A NOTE ON THE REVERSE MATHEMATICS OF THE SORITES

DAMIR D. DZHAFAROV

Abstract. Sorites is an ancient piece of paradoxical reasoning pertaining to sets with the following properties: (Supervenience) elements of the set are mapped into some set of "attributes"; (Tolerance) if an element has a given attribute then so are the elements in some vicinity of this element; and (Connectedness) such vicinities can be arranged into pairwise overlapping finite chains connecting two elements with different attributes. Obviously, if Supervenience is assumed, then (1) Tolerance implies lack of Connectedness, and (2) Connectedness implies lack of Tolerance. Using a very general but precise definition of "vicinity", Dzhafarov and Dzhafarov (2010) offered two formalizations of these mutual contrapositions. Mathematically, the formalizations are equally valid, but in this paper, we offer a different basis by which to compare them. Namely, we show that the formalizations have different proof-theoretic strengths when measured in the framework of reverse mathematics: the formalization of (1) is provable in RCA_0, while the formalization of (2) is equivalent to ACA_0 over RCA_0. Thus, in a certain precise sense, the approach of (1) is more constructive than that of (2).

1. Introduction

In November of 2009, the *Reverse Mathematics: Foundations and Applications* workshop at the University of Chicago asked about using mathematical logic as a possible new basis for judging and comparing alternative and competing quantitative approaches to problems in cognitive science. There have been several papers written in this direction (e.g., [5], [1]), and this note is a further such contribution. In it, we show that two ostensibly equivalent mathematical approaches for a certain problem in cognitive science can be teased apart in terms of their logical (or proof-theoretic) strength.

Sorites is a "paradox" attributed to Eubulites of Miletus, a philosopher of the Megarian School in the 4th century BCE. It continues to be of interest to philosophers of mind and cognitive scientists. The essence of the two original versions of Sorites (known as The Heap and The Bald Man) is that if natural numbers can be classified as very large and not very large, and if N is a very large number (a heap of grains, a full head of hair), then so is $N-1$; but by repeated subtractions of 1 (removing grains or hairs one-by-one) one can get from N to any n that is not a very large number. There are obvious analogues of Sorites in a continuum of real numbers (e.g., growing a very short person by a sufficiently small amount would not make this person not very short), and spaces without linear orders (e.g., in the set of spectra of color patches, a sufficiently small change in the spectral composition of a patch judged to be red would not change its redness). There is an opinion that due to the use of the notion of "small changes" (sufficiently small, or as small as

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possible) Sorites requires a metric space [11] or at least a full-fledged topological space [10]. However, it has been shown in [3] that the most general formulation of Sorites only requires a variant of the pre-topological structure proposed by Fréchet and dubbed by him *V-spaces* (see, e.g., [7]).

Definition 1.1. A *V-space* is a pair $(X, \{\mathcal{V}_x : x \in X\})$, where X is a set, and for each $x \in X$, $\mathcal{V}_x$ is a non-empty collection of subsets of X containing x.

When we have fixed a particular V-space $(X, \{\mathcal{V}_x : x \in X\})$, we call each $V \in \mathcal{V}_x$ a *vicinity* of x. Each $x \in X$ has at least one vicinity, and one can think of each such vicinity as representing a "sense" in which the elements of that vicinity are close to x. Since x belongs to each of its vicinities, it is therefore close to itself in every sense. On the other hand, $\{x\}$ need not belong to $\mathcal{V}_x$, and more generally, if $V_x \in \mathcal{V}_x$ and $V' \subset V_x$, V need not belong to $\mathcal{V}_x$. Furthermore, if some $y \in X$ belongs to some vicinity of x, it need not be the case that x belongs to some vicinity of y. In other words, y can be close to x in some sense, without x needing to be close to y in any sense; the notion of "being close to in some sense" is not necessarily symmetric. The vicinities of a V-space can be used for the following natural definition of connectedness.

Definition 1.2. Let $(X, \{\mathcal{V}_x : x \in X\})$ be a V-space.

(1) A *cover* of this V-space is a sequence $\{V_x : x \in X\}$ such that $V_x \in \mathcal{V}_x$ for each $x \in X$.
(2) Two elements $a, b \in X$ are *connected* in this V-space if for every cover $\{V_x : x \in X\}$ there is a finite sequence $x_0, \dots, x_k$ of elements of X such that $a = x_0$, $b = x_k$, and $V_{x_j} \cap V_{x_{j+1}} \neq \emptyset$ for each $j < k$.
(3) If $a, b \in X$ are not connected, then we say a cover $\{V_x : x \in X\}$ for which there is no finite sequence $x_0, \dots, x_k$ as in (2) *witnesses* that a and b are not connected.

Using the language of V-spaces, Dzhafarov and Dzhafarov [3] formulated the following theorem central to our analysis.

Theorem 1.3 ([3, Theorem 3.5])**.** *Let $(X, (\mathcal{V}_x : x \in X))$ be a V-space, Y a set, and $\pi : X \to Y$ a function. If $a, b \in X$ are connected and $\pi(a) \neq \pi(b)$, then there exists a $x \in X$ such that π is not constant on any vicinity of x.*

The significance of this theorem for Sorites is as follows. The soritical reasoning is based on the assumption that the function $\pi : X \to Y$ can be chosen so that every $x \in X$ has a vicinity V_x with $\pi(y) = \pi(x)$ for any $y \in V_x$. This assumption is called Tolerance. The assumption that there are connected $a, b \in X$ with $\pi(a) \neq \pi(b)$ is called Connectedness. Using this terminology, the theorem above says that, for any V-space and any function π, Connectedness implies lack of Tolerance.

Dzhafarov and Dzhafarov [3] also considered the following alternative formulation, in which tolerance is assumed a priori. The theorem below can, by analogy with Theorem 1.3, be seen as a formalization of the statement that Tolerance implies lack of Connectedness.

Definition 1.4. Let X and Y be sets, and $\pi : X \to Y$ a function. The *V-space induced by π* is $(X, \{\mathcal{V}_x : x \in X\})$, where for each $x \in X$, $\mathcal{V}_x$ contains the single vicinity $\{y \in X : \pi(x) = \pi(y)\}$.

Theorem 1.5 ([3, Theorem 3.10])**.** *Let X and Y be sets and $\pi : X \to Y$ a function. If $\pi(a) \neq \pi(b)$ for some $a, b \in X$, then a and b are not connected in the V-space induced by π.*

It should be noted for completeness, that in many specific versions of Sorites the main culprit of the ensuing contradiction is the very assumption that there is a function mapping a specific set X, such as the set of heights, into a set of attributes Y, such as "tall, not tall." However, this assumption (called Supervenience) often can be saved or at least made plausible by redefining the set X (e.g., replacing its elements by sequences in which they are listed) or the set of the attributes Y (e.g., replacing such attributes as "tall, not tall" with probability distributions thereof). With this in mind, we can say that the two theorems above tell us that Supervenience can only be achieved by dispensing with either Tolerance or Connectedness. Which of them it is in specific cases can be revealed by adopting the "behavioral" approach to Sorites [2, 3]. We do not need to discuss this approach here: our focus in the present paper is the proof-theoretic strength of Theorems 1.3 and 1.5.

2. Reverse mathematical analysis

Reverse mathematics is an area of mathematical logic devoted to classifying mathematical theorems according to their proof theoretic strength. The goal is to calibrate this strength according to how much comprehension is needed to establish the existence of the sets needed to prove the theorem (i.e., according to how complex the formulas specifying such sets must be). This is a two-step process. The first involves searching for a comprehension scheme sufficient to prove the theorem, while the second gives sharpness by showing that the theorem is in fact equivalent to this comprehension scheme over some base (minimal) theory. In practice, we use for these comprehension schemes certain subsystems of second-order arithmetic. As our base theory, we use a weak subsystem called RCA_0, which suffices to prove the existence of the computable sets, but not of any non-computable ones. As such, RCA_0 corresponds to computable or constructive mathematics. A strictly stronger system is ACA_0, which adds to RCA_0 comprehension for sets described by arithmetical formulas. ACA_0 is considerably stronger than RCA_0, sufficing to prove the existence of, e.g., the halting set, and many other non-computable sets. There is, more generally, a rich and fruitful relationship between reverse mathematics on the one hand, and computability theory on the other (see, e.g., [6] for a discussion). We refer the reader to Simpson [8] or Hirschfeldt [4] for background on reverse mathematics, and to Soare [9] for background on computability.

In this section, we provide the computability-theoretic and reverse mathematical analysis of Theorems 1.3 and 1.5. We begin by formalizing the concepts from Definition 1.1 in a countable setting.

Definition 2.1. Let X be a non-empty subset of ω.

(1) A *weak system of vicinities for x in X* is a sequence $\mathcal{W} = \langle W_n : n \in \omega \rangle$ such that $x \in W_n \subseteq X$ for all n.
(2) A *weak V-space* is a pair $(X, \{\mathcal{W}_x : x \in X\})$, where for each $x \in X$, $\mathcal{W}_x$ is a weak system of vicinities for x in X.

(3) A *strong system of vicinities for x in X* is a sequence $\mathcal{S} = \langle S_n : n \in I \rangle$, where I is a non-empty (possibly finite) initial segment of ω, $x \in S_n \subseteq X$ for all $n \in I$, and $S_n \neq S_m$ for all $n, m \in I$ with $n \neq m$.
(4) A *strong V-space* is a pair $(X, \{\mathcal{S}_x : x \in X\})$, where for each $x \in X$, $\mathcal{S}_x$ is a strong system of vicinities for x in X.

Note that every strong system of vicinities for x in X computes a weak such system. Namely, if $\langle S_n : n \in I \rangle$ is a strong system of vicinities for x, define a weak system of vicinities $\langle W_n : n \in \omega \rangle$ for x by setting $W_n = S_n$ for all $n \in I$, and $W_n = S_0$ for all $n \notin I$. The converse is false, because in a weak system of vicinities $\langle W_n : n \in \omega \rangle$ it could in principle be that $W_n = W_m$ for some $n \neq m$, and there is no computable way to tell when this is the case.

Proposition 2.2. *Let $(X, \{\mathcal{W}_x : x \in X\})$ be a computable weak V-space, Y a computable set, and $\pi : X \to Y$ a computable function. If there exist $a, b \in X$ with $\pi(a) \neq \pi(b)$, but every $x \in X$ has a vicinity on which π is constant, then there is a $\emptyset'$-computable cover witnessing that a and b are not connected.*

Proof. For each $x \in X$, write $\mathcal{W}_x = \langle W_{x,n} : n \in \omega \rangle$. Now given x, search computably in $\emptyset'$ for the least n such that $\pi(y) = \pi(x)$ for all $y \in W_{x,n}$, which exists by assumption, and define $V_x = W_{x,n}$. Then $\langle V_x : x \in X \rangle$ is a cover of X, and π is constant on each V_x. Since $\pi(a) \neq \pi(b)$, this cover obviously witnesses that a and b are not connected. □

In the proof of the following result, we fix an computable enumeration $\emptyset'[s]$ of $\emptyset'$. So for all x, we have $x \in \emptyset'$ if and only if $x \in \emptyset'[s]$ for some s, in which case also $x \in \emptyset'[t]$ for all $t \geq s$. We write $x \searrow \emptyset'[s]$ if $x \in \emptyset'[s]$ and $x \notin \emptyset'[t]$ for any $t < s$.

Proposition 2.3. *There exists a computable strong V-space $(X, \{\mathcal{S}_x : x \in X\})$, a computable function $\pi : X \to \{0, 1\}$, and $a, b \in X$ with the following properties:*

(1) $\pi(a) \neq \pi(b)$;
(2) every $x \in X$ has an vicinity on which π is constant;
(3) every cover witnessing that a and b are not connected computes $\emptyset'$.

Proof. We work with $X = \omega$. Let $a < b$ be any two numbers not in $\emptyset'$. For every $x \in X$, we uniformly construct a computable strong system of vicinities. Each of a and b will have a single vicinity, V_a and V_b, respectively, while every other x will have infinitely many vicinities, $V_{x,0}, V_{x,1}, \ldots$. Specifically, we let

$$V_a = \{a\} \cup \{\langle x, s \rangle \in \omega^2 : x \searrow \emptyset'[s]\},$$

and

$$V_b = \{b\} \cup \{\langle x, s \rangle \in \omega^2 : s \geq 1 \wedge x \searrow \emptyset'[s-1]\},$$

and for every $x \notin \{a, b\}$ and for every n, we let

$$V_{x,n} = \{x\} \cup \{\langle x, t \rangle \in \omega^2 : t \geq n\}.$$

Obviously, each of these vicinities is such that the resulting V-space is computable. Note that if x is different from a and b then $V_{x,n} \neq V_{x,m}$ for all $n \neq m$. Also, if x is enumerated into $\emptyset'$ at some stage s, then $\langle x, s \rangle$ belongs to V_a and $\langle x, s+1 \rangle$ to V_b, so for any $n \leq s$, $V_a \cap V_{x,n} \neq \emptyset$ and $V_b \cap V_{x,n} \neq \emptyset$. It follows that, for any such n, no cover containing V_a, V_b, and $V_{x,n}$ can witness that a and b are not connected. We shall make use of this fact below.

We next define the function $\pi : X \to \{0,1\}$. To begin, set $\pi(a) = 0$ and $\pi(b) = 1$. For all other numbers, we proceed by induction. Unless already defined, set $\pi(0) = 0$ and $\pi(1) = 0$. Now fix $y > 1$ such that $\pi(y)$ is undefined, and assume we have defined π on all smaller numbers. Say $y = \langle x, s\rangle$, so that in particular $x < y$. If $x \searrow \emptyset'[s]$, set $\pi(y) = 0$; if $s \geq 1$ and $x \searrow \emptyset'[s-1]$, set $\pi(y) = 1$; and otherwise, set $\pi(y) = \pi(x)$. This completes the definition. It is immediate that π is constant on each of V_a and V_b. For every x different from a and b, if $x \notin \emptyset'$ then π is also constant on each $V_{x,n}$. And if x is enumerated into $\emptyset'$, say at stage s, then x is constant on every $V_{x,n}$ for $n \geq s+2$.

It follows by construction that our V-space and function π satisfy conditions (1) and (2) in the statement of the proposition. We conclude by verifying property (3). Fix any cover witnessing that a and b are not connected. Since a and b each have just one vicinity, the cover must contain V_a and V_b. As noted above, if $x \in \emptyset'$, then this cover cannot contain $V_{x,n}$ for any $n \leq s$, where s is the stage at which x is enumerated into $\emptyset'$. It follows that $x \in \emptyset'$ if and only if x is different from a and b, and $x \in \emptyset'[t]$ for the least t such that $\langle x, t\rangle$ belongs to the vicinity of x in the cover. Thus, $\emptyset'$ is computable from the cover, as desired. □

Combining the above results allows us to characterize the proof-theoretic strength of Theorem 1.3.

Theorem 2.4. *The following are equivalent over* RCA_0.

(1) ACA_0.
(2) Let $(X, \{\mathcal{W}_x : x \in X\})$ *be a weak V-space,* Y *a set, and* $\pi : X \to Y$ *a function. Suppose* $a, b \in X$ *are connected in this V-space, and* $\pi(a) \neq \pi(b)$. *Then there exists an* $x \in X$ *such that* π *is not constant on any* $W \in \mathcal{W}_x$.
(3) Let $(X, \{\mathcal{S}_x : x \in X\})$ *be a strong V-space,* Y *a set, and* $\pi : X \to Y$ *a function. Suppose* $a, b \in X$ *are connected in this V-space, and* $\pi(a) \neq \pi(b)$. *Then there exists an* $x \in X$ *such that* π *is not constant on any* $S \in \mathcal{S}_x$.

Proof. The implication from part 1 to part 2 follows by formalizing the proof of Proposition 2.2 in ACA_0. The implication from 2 to 3 is immediate by the remark following Definition 2.1. The implication 3 to 1 follows by formalizing the proof of Proposition 2.3 in RCA_0. □

We obtain a very contrasting result concerning the strength of Theorem 1.5. It is an easy observation that if X, Y, and $\pi : X \to Y$ are all computable, then the V-space induced by π is a computable strong V-space. Formalizing this, we have that, given sets X and Y and a function $\pi : X \to Y$, RCA_0 can prove the existence of the V-space (as a storng V-space) induced by π.

Theorem 2.5. RCA_0 *proves the following statement. Let* X *and* Y *be sets, and* $\pi : X \to Y$ *a function with* $\pi(a) \neq \pi(b)$ *for some* $a, b \in X$. *Then* a *and* b *are not connected in the V-space induced by* π.

Proof. We argue in RCA_0. The only cover in the V-space induced by π is $\langle S_x : x \in X\rangle$, where S_x is the (unique) vicinity of x, $\{y \in X : \pi(x) = \pi(y)\}$. Suppose $\langle x_0, \ldots, x_k\rangle$ is a finite sequence of elements of X such that $a = x_0$, $b = x_k$, and $S_{x_j} \cap S_{x_{j+1}} \neq \emptyset$ for each $j < k$. Define $\langle y_j : j < k\rangle$ such that y_j is the least element of $S_{x_j} \cap S_{x_{j+1}}$ for all $j < k$, which exists by Δ^0_1 comprehension, and is well-defined by assumption. Thus, $\pi(y_j) = \pi(y_{j+1})$ by assumption, so $\pi(y_j) = \pi(a)$ for all $j \leq k$ by Σ^0_1 induction. But then $\pi(a) = \pi(y_{k-1}) = \pi(b)$, a contradiction. □

3. Conclusion

In the analysis of Sorites, Supervenience implies incompatibility of Connectedness and Tolerance. We have shown that the two implications forming this incompatibility, Connectedness $\to \neg$Tolerance and Tolerence $\to \neg$Connectedness, when formalized using the general framework of Fréchet spaces as in Dzhafarov and Dzhafarov [3], have different proof-theoretic strength: the formalization of the former implication has the strength of RCA_0, while the formalization of the latter has the strength of ACA_0. In this sense, the implication Connectedness $\to \neg$Tolerance can be formalized by more constructive methods than its contrapositive.

References

[1] Damir D. Dzhafarov, Infinite saturated orders, Order, 28, 163–172, 2011.
[2] Damir D. Dzhafarov and Ehtibar N. Dzhafarov, Classificatory sorites, probabilistic supervenience, and rule-making, in A. Abasnezhad and O. Bueno (editors), On the Sorites Paradox, Springer, to appear.
[3] Ehtibar N. Dzhafarov and Damir D. Dzhafarov, Sorites without vagueness I: Classificatory sorites, Theoria, 76, 4–24, 2010.
[4] Denis R. Hirschfeldt, Slicing the Truth: On the Computable and Reverse Mathematics of Combinatorial Principles, Lecture notes series / Institute for Mathematical Sciences, National University of Singapore, World Scientific Publishing Company Incorporated, 2014.
[5] Alberto Marcone, Interval orders and reverse mathematics, Notre Dame J. Formal Logic, 48, 425–448 (electronic), 2007.
[6] Richard A. Shore, Reverse mathematics: the playground of logic, Bull. Symbolic Logic, 16, 378–402, 2010.
[7] Waclaw Sierpinski, General topology, Mathematical Expositions, No. 7, University of Toronto Press, Toronto, translated by C. Cecilia Krieger, 1952.
[8] Stephen G. Simpson, Subsystems of second order arithmetic, Perspectives in Logic, second edition, Cambridge University Press, Cambridge, 2009.
[9] Robert I. Soare, Turing Computability: Theory and Applications, first edition, Springer Publishing Company, Incorporated, 2016.
[10] Zach Weber and Mark Colyvan, A topological sorites, Journal of Philosophy, 107, 311–325, 2010.
[11] Timothy Williamson, Vagueness, Routledge, 1994.

Department of Mathematics, University of Connecticut, Storrs, Connecticut U.S.A.
E-mail address: damir@math.uconn.edu