq \mathbf{K}$ (by Definition 3.6), and $\{\beta_1, \ldots, \beta_n\} \vdash \delta$, it follows from the fact that $\mathbf{K}$ is a belief set that $\delta \in \mathbf{K}$. Next we must show that $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee \delta$. From $\beta_1, \ldots, \beta_n \in \mathbf{K} \div B$ and $B \cap Cn(\emptyset) = \emptyset$ it follows that $\forall \beta_i \in S\, \forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee \beta_i$. Hence, for any given $\alpha_l \in B_{\leq}$ we have that $\alpha_l < \alpha_l \vee \beta_1$ and $\alpha_l < \alpha_l \vee \beta_2$. From these two conditions it follows that $\alpha_l < (\alpha_l \vee \beta_1) \wedge (\alpha_l \vee \beta_2)$. Then, since $\vdash (\alpha_l \vee \beta_1) \wedge (\alpha_l \vee \beta_2) \leftrightarrow (\alpha_l \vee (\beta_1 \wedge \beta_2))$, by intersubstitutivity we obtain that $\alpha_l < \alpha_l \vee (\beta_1 \wedge \beta_2)$. Hence $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee (\beta_1 \wedge \beta_2)$. By iteration of the previous procedure we can conclude that $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee (\beta_1 \wedge \ldots \wedge \beta_n)$. Now we note that, since $\beta_1 \wedge \ldots \wedge \beta_n \vdash \delta$ we have that for any $\alpha_l \in B_{\leq}$ it holds that $\alpha_l \vee (\beta_1 \wedge \ldots \wedge \beta_n) \vdash \alpha_l \vee \delta$. So, it follows from (EE2) that $\forall \alpha_j \in B_{\leq} \alpha_j \vee (\beta_1 \wedge \ldots \wedge \beta_n) \leq \alpha_j \vee \delta$. Finally by (EE1) we obtain that $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee \delta$ and this finishes the proof for this case.

Case 2, $\mathbf{K} \div B = \emptyset$. This means that $\vdash \delta$. Then, that $\delta \in \mathbf{K}$ follows from the fact that $\mathbf{K}$ is a belief set. Hence it remains to prove that $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee \delta$. Hence, let α_j be an arbitrary element of $B_{\leq}$. That $\alpha_j \leq \alpha_j \vee \delta$ follows immediately from (EE2). It remains to prove that $\alpha_j \vee \delta \not\leq \alpha_j$. From $\vdash \delta$ it follows that $\vdash \alpha_j \vee \delta$. So, by (EE2), $\varepsilon \leq \alpha_j \vee \delta$ for all $\varepsilon \in \mathcal{L}$. On the other hand, since $\not\vdash \alpha_j$, it follows from (EE5) that $\varepsilon \not\leq \alpha_j$ for some $\varepsilon \in \mathcal{L}$. Then, by (EE1) $\alpha_j \vee \delta \not\leq \alpha_j$ and this finishes the proof. $\square$

LEMMA 4.6. *Let* $\mathbf{K}$ *be a belief set,* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*,* $\div_{\leq}$ *be the* $\leq$*-based multiple contraction on* $\mathbf{K}$ *and* B *be a finite set of sentences. Then:*

(i) $B \cap Cn(\emptyset) \neq \emptyset$ *if and only if* $(B_N \cap \mathbf{K}) \cap Cn(\emptyset) \neq \emptyset$.
(ii) $\mathbf{K} \div_{\leq} B = \mathbf{K} \div_{\leq} (B_N \cap \mathbf{K})$.

Proof. We start by showing that condition (i) in the statement of the lemma holds. The *right-to-left* implication follows trivially from the fact that $B_N \cap \mathbf{K} \subseteq B$. Now we prove that the converse implication also holds. Assume $B \cap Cn(\emptyset) \neq \emptyset$ and let $\beta \in B \cap Cn(\emptyset)$. Then $\beta \in B$ and $\beta \in Cn(\emptyset)$. Now, on the one hand, it follows immediately from the definition of B_N that $\beta \in B_N$ and, on the other hand, from the fact that $\mathbf{K}$ is a belief set we obtain that $\beta \in \mathbf{K}$, hence that $\beta \in B_N \cap \mathbf{K}$, and finally that $(B_N \cap \mathbf{K}) \cap Cn(\emptyset) \neq \emptyset$. So, (i) is proved.

Now we prove (ii), that is, that $\mathbf{K} \div_{\leq} B = \mathbf{K} \div_{\leq} (B_N \cap \mathbf{K})$. Since (i) holds, according to Definition 3.6, to show that the above equality holds it suffices to prove that if $\beta \in \mathbf{K}$ then

$$\forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \beta \text{ iff } \forall \alpha_l \in (B_N \cap \mathbf{K})_{\leq}\ \alpha_l < \alpha_l \vee \beta. \tag{1}$$

Let $\beta \in \mathbf{K}$. We start by remarking that (1) is equivalent to

$$\forall \alpha_j \in (B_{\leq} \cap \mathbf{K})\ \alpha_j < \alpha_j \vee \beta \text{ iff } \forall \alpha_j \in (B_N \cap \mathbf{K})_{\leq}\ \alpha_j < \alpha_j \vee \beta. \tag{2}$$

To see that such equivalence indeed holds it is enough to notice that given $\alpha_j \in B_{\leq} \setminus \mathbf{K}$ it follows from the fact that $\mathbf{K}$ is a belief set that $\alpha_j \vee \beta \in \mathbf{K}$ and, therefore, by Lemma 2.5-(i) we can conclude that $\alpha_j < \alpha_j \vee \beta$.

Having seen this, to finish the proof we only need to show that (2) holds.

To prove that, we start by remarking that it follows from Lemma 3.9-(i) and (iii) that $(B_N \cap \mathbf{K})_{\leq} = ((B \cap \mathbf{K})_{\leq})_N$. Hence, it holds that $\forall \alpha_j \in (B_N \cap \mathbf{K})_{\leq}\ \alpha_j < \alpha_j \vee \beta$ if and only if $\forall \alpha_j \in ((B \cap \mathbf{K})_{\leq})_N\ \alpha_j < \alpha_j \vee \beta$. On the other hand, according to Lemma 3.9-(iv) we have that $\forall \alpha_j \in ((B \cap \mathbf{K})_{\leq})_N\ \alpha_j < \alpha_j \vee \beta$ if and only if $\forall \alpha_j \in (B \cap \mathbf{K})_{\leq}\ \alpha_j < \alpha_j \vee \beta$. Furthermore, 3.9-(ii) yields $\forall \alpha_j \in (B \cap \mathbf{K})_{\leq}\ \alpha_j < \alpha_j \vee \beta$ if and only if $\forall \alpha_j \in (B_{\leq} \cap \mathbf{K})$ $\alpha_j < \alpha_j \vee \beta$.

Thus, from all the above stated equivalences it follows immediately that $\forall \alpha_j \in (B_{\leq} \cap \mathbf{K})$ $\alpha_j < \alpha_j \vee \beta$ if and only if $\forall \alpha_j \in (B_N \cap \mathbf{K})_{\leq}\ \alpha_j < \alpha_j \vee \beta$, which is the desired conclusion. □

Construction-to-postulates:

Let $\mathbf{K}$ be a belief set, $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$, $\div$ be the $\leq$-based multiple contraction on $\mathbf{K}$ and B and C be two arbitrary finite sets of sentences. Recall that $B_N = \{\alpha_i \in B : \forall \alpha_j \in B,\ \not\vdash \alpha_i \to \alpha_j \text{ or } \vdash \alpha_j \to \alpha_i\}$.

Package inclusion Follows immediately from the definition of the $\leq$-based multiple contraction on $\mathbf{K}$ (see Definition 3.6).

Package success Let $B \cap Cn(\emptyset) = \emptyset$. We must show that $B \cap \mathbf{K}\div B = \emptyset$. Let α be an arbitrary element of B. If $\alpha \notin \mathbf{K}$, it follows immediately from $(CM_{\div_{\leq}})$ that $\alpha \notin \mathbf{K}\div B$. Hence, we assume that $\alpha \in \mathbf{K}$. Since $B \cap Cn(\emptyset) = \emptyset$, it follows from $(CM_{\div_{\leq}})$ that it will only be the case that $\alpha \in \mathbf{K}\div B$ if $\forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \alpha$.
Case 1, $\alpha \in B_{\leq}$. Then, since $\alpha \vee \alpha \vdash \alpha$, it follows from (EE2) that $\alpha \vee \alpha \leq \alpha$. Hence $\alpha \not< \alpha \vee \alpha$ and condition $\forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \alpha$ fails.
Case 2, $\alpha \notin B_{\leq}$. Then, according to Lemma 3.3, there is some $\alpha_i \in B_{\leq}$ such that $\alpha_i < \alpha_i \vee \neg\alpha$. Assume by *reductio* that $\alpha_i < \alpha_i \vee \alpha$. From the last two conditions it follows by conjunction up (Lemma 2.5-(vi)) that $\alpha_i < (\alpha_i \vee \neg\alpha) \wedge (\alpha_i \vee \alpha)$. Finally, from $\vdash (\alpha_i \vee \neg\alpha) \wedge (\alpha_i \vee \alpha) \leftrightarrow \alpha_i$ and intersubstitutivity (Lemma 2.5-(v)) we obtain that $\alpha_i < \alpha_i$ which is absurd.

Package uniformity Assume that every subset X of $\mathbf{K}$ implies some element of B if and only if X implies some element of C. We must prove that $\mathbf{K}\div B = \mathbf{K}\div C$. According to Lemma 4.6, such equality is equivalent to the following one $\mathbf{K}\div(B_N \cap \mathbf{K}) = \mathbf{K}\div(C_N \cap \mathbf{K})$. So, in what follows we will show that this latter equality is satisfied. To do that, according to Definition 3.6, it is enough to prove that the two following statements hold:

(a) $(B_N \cap \mathbf{K}) \cap Cn(\emptyset) \neq \emptyset$ if and only if $(C_N \cap \mathbf{K}) \cap Cn(\emptyset) \neq \emptyset$.
(b) If $\beta \in \mathbf{K}$, then $\forall \alpha_j \in (B_N \cap \mathbf{K})_{\leq}\ \alpha_j < \alpha_j \vee \beta$ if and only if $\forall \alpha_l \in (C_N \cap \mathbf{K})_{\leq}\ \alpha_l < \alpha_l \vee \beta$

To show that (a) is satisfied we first notice that, since $\emptyset \subseteq \mathbf{K}$, it follows from the hypothesis that $B \cap Cn(\emptyset) \neq \emptyset$ if and only if $C \cap Cn(\emptyset) \neq \emptyset$. Now, combining this latter equivalence with Lemma 4.6-(i), we can assert that (a) holds.
Finally, in order to prove that (b) also holds, we recall that it follows from Lemma 3.12 that for every element of $(B_N \cap \mathbf{K})_{\leq}$ there is a logically equivalent element of $(C_N \cap \mathbf{K})_{\leq}$, and vice versa. From this it follows immediately, by intersubstitutivity (Lemma 2.5-(v)), that (b) holds, and this finishes the proof.

Package relevance Let $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K}\div B$. Then it follows from Definition 3.6 that $B \cap Cn(\emptyset) = \emptyset$ and $\exists \alpha_j \in B_{\leq}$, such that

$$\alpha_j \vee \beta \leq \alpha_j. \tag{3}$$

We must show that there is a set K' such that:

(i) $\mathbf{K}\div B \subseteq K' \subseteq \mathbf{K}$
(ii) $\forall \alpha_i \in B\ \alpha_i \notin Cn(K')$
(iii) $\exists \alpha_r \in B : \neg\beta \vee \alpha_r \in Cn(K')$.

Consider $K' = \mathbf{K}\div B \cup \{\neg\beta \vee \alpha_j\}$, where α_j is an element of $B_{\leq}$ that satisfies condition (3). We have that K' trivially satisfies (iii) and $\mathbf{K}\div B \subseteq K'$. Now, in order to verify that it also satisfies $K' \subseteq \mathbf{K}$, the remaining inclusion stated in (i), first recall that it follows from *Package inclusion*, already proven above, that $\mathbf{K}\div B \subseteq \mathbf{K}$. It remains to show that $\neg\beta \vee \alpha_j \in \mathbf{K}$. Assume by *reductio* that $\alpha_j \notin \mathbf{K}$. It follows by (EE4) that $\forall \delta \in \mathcal{L}, \alpha_j \leq \delta$. Hence, using (EE2), (3), and (EE1) we obtain that $\forall \delta \in \mathcal{L}, \beta \leq \delta$. But, from this it follows by (EE4) that $\beta \notin \mathbf{K}$, which contradicts our assumption. Therefore $\alpha_j \in \mathbf{K}$ and, since $\mathbf{K}$ is a belief set, we can also conclude that $\neg\beta \vee \alpha_j \in \mathbf{K}$ as required. It remains to prove (ii). Assume by *reductio* that $\alpha_i \in B$ is such that $K' \vdash \alpha_i$. Since $K' = \mathbf{K}\div B \cup \{\neg\beta \vee \alpha_j\}$, it follows by deduction that $\mathbf{K}\div B \vdash (\neg\beta \vee \alpha_j) \rightarrow \alpha_i$. Hence, by *Package closure* (which is attested to hold by Lemma 4.5), $(\beta \wedge \neg\alpha_j) \vee \alpha_i \in \mathbf{K}\div B$. Then, according to the definition of $\div$, we have that

$$\forall \alpha_k \in B_{\leq}\ \alpha_k < ((\beta \wedge \neg\alpha_j) \vee \alpha_i) \vee \alpha_k$$

and, since $\vdash ((\beta \wedge \neg\alpha_j) \vee \alpha_i) \vee \alpha_k \leftrightarrow (\beta \vee \alpha_i \vee \alpha_k) \wedge (\neg\alpha_j \vee \alpha_i \vee \alpha_k)$, it follows from intersubstitutivity (Lemma 2.5-(v)), (EE2), and (EE1) that

$$\forall \alpha_k \in B_{\leq}\ \alpha_k < \beta \vee \alpha_i \vee \alpha_k \tag{4}$$

and

$$\forall \alpha_k \in B_{\leq}\ \alpha_k < \neg\alpha_j \vee \alpha_i \vee \alpha_k. \tag{5}$$

Now we distinguish two cases.
Case 1, $\alpha_i \in B_{\leq}$. Since $\alpha_j \in B_{\leq}$, according to Proposition 3.2, we have that $\alpha_i \vee \neg\alpha_j \leq \alpha_i$ or $\alpha_j < \alpha_j \vee \neg\alpha_i$.
Case 1.1, $\alpha_i \vee \neg\alpha_j \leq \alpha_i$. This is a contradiction because, since $\alpha_i \in B_{\leq}$, it follows from (5), $\vdash \neg\alpha_j \vee \alpha_i \vee \alpha_i \leftrightarrow \neg\alpha_j \vee \alpha_i$, and intersubstitutivity (Lemma 2.5-(v)), that $\alpha_i < \neg\alpha_j \vee \alpha_i$.
Case 1.2, $\alpha_j < \alpha_j \vee \neg\alpha_i$. From this and the fact that $\alpha_j \vee \neg\alpha_i \vdash \alpha_j \vee \neg\alpha_i \vee \beta$ making use of (EE2) and (EE1) we obtain that

$$\alpha_j < \alpha_j \vee \neg\alpha_i \vee \beta. \tag{6}$$

Since $\alpha_j \in B_{\leq}$, from (4) we have that $\alpha_j < \beta \vee \alpha_i \vee \alpha_j$. From the latter condition and (6) it follows by conjunction up (Lemma 2.5-(vi)), that $\alpha_j < (\alpha_j \vee \neg\alpha_i \vee \beta) \wedge (\beta \vee \alpha_i \vee \alpha_j)$. Finally from $\vdash (\alpha_j \vee \neg\alpha_i \vee \beta) \wedge (\beta \vee \alpha_i \vee \alpha_j) \leftrightarrow \alpha_j \vee \beta$ and intersubstitutivity (Lemma 2.5-(v)) we obtain that $\alpha_j < \alpha_j \vee \beta$, which contradicts (3).
Case 2, $\alpha_i \notin B_{\leq}$. Then, let $\alpha_x \in B_{\leq}$ be such that[5]

$$\alpha_x < \alpha_x \vee \neg\alpha_i. \tag{7}$$

Since $\alpha_x \in B_{\leq}$ it follows from (4) and (5), respectively, that:

$$\alpha_x < \beta \vee \alpha_i \vee \alpha_x \tag{8}$$

$$\alpha_x < \neg\alpha_j \vee \alpha_i \vee \alpha_x. \tag{9}$$

[5] Notice that it follows from Proposition 3.3 that there is some $\alpha_x \in B_{\leq}$ in such conditions.

From (7) and the fact that $\alpha_x \vee \neg\alpha_i \vdash \alpha_x \vee \neg\alpha_i \vee \beta$ making use of (EE2) and (EE1) we obtain that $\alpha_x < \alpha_x \vee \neg\alpha_i \vee \beta$. From the latter condition and (8) it follows by conjunction up (Lemma 2.5-(vi)) that $\alpha_x < (\alpha_x \vee \neg\alpha_i \vee \beta) \wedge (\beta \vee \alpha_i \vee \alpha_x)$. Hence, from $\vdash (\alpha_x \vee \neg\alpha_i \vee \beta) \wedge (\beta \vee \alpha_i \vee \alpha_x) \leftrightarrow \alpha_x \vee \beta$ and intersubstitutivity (Lemma 2.5-(v)) we obtain that

$$\alpha_x < \alpha_x \vee \beta. \tag{10}$$

From (7) and the fact that $\alpha_x \vee \neg\alpha_i \vdash \alpha_x \vee \neg\alpha_i \vee \neg\alpha_j$ making use of (EE2) and (EE1) we obtain that $\alpha_x < \alpha_x \vee \neg\alpha_i \vee \neg\alpha_j$. From the latter condition and (9) it follows by conjunction up (Lemma 2.5-(vi)) that $\alpha_x < (\alpha_x \vee \neg\alpha_i \vee \neg\alpha_j) \wedge (\neg\alpha_j \vee \alpha_i \vee \alpha_x)$. Hence, from $\vdash (\alpha_x \vee \neg\alpha_i \vee \neg\alpha_j) \wedge (\neg\alpha_j \vee \alpha_i \vee \alpha_x) \leftrightarrow \alpha_x \vee \neg\alpha_j$ and intersubstitutivity (Lemma 2.5-(v)) we obtain that

$$\alpha_x < \alpha_x \vee \neg\alpha_j. \tag{11}$$

According to Lemma 2.5-(ii), it follows from (11) that $\alpha_j \leq \alpha_x$.
Now we note that, since $\alpha_j, \alpha_x \in B_{\leq}$, it follows from Propositions 3.2 and 3.3 that either $\alpha_x \vee \neg\alpha_j \leq \alpha_x$ or $\alpha_j < \alpha_j \vee \neg\alpha_x$.
Case 2.1, $\alpha_x \vee \neg\alpha_j \leq \alpha_x$. This contradicts (11).
Case 2.2, $\alpha_j < \alpha_j \vee \neg\alpha_x$. From this and the fact that $\alpha_j \vee \neg\alpha_x \vdash \alpha_j \vee \neg\alpha_x \vee \beta$, it follows by (EE2) and (EE1) that

$$\alpha_j < \alpha_j \vee \neg\alpha_x \vee \beta. \tag{12}$$

From $\alpha_j \leq \alpha_x$, (10) and (EE1) we obtain that $\alpha_j < \alpha_x \vee \beta$. Hence, from $\alpha_x \vee \beta \vdash \alpha_x \vee \beta \vee \alpha_j$, (EE2) and (EE1) we obtain that

$$\alpha_j < \alpha_x \vee \beta \vee \alpha_j. \tag{13}$$

From (12) and (13) it follows by conjunction up (Lemma 2.5-(vi)) that $\alpha_j < (\alpha_j \vee \neg\alpha_x \vee \beta) \wedge (\alpha_x \vee \beta \vee \alpha_j)$. Hence, from $\vdash (\alpha_j \vee \neg\alpha_x \vee \beta) \wedge (\alpha_x \vee \beta \vee \alpha_j) \leftrightarrow \alpha_j \vee \beta$ and intersubstitutivity (Lemma 2.5-(v)) we obtain that $\alpha_j < \alpha_j \vee \beta$, which contradicts (3).

Package conjunctive overlap and Package conjunctive inclusion According to Proposition 3.7, for all sentences α it holds that $\mathbf{K}-\alpha = \mathbf{K}\div\{\alpha\}$, where $-$ is the $\leq$-based singleton contraction on $\mathbf{K}$. On the other hand, it follows from Gärdenfors & Makinson (1988, Theorem 4) that $-$ satisfies *Conjunctive overlap* and *Conjunctive inclusion*. Hence, by Proposition 4.2 (ii), we can conclude that $\div$ satisfies *Package conjunctive overlap and Package conjunctive inclusion*.

Package singleton reduction Let B be an arbitrary set of sentences such that $B \cap Cn(\emptyset) = \emptyset$. As we have already proven above, $\div$ satisfies *Package inclusion*, *Package success*, *Package uniformity*, *Package relevance*, *Package conjunctive overlap*, and *Package conjunctive inclusion*.
Now let $\leq_{\div}$ be the epistemic entrenchment relation with respect to $\mathbf{K}$ defined by

$$\alpha \leq_{\div} \beta \text{ if and only if } \alpha \notin \mathbf{K}\div\{\alpha \wedge \beta\} \text{ or } \vdash \alpha \wedge \beta.$$

We only need to show that $\mathbf{K}\div B = \bigcap_{\alpha_i \in B_{\leq_{\div}}} \mathbf{K}\div\{\alpha_i\}$, where $B_{\leq_{\div}}$ is the $\leq_{\div}$-based filtration of B.
Let $-$ be the $\leq$-based singleton contraction on $\mathbf{K}$. On the one hand, according to Proposition 3.7, for any sentence α, $\mathbf{K}-\alpha = \mathbf{K}\div\{\alpha\}$. On the other hand, it follows from Gärdenfors & Makinson (1988, Theorem 4) that $-$ satisfies condition $(C_{\leq})$. Hence, by Proposition 4.2-(i), we can conclude that $\div$ satisfies condition $(CM_{\leq})$.

Therefore the binary relation $\leq_{\div}$ introduced above coincides with the epistemic entrenchment relation $\leq$ (on which the contraction function $\div$ is based).

Thus the equality that we need to prove is indeed equivalent to following one: $\mathbf{K}\div B = \bigcap_{\alpha_i \in B_{\leq}} \mathbf{K}\div\{\alpha_i\}$. But this latter equality follows immediately from Proposition 3.7, and this part of the proof is complete.

Postulates-to-construction:

Let $\div$ be a multiple contraction function on $\mathbf{K}$ that satisfies *Package inclusion*, *Package success*, *Package uniformity*, *Package relevance*, *Package conjunctive overlap*, *Package conjunctive inclusion*, and *Package singleton reduction* and consider the epistemic entrenchment relation $\leq$ with respect to $\mathbf{K}$ defined by condition $(CM_{\leq})$.[6]

In order to prove that $\div$ is an epistemic entrenchment-based multiple contraction on $\mathbf{K}$ it is enough to show that, for any set of sentences B, $\mathbf{K}\div B = \mathbf{K}\div_{\leq} B$, where $\div_{\leq}$ is the $\leq$-based multiple contraction defined by $(CM_{\div_{\leq}})$.

Thus, in what follows we will prove that

$$\mathbf{K}\div B = \begin{cases} \{\beta \in \mathbf{K} : \forall \alpha_i \in B_{\leq}\ \alpha_i < \alpha_i \vee \beta\} & \text{, if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{, if } B \cap Cn(\emptyset) \neq \emptyset \end{cases}.$$

If $B \cap Cn(\emptyset) \neq \emptyset$, then it follows immediately from *Package inclusion* and *Package relevance* that $\mathbf{K}\div B = \mathbf{K}$ and we are done.

So, it only remains to show that if $B \cap Cn(\emptyset) = \emptyset$, then $\mathbf{K}\div B = \{\beta \in \mathbf{K} : \forall \alpha_i \in B_{\leq}\ \alpha_i < \alpha_i \vee \beta\}$. Hence, in what follows we assume that $B \cap Cn(\emptyset) = \emptyset$ and show that (under that assumption) the above equality is satisfied.

We start by noticing that it follows from *Package singleton reduction* that

$$\mathbf{K}\div B = \bigcap_{\alpha_i \in B_{\leq}} \mathbf{K}\div\{\alpha_i\}, \tag{14}$$

where $B_{\leq}$ is the $\leq$-based filtration of B. In fact, in the present conditions, according to that postulate, it holds that $\mathbf{K}\div B = \bigcap_{\alpha_i \in B_{\leq_{\div}}} \mathbf{K}\div\{\alpha_i\}$, where $\leq_{\div}$ is the epistemic entrenchment relation with respect to $\mathbf{K}$ defined by

$$\alpha \leq_{\div} \beta \text{ if and only if } \alpha \notin \mathbf{K}\div\{\alpha \wedge \beta\} \text{ or } \vdash \alpha \wedge \beta,$$

and $B_{\leq_{\div}}$ is the $\leq_{\div}$-based filtration of B. So, since it follows from condition $(CM_{\leq})$ that $\leq_{\div}$ coincides with $\leq$, we can conclude that the equality (14) indeed holds.

Now let $-$ be the singleton contraction function on $\mathbf{K}$ defined in the following way:

$$\forall \alpha \in \mathcal{L},\ \mathbf{K}-\alpha = \mathbf{K}\div\{\alpha\}.$$

At this point we remark that it follows from Proposition 4.1 that $\div$ also satisfies *Package closure*, *Package vacuity*, *Package extensionality*, and *Package recovery*.

Therefore we can conclude by Proposition 4.2 that: (a) Conditions $(C_{\leq})$ and $(CM_{\leq})$ define the same binary relation $\leq$ on $\mathcal{L}$, and (b) the singleton contraction $-$ satisfies all the basic and supplementary AGM postulates for belief set contraction (namely, *closure*, *inclusion*, *vacuity*, *success*, *recovery*, *extensionality*, *Conjunctive overlap*, and *Conjunctive inclusion*). Hence, it follows from Gärdenfors & Makinson (1988, Theorem 5) that

[6] Notice that Proposition 4.3 assures that under the above assumptions it holds that the binary relation $\leq$ thus defined is indeed an epistemic entrenchment relation with respect to $\mathbf{K}$.

$\mathbf{K}-\alpha = \mathbf{K}-_{\leq}\alpha$ for any sentence α, where $-_{\leq}$ is the $\leq$-based contraction defined by condition $(C_{-_{\leq}})$.

Thus, we can conclude that

$$\forall \alpha_i \in B,\ \mathbf{K} \div \{\alpha_i\} = \{\beta \in \mathbf{K} : \alpha_i < \alpha_i \vee \beta\}. \tag{15}$$

Finally, combining (14) with (15) we get $\mathbf{K} \div B = \bigcap_{\alpha_i \in B_{\leq}} \{\beta \in \mathbf{K} : \alpha_i < \alpha_i \vee \beta\} = \{\beta \in \mathbf{K} : \forall \alpha_i \in B_{\leq}\ \alpha_i < \alpha_i \vee \beta\}$ as required. □

Combining the above theorem with Proposition 2.10 we can immediately conclude the following corollary:

COROLLARY 4.7. *Let* $\mathbf{K}$ *be a belief set. An epistemic entrenchment-based multiple contraction on* $\mathbf{K}$ *is a partial meet multiple contraction on* $\mathbf{K}$.

§5. Interrelation Between System of Spheres-based Multiple Contractions and Epistemic Entrenchment-based Multiple Contractions. In this section our main goal is to prove that the class of epistemic entrenchment-based multiple contractions introduced in section §3. coincides with the class of system of spheres-based multiple contractions presented in Reis (2011) and Fermé & Reis (2012).

We start by noticing that it follows from Propositions 2.6 and 2.7 that for every epistemic entrenchment relation $\leq$ there is a system of spheres $\mathbb{S}$ such that condition $(\leq -\mathbb{S})$ is satisfied and, conversely, for every system of spheres $\mathbb{S}$ there is an epistemic entrenchment relation $\leq$ for which that same condition holds. Therefore, in order to assure that the class of epistemic entrenchment-based multiple contractions coincides with the class of system of spheres-based multiple contractions it is enough to assure that given an arbitrary epistemic entrenchment relation $\leq$ the $\leq$-based multiple contraction $\div_{\leq}$ is such that it coincides with the $\mathbb{S}$-based multiple contraction $\div_{\mathbb{S}}$, where $\mathbb{S}$ is any system of spheres such that $\leq$ and $\mathbb{S}$ satisfy condition $(\leq -\mathbb{S})$. Now, let $\mathbf{K}$ be a belief set, $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$, and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ such that $\leq$ and $\mathbb{S}$ satisfy condition $(\leq -\mathbb{S})$ (cf. Proposition 2.7). Having in mind the argument exposed in the previous paragraph, in what follows we will show that, for all sets B:

$$\mathbf{K} \div_{\leq} B = \mathbf{K} \div_{\mathbb{S}} B$$

So, let B be an arbitrary set of sentences. The following proposition asserts that, under the above assumptions, $B_{\mathbb{S}} = B_{\leq}$.

PROPOSITION 5.1. *Let* $\mathbf{K}$ *be a belief set,* B *be a set of sentences and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$ *and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *such that condition* $(\leq -\mathbb{S})$ *holds. Then*

$$B_{\mathbb{S}} = B_{\leq},$$

where $B_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based filtration of* B *(cf. Definition* 2.11*) and* $B_{\leq}$ *is the* $\leq$*-based filtration of* B *(cf. Definition* 3.1).

Proof. If B is such that $B \subseteq Cn(\emptyset)$ then $B_{\mathbb{S}} = B_{\leq} = \emptyset$. So, in what follows we assume $B \setminus Cn(\emptyset) \neq \emptyset$.

Consider the two following Lemmas:

LEMMA 5.2 (Reis, 2011). *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$. *If* $\mathbb{S}$ *and* $\leq$ *satisfy* *condition ($\leq -\mathbb{S}$), then for any $\alpha \in \mathcal{L} \setminus Cn(\emptyset)$ and any $\beta \in \mathcal{L}$ it holds that*

$$\alpha < \alpha \vee \beta \textit{ iff } f_{\mathbb{S}}(\neg\alpha) \subseteq \|\beta\|.$$

LEMMA 5.3. *Let $\mathbf{K}$ be a belief set, $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$, and $\alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset)$. Then the following statements hold:*

(i) *If $\mathbb{S}_{\neg\beta} \subset \mathbb{S}_{\neg\alpha}$, then*

$$f_{\mathbb{S}}(\neg\alpha) \not\subset \|\neg\beta\| \textit{ iff } f_{\mathbb{S}}(\neg\alpha) \not\subseteq \|\neg\beta\|.$$

(ii) *If $\mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$, then*

$$f_{\mathbb{S}}(\neg\alpha) \not\subset f_{\mathbb{S}}(\neg\beta) \textit{ iff } (f_{\mathbb{S}}(\neg\alpha) \not\subseteq \|\neg\beta\| \textit{ or } f_{\mathbb{S}}(\neg\beta) \subseteq \|\neg\alpha\|).$$

Proof. Assume $\mathbf{K}$, $\mathbb{S}$, α, and β are as mentioned in the statement of the lemma. Statement (i) is obviously true. Now we prove that (ii) also holds.

Assume $\mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$. Since $f_{\mathbb{S}}(\neg\alpha) \not\subset f_{\mathbb{S}}(\neg\beta) \Leftrightarrow f_{\mathbb{S}}(\neg\alpha) \not\subseteq f_{\mathbb{S}}(\neg\beta)$ or $f_{\mathbb{S}}(\neg\beta) \subseteq f_{\mathbb{S}}(\neg\alpha)$, the thesis of the statement (ii) is equivalent to $(f_{\mathbb{S}}(\neg\alpha) \not\subseteq f_{\mathbb{S}}(\neg\beta)$ or $f_{\mathbb{S}}(\neg\beta) \subseteq f_{\mathbb{S}}(\neg\alpha)) \Leftrightarrow (f_{\mathbb{S}}(\neg\alpha) \not\subseteq \|\neg\beta\|$ or $f_{\mathbb{S}}(\neg\beta) \subseteq \|\neg\alpha\|)$.

Then it is enough to show that $f_{\mathbb{S}}(\neg\alpha) \not\subseteq f_{\mathbb{S}}(\neg\beta) \Leftrightarrow f_{\mathbb{S}}(\neg\alpha) \not\subseteq \|\neg\beta\|$ and $f_{\mathbb{S}}(\neg\beta) \subseteq f_{\mathbb{S}}(\neg\alpha) \Leftrightarrow f_{\mathbb{S}}(\neg\beta) \subseteq \|\neg\alpha\|$. Moreover, by symmetry it is indeed enough to prove one of the two mentioned equivalences. Hence, in what follows we will show that $f_{\mathbb{S}}(\neg\beta) \subseteq f_{\mathbb{S}}(\neg\alpha) \Leftrightarrow f_{\mathbb{S}}(\neg\beta) \subseteq \|\neg\alpha\|$.

($\Rightarrow$) Follows immediately from the fact that $f_{\mathbb{S}}(\neg\alpha) \subseteq \|\neg\alpha\|$.

($\Leftarrow$) Let $f_{\mathbb{S}}(\neg\beta) \subseteq \|\neg\alpha\|$. Since $f_{\mathbb{S}}(\neg\beta) \subseteq \mathbb{S}_{\neg\beta}$ and $\mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$, we can conclude that $f_{\mathbb{S}}(\neg\beta) \subseteq \|\neg\alpha\| \cap \mathbb{S}_{\neg\alpha} (= f_{\mathbb{S}}(\neg\alpha))$. □

Now let $C_1, \ldots, C_m, C'_1, \ldots, C'_m, C''_1, \ldots, C''_m, B_1, \ldots, B_m$ be the subsets of B constructed as described in Definition 2.11. From the two above lemmas it follows trivially that:

(a) For all $l \in \{1, \ldots, m\}$ it holds that

$$\begin{aligned} C''_l &= \{\alpha_i \in C'_l : \forall \alpha_j \in C'_l\ f_{\mathbb{S}}(\neg\alpha_j) \not\subseteq \|\neg\alpha_i\| \text{ or } f_{\mathbb{S}}(\neg\alpha_i) \subseteq \|\neg\alpha_j\|\} \\ &= \{\alpha_i \in C'_l : \forall \alpha_j \in C'_l\ \alpha_j \vee \neg\alpha_i \leq \alpha_j \text{ or } \alpha_i < \alpha_i \vee \neg\alpha_j\}. \end{aligned}$$

(b) If $m > 1$, then for all $l \in \{2, \ldots, m\}$ the following identities are satisfied:

$$\begin{aligned} C'_l &= \{\alpha_i \in C_l : \forall \alpha_j \in B_{l-1}\ f_{\mathbb{S}}(\neg\alpha_j) \not\subseteq \|\neg\alpha_i\|\}; \\ &= \{\alpha_i \in C_l : \forall \alpha_j \in B_{l-1}\ \alpha_j \vee \neg\alpha_i \leq \alpha_j\}. \end{aligned}$$

Finally, recalling the definition of the sets $C_1, \ldots, C_m, C'_1, \ldots, C'_m, C''_1, \ldots, C''_m, B_1, \ldots, B_m$ which occur in Definition 3.1, and having in mind that we are assuming that $\mathbb{S}$ and $\leq$ satisfy condition ($\leq -\mathbb{S}$), we can immediately conclude that $B_{\mathbb{S}} = B_{\leq}$, and this finishes the proof. □

We are now in a position to assert that if $\mathbb{S}$ and $\leq$ satisfy condition ($\leq -\mathbb{S}$), then the $\mathbb{S}$-based multiple contraction coincides with the $\leq$-based multiple contraction. Formally:

THEOREM 5.4. *Let $\mathbf{K}$ be a belief set, $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$, and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. If $\leq$ and $\mathbb{S}$ satisfy condition ($\leq -\mathbb{S}$), then for any set of sentences B,*

$$\mathbf{K} \div_{\leq} B = \mathbf{K} \div_{\mathbb{S}} B,$$

where $\div_{\leq}$ is the $\leq$-based multiple contraction on $\mathbf{K}$ *and* $\div_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based multiple contraction on* $\mathbf{K}$.

Proof. Assume $\mathbf{K}$, $\mathbb{S}$, and $\leq$ are as mentioned in the statement of the theorem and let B be any finite set of sentences.

We must show that $\mathbf{K}\div_{\leq} B = \mathbf{K}\div_{\mathbb{S}} B$.

If $B \cap Cn(\emptyset) \neq \emptyset$, then $\mathbf{K}\div_{\leq} B = \mathbf{K}\div_{\mathbb{S}} B = \mathbf{K}$ and we are done.

Now we consider the case $B \cap Cn(\emptyset) = \emptyset$. In this case we must prove that

$$\{\beta \in \mathbf{K} : \forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \beta\} = Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_j \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_j)\right)\right).$$

We start by noticing that, according to Proposition 5.1 it holds that $B_{\leq} = B_{\mathbb{S}}$. Now we prove the above mentioned required equality of sets.

$$\begin{aligned}
& \beta \in \{\beta \in \mathbf{K} : \forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \beta\} \\
\text{iff}\quad & \beta \in \mathbf{K} \text{ and } \forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \beta \\
\text{iff}\quad & \beta \in \bigcap \|\mathbf{K}\| \text{ and } \forall \alpha_j \in B_{\mathbb{S}}\ f_{\mathbb{S}}(\neg\alpha_j) \subseteq \|\beta\| \text{ (by Lemma 5.2,} \\
& \text{Grove, 1988, pp. 158, Property (1) and the equality } B_{\leq} = B_{\mathbb{S}}) \\
\text{iff}\quad & \beta \in \bigcap \|\mathbf{K}\| \text{ and } \forall \alpha_j \in B_{\mathbb{S}}\ \beta \in \bigcap f_{\mathbb{S}}(\neg\alpha_j) \\
\text{iff}\quad & \beta \in \bigcap \left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_j \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_j)\right)\right) \\
\text{iff}\quad & \beta \in Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_j \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_j)\right)\right).
\end{aligned}$$

□

We can observe that the converse of the statement of Theorem 5.4 is also satisfied and, therefore, the following stronger result, which generalizes (to the multiple contractions case) Proposition 2.8, also holds.

COROLLARY 5.5. *Let* $\mathbf{K}$ *be a belief set,* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$ *and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$. *Then, for any finite set of sentences* B,

$$\mathbf{K}\div_{\leq} B = \mathbf{K}\div_{\mathbb{S}} B,$$

if and only if $\leq$ *and* $\mathbb{S}$ *satisfy condition* ($\leq -\mathbb{S}$).

Proof. Let $\mathbf{K}$, $\mathbb{S}$, and $\leq$ be as stated above.

Now assume that $\mathbf{K}\div_{\leq} B = \mathbf{K}\div_{\mathbb{S}} B$, for any finite set of sentences B.

Then, in particular, $\mathbf{K}\div_{\leq} B = \mathbf{K}\div_{\mathbb{S}} B$, for any singleton set $B \subseteq \mathcal{L}$ and it follows from Propositions 3.7 and 2.13 that $\mathbf{K}-_{\leq}\alpha = \mathbf{K}-_{\mathbb{S}}\alpha$, for any sentence $\alpha \in \mathcal{L}$. Hence, according to Theorem 2.8, condition ($\leq -\mathbb{S}$) is satisfied.

Hence we have just shown that if $\mathbf{K}\div_{\leq} B = \mathbf{K}\div_{\mathbb{S}} B$, for any finite set of sentences B, then condition ($\leq -\mathbb{S}$) holds.

The converse implication is given by Theorem 5.4. □

The next theorem points out that the class of epistemic entrenchment-based multiple contractions coincides with the class of systems of spheres-based multiple contractions. In fact, having Theorem 5.4 in mind and recalling that given a system of spheres $\mathbb{S}$ there is an epistemic entrenchment relation $\leq$ such that condition ($\leq -\mathbb{S}$) holds, and vice versa (see Propositions 2.6 and 2.7), we can conclude that the following theorem holds:

THEOREM 5.6. *Let* **K** *be a belief set. A multiple contraction function on* **K** *by finite sets of sentences is an epistemic entrenchment-based multiple contraction on* **K** *if and only if it is a system of spheres-based multiple contraction on* **K**.

Combining the above theorem with Theorem 4.4 we can immediately we obtain the following axiomatic characterization for the system of spheres-based multiple contractions:

COROLLARY 5.7. *Let* **K** *be a belief set and* $\div$ *be a multiple contraction function on* **K**. *Then the following statements are equivalent:*

1. $\div$ *is a system of spheres-based multiple contraction on* **K**.
2. $\div$ *satisfies Package success, Package inclusion, Package relevance, Package uniformity, Package conjunctive overlap, Package conjunctive inclusion, Package singleton reduction.*

§6. Conclusions and Discussion. We have introduced a new class of multiple contraction operations—the *epistemic entrenchment-based multiple contractions*—which: (a) is formed by a kind of multiple contraction functions which are a generalization of the epistemic entrenchment-based (singleton) contractions introduced in Gärdenfors (1988) and Gärdenfors & Makinson (1988) to the case of contraction by sets of sentences and (b) coincides with the class of system of spheres-based multiple contractions presented in Reis (2011) and Fermé & Reis (2012), analogously to what is the case in what concerns the interrelation between the singleton contraction counterparts of such two classes of functions.

Furthermore we have obtained an axiomatic characterization for the class of epistemic entrenchment-based multiple contractions and, consequently, also for the class of system of spheres-based multiple contractions.

In what follows we briefly discuss each one of the postulates included in that axiomatic characterization:

- The postulates of *Package success*, *Package inclusion*, *Package relevance*, and *Package uniformity* are precisely the postulates included in the well-known axiomatic characterization of the partial meet multiple contraction that was presented in Fuhrmann & Hansson (1994).
- The postulates of *Package conjunctive overlap* and *Package conjunctive inclusion* are straightforward adaptations of the supplementary AGM postulates for (singleton) contraction to the case of multiple contraction. Besides, their role in the presented axiomatic characterization is analogous to the (central) role played by their singleton counterparts in the obtention of the axiomatic characterization presented in Gärdenfors & Makinson (1988) for the epistemic entrenchment-based (singleton) contraction. More precisely, those two postulates are essential in order to assure that the binary relation defined from a multiple contraction function $\div$ by means of condition ($CM_{\leq}$) is an epistemic entrenchment relation (cf. Proposition 4.3).
- Finally, the postulate of *Package singleton reduction* is a less intuitive property which, however, was necessary in our proof of the postulates-to-construction part of the mentioned representation theorem. Nevertheless, in spite of its arguably

unnatural formulation, such postulate was intuitively motivated by our observations clarifying that and how an epistemic entrenchment-based multiple contraction can be reduced to (or, in other words, defined by means of) the corresponding singleton contraction operation. Furthermore, that postulate is essentially a weaker version of the following property of a multiple contraction function $\div$: "$K \div B = \bigcap_{\alpha_i \in B} \mathbf{K} \div \{\alpha_i\}$," that is, indeed, a more intuitively appealing property[7] than *Package singleton reduction* but which, as we have clarified in Section §4., has already been proven not to be fulfilled by the class of partial meet multiple contractions (which subsumes the newly defined class of epistemic entrenchment-based multiple contractions). To finish, yet in support of that postulate we remark that it can be straightforwardly noticed that its *translation* into a singleton contraction postulate results in a property which is trivially satisfied by any contraction function which satisfies both the basic and the supplementary AGM postulates for contraction.

At this point it is worth remarking that Fuhrmann & Hansson (1994) have conjectured that the following two properties of a multiple contraction function $\div$ might be adequate generalizations of the supplementary singleton contraction postulates (of Alchourrón *et al.*, 1985), namely conjunctive overlap and conjunctive inclusion, respectively:[8]

Package conjunctive overlap' $\mathbf{K} \div \{\alpha, \alpha \wedge \beta\} \cap \mathbf{K} \div \{\beta, \alpha \wedge \beta\} \subseteq \mathbf{K} \div \{\alpha \wedge \beta\}$
Package conjunctive inclusion' If $\alpha \notin \mathbf{K} \div \{\alpha \wedge \beta\}$, then $\mathbf{K} \div \{\alpha \wedge \beta\} \subseteq \mathbf{K} \div \{\alpha, \alpha \wedge \beta\}$.

More precisely, having in mind the role that is played by the supplementary postulates in the axiomatic characterization of the *transitively relational partial meet (singleton) contractions* (introduced in Alchourrón *et al.*, 1985), Fuhrmann & Hansson (1994) pointed out as a future work topic the investigation of whether and/or how the two properties above (or some generalizations of them) are connected to the class of *transitively relational partial meet multiple contractions* (which has not yet been axiomatically characterized).

In order to relate the two above postulates with the ones that we have proposed we start by noticing that every epistemic entrenchment-based multiple contraction satisfies *Package conjunctive overlap'* and *Package conjunctive inclusion'*.[9] To see that this indeed holds observe that, on the one hand, it follows trivially from Lemma 4.6 that, if $\div$ is an epistemic entrenchment-based multiple contraction, then $\mathbf{K} \div \{\alpha, \alpha \wedge \beta\} = \mathbf{K} \div \{\alpha\}$ and $\mathbf{K} \div \{\beta, \alpha \wedge \beta\} = \mathbf{K} \div \{\beta\}$. And, on the other hand, it follows from the two latter identities that *Package conjunctive overlap'* and *Package conjunctive inclusion'* are implied by the postulates of *Package conjunctive overlap* and *Package conjunctive inclusion*, respectively.

Furthermore, the above reasoning allows us to conclude that if a multiple contraction function satisfies the following postulate:

Package normalization $\mathbf{K} \div B = \mathbf{K} \div (B_N \cap \mathbf{K})$,

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[7] In fact, based on a ranking theoretic approach, Spohn (2010) suggested a multiple contraction function defined precisely in that way.

[8] In Fuhrmann & Hansson (1994) these two properties have been presented without any specific designation however, for convenience, here we have named each of them.

[9] We notice, however, that although the epistemic entrenchment-based multiple contractions satisfy *Package conjunctive overlap'* and *Package conjunctive inclusion'* it can be easily shown that such contraction functions do not satisfy the following two postulates presented in (Fuhrmann & Hansson, 1994) as generalizations of those two properties:

1. $\mathbf{K} \div A \cap \mathbf{K} \div B \subseteq \mathbf{K} \div (A \cap B)$
2. If $A \cap \mathbf{K} \div B = \varnothing$, then $\mathbf{K} \div B \subseteq \mathbf{K} \div (A \cup B)$.

then $\div$ satisfies *Package conjunctive overlap* (respectively, *Package conjunctive inclusion*) if and only if it satisfies *Package conjunctive overlap'* (respectively, *Package conjunctive inclusion'*).

Finally, we highlight that, combining all the remarks above with Theorem 4.4 we can obtain the following alternative representation theorem for the epistemic entrenchment-based multiple contractions:

THEOREM 6.1. *Let* $\mathbf{K}$ *be a belief set. Then a multiple contraction function* $\div$ *on* $\mathbf{K}$ *is an epistemic entrenchment-based multiple contraction if and only if it satisfies Package success, Package inclusion, Package relevance, Package uniformity, Package normalization, Package conjunctive overlap', Package conjunctive inclusion', and Package singleton reduction.*

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CENTRO DE CIÊNCIAS EXACTAS E DA ENGENHARIA
UNIVERSIDADE DA MADEIRA
CAMPUS UNIVERSITÁRIO DA PENTEADA, 9020-105 FUNCHAL, PORTUGAL
E-mail: ferme@uma.pt
E-mail: m_reis@uma.pt