ANOTHER LOOK AT THE SECOND INCOMPLETENESS THEOREM

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Abstract. In this paper we study proofs of some general forms of the Second Incompleteness Theorem. These forms conform to the Feferman format, where the proof predicate is fixed and the representation of the set of axioms varies. We extend the Feferman framework in one important point: we allow the interpretation of number theory to vary.

§1. A dialogue.

Alcibiades: Hi, Socrates. You don't know how happy I am to see you. I am thoroughly confused, and you're just the man to liberate me of this annoying puzzlement.

Socrates: I am flattered that such a popular young person still needs an old man who is not even on twitter. As you know, I think confusion is a good thing. It is an important step on the road to insight. What is your puzzlement about?

Alcibiades: Well, you remember that we were taught about the Second Incompleteness Theorem in the Lyceum? A theory cannot prove its own consistency? Arithmetization? Great stuff. I worked hard, and I dare say that I obtained a decent understanding of the proof.

Socrates: I do remember you did very well on the exam.

Alcibiades: However, now I have been reading Feferman's paper *Arithmetization of Metamathematics*. He gives an example of an axiomatization of Peano Arithmetic such that Peano Arithmetic can prove its own consistency with respect to that axiomatization.

Socrates: I commend you on your diligence. Reading the *Arithmetization* is an important step on the road to wisdom. Let no one say ever again that Alcibiades is only an irresponsible rascal and party animal. But, to be honest, I still do not see the source of your puzzlement. The Second Incompleteness Theorem is applicable when certain conditions are fulfilled and

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Feferman's clever axiomatization does not fulfill these conditions. That's how it is able to escape Gödel's Second.

Alcibiades: But, you see, Socrates, I seem to be able to prove that any axiomatization, under minimal conditions, must obey Gödel's Second. Moreover, the proof is very simple, just an application of the Compactness Theorem.

Socrates: Bring it on.

Alcibiades: Here it is. Suppose a consistent theory, for the given axiomatization, proves its own consistency. Let us call this theory *Theory*. Then, by compactness, there must be a finitely axiomatized sub-theory that already proves the consistency statement for our original theory. I will call this sub-theory simply *Sub-theory*. Since Sub-theory proves the consistency of Theory, it must also prove its own consistency. So, we have a finitely axiomatized theory that proves its own consistency. But, with the finite axiomatization, we can have no Fefermanian funny business, so, the Second Incompleteness Theorem applies in its full glory and we have a contradiction. It follows that Theory cannot prove its own consistency after all.

Socrates: I can see why you are puzzled. But, here, can you explain to me why we can infer that Sub-theory proves its own consistency, from the fact that it proves the consistency of Theory?

Alcibiades: Isn't that obvious? The consistency of the whole implies the consistency of the part.

Socrates: You are certainly right about that. It is not only true that the consistency of the whole implies the consistency of the part, what is more: theories with a modicum of arithmetic verify this important principle.

Alcibiades: But if this is right, then what can be wrong with my argument?

Socrates: To understand these matters, we must carefully distinguish the internal perspectives of the theories we are considering from our own external perspective. There are three perspectives here: ours, the perspective of Theory, and the perspective of Sub-theory.

What we have seen is that the principle that the consistency of the part is implied by the consistency of the whole is validated from all perspectives.

We also know that Sub-theory is part of Theory in our perspective. What we need is that Sub-theory knows —this is the relevant perspective for your inference— that it is a part of Theory. How does it know that?

Alcibiades: It knows that by proving, for each of its axioms, that the axiom is in the internally represented set of axioms of Theory. To be able to speak about provability-in-Theory inside Theory at all, we have to agree that our internal representation of the set of axioms is such that, for any axiom of Theory, it proves that the axiom is in the set of axioms as internally represented.

Socrates: Admirably said. So, Theory knows of each of its axioms that it is an axiom according to the internal representation. Thus, Theory knows that Sub-theory is part of Theory. But how does it follow that Sub-theory knows of each of its own axioms that that axiom is also an axiom of Theory? One would expect that Sub-theory, being a finite part, cannot automatically do everything that Theory can.

Alcibiades: I start to see some light at the end of the tunnel, but let me still try to push the argument a bit further. Clearly, Sub-theory, when it is very weak, need not be able to do this. But, for my argument, we can take the finite part as large as we want. So simply take a finite sub-theory that can verify that its axioms are axioms of the original theory according to the internal representation.

Socrates: But if you extend the finite sub-theory, you also make the task of the finite sub-theory heavier: it has to prove of more axioms that they are in the set of axioms as internally presented. Could it not happen that no finite sub-theory can do this?

Alcibiades: By the dog and by Zeus, Socrates, it seems that you unraveled the mystery. We can escape my argument in case, for every given finite set of axioms of Theory that proves the consistency of Theory, we need more axioms than there are in the given set to verify that the axioms of the given set are indeed axioms according to the given presentation.

Socrates: If you analyze Feferman's clever example, you will see that this is precisely what is going on there.

Alcibiades: I am relieved. Now I can go to Aristophanes' party this evening without having to think about the darned problem all the time.

Socrates: I am glad that I was able to help with that.

§2. Introduction. The present paper is, in a sense, a footnote to Feferman's great paper on arithmetization [5]. More precisely, it is a footnote to the part of Feferman's paper that is concerned with the Second Incompleteness Theorem. In fact the present paper started with the puzzlement voiced by Alcibiades in the dialogue.

We present some general versions of the Second Incompleteness Theorem. Clearly, the Second Incompleteness Theorem can be generalized in many ways and in many directions. For example, the Löb Conditions provide one such generalization: roughly, suppose N is an interpretation of the Tarski-Mostowski-Robinson theory R in U, in symbols $N : \mathsf{R} \lhd U$, and suppose the U-predicate $\Box$ satisfies the Löb Conditions w.r.t. sentences coded in the N-numerals, then $\Box$ also satisfies Löb's Principle. Another important generalization is Feferman's Theorem of the interpretability of inconsistency. Incompleteness, in this generalization, is not failure to prove consistency, but rather the ability to build an internal model of the theory itself plus its inconsistency statement.

Our generalization takes a different direction. We keep —as Feferman did— our proof-predicate and its arithmetization fixed but vary the formula α representing a set of axioms. There is one extra feature that we allow to vary which is constant in Feferman's paper: the interpretation of arithmetic. Thus, we do not just have to specify the predicate α that defines the set of axioms but also an interpretation that tells us where the numbers live. This leads us to a device $\mathcal{A}$, *the presentation*, that translates the language of arithmetic plus an extra predicate ax into the language of the given theory. The notion of presentation is worked out in Section 4.

We provide, in Section 5, a sufficient condition on the presentation $\mathcal{A}$ for the validity of the Second Incompleteness Theorem for $\mathcal{A}$, to wit: *being a uniform* *semi-numeration*, roughly, there are arbitrarily large finite approximations U_0 of the theory U, such that $\mathcal{A}$ semi-numerates the axioms of U_0 in U_0.

A major special case is formed by the Σ^0_1-presentations. Here the set of axioms of U is semi-numerated in U by a Σ^0_1-formula σ relativized to a suitable interpretation N of arithmetic. We discuss Σ^0_1-presentations in Section 6. We zoom in on the case where the arithmetic involved is EA. We provide some examples to liberate the reader of the impression that the case at hand is already completely clear.

- We give an example of a finitely axiomatized theory B and a Σ^0_1-formula γ that numerates the axiom set in $N_0 : \mathsf{EA} \lhd B$ but not in $N_1 : \mathsf{EA} \lhd B$. In the case of N_0 the Second Incompleteness Theorem applies, but in the case of N_1, we have that B proves the consistency of the theory axiomatized by γ. See Example 6.3.
- We give an example of a Σ^0_1-presentation that numerates the axioms of a given theory U in U, but not uniformly. See Subsection 6.1.
- We give an example that shows that theories U, where the axiom set is numerated in U by a Σ^0_1-presentation, need not be recursively enumerable but can be arbitrarily complex. See Subsection 6.1.
- We give an example of a Σ^0_1-presentation of Elementary Arithmetic EA that defines a finite set of axioms in the standard model and is believed by EA to define a finite set of axioms, for which the Löb conditions fail. Specifically, EA cannot prove the formalized Second Incompleteness Theorem for this presentation.[1] See Subsection 6.2.

We use a variant of our main theorem to prove the full second incompleteness theorem for Σ^0_1-numerations of the set of axioms for the case that the interpreted number theory is (at least) EA.

In Section 7, we have a brief look at some salient examples of non-Σ^0_1-numerations over PA. We discuss the well-known Feferman predicate. We give an example where we have a uniform numeration for which the Löb Conditions cannot be verified.

In Appendix A, we provide the basics of translations and interpretations.

§3. Preliminaries.

In this section, we present some basic definitions.

3.1. Theories. In the present paper, a theory is given by a signature and a set of sentences of that signature closed under deduction. We only consider finite signatures. I guess we can allow countable signatures, but if we allow these, we need some constraints on the effectiveness of the presentation. Our signatures are officially relational but, since we have a p-time term-elimination algorithm, for most purposes, we can pretend that we have terms. See, for a treatment of term-elimination, e.g., [23, Appendix 7.3]. We will assume that we eliminate terms using the small scope reading.

[1] This example is a good caveat for Alcibiades' naive use of 'finitely axiomatized theory'. What he has in mind is something like a representation of the set of axioms that involves a finite disjunction of formulas of the form $(x = \ulcorner A \urcorner)^N$. For this very special presentation the relevant part of his argument works, but, as the example illustrates, there are readings of 'finitely axiomatizable' that share all the pitfalls of intensionality.

We allow the theory to be of any complexity. It might be Π^{77}_{88}, or it might be outside any known hierarchy.

A set of axioms X for a theory U is simply a set of sentences of the signature of the theory such that the deductive closure $\overline{X}$ of X is equal to U. The axioms for identity will be treated as part of the theory and not of the logic. Thus, for example, an axiomatization of Elementary Arithmetic is supposed to include axioms that imply the usual axioms for identity.

We employ some special theories in this paper. These are the Tarski-Mostowski-Robinson theory R (see [21]), Robinson's arithmetic Q (see [21]), Buss' theory S^1_2 (see [4], [6]), Elementary Arithmetic EA (also known as EFA or $\mathrm{I}\Delta_0 + \mathsf{exp}$, see [6]) and Peano Arithmetic PA (see [6], [9]). For all these theories we work with variants that are formulated in the signature Ar of arithmetic with a unary relation symbol Z for zero, a binary relation symbol S for successor, and two ternary relation symbols A and M for addition and multiplication.

3.2. Translations and interpretations. We treat the definitions of this subsection in more detail in Appendix A.

A *translation* τ from signature Ξ to signature Θ is basically a mapping from the predicates of Ξ to Θ-formulas. We allow identity to be translated to a formula other than identity. The translation can be lifted to the whole Ξ-language such that the result commutes with the connectives of predicate logic.

There are two extra features. The first is domain relativization. The translation τ provides a domain given by a formula δ_τ. When we lift the translation to the full language the translated quantifiers are relativized to this domain.

The second feature is the treatment of variables. The simplest possibility is that our translation is 1-dimensional and parameter-free. In this case, we simply require that the translation of an n-ary P is of the form $A(v_0, \ldots, v_{n-1})$, where all free variables are among those shown and each v_i is from a fixed infinite list $v_0, v_1, \ldots$. We then translate $P(x_0, \ldots, x_{n-1})$ into $A(x_0, \ldots, x_{n-1})$ (under an appropriate convention to handle variable-clashes). In case we allow parameters, the translation of P has the form $A(w_0, \ldots, w_{k-1}, v_0, \ldots, v_{n-1})$, for a fixed k. The w_i are supposed to be distinct from the v_j. When we lift the translation to the full Ξ-language, we have to take care that none of the parameters gets bound. Finally, a translation could have dimension $m > 1$. In this case we send P to $A(\vec{v}_0, \ldots, \vec{v}_{n-1})$, where the $\vec{v}_i$ are pairwise disjoint sequences of variables of length m. In this case $P(x_0, \ldots, x_{n-1})$ is translated into $A(\vec{x}_0, \ldots, \vec{x}_{n-1})$, and we need some bookkeeping to assign sequences of variables of the Θ-language to variables of the Ξ-language. We can combine more-dimensionality with parameters in the obvious way.

We write B^τ for the τ-translation of B of the Ξ-language in the Θ-language.

An *interpretation* K of a theory U in a theory V is a triple $\langle U, \tau, V\rangle$, where τ translates the signature of U into the signature of V. We require that, for all U-sentences A, if $U \vdash A$, then $V \vdash A^\tau$.[2] We write $K : U \lhd V$ or $K : V \rhd U$ in case $\langle U, \tau, V\rangle$ is an interpretation.

[2] In case we have parameters the definition should be slightly expanded. We need a parameter domain.

3.3. Arithmetization. We follow Feferman in that the arithmetization of provability is fixed. It is somewhat ironic that the arithmetization given by Feferman as the one that should be fixed once and for all is one that we cannot adopt. The size of a Feferman code is superexponential(!) in the length of the formula. We want a code to be of order $2^{P(n)}$, where P is a polynomial and n is the length of the formula. In this way we can work naturally with our coding in Buss' theory S^1_2. Thus, we fix an efficient coding. The codings of [4] or of [29] would do, or, more precisely, a reworking of those codings for our arithmetical language of signature Ar.[3]

The default in this paper is to use efficient numerals: these simulate dyadic notation. For example, the number 3 is '11' in dyadic and the corresponding efficient numeral is $\mathsf{S}(\mathsf{SS0} \times \mathsf{S0})$.

We use $\ulcorner A \urcorner$ ambiguously for the Gödel number of A and for the numeral of the Gödel number of A. We will employ Smoryński's dot notation. E.g., $\ulcorner A(\dot{x}) \urcorner$ stands for the arithmetization of the function that sends a formula $A(x)$ and a number n to the Gödel number of the result of substituting the numeral $\underline{n}$ of n for x in $A(x)$.

Let a signature Θ be given. Our default situation is that we study provability *in* a theory U formulated in a Θ-language *for* a theory formulated in a Θ-language. We follow Feferman in that the notion of Θ-proof has a fixed arithmetization. We have access to this arithmetization in U via a translation $\tau : \mathsf{Ar} \to \Theta$. The definition of the axiom set is allowed to use the full Θ-language. Thus, it need not be restricted to the image of the arithmetical language under τ. This leads us to the notation $\Box^\tau_\alpha A$.

Let us explain the notation $\Box^\tau_\alpha A$ a bit more explicitly. Let $\mathsf{proof}(p)$ be the arithmetization of 'p is a predicate logical proof-from-assumptions in the Θ-language'. We note that, *par abus de langage*, we notationally suppress Θ.[4] We assume that the proof is set up in such a way that the assumptions are explicitly labeled as assumptions. Let ass represent the function that extracts the set of assumptions from p and let conc represent the function that extracts the conclusion from p. Let n be the Gödel number of A. Then,

- $\mathsf{prov}^\tau_\alpha(x) :\leftrightarrow \exists p \in \delta_\tau\, (\forall a \in \delta_\tau\, ((a \in \mathsf{ass}(p))^\tau \to a \in \alpha) \;\wedge$ $(\mathsf{proof}(p) \wedge \mathsf{conc}(p) = x)^\tau)$.
- $\Box^\tau_\alpha A :\leftrightarrow \mathsf{prov}^\tau_\alpha(\underline{n})$.

We note that our convention automatically makes $\underline{n}$ a τ-numeral. In case the ambient theory proves the functionality and totality of successor and addition inside τ, we find that $\Box^\tau_\alpha A$ is equivalent to the large scope reading

$$\exists x \in \delta_\tau((x = \underline{n})^\tau \wedge \mathsf{prov}^\tau_\alpha(x)).$$

We also note that α may contain junk elements that are not in the Θ-language. These are, in our context, don't care, since they are always automatically ignored.

Suppose Θ is an extension of Ar and $\tau = \mathsf{emb}_{\mathsf{Ar},\Theta}$, the identical embedding of Ar in Θ. In this case, we write $\mathsf{prov}_\alpha(x)$ for $\mathsf{prov}^\tau_\alpha(x)$ and $\Box_\alpha A$ for $\Box^\tau_\alpha A$.

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[3] Of course, the choice of an arithmetical basis rather than a set theory or a theory of strings or a theory of binary trees is for a large part a legacy thing. However, some methods like Craig's Trick work most naturally with our choice. The same holds for Rosser-style arguments.

[4] An alternative strategy would be to let α give both axioms and signature.

Here is an example. Let Θ be the signature of set theory and let neu be the translation of arithmetic in ZF that is based on the finite von Neumann ordinals. Let ζ be a suitable arithmetical predicate that defines the axioms of ZF. Let true be the set theoretical truth predicate for arithmetic. Then, $(\Box_\zeta A)^{\mathsf{neu}}$ and $(\Box^{\mathsf{neu}}_{\zeta^{\mathsf{neu}}} A)$ both represent ZF-provability in ZF. Let ν be the arithmetization of the mapping $B \mapsto B^{\mathsf{neu}}$ and let

$$\zeta^*(x) :\leftrightarrow \zeta^{\mathsf{neu}}(x) \vee \exists y((y \in \mathsf{sent}_{\mathsf{Ar}} \wedge x = \nu(y))^{\mathsf{neu}} \wedge \mathsf{true}(y)).$$

Then, $\Box^{\mathsf{neu}}_{\zeta^*} A$ represents provability from ZF plus arithmetical truth in ZF.

In Section 4, we will present a slightly different approach that avoids the notation $\Box^\tau_\alpha A$.

§4. A framework. In this section we introduce presentations and develop a basic framework for working with presentations. Let a signature Θ be given.

Let the signature of arithmetic be Ar and let the signature of arithmetic extended with a unary predicate ax be Ar^+. *A presentation* $\mathcal{A}$ is a translation from Ar^+ to Θ. The predicate ax stands for the set of axioms. To keep our exposition simple, we work with a parameter-free $\mathcal{A}$, but it is easy to adapt the development to the case with parameters.

We note that we can read off the signature Θ from $\mathcal{A}$. Moreover, $\mathcal{A}$ is an extension of the intended translation for arithmetic and tells us what the intended predicate representing the axiom set is, to wit: $\mathsf{ax}^{\mathcal{A}}$.

We write $(\mathcal{A})^-$ for the restriction of $\mathcal{A}$ to the arithmetical language.

REMARK 4.1. We note that $(\Box_{\mathsf{ax}} B)^{\mathcal{A}} = \Box^{(A)^-}_{\mathsf{ax}^{\mathcal{A}}} B$. ❍

Two presentations $\mathcal{A}$ and $\mathcal{B}$ are *compatible* if $(\mathcal{A})^- = (\mathcal{B})^-$. For $\tau : \mathsf{Ar} \to \Theta$, Δ_τ is the set of all presentations $\mathcal{A} : \mathsf{Ar}^+ \to \Theta$ such that $(\mathcal{A})^- = \tau$.

4.1. Presentations and set of axioms. We write sent_Θ for the Θ-sentences. Let $X \subseteq \mathsf{sent}_\Theta$. We define:

- $\mathsf{G}_{\mathcal{A}}(X) := \{B \in \mathsf{sent}_\Theta \mid X \vdash (\mathsf{ax}(\ulcorner B \urcorner))^{\mathcal{A}}\}$.
- $\mathsf{H}_{\mathcal{A}}(X) := \mathsf{G}_{\mathcal{A}}(X) \cap X$.

We note that, since we assumed nothing about the translation $\mathcal{A}$ and allow any input X, the function $\mathsf{G}_{\mathcal{A}}$ might, for certain values, be completely silly. Numerals may even fail to be defined. E.g., $X \vdash (\exists x\ \ x = \underline{3})^{\mathcal{A}}$ may fail.

REMARK 4.2. We remind the reader that, by our conventions, modulo provability in predicate logic, e.g., $(\exists y_0\ \ y_0 = \underline{3})^{\mathcal{A}}$ unravels to:

$$\begin{aligned} &\exists y_0 \in \delta_{\mathcal{A}}\, \exists y_1 \in \delta_{\mathcal{A}}\, \exists y_2 \in \delta_{\mathcal{A}}\, \exists y_3 \in \delta_{\mathcal{A}}\, \exists y_4 \in \delta_{\mathcal{A}}\, \exists y_5 \in \delta_{\mathcal{A}}\, \exists y_6 \in \delta_{\mathcal{A}} \\ &\qquad (\mathsf{S}^{\mathcal{A}}(y_1, y_0) \wedge \mathsf{M}^{\mathcal{A}}(y_2, y_3, y_1) \wedge \mathsf{S}^{\mathcal{A}}(y_4, y_2) \wedge \\ &\qquad\qquad \mathsf{S}^{\mathcal{A}}(y_5, y_4) \wedge \mathsf{Z}^{\mathcal{A}}(y_5) \wedge \mathsf{S}^{\mathcal{A}}(y_6, y_3) \wedge \mathsf{Z}^{\mathcal{A}}(y_6)). \end{aligned}$$

❍

We write $X_0 \subseteq_{\mathsf{fin}} X$ for $X_0 \subseteq X$ and X_0 is finite.

Clearly, $\mathsf{G}_{\mathcal{A}}(X) = \mathsf{G}_{\mathcal{A}}(\overline{X})$. We also have, by the compactness property of predicate logic, that $\mathsf{G}_{\mathcal{A}}(X) = \bigcup\{\, \mathsf{G}_{\mathcal{A}}(X_0) \mid X_0 \subseteq_{\mathsf{fin}} X\}$. In other words, $\mathsf{G}_{\mathcal{A}}$ is

Scott-continuous. It follows that $\mathsf{G}_{\mathcal{A}}$ is monotonic and that $\mathsf{G}_{\mathcal{A}}$ commutes with unions of directed sets of sets of sentences. Similarly, for $\mathsf{H}_{\mathcal{A}}$.

- A set of Θ-sentences X is *$\mathcal{A}$-complete* iff $X \subseteq \mathsf{G}_{\mathcal{A}}(X)$.[5]
- A set of Θ-sentences X is *$\mathcal{A}$-sound* iff $\mathsf{G}_{\mathcal{A}}(X) \subseteq X$.

We observe that $\mathcal{A}$-complete sets are closed under arbitrary unions.

It is natural to lift Feferman's notion of numeration to the current setting. For us it is useful to also have the weaker notion of semi-numeration. Let U be a theory of signature Θ and let Z be a set of numbers. We say that:

- *$\mathcal{A}$ semi-numerates Z in U* if $n \in Z$ implies $U \vdash (\mathsf{ax}(\underline{n}))^{\mathcal{A}}$.
- *$\mathcal{A}$ numerates Z in U* iff we have: $n \in Z$ iff $U \vdash (\mathsf{ax}(\underline{n}))^{\mathcal{A}}$.
- *$\mathcal{A}$ is a (semi-)numeration [of axioms] for U* iff $\mathcal{A}$ (semi-)numerates in U some set X of (Gödel numbers of) axioms for U.[6]

Thus, a set of axioms X is $\mathcal{A}$-complete iff $\mathcal{A}$ semi-numerates (the set of Gödel numbers of the elements of) X in $\overline{X}$. A set of axioms X is $\mathcal{A}$-sound and $\mathcal{A}$-complete iff $\mathcal{A}$ numerates (the set of Gödel numbers of the elements of) X in $\overline{X}$.

Here are some basic insights.

THEOREM 4.3. *Suppose U is axiomatized by X and X is $\mathcal{A}$-complete. Then $X \subseteq \mathsf{H}_{\mathcal{A}}(U)$ and $\mathsf{H}_{\mathcal{A}}(U)$ is $\mathcal{A}$-complete. In other words, if U has an $\mathcal{A}$-complete axiomatization, then $\mathsf{H}_{\mathcal{A}}(U)$ is the maximal $\mathcal{A}$-complete axiomatization of U.*

PROOF. Suppose $\overline{X} = U$ and $X \subseteq \mathsf{G}_{\mathcal{A}}(X)$. We have $X = \mathsf{H}_{\mathcal{A}}(X) \subseteq \mathsf{H}_{\mathcal{A}}(U)$. Moreover, $\mathsf{G}_{\mathcal{A}}(\mathsf{H}_{\mathcal{A}}(U)) \supseteq \mathsf{G}_{\mathcal{A}}(\mathsf{H}_{\mathcal{A}}(X)) = \mathsf{G}_{\mathcal{A}}(X) = \mathsf{G}_{\mathcal{A}}(U) \supseteq \mathsf{H}_{\mathcal{A}}(U)$. ❑

THEOREM 4.4. *Suppose X is $\mathcal{A}$-complete and X axiomatizes U. Suppose further that U is finitely axiomatizable. Then, there is a finite $\mathcal{A}$-complete $X_0 \subseteq X$ that axiomatizes U.*

PROOF. Suppose Y_0 is a finite set of axioms for U. Since, X axiomatizes U, there is, by compactness, a finite set $X_0 \subseteq X$ that implies Y_0. Thus, X_0 axiomatizes U. It follows that

$$\mathsf{G}_{\mathcal{A}}(X_0) = \mathsf{G}_{\mathcal{A}}(\overline{X_0}) = \mathsf{G}_{\mathcal{A}}(\overline{X}) = \mathsf{G}_{\mathcal{A}}(X) \supseteq X \supseteq X_0.$$

So, X_0 is $\mathcal{A}$-complete. ❑

We say that:

- X is *uniformly $\mathcal{A}$-complete* when X is the union of all its $\mathcal{A}$-complete finite subsets, in other words, if $X = \bigcup\{X_0 \subseteq_{\mathsf{fin}} X \mid X_0 \subseteq \mathsf{G}_{\mathcal{A}}(X_0)\}$.

We note that uniform $\mathcal{A}$-completeness implies $\mathcal{A}$-completeness. Moreover, uniform $\mathcal{A}$-completeness is closed under arbitrary unions.

Let X be a set of (Gödel numbers of) axioms for U. We say that:

- $\mathcal{A}$ is *a uniform semi-numeration of X in U* if X is uniformly $\mathcal{A}$-complete.

[5]The expression 'completeness' is used in analogy with 'Σ_1-completeness' and 'Completeness Theorem'. Our completeness says roughly: if something is an axiom, then it is provably an axiom.

[6]For brevity, we will omit the qualification 'of axioms' when speaking about semi-numerations or numerations *for* a theory.

- $\mathcal{A}$ is *a uniform numeration of* X *in* U if X is uniformly $\mathcal{A}$-complete and X is $\mathcal{A}$-sound.
- $\mathcal{A}$ is *a uniform (semi-)numeration [of axioms] for* U iff $\mathcal{A}$ uniformly (semi-) numerates in U a set of (Gödel numbers of) axioms X for U.[6]

We first prove the analogue of Theorem 4.3

THEOREM 4.5. *Suppose* U *is axiomatized by* X *and* X *is uniformly* $\mathcal{A}$*-complete. Then* $X \subseteq \mathsf{H}_{\mathcal{A}}(U)$ *and* $\mathsf{H}_{\mathcal{A}}(U)$ *are uniformly* $\mathcal{A}$*-complete. Thus, if* $\mathcal{A}$ *is a uniform semi-numeration for* U*, then* $\mathsf{H}_{\mathcal{A}}(U)$ *is the maximal uniformly* $\mathcal{A}$*-complete axiomatization.*

PROOF. Suppose U is axiomatized by X and X is uniformly $\mathcal{A}$-complete. Clearly, $X \subseteq \mathsf{H}_{\mathcal{A}}(U)$. Suppose $A \in \mathsf{H}_{\mathcal{A}}(U)$. Since X axiomatizes U, it follows that there are $B_0, \ldots, B_{n-1}$ in X such that $B_0, \ldots, B_{n-1} \vdash (\mathsf{ax}(\ulcorner A \urcorner))^{\mathcal{A}}$. By uniform $\mathcal{A}$-completeness, we can find $\mathcal{A}$-complete $Z_i \subseteq_{\mathsf{fin}} X$, such that $B_i \in Z_i$. Let X_0 be the union of the Z_i, for $i < n$. Then X_0 is a finite, $\mathcal{A}$-complete subset of X. It follows that $X_0 \cup \{A\}$ is a finite, $\mathcal{A}$-complete subset of $\mathsf{H}_{\mathcal{A}}(U)$. ❑

THEOREM 4.6. $\mathcal{A}$ *is a uniform semi-numeration for* U *iff, for every* $A \in U$*, there is an* $\mathcal{A}$*-complete* $X_0 \subseteq_{\mathsf{fin}} U$ *such that* $X_0 \vdash A$*.*

PROOF. "$\Rightarrow$" Suppose $\mathcal{A}$ is a uniform semi-numeration for U. Let X be a uniformly $\mathcal{A}$-complete axiomatization of U. Suppose $A \in U$. It follows that $X \vdash A$. Hence, reasoning as in the proof of Theorem 4.5, we find an $\mathcal{A}$-complete $X_0 \subseteq_{\mathsf{fin}} X$ such that $X_0 \vdash A$.

"$\Leftarrow$" Suppose, for every A in U, there is an $\mathcal{A}$-complete $X_0 \subseteq_{\mathsf{fin}} U$ such that $X_0 \vdash A$. Let X be the union of all finite $\mathcal{A}$-complete $X_0 \subseteq U$. Clearly, X axiomatizes U. Moreover, by definition, X is uniformly $\mathcal{A}$-complete. ❑

4.2. Ordering on presentations. We will use $\Diamond$ for $\neg\Box\neg$. Thus, $\Diamond_{\mathsf{ax}}\top$ will stand for a consistency statement.

Suppose we have a Θ-theory U and a translation $\tau : \mathsf{Ar} \to \Theta$ such that $U \vdash (\mathsf{S}^1_2)^\tau$. We define a preordering of presentations in Δ_τ as follows:

- $\mathcal{A} \preceq_{U,\tau} \mathcal{B}$ iff $U \vdash \forall \vec{x} \in \delta_\tau\, ((\mathsf{prov}_{\mathsf{ax}})^{\mathcal{A}}(\vec{x}) \to (\mathsf{prov}_{\mathsf{ax}})^{\mathcal{B}}(\vec{x}))$.
- $\mathcal{A} =_{U,\tau} \mathcal{B}$ iff $\mathcal{A} \preceq_{U,\tau} \mathcal{B}$ and $\mathcal{A} \succeq_{U,\tau} \mathcal{B}$.

We could drop the restriction to τ by defining a category instead of a preordering, where an arrow is a U-definable embedding between presentations. However, for the present paper such a category does not seem to be relevant.

4.3. Operations on presentations. In the present subsection, we study ways of transforming presentations into other presentations.

Let $\tau : \mathsf{Ar} \to \Theta$ be a translation. We write $\tau[\mathsf{ax} := \alpha]$ for the translation from Ar^+ to Θ, which is equal to τ on Ar and where $\mathsf{ax}(x)$ is translated to $\delta_\tau(\vec{x}) \wedge \alpha(\vec{x})$.

We extend the $[\mathsf{ax} := \alpha]$ notation to presentations, by writing: $\mathcal{A}[\mathsf{ax} := \alpha]$ for $(\mathcal{A})^-[\mathsf{ax} := \alpha]$. So here $[\mathsf{ax} := \alpha]$ becomes a *reset to* rather than a *set to.*

4.3.1. *Finite sets of axioms.* Let $X_0 := \{A_0, \ldots, A_{k-1}\}$ be a finite set of Θ-sentences. We write β_{X_0} for $\bigvee_{i<k} x = \ulcorner A_i \urcorner$. Here, of course we should fix some order for the elements of X_0 and have some convention on how to put the brackets. However, since all such choices lead to formulas that are equivalent over predicate logic, we will not worry about them. Consider the Tarski-Mostowski-Robinson theory R. We have:

THEOREM 4.7. *Let X_0 be a finite set of Θ-sentences. Suppose $X_0 \vdash \mathsf{R}^\tau$. Then, X_0 is $\tau[\mathsf{ax} := \beta^\tau_{X_0}]$-complete.*

We omit the obvious proof. We also have:

THEOREM 4.8. *Suppose X_0 is a finite set of Θ-sentences. Let U be a Θ-theory. Let $\tau := (\mathcal{A})^-$. Suppose $U \vdash (\mathsf{S}^1_2)^\tau$ and X_0 is $\mathcal{A}$-complete. Then, $\mathcal{A}[\mathsf{ax} := \beta^\tau_{X_0}] \preceq_{U,\tau} \mathcal{A}$.*

PROOF. Suppose $U \vdash (\mathsf{S}^1_2)^\tau$ and X_0 is $\mathcal{A}$-complete.
We find $U \vdash \bigwedge_{C \in X_0} (\mathsf{ax}(\ulcorner C \urcorner))^{\mathcal{A}}$. So, $U \vdash (\forall x\, (\beta_{X_0}(x) \to \mathsf{ax}(x)))^{\mathcal{A}}$. Hence, *a fortiori*, $U \vdash (\forall x \in \mathsf{sent}_\Theta\, (\mathsf{prov}_{\beta_{X_0}}(x) \to \mathsf{prov}_{\mathsf{ax}}(x)))^{\mathcal{A}}$. ❑

4.3.2. *Union of presentations.* The operation $+$ is defined on compatible pairs $\mathcal{A}$, $\mathcal{B}$. It is simply the intensional counterpart of union of sets of axioms. Of course, to do this meaningfully, we cannot switch numbers. We define:

- $\mathcal{A} + \mathcal{B} := \mathcal{A}[\,\mathsf{ax} := ((\exists y \leq x\, (x = \mathsf{disj}(\ulcorner \bot \urcorner, y) \wedge \mathsf{ax}(y)))^{\mathcal{A}} \vee$
  $(\exists y \leq x\, (x = \mathsf{conj}(\ulcorner \top \urcorner, y) \wedge \mathsf{ax}(y)))^{\mathcal{B}})]$.

The disjunction with $\bot$ and the conjunction with $\top$ are added to make addition a bi-functor with respect to $\preceq_{U,\tau}$. To have this we need to be able to distinguish the sources of the axioms effectively in the proof. See the proof of Theorem 4.9. (It would be nice to have an example to show that the naive definition does not work.) We have:

THEOREM 4.9. *Suppose $\tau : \mathsf{Ar} \to \Theta$ and $U \vdash (\mathsf{S}^1_2)^\tau$. We have, for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \Delta_\tau$,*

a. The operation $+$ restricted to Δ_τ is monotonic w.r.t. $\preceq_{U,\tau}$.
b. $\mathcal{A} \preceq_{U,\tau} \mathcal{A} + \mathcal{B}$ and $\mathcal{B} \preceq_{U,\tau} \mathcal{A} + \mathcal{B}$.
c. $\mathcal{A} + \mathcal{A} =_{U,\tau} \mathcal{A}$.
d. $(\mathcal{A} + \mathcal{B}) + \mathcal{C} =_{U,\tau} \mathcal{A} + (\mathcal{B} + \mathcal{C})$.
e. $\mathcal{A} + \mathcal{B} =_{U,\tau} \mathcal{B} + \mathcal{A}$.

PROOF. The only item that deserves some attention here is (a). Suppose, e.g., we have $\mathcal{A} \preceq_{U,\tau} \mathcal{A}'$ and $\mathcal{B} \preceq_{U,\tau} \mathcal{B}'$. We extend the signature of arithmetic with ax_0, ax_1, ax'_0, ax'_1. Let $\mathcal{C}$ be the translation from the new signature to Θ that is τ when restricted to Ar, where $\mathsf{ax}_0^{\mathcal{C}} := \mathsf{ax}^{\mathcal{A}}$, $\mathsf{ax}_1^{\mathcal{C}} := \mathsf{ax}^{\mathcal{B}}$, ${\mathsf{ax}'_0}^{\mathcal{C}} := \mathsf{ax}^{\mathcal{A}'}$, ${\mathsf{ax}'_1}^{\mathcal{C}} := \mathsf{ax}^{\mathcal{B}'}$. We write inside $\mathcal{C}$,

$$(\alpha + \beta)(x) := \exists y \leq x\, ((x = \mathsf{disj}(\ulcorner \bot \urcorner, y) \wedge \alpha(y)) \vee (x = \mathsf{conj}(\ulcorner \top \urcorner, y) \wedge \beta(y))).$$

Note that $U \vdash \forall \vec{x} \in \delta_\tau\, (\mathsf{ax}^{\mathcal{A}+\mathcal{B}}(\vec{x}) \leftrightarrow (\mathsf{ax}_0 + \mathsf{ax}_1)^{\mathcal{C}}(\vec{x}))$, and similarly for $\mathcal{A}' + \mathcal{B}'$, $\mathsf{ax}'_0 + \mathsf{ax}'_1$.

We reason in U inside $\mathcal{C}$. Suppose $\mathsf{prov}_{\mathsf{ax}_0+\mathsf{ax}_1}(x)$. Let p be a witnessing proof. Let z be a conjunction of the C such that an axiom of the form $(\bot \vee C)$ is used in p and let w be a conjunction of the D such that an axiom of the form $(\top \wedge D)$ is used in p. We note that, by definition, the C are ax_0-axioms and the D are ax_1-axioms. It is easy to see that these conjunctions exist in S^1_2. We now transform p into a proof q in predicate logic of $\mathsf{imp}(\mathsf{conj}(z,w),x)$. It is easy to check that the transformation of p to q is available in S^1_2. Clearly, we have $\mathsf{prov}_{\mathsf{ax}_0}(z)$ and $\mathsf{prov}_{\mathsf{ax}_1}(w)$. It follows that $\mathsf{prov}_{\mathsf{ax}'_0}(z)$ and $\mathsf{prov}_{\mathsf{ax}'_1}(w)$. Ergo, we can construct a proof witnessing $\mathsf{prov}_{\mathsf{ax}'_0+\mathsf{ax}'_1}(x)$. ❑

THEOREM 4.10. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are compatible. Suppose further $\mathcal{A}$ and $\mathcal{B}$ are semi-numerations for U and V. Then, $\mathcal{A}+\mathcal{B}$ is a semi-numeration for $\overline{(U \cup V)}$. Similarly, if $\mathcal{A}$ and $\mathcal{B}$ are uniform semi-numerations for U and V, then, $\mathcal{A}+\mathcal{B}$ is a uniform semi-numeration for $\overline{(U \cup V)}$.*

We leave the trivial proofs to the reader.

Let $\mathcal{A}$ be a presentation and let A be a Θ-sentence. We write:

- $\mathcal{A} \oslash A$ for $\mathcal{A} + \mathcal{A}[\mathsf{ax} := \beta^{\mathcal{A}}_{\{A\}}]$.

We note that, for any translation $\tau : \mathsf{Ar} \to \Theta$ and any finite set $X_0 = \{A_0, \ldots, A_{k-1}\}$ of Θ-sentences, we have that $\tau[\mathsf{ax} := \beta^{\tau}_{X_0}]$ is equivalent over predicate logic to $\tau[\mathsf{ax} := \bot] \oslash A_0 \ldots \oslash A_{k-1}$.[7]

We have, by a trivial proof:

THEOREM 4.11. *Suppose $U \vdash (\mathsf{S}^1_2)^{\mathcal{A}}$. Then, $U \vdash (\Box_{\mathsf{ax}} B)^{\mathcal{A} \oslash A} \leftrightarrow (\Box_{\mathsf{ax}}(A \to B))^{\mathcal{A}}$.*

4.3.3. *Deductive closure.* We define the deductive closure of $\mathcal{A}$ as

$$\overline{\mathcal{A}} := \mathsf{Th}(\mathcal{A}) := \mathcal{A}[\mathsf{ax} := (\mathsf{prov}_{\mathsf{ax}})^{\mathcal{A}}(\vec{x})].$$

We have the obvious:

THEOREM 4.12. *Suppose $U \vdash (\mathsf{S}^1_2)^{\mathcal{A}}$. Let $\tau := (\mathcal{A})^{-}$. Then,*

a. The operation Th restricted to Δ_τ is monotonic w.r.t. $\preceq_{U,\tau}$. In other words, Th is an endo-functor of the preorder category given by $\preceq_{U,\tau}$.
b. $U \vdash \forall \vec{x} \in \delta_{\mathcal{A}}\, (\mathsf{ax}^{\mathcal{A}}(\vec{x}) \to \mathsf{ax}^{\mathsf{Th}(\mathcal{A})}(\vec{x}))$, and, hence $\mathcal{A} \preceq_{U,\tau} \mathsf{Th}(\mathcal{A})$.
c. If $\mathcal{A}$ is a semi-numeration for U, then so is $\mathsf{Th}(\mathcal{A})$.
d. If $\mathcal{A}$ is a uniform semi-numeration for U, then so is $\mathsf{Th}(\mathcal{A})$.

Suppose $U \vdash (\mathsf{S}^1_2)^{\mathcal{A}}$. Let $\tau := (\mathcal{A})^{-}$. It is important to note that we do not generally have: $\mathsf{Th}(\mathcal{A}) \preceq_{U,\tau} \mathcal{A}$. A counterexample will be given in Example 4.17.

We have the following strengthening of Theorem 4.8.

THEOREM 4.13. *Suppose X_0 is a finite set of Θ-sentences. Let U be a Θ-theory. Let $\tau := (\mathcal{A})^{-}$. Suppose $U \vdash (\mathsf{S}^1_2)^{\tau}$ and X_0 is $\mathsf{Th}(\mathcal{A})$-complete. Then, we have $\mathcal{A}[\mathsf{ax} := \beta^{\tau}_{X_0}] \preceq_{U,\tau} \mathcal{A}$.*

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[7] Here $\tau[\mathsf{ax} := \bot] \oslash A_0 \ldots \oslash A_{k-1}$ is read as $((\ldots(\tau[\mathsf{ax} := \bot] \oslash A_0)\ldots) \oslash A_{k-1})$.

Proof. Suppose $U \vdash (\mathsf{S}^1_2)^\tau$ and X_0 is $\mathsf{Th}(\mathcal{A})$-complete. Thus, for any $C \in X_0$, we have $U \vdash (\Box_{\mathsf{ax}}\, C)^{\mathcal{A}}$. It follows, by a feasible transformation of proofs, using that X_0 is standardly finite, that: $U \vdash (\forall x \in \mathsf{sent}_\Theta\, (\mathsf{prov}_{\beta_{X_0}}(x) \to \mathsf{prov}_{\mathsf{ax}}(x)))^{\mathcal{A}}$. ❑

4.3.4. *Craigification.* We consider a presentation $\mathcal{A} : \mathsf{Ar}^+ \to \Theta$. We will say that $\mathcal{A}$ is *an E-presentation* if $\mathsf{ax}^{\mathcal{A}}(\vec{x})$ has the form $\alpha(\vec{x}\,) := \exists \vec{y} \in \delta_{\mathcal{A}}\, B(\vec{y}, \vec{x})$. Throughout this subsubsection we will assume that $\mathcal{A}$ is an E-presentation.

We extend Ar^+ with a new binary predicate symbol B, which we translate as B. Let the resulting translation be $\mathcal{A}^*$. We can now write $\alpha(\vec{x})$ as $(\exists y\, \mathsf{B}(y, x))^{\mathcal{A}^*}$, where we assume that the single variable x is translated to the sequence of variables $\vec{x}$. We define:

$$\alpha^\star(\vec{x}) :\leftrightarrow (\exists u \leq x\, \exists y \leq x\, (x = \mathsf{conj}(u, \ulcorner \dot{y} = \dot{y} \urcorner) \wedge \mathsf{B}(y, u)))^{\mathcal{A}^*}.$$

We define the Craigification of $\mathcal{A}$ by: $\mathsf{Cr}(\mathcal{A}) := \mathcal{A}[\mathsf{ax} := \alpha^\star]$.

Unfortunately, Craigification is irredeemably syntactic. It is generally not a functor w.r.t. $\preceq_{U,\tau}$.

It will be convenient for readability to have the following presentation $\mathcal{A}^\star$ at hand. We expand Ar with ax, B, and $\mathsf{ax}^\star$ and extend $\mathcal{A}^*$ to $\mathcal{A}^\star$ on the new signature by setting the translation of $\mathsf{ax}^\star$ to $\alpha^\star$.

Theorem 4.14. *Suppose $\mathcal{A}$ is an E-presentation and $U \vdash (\mathsf{S}^1_2)^{\mathcal{A}}$. Let τ be $(\mathcal{A})^-$. Then, $\mathsf{Cr}(\mathcal{A}) \preceq_{U,\tau} \mathcal{A}$. In terms of $\mathcal{A}^\star$, this says:*

$$U \vdash (\forall x\, (\mathsf{prov}_{\mathsf{ax}^\star}(x) \to \mathsf{prov}_{\mathsf{ax}}(x)))^{\mathcal{A}^\star}.$$

Proof. We reason in U inside $\mathcal{A}^\star$. Suppose $\mathsf{prov}_{\mathsf{ax}^\star}(x)$. Let p be a witnessing proof. We zoom in on the occurrence of an axiom C from $\mathsf{ax}^\star$ in p. This axiom is of the form $\ulcorner D \wedge \dot{y} = \dot{y} \urcorner$, where D is an axiom from ax. We replace the sub-proof consisting of C by the obvious sub-proof of C from D, and similarly for all other occurrences of axioms in p, thus obtaining an ax-proof p' of x. To show that this is possible, we have to verify that the transformation $p \mapsto p'$ is p-time. To see this, we note that the length of the subproof replacing C is just twice the length of C plus some standard overhead m. Thus, the length of p' will be estimated by two times the length of p plus the length of p times m. In other words, the length of p' is bounded by $m+1$ times the length of p. This yields a polynomial bound on p'. ❑

The converse of Theorem 4.14 does not hold, as will be illustrated by the Example 4.17 at the end of this subsubsection. We collect some further properties of Craigification.

Theorem 4.15. *Suppose $\mathcal{A}$ is an E-presentation and $U \vdash (\mathsf{S}^1_2)^{\mathcal{A}}$. We have:*

a. *Suppose U has a $\mathcal{A}$-complete axiomatization, then U is $\mathsf{Th}(\mathsf{Cr}(\mathcal{A}))$-complete; in other words, U is closed under the necessitation rule for $(\Box_{\mathsf{ax}}(\cdot))^{\mathsf{Cr}(\mathcal{A})}$.*
b. *Suppose U has a uniformly $\mathcal{A}$-complete axiomatization, then U is uniformly $\mathsf{Th}(\mathsf{Cr}(\mathcal{A}))$-complete.*

PROOF. We expand $\mathcal{A}$ to $\mathcal{A}^\star$ as in the proof of Theorem 4.14.

Ad (a): Let X be some axiomatization of U that is $\mathcal{A}$-complete. So we have: if $A \in X$, then $U \vdash (\mathsf{ax}(\ulcorner A \urcorner))^{\mathcal{A}^\star}$. In other words, $U \vdash (\exists y\, \mathsf{B}(y, \ulcorner A \urcorner))^{\mathcal{A}^\star}$.

Suppose $C \in U$. Then, for some $A_0, \dots, A_{n-1} \in X$, we have $A_0, \dots, A_{n-1} \vdash C$. Say the Gödelnumber of the proof is p. We find:

$$U \vdash (\bigwedge_{i<n} \exists y_i\, \mathsf{B}(y_i, \ulcorner A_i \urcorner))^{\mathcal{A}^\star} \text{ and } U \vdash (\mathsf{proof}_{\beta_{\{A_0,\dots,A_{n-1}\}}}(\underline{p}, \ulcorner C \urcorner))^{\mathcal{A}^\star}.$$

We want to show $U \vdash (\Box_{\mathsf{ax}^\star}\, C)^{\mathcal{A}^\star}$.

We reason in U inside $\mathcal{A}^\star$. Let y_i be such that $\mathsf{B}(y_i, \ulcorner A_i \urcorner)$ for $i < n$. It follows that $\ulcorner A_i \wedge \dot{y}_i = \dot{y}_i \urcorner$ is in $\mathsf{ax}^\star$. We note that the map $y \mapsto \ulcorner A_i \wedge \dot{y}_i = \dot{y}_i \urcorner$ is p-time, thanks to our use of efficient numerals. We now transform $\underline{p}$ to an $\mathsf{ax}^\star$-proof q by replacing each sub-proof of $\underline{p}$ that consists of an ax-axiom A_i by a proof of A_i from $\ulcorner A_i \wedge \dot{y}_i = \dot{y}_i \urcorner$. We note that the length of q is bounded by the length of $y^\star$, i.e., the maximum of the y_i's times a standard constant. We know that $y^\star$ exists since p is standard. Thus, the transformation $\underline{p} \mapsto q$ is available within the resources of S^1_2.

Ad (b). Let X be an uniformly $\mathcal{A}$-complete axiomatization of U. Let C be any sentence of U. Let $X_0 \subseteq_{\mathsf{fin}} X$ be such that $X_0 \vdash C$, $X_0 \vdash (\mathsf{S}^1_2)^{\mathcal{A}}$, and X_0 is $\mathcal{A}$-complete. It follows from (a) that $\overline{X}_0$ is $\mathsf{Th}(\mathsf{Cr}(\mathcal{A}))$-complete, and hence that $X_0 \cup \{C\}$ is $\mathsf{Th}(\mathsf{Cr}(\mathcal{A}))$-complete. From this it is immediate that U is uniformly $\mathsf{Th}(\mathsf{Cr}(\mathcal{A}))$-complete. ❑

The next property shows that Cr is a kind of left-inverse of Th. Regrettably, this does not have functorial meaning because of the irredeemably syntactic character of Cr. We note that $\mathsf{Th}(\mathcal{A})$ is an E-presentation, so that it is in the range of Cr.

THEOREM 4.16. *Suppose* $U \vdash (\mathsf{S}^1_2)^{\mathcal{A}}$. *Let* $\tau = (\mathcal{A})^-$. *Then,* $\mathsf{Cr}(\mathsf{Th}(\mathcal{A})) =_{U,\tau} \mathcal{A}$.

PROOF. We extend the signature Ar^+ with an extra predicate $\widetilde{\mathsf{ax}}$ that is interpreted as $\mathsf{ax}^{\mathsf{Cr}(\mathsf{Th}(\mathcal{A}))}$. Thus we obtain a translation $\widetilde{\mathcal{A}}$. We reason in U, inside $\widetilde{\mathcal{A}}$. We note that:

$$\widetilde{\mathsf{ax}}(B) \leftrightarrow \exists C \leq B\, \exists p \leq B\, (B = \ulcorner C \wedge \dot{p} = \dot{p} \urcorner \wedge \mathsf{proof}_{\mathsf{ax}}(p, C)).$$

We have to show that, for any A, $\Box_{\widetilde{\mathsf{ax}}}\, A$ iff $\Box_{\mathsf{ax}}\, A$.

From left to right: Suppose q is an $\widetilde{\mathsf{ax}}$-proof of A. Consider any occurrence of an axiom B_i in q. B_i is of the form $\ulcorner C_i \wedge \dot{p}_i = \dot{p}_i \urcorner$, where p_i is an ax-proof of C_i. We replace the occurrence of B_i by the ax-proof p_i of C_i followed by the inference from C_i to $\ulcorner C_i \wedge \dot{p}_i = \dot{p}_i \urcorner$. We may assume that the length of p_i is of the same order as the length of $\ulcorner \dot{p}_i \urcorner$. It follows that the length of the replacement is bounded by four times the length of p_i with some standard overhead.

Now when we replace all occurrences of $\widetilde{\mathsf{ax}}$-axioms in the manner prescribed, we see that the length of the new ax-proof, say $p^\star$, will be bound by k times the length of q, for some standard k. Thus, the transformation $q \mapsto p^\star$ is available with the resources of S^1_2.

From right to left: Let p be an ax-proof of A. It follows that $\ulcorner A \wedge \dot{p} = \dot{p} \urcorner$ is in $\widetilde{\mathsf{ax}}$. So we can take as $\widetilde{\mathsf{ax}}$-proof the inference from $\ulcorner A \wedge \dot{p} = \dot{p} \urcorner$ to A. ❑

Here is an example that separates a Craigification from the original axiomatization and an original axiomatization from its theoretization.

EXAMPLE 4.17. On the one hand, in [27, Theorem 4.1], we proved the following. Suppose that C is a single axiom that axiomatizes EA. We take $\beta(x) := (x = \ulcorner C \urcorner)$ and $\mathcal{B} := \mathsf{Id}_{\mathsf{Ar}}[\mathsf{ax} := \beta^{\mathsf{Id}_{\mathsf{Ar}}}]$. Let B be a single statement that is equivalent to Σ^0_1-collection over EA. Then $\mathsf{EA} + \neg B \vdash (\Box_{\mathsf{ax}} \bot)^{\mathsf{Th}(\mathcal{B})}$.

On the other hand, we have $\mathsf{Cr}(\mathsf{Th}(\mathcal{B})) =_{\mathsf{EA},\mathsf{Id}_{\mathsf{Ar}}} \mathcal{B}$. But, $\mathsf{EA} + \neg B \nvdash (\Box_{\mathsf{ax}} \bot)^{\mathcal{B}}$, since $\mathsf{EA} + \neg B$ is Π_3-conservative over EA, by the results of Paris & Kirby in [14].

It follows that $\mathsf{EA} \nvdash (\Box_{\mathsf{ax}} \bot)^{\mathsf{Th}(\mathcal{B})} \to (\Box_{\mathsf{ax}} \bot)^{\mathsf{Cr}(\mathsf{Th}(\mathcal{B}))}$, and, thus,

$$\mathsf{Cr}(\mathsf{Th}(\mathcal{B})) =_{\mathsf{EA},\mathsf{Id}_{\mathsf{Ar}}} \mathcal{B} \precnsim_{\mathsf{EA},\mathsf{Id}_{\mathsf{Ar}}} \mathsf{Th}(\mathcal{B}).$$

Thus, taking $\mathsf{Th}(\mathcal{B})$ as our original axiomatization, we find that the Craigification is strictly below the original axiomatization. In Example 6.19, we provide a second example of this phenomenon. We also see that, if we take $\mathcal{B}$ as our original axiomatization, we the original axiomatization is strictly below its theoretization. ❍

§5. The Second Incompleteness Theorem à la Alcibiades. We are ready and set to give the corrected Alcibiades argument. This will be done in Subsection 5.1. In the succeeding subsections, we will provide some variations and strengthenings.

5.1. The basic version. We have:

THEOREM 5.1. *Suppose the Θ-theory U has a uniformly $\mathcal{A}$-complete axiomatization. Then, whenever $U \vdash (\mathsf{S}^1_2 + \Diamond^{\Theta}_{\mathsf{ax}} \top)^{\mathcal{A}}$, the theory U is inconsistent.*

In a different formulation: *suppose $K : U \rhd (\mathsf{S}^1_2 + \Diamond^{\Theta}_{\mathsf{ax}} \top)$. Suppose further that U has uniformly τ_K-complete axiomatization. Then, U is inconsistent.*

PROOF. The theory S^1_2 is finitely axiomatizable, say B is a single axiom for it. So, if $U \vdash (\mathsf{S}^1_2 + \Diamond^{\Theta}_{\mathsf{ax}} \top)^{\mathcal{A}}$, it follows that $U \vdash (B \wedge \Diamond_{\mathsf{ax}} \top)^{\mathcal{A}}$. Then, for some $\mathcal{A}$-complete $X_0 \subseteq_{\mathsf{fin}} U$, we have $X_0 \vdash (B \wedge \Diamond_{\mathsf{ax}} \top)^{\mathcal{A}}$. It follows, by Theorem 4.8 (with X_0 in the role of U), that $X_0 \vdash (B \wedge \Diamond_{\beta_{X_0}} \top)^{\mathcal{A}}$. We now apply the Second Incompleteness Theorem for finitely axiomatized theories and find that $\overline{X_0}$ is inconsistent. Hence, *a fortiori*, U is inconsistent. ❑

This is, of course, precisely the proof that Alcibiades had in mind, where the hidden assumption is made explicit. We note that the above proof is fully constructive.

5.2. A slightly stronger version. In this subsection, we prove a strengthening of Theorem 5.1. In Subsection 6.3, we will see a case where Theorem 5.2 rather than Theorem 5.1 is needed.

THEOREM 5.2. *Suppose the* Θ*-theory* U *has an uniformly* $\mathsf{Th}(\mathcal{A})$*-complete axiomatization. Then, whenever* $U \vdash (\mathsf{S}^1_2 + \Diamond^{\Theta}_{\mathsf{ax}} \top)^{\mathcal{A}}$*, the theory* U *is inconsistent.*

In a different formulation: *suppose* $K : U \rhd (\mathsf{S}^1_2 + \Diamond^{\Theta}_{\mathsf{ax}} \top)$*. Suppose further that* U *has a uniformly* $\mathsf{Th}(\tau_K)$*-complete axiomatization. Then,* U *is inconsistent.*

PROOF. Let B be a single axiom for S^1_2. Suppose $U \vdash (\mathsf{S}^1_2 + \Diamond_{\mathsf{ax}} \top)^{\mathcal{A}}$. It follows that $U \vdash (B \wedge \Diamond_{\mathsf{ax}} \top)^{\mathcal{A}}$. Thus, for some $\mathsf{Th}(\mathcal{A})$-complete $X_0 \subseteq_{\mathsf{fin}} U$, we have $X_0 \vdash (B \wedge \Diamond_{\mathsf{ax}} \top)^{\mathcal{A}}$. By Theorem 4.13 (with X_0 in the role of U), we find: $X_0 \vdash (\Diamond_{\mathsf{ax}} \top \to \Diamond_{\beta_{X_0}} \top)^{\mathcal{A}}$, and, thus, $X_0 \vdash (\mathsf{S}^1_2 + \Diamond_{\beta_{X_0}} \top)^{\mathcal{A}}$. We now apply the Second Incompleteness Theorem for finitely axiomatized theories and find that X_0 is inconsistent. Hence, *a fortiori*, U is inconsistent. ❑

5.3. Löb's Rule. We prove closure under Löb's Rule under the appropriate conditions.

THEOREM 5.3. *Suppose the* Θ*-theory* U *has a uniformly* $\mathcal{A}$*-complete axiomatization and* $U \vdash (\mathsf{S}^1_2)^{\mathcal{A}}$*. Then, whenever* $U \vdash (\Box_{\mathsf{ax}} C)^{\mathcal{A}} \to C$*, we have* $U \vdash C$*.*

We give two proofs. The first proof is in essence the usual proof of closure under Löb's Rule from the Second Incompleteness Theorem. This proof is attributed to Saul Kripke.

FIRST PROOF. Suppose the Θ-theory U has a uniformly $\mathcal{A}$-complete axiomatization and $U \vdash (\mathsf{S}^1_2)^{\mathcal{A}}$. Suppose further that $U \vdash (\Box_{\mathsf{ax}} C)^{\mathcal{A}} \to C$. It follows that $U + \neg C \vdash (\Diamond_{\mathsf{ax}} \neg C)^{\mathcal{A}}$. By Theorems 4.7 and 4.10, the theory $U + \neg C$ is uniformly $\mathcal{A} \oslash \neg C$-complete. By Theorem 4.11, we have $U \vdash (\Diamond_{\mathsf{ax}} \neg C)^{\mathcal{A}} \leftrightarrow (\Diamond_{\mathsf{ax}} \top)^{\mathcal{A} \oslash \neg C}$. Thus, we may conclude that $U + \neg C \vdash (\Diamond_{\mathsf{ax}} \top)^{\mathcal{A} \oslash \neg C}$ and $\mathcal{A} \oslash \neg C$ is a uniform semi-numeration for $U + \neg C$. It follows, by Theorem 5.1, that $U + \neg C \vdash \bot$, and, hence, $U \vdash C$. ❑

We can also prove the desired result directly.

SECOND PROOF. Suppose the Θ-theory U has a uniformly $\mathcal{A}$-complete axiomatization and $U \vdash (\mathsf{S}^1_2)^{\mathcal{A}}$. Let B be a single axiom for S^1_2. Suppose that $U \vdash (\Box_{\mathsf{ax}} C)^{\mathcal{A}} \to C$. So, $U \vdash B^{\mathcal{A}} \wedge ((\Box_{\mathsf{ax}} C)^{\mathcal{A}} \to C)$. Then, for some $\mathcal{A}$-complete $X_0 \subseteq_{\mathsf{fin}} U$, we have $X_0 \vdash B^{\mathcal{A}} \wedge ((\Box_{\mathsf{ax}} C)^{\mathcal{A}} \to C)$. It follows that $X_0 \vdash B^{\mathcal{A}} \wedge ((\Box_{\beta_{X_0}} C)^{\mathcal{A}} \to C)$. We now apply Löb's Rule for finitely axiomatized theories and find that $X_0 \vdash C$. Hence, *a fortiori*, $U \vdash C$. ❑

We note that the second proof has the advantage that it is fully constructive. We can strengthen the previous theorem a bit.

THEOREM 5.4. *Suppose the* Θ*-theory* U *has a uniformly* $\mathsf{Th}(\mathcal{A})$*-complete axiomatization and* $U \vdash (\mathsf{S}^1_2)^{\mathcal{A}}$*. Then, whenever* $U \vdash (\Box_{\mathsf{ax}} C)^{\mathcal{A}} \to C$*, we have* $U \vdash C$*.*

The proof is a slight modification of the proofs above using either Theorem 5.2 or a variation on the proof of Theorem 5.2.

§6. Σ^0_1-presentations. In this section, we will consider the case of Σ^0_1-numerations of the set of axioms. This is, of course, in part, the case discussed in Feferman's [5]. Before proceeding, let us state and prove our version of the traditional version of the Second Incompleteness Theorem for Σ^0_1-numerations.

THEOREM 6.1. *Suppose $\sigma(x)$ is a Σ^0_1-formula that numerates the axioms of U in the standard model. Then, if $U \rhd (\mathsf{S}^1_2 + \Diamond_\sigma \top)$, we have that U is inconsistent.*

PROOF. Suppose $N : U \rhd (\mathsf{S}^1_2 + \Diamond_\sigma \top)$. It follows that, for some finite sub-theory U_0 of U, we have $U_0 \vdash (\mathsf{S}^1_2)^{\tau_N}$. It immediately follows, by Σ^0_1-completeness, that $\tau_N[\mathsf{ax} := \sigma^{\tau_N}]$ semi-numerates the axioms of U in U_0. So, *a fortiori*, the axioms of U are uniformly $\tau_N[\mathsf{ax} := \sigma^{\tau_N}]$-complete in U. ❑

In Appendix B, we give four alternative proofs of Theorem 6.1.

Since Feferman's set-up was less general than ours, the result does not summarize everything to be said. There are three issues that need to be addressed.

The first issue is that Σ^0_1-numerations of the set of axioms in the given theory need not be uniform. We provide an example of this phenomenon in Subsection 6.1. There we will also show that a theory, whose axioms are numerated by a Σ^0_1-formula, can be as complex as we like: for every set of numbers Z there is a such a theory U such that Z is reducible to U.

The second issue is that we do not know whether we have provable Σ^0_1-completeness in weak theories like S^1_2. This problem is connected to questions concerning the collapse of the polynomial hierarchy. We want to sidestep this issue. There are two ways to do this. The first is to replace Σ^0_1 by $\exists\Sigma^{\mathsf{b}}_1$. In the context of the stronger theory EA, Σ^0_1 and $\exists\Sigma^{\mathsf{b}}_1$ coincide modulo provable equivalence. So this approach does not differ from the classical one as soon as we have EA. The second way is more simple-minded: just work with EA as our basic theory. We will choose this last option.

The third issue is that, in the absence of Σ^0_1-collection, Σ^0_1-axiomatized theories need not satisfy the Löb conditions. We provide an example of this fact in Subsection 6.2. In case we have Σ^0_1-collection, we do have the following theorem.[8]

THEOREM 6.2. *Suppose $\sigma(x)$ is Σ^0_1 and $N : U \rhd (\mathsf{EA} + \mathrm{B}\Sigma^0_1 + \Diamond_\sigma \top)$. Suppose further that $\mathcal{S} := \tau_N[\mathsf{ax} := \sigma^{\tau_N}]$ semi-numerates the axioms of U in U, then U is inconsistent.*

PROOF. The predicate $(\Box_{\mathsf{ax}} A)^{\mathcal{S}}$ satisfies the Löb conditions. ❑

Finally, in Subsection 6.3, we improve Theorem 6.2 by presenting two proofs of the Second Incompleteness Theorem for Σ^0_1-numerations in the EA-case.

To wet the reader's appetite, here is a first example of the behaviour of Σ_1-numerations.

[8]Theorem 6.2 is the direct generalization of Feferman's Theorem 5.6 in [5]. We note that Feferman also does not require the theory U to be recursively enumerable.

EXAMPLE 6.3. Let C be a single axiom for EA and let $\beta(x) := (x = \ulcorner C \urcorner)$. We consider the theory $A := \mathsf{EA} + \Box_\beta \bot$. Since, A is a finitely axiomatized sequential theory, there is a faithful interpretation $K : A \rhd_{\mathsf{faith}} \mathsf{EA}$. (This result is due to Harvey Friedman. See [24] for an exposition.) Let $\tau := \tau_K$ and let $B := A + (\Diamond_\beta \top)^\tau$. So, B is a consistent theory. We take:

$$\gamma(x) := \beta(x) \vee (\Box_\beta \bot \wedge (x = \ulcorner \Box_\beta \bot \urcorner \vee x = \ulcorner (\Diamond_\beta \top)^\tau \urcorner)).$$

We set $\mathfrak{S} := \mathsf{Id}_{\mathsf{Ar}}[\mathsf{ax} := \gamma]$ and $\mathfrak{T} := \tau[\mathsf{ax} := \gamma^\tau]$.

Clearly, $\mathfrak{S}$ numerates the set of axioms $\{C, \Box_\beta \bot, (\Diamond_\beta \top)^\tau\}$ of B in B. Hence, $B \nvdash (\Diamond_{\mathsf{ax}} \top)^{\mathfrak{S}}$.

On the other hand, $B \vdash (\forall x\, (\gamma(x) \leftrightarrow \beta(x)))^\tau$, and, hence, $B \vdash (\Diamond_\gamma \top)^\tau$, or, in other words, $B \vdash (\Diamond_{\mathsf{ax}} \top)^{\mathfrak{T}}$. ❍

6.1. A non-uniform Σ_1-numeration. We remind the reader of witness comparison notation in the context of arithmetic. Suppose $A = \exists x\, A_0(x)$ and $B = \exists y\, B_0(y)$. Then,

- $A \leq B :\leftrightarrow \exists x\, (A_0(x) \wedge \forall y < x\, \neg B_0(y))$,
- $A < B :\leftrightarrow \exists x\, (A_0(x) \wedge \forall y \leq x\, \neg B_0(y))$.

Let C be a single axiom for EA and let $\beta := (x = \ulcorner C \urcorner)$. By the Gödel Fixed-Point Theorem, we find a formula $R(x)$ such that:

$$\mathsf{EA} \vdash R(x) \leftrightarrow \Box_{\beta + \bigwedge_{y<x} R(\dot{y})} \neg R(\dot{x}) \leq \Box_{\beta + \bigwedge_{y<x} R(\dot{y})} R(\dot{x}).$$

We consider the theory U axiomatized by $X := \{C\} \cup \{R(\underline{n}) \mid n \in \omega\}$.

The theory U is consistent, since $R(\underline{n})$ is a Rosser sentence for $\mathsf{EA} + \{R(\underline{k}) \mid k < n\}$.

Let α be the following predicate: $\alpha(x) := \beta(x) \vee \exists y < x\, (x = \ulcorner R(\dot{y}) \urcorner \wedge R(y+1))$. Let $\mathfrak{A} := \mathsf{Id}_{\mathsf{Ar}}[\mathsf{ax} := \alpha^{\mathsf{Id}_{\mathsf{Ar}}}]$. We clearly have: $\mathsf{EA} \vdash \alpha(\ulcorner R(\underline{n}) \urcorner) \leftrightarrow R(\underline{n+1})$. Since U is consistent, it follows that $\mathfrak{A}$ numerates X in U.

THEOREM 6.4. *For $\mathfrak{A}$, U and X as defined above, we have: $\mathfrak{A}$ is not a uniform semi-numeration of X in U.*

PROOF. Let X_0 be a finite subset of X. Without loss of generality, we may assume that X_0 contains an axiom larger than C. Let $R(\underline{n})$ be the largest axiom in X_0. Suppose $X_0 \vdash \alpha(\ulcorner R(\underline{n}) \urcorner)$. It follows that $\mathsf{EA} + \{R(\underline{k}) \mid k \leq n\} \vdash \alpha(\ulcorner R(\underline{n}) \urcorner)$. Hence, $\mathsf{EA} + \{R(\underline{k}) \mid k \leq n\} \vdash R(\underline{n+1})$. *Quod non.* ❑

REMARK 6.5. We can adapt the ideas around the construction of the non-uniform Σ^0_1-axiomatization to produce a very complex U which still numerates its own axioms with a Σ^0_1-formula. Let C and β be as in the proof of Theorem 6.4.

Let Z be any set of natural numbers. By a result due to, independently, Mostowski, Feferman, Scott, and Kripke (see, e.g., [10]), there is a Σ^0_1-formula $S^*(x)$ such that

$$\mathsf{EA} + \{S^*(\underline{n}) \mid n \in Z\} + \{\neg S^*(\underline{m}) \mid m \notin Z\}$$

is consistent. We consider the theory $X := C + \{S^*(\underline{n}) \mid n \in Z\}$. It follows that $n \in Z$ iff $X \vdash S^*(\underline{n})$. It is easy to see that

$$\zeta(x) := (\beta(x) \vee \exists y < x\, (x = \ulcorner S^*(\dot{y}) \urcorner \wedge S^*(y)))$$

numerates X in $U := \overline{X}$. ❍

6.2. Failure of the Löb conditions for a Σ^0_1-axiomatization of EA. In this subsection, we provide a curious example. We provide a Σ^0_1-formula $\sigma(x)$ with the following properties.

- The formula σ defines the axioms of EA, and, hence, numerates the axioms of EA in EA.
- EA knows that σ defines a finite set of axioms.
- EA knows that the theory defined by σ is between EA and $\mathsf{EA}+\Diamond_\beta \top$, where β is a standard axiomatization of EA.
- EA does not prove the Löb condtions for $\Box_\sigma$, and, what is more, EA does not prove the formalized Second Incompleteness Theorem for σ.

We work, for the moment, in EA. Our first order of business is to define a Kripke model $\mathcal{K}$. Let p be any (possibly non-standard) number. Our model has nodes $0, \ldots, p+1$. We set $x \prec y$ iff $x = 0$ and $1 \leq y \leq p+1$. Let C be a single axiom for EA and let $\beta(x) := (x = \ulcorner C \urcorner)$. We define the usual Solovay function h_p on $\mathcal{K}$ for β.[9]

- $\ell_p = 0$ iff $\forall x\, h_p(x) = 0$.
- $\ell_p = y$, if $0 \prec y$ and $\exists x\, h_p(x) = y$.
- $h_p(0) = 0$,
- $h_p(y+1) := \begin{cases} x & \text{if } h_p(y) \prec x \text{ and } \mathsf{proof}_\beta(y, \ulcorner \ell_{\dot{p}} \neq \dot{x} \urcorner) \\ h_p(y) & \text{otherwise} \end{cases}$

We find the following:

Lemma 6.6. *We have:*

a. $\mathsf{EA} \vdash \Box_\beta(\Box_\beta \bot \leftrightarrow \bigvee_{x \leq p} \ell_{\dot{p}} = (x \dot{+} 1))$.
b. $\mathsf{EA} \vdash (x \leq p \wedge \Box_\beta\, \ell_{\dot{p}} \neq x \dot{+} 1) \to \Box_\beta \bot$.

Proof. The proof follows the usual lines of a proof of Solovay's Theorem. Since our model is so simple, some short cuts in the proof are possible. ❑

We use that over EA we have a Σ_1-predicate $\mathsf{def}(y,z)$ such that an element a is Σ_1-definable iff, for some number k, $\mathsf{def}(\underline{k}, z)$ defines a. We follow Paris & Kirby, [14], in defining def as follows.[10] Let $\mathsf{T}(e,w,x)$ be Kleene's T-predicate where T is Δ_0. Here 'e' is the place for the index of a partial recursive function, 'w' is the place for the sequence of arguments, and 'x' is the place for the computation. We use ε for the (code of) the empty sequence. We assume that there is a result-extracting rudimentary U such that $\mathsf{U}(x)$ is the result of the computation. We take:

$$\mathsf{def}(s,y) := \exists v\, (\mathsf{T}(s,\varepsilon,v) \wedge \mathsf{U}(v) = y).$$

[9] The proof of Solovay's Theorem can nowadays be found in many places. There is Solovay's original article [20]. There are two excellent textbooks: [18] and [3]. There are two great handbook articles: [8] and [1].
[10] Paris & Kirby defined a somewhat more general version with parameters.

We proceed to specify σ. Let π be the usual axiomatization of PA.

$$\begin{aligned}\sigma(x) := \beta(x)\ \vee \\ &\exists p\,(\,\mathsf{proof}_\pi(p,\bot) \wedge \forall q < p\,\neg\,\mathsf{proof}_\pi(q,\bot)\ \wedge \\ &\exists y \leq p\,(x = \ulcorner \ell_{\dot p} \neq (y \dot{+} 1)\urcorner \wedge \exists s < p\ \mathsf{def}(s,y)))\end{aligned}$$

It is convenient to have a partial term $\mathfrak{p}$ (as defined symbol) that stands for the smallest inconsistency proof of PA (as given by π) if it exists. We note that $\mathfrak{p}$ is 'rigid' over EA:

Lemma 6.7. $\mathsf{EA} \vdash \mathfrak{p} = x \to \Box_\beta\ \mathfrak{p} = \dot{x}$.

With our new notation we can rewrite σ as:

$$\sigma(x) := \beta(x) \vee (\mathfrak{p}\downarrow \wedge \exists y \leq \mathfrak{p}\,(x = \ulcorner \ell_{\mathfrak{p}} \neq (y \dot{+} 1)\urcorner \wedge \exists s < \mathfrak{p}\ \mathsf{def}(s,y))).$$

Without any worries about the meaning of the second conjunct of the definition of σ, we can already prove some important claims about σ. Let $\mathfrak{S}_0 := \mathsf{Id}_{\mathsf{Ar}}[\mathsf{ax} := \beta]$ and $\mathfrak{S} := \mathsf{Id}_{\mathsf{Ar}}[\mathsf{ax} := \sigma]$. We write $\preceq\ := \ \preceq_{\mathsf{EA},\mathsf{Id}_{\mathsf{Ar}}}$.

Lemma 6.8. *We have:*

a. $\mathfrak{S}$ *is a uniform numeration of* $\{C\}$ *in* EA*. Consequently, we have* $\mathsf{EA} \nvdash \Diamond_\sigma \top$.
b. $\mathsf{EA} \vdash \forall A\,((\Box_\beta A \to \Box_\sigma A) \wedge (\Box_\sigma A \to \Box_\beta(\Diamond_\beta \top \to A)))$.
In other words, $\mathfrak{S}_0 \preceq \mathfrak{S} \preceq \mathfrak{S}_0 \oslash \Diamond_\beta \top$.

Proof. Ad (a): Clearly, we have $\mathsf{EA} \vdash \sigma(\ulcorner C \urcorner)$. Conversely suppose $\mathsf{EA} \vdash \sigma(\ulcorner D \urcorner)$ and $C \neq D$. It follows that $\mathsf{EA} \vdash \mathfrak{p}\downarrow$. *Quod non.*

Ad (b). We reason in EA. The first conjunct is immediate. Suppose $\Box_\sigma A$. In case $\mathfrak{p}\uparrow$, it follows that β and σ coincide, and hence $\Box_\beta A$ and, *a fortiori*, $\Box_\beta(\Diamond_\beta \top \to A)$. Suppose $\mathfrak{p}\downarrow$. It follows, by Lemma 6.6(a), that $\Box_\beta(\Diamond_\beta \top \to \bigwedge_{x \leq \mathfrak{p}} \ell_{\mathfrak{p}} \neq (x+1))$. Hence, $\Box_{\beta + \Diamond_\beta \top}$ extends $\Box_\sigma$. ❑

We now proceed to 'compute' $\Box_\sigma \bot$ and $\Box_\sigma \Diamond_\sigma \top$. The result of the computation will be expressed as closed terms constructed from $\Box_\beta$ and a special propositional constant $\mathsf{S}^\star$. We define $\mathsf{S}^\star$ as follows.

$$\begin{aligned}\mathsf{S}^\star := \exists p\,(\,&\mathsf{proof}_\pi(p,\bot) \wedge \forall q < p\,\neg\,\mathsf{proof}_\pi(q,\bot)\ \wedge \\ &\forall y \leq p\,\exists s < p\ \mathsf{def}(s,y)\,).\end{aligned}$$

In our $\mathfrak{p}$-notation: $\mathsf{S}^\star := (\mathfrak{p}\downarrow \wedge \forall y \leq \mathfrak{p}\,\exists s < \mathfrak{p}\ \mathsf{def}(s,y))$. We note that $\mathsf{S}^\star$ is $\Sigma^0_{1,1}$, i.e., it can be rewritten, modulo provable equivalence, as a formula of the form $\exists x\,\forall y \leq t(x)\exists z\, S^\star_0(x,y,z)$, where $S^\star_0(x,y,z)$ is elementary. Here is a basic insight about $\mathsf{S}^\star$.

Lemma 6.9. $\mathsf{EA} + \mathfrak{p}\downarrow\ \vdash \Box_\beta \neg \mathsf{S}^\star$*, and, hence,* $\mathsf{EA} + \mathsf{S}^\star \vdash \Box_\beta \neg \mathsf{S}^\star$.

Proof. Let us write $\mathsf{def}^z(s,y)$ for $\exists v \leq z\,(\mathsf{T}(s,\varepsilon,v) \wedge \mathsf{U}(v) = y)$. Let

$$D(x) := (\forall y \leq x\,\exists s < x\ \mathsf{def}(s,y) \to \exists z\,\forall y \leq x\,\exists s < x\ \mathsf{def}^z(s,y)).$$

We work in EA. It is easily seen that D is closed under zero and successor. Let I be a cut shortening D. Here we assume that I is downward closed w.r.t. $\leq$ and is closed under successor, addition, multiplication and ω_1.

We note that $D(x)$ is equivalent to $\neg(\forall y \leq x\, \exists s < x\; \mathsf{def}(s,y))$, since EA verifies the Δ_0-pigeon hole principle. See, e.g., [6], p. 42.

We have, by a result of Pudlák (the *outside large, inside small* principle), that $\forall x\, \Box_\beta\, x \in I$. See [15]. Suppose $\mathfrak{p}\downarrow$. It follows that $\Box_\beta\, \mathfrak{p} \in I$. Hence, $\Box_\beta \neg \mathsf{S}^\star$. ❑

Remark 6.10. Jumping ahead for a moment: the considerations from the proof of Theorem 6.15 show that $\mathsf{EA} \nvdash \mathsf{S}^\star \to \Box_\beta \bot$. Thus, it follows that $\mathsf{EA} \nvdash \mathsf{S}^\star \to \Box_\beta \mathsf{S}^\star$. We may conclude that EA does not prove $\Sigma_{1,1}$-completeness. ❍

The following result characterizes $\Box_\sigma$ under the assumption $\mathsf{S}^\star$.

Lemma 6.11. *$\mathsf{EA} + \mathsf{S}^\star \vdash \Box_\sigma A \leftrightarrow \Box_\beta(\Diamond_\beta \top \to A)$.*

Proof. We reason in $\mathsf{EA} + \mathsf{S}^\star$. We remind the reader that:

$$\sigma(x) := \beta(x) \vee (\mathfrak{p}\downarrow \wedge\, \exists y \leq \mathfrak{p}\; x = \ulcorner \ell_{\mathfrak{p}} \neq (y \dot{+} 1)\urcorner \wedge \exists s < \mathfrak{p}\; \mathsf{def}(s,y)).$$

In combination with $\mathsf{S}^\star$ this yields:

$$\sigma(x) \leftrightarrow \beta(x) \vee \exists y \leq \mathfrak{p}\; x = \ulcorner \ell_{\mathfrak{p}} \neq (y \dot{+} 1)\urcorner.$$

It follows that $\Box_\sigma A$ iff $\Box_{\beta + \bigwedge_{y \leq \mathfrak{p}}(\ell_{\mathfrak{p}} \neq y \dot{+} 1)} A$, hence, we find by Lemma 6.6(a), $\Box_\sigma A$ iff $\Box_\beta(\Diamond_\beta \top \to A)$. ❑

We will need a simple observation. We have:

Lemma 6.12. *$\mathsf{EA} \vdash \Box_\beta^n \bot \to \Box_\pi \bot$.*

Proof. This insight uses the fact that PA is essentially reflexive, i.e., it proves reflection for each of its finitely axiomatized sub-theories. See, e.g., [6, Chapter III, Theorem 2.35, p168].

We use induction on n. In case $n = 0$ or $n = 1$, we are immediately done. Suppose $n = k + 2$. We find:

$$\begin{aligned} \mathsf{EA} \vdash \Box_\beta^{k+2} \bot &\to \Box_\pi \Box_\beta^{k+1} \bot \\ &\to \Box_\pi \Box_\beta^{k} \bot \\ &\to \Box_\pi \bot. \end{aligned}$$
❑

The following lemma gives our calculation of $\Box_\sigma \bot$.

Lemma 6.13. *We have:*

a. $\mathsf{EA} + \mathsf{S}^\star \vdash \Box_\sigma \bot \leftrightarrow \Box_\beta \Box_\beta \bot$.
b. $\mathsf{EA} + \neg \mathsf{S}^\star \vdash \Box_\sigma \bot \leftrightarrow \Box_\beta \bot$.
c. $\mathsf{EA} \vdash \Box_\sigma \bot \leftrightarrow ((\mathsf{S}^\star \wedge \Box_\beta \Box_\beta \bot) \vee \Box_\beta \bot)$.

Proof. (a): This is immediate from Lemma 6.11.

(b): The right-to-left direction is immediate. We prove the left-to-right direction. We work in $\mathsf{EA} + \neg \mathsf{S}^\star + \Box_\sigma \bot$. By Lemma 6.8(b), we get $\Box_\beta(\Diamond_\beta \top \to \bot)$. Hence, $\Box_\beta \Box_\beta \bot$, and so $\Box_\pi \bot$. Thus, $\mathfrak{p}\downarrow$.

Now $\neg \mathsf{S}^\star$, in combination with $\mathfrak{p}\downarrow$, tells us that there is an $i \leq \mathfrak{p}$, such that, for all $s < \mathfrak{p}$, we have $\neg \mathsf{def}(s,i)$. It follows that every σ axiom is either

C or of the form $\ell_{\mathfrak{p}} \neq (j+1)$, where $j \leq p$ and $j \neq i$. Thus, $\Box_\sigma \bot$ yields: $\Box_\beta \neg \bigwedge_{j \leq \mathfrak{p},\, j \neq i} \ell_{\mathfrak{p}} \neq (j+1)$. It follows that $\Box_\beta\, \ell_{\mathfrak{p}} \neq (i+1)$. By Lemma 6.6(b), we find, as promised, $\Box_\beta \bot$.

(c) is immediate from (a) and (b). ❑

Here is our calculation of $\Box_\sigma \Diamond_\sigma \top$.

LEMMA 6.14. *We have:*

a. $\mathsf{EA} + \mathsf{S}^\star \vdash \Box_\sigma \Diamond_\sigma \top$.
b. $\mathsf{EA} + \neg \mathsf{S}^\star \vdash \Box_\sigma \Diamond_\sigma \top \leftrightarrow \Box_\beta \bot$.
c. $\mathsf{EA} \vdash \Box_\sigma \Diamond_\sigma \top \leftrightarrow (\mathsf{S}^\star \vee \Box_\beta \bot)$.

PROOF. Ad (a): We work in $\mathsf{EA} + \mathsf{S}^\star$. By Lemma 6.11, we find that $\Box_\sigma \Diamond_\sigma \top$ is equivalent with $\Box_\beta(\Diamond_\beta \top \to \Diamond_\sigma \top)$. By Lemma 6.9, we have $\Box_\beta \neg \mathsf{S}^\star$. Hence, by Lemma 6.13(b) and necessitation for $\Box_\beta$, we have $\Box_\beta(\Diamond_\sigma \top \leftrightarrow \Diamond_\beta \top)$. We may conclude that $\Box_\sigma \Diamond_\sigma \top$.

Ad (b): We reason in $\mathsf{EA} + \neg \mathsf{S}^\star$. The right-to-left direction is easy. We treat left-to-right. Suppose $\Box_\sigma \Diamond_\sigma \top$.

Suppose $\mathfrak{p} \uparrow$. In this case we find $\Box_\beta \Diamond_\sigma \top$ and hence $\Box_\beta \Diamond_\beta \top$. We may conclude $\Box_\beta \bot$.

Suppose $\mathfrak{p} \downarrow$. Since $\neg \mathsf{S}^\star$, it follows that, for some $i \leq \mathfrak{p}$, we have

$$\Box_\beta\Big(\bigwedge_{j \leq \mathfrak{p}, j \neq i} \ell_{\mathfrak{p}} \neq (j+1) \to \Diamond_\sigma \top\Big).$$

Ergo, $\Box_\beta(\ell_{\mathfrak{p}} = (i+1) \to \Diamond_\sigma \top)$. On the other hand, $\Box_\beta(\ell_{\mathfrak{p}} = (i+1) \to \Box_\beta \bot)$. Hence, $\Box_\beta(\ell_{\mathfrak{p}} = (i+1) \to \Box_\sigma \bot)$. We may conclude $\Box_\beta(\ell_{\mathfrak{p}} \neq (i+1))$ and, thus, $\Box_\beta \bot$.

Ad (c): (c) is immediate from (a) and (b). ❑

THEOREM 6.15. $\mathsf{EA} \nvdash \Box_\sigma \Diamond_\sigma \top \to \Box_\sigma \bot$.

Our proof is a simple adaptation of a proof of Paris & Kirby, [14]. See also [9].

PROOF. Let $\mathcal{M}$ be a model of $\mathsf{PA} + \Box_\pi \bot$. Let $\mathcal{N}$ be the model given by all Σ_1-definable elements of $\mathcal{M}$. We can easily see that $\mathcal{N}$ is a Σ^0_1-elementary submodel of $\mathcal{M}$. It follows that Π^0_2-sentences are downwards preserved from $\mathcal{M}$ to $\mathcal{N}$. Hence, $\mathcal{N} \models \mathsf{EA}$. Reflecting on the construction of $\mathcal{N}$, we have $\mathcal{N} \models \mathsf{S}^\star$.

It follows, by Lemma 6.14(a), that $\mathcal{N} \models \Box_\sigma \Diamond_\sigma \top$. If we would have $\mathcal{N} \models \Box_\sigma \bot$, it would follow, by Lemma 6.13(a), that $\mathcal{N} \models \Box_\beta \Box_\beta \bot$. Since $\Box_\beta \Box_\beta \bot$ is Σ^0_1, we would find that $\mathcal{M} \models \Box_\beta \Box_\beta \bot$.[11] But this is impossible, since $\mathcal{M} \models \mathsf{PA}$ and PA proves reflection for $\Box_\beta$. ❑

Let G be the Gödel sentence for $\Box_\sigma$.

COROLLARY 6.16. $\mathsf{EA} \nvdash \Box_\sigma G \to \Box_\sigma \Box_\sigma G$.

PROOF. If we would have $\mathsf{EA} \vdash \Box_\sigma G \to \Box_\sigma \Box_\sigma G$, then the usual reasoning for the formalized proof of the Second Incompleteness Theorem for $\Box_\sigma$ would go through, contradicting Theorem 6.15. ❑

[11] We note that $\Box_\sigma \bot$ is not *prima facie* Σ^0_1 since it has $\Sigma^0_{1,1}$-form.

REMARK 6.17. We note that Corollary 6.16 does not contradict the fact that $\mathsf{EA} \vdash S \to \Box_\sigma S$, for $S \in \Sigma^0_1 = \Sigma^0_{1,0}$. The reason is that a Σ^0_1-sentence S is of the form $\exists x\, S_0(x)$, where $S_0 \in \Delta_0$ (or, if you wish, $S_0 \in \Delta_0(\mathsf{exp})$). The formula $\Box_\sigma G$ is $\Sigma^0_{1,1}$, i.e., it is of the form $\exists x\, \forall y < t(x)\, \exists y\, S_0(x,y)$, where $S_0 \in \Delta_0$ (or, if you wish, $S_0 \in \Delta_0(\mathsf{exp})$). Our result illustrates that in EA there are $\Sigma^0_{1,1}$-sentences that are not equivalent to a $\Sigma^0_{1,0}$-sentence. See Remark 6.10 of the present paper, [26], and [27] for more on this phenomenon. ❍

OPEN QUESTION 6.18. Here are four questions.

A. What are the possible provability logics of Σ^0_1-numerations of the axioms of EA over EA?
B. What are the possible closed fragments of provability logics of Σ^0_1-numerations of the axioms of EA over EA?
C. What is the provability logic of $\Box_\sigma$ over EA?
D. What is the closed fragment of the provability logic of $\Box_\sigma$ over EA?

❍

EXAMPLE 6.19. Let $\mathfrak{S}$ be the presentation based on σ constructed in this subsection. We note that $\mathsf{Cr}(\mathfrak{S})$ is an elementary axiomatization. Hence, $\mathsf{Cr}(\mathfrak{S})$ does satisfy the formalized Second Incompleteness Theorem over EA. It follows that $\mathfrak{S} \neq_{\mathsf{EA},\mathsf{Id}_{\mathsf{Ar}}} \mathsf{Cr}(\mathfrak{S})$. ❍

6.3. The Second Incompleteness Theorem for Σ^0_1-semi-numerations. We prove the Second Incompleteness Theorem for Σ^0_1-semi-numerations of U in U. As we have seen, in Subsection 6.1, Σ^0_1-semi-numerations need not be uniform. Thus, we cannot use Theorem 5.1 to prove the Second Incompleteness Theorem. Fortunately, if $\mathcal{S}$ is a Σ^0_1-semi-numeration, then $\mathsf{Th}(\mathcal{S})$ is a uniform Σ^0_1-semi-numeration. So, we may apply Theorem 5.2 to obtain the desired result.

THEOREM 6.20. *Suppose U is consistent and $N : U \rhd \mathsf{EA}$. Let $\sigma(x)$ be a Σ^0_1-formula and suppose that $\mathcal{S} := \tau_N[\mathsf{ax} := \sigma^{\tau_N}]$ is a semi-numeration of U in U. Then, $U \nvdash (\Diamond_{\mathsf{ax}} \top)^{\mathcal{S}}$.*

PROOF. By Theorem 5.2, it is sufficient to show that $\mathsf{Th}(\mathcal{S})$ is a uniform semi-numeration of U in U.

Suppose X axiomatizes U and $\mathcal{S}$ semi-numerates X in U. Let C be a single axiom for EA. Let Y_0 be a finite subset of U. Let X_0 be a finite subset of X-axioms sufficient for deriving the sentences in $Y_0 \cup \{C^N\}$. We take

$$Y_1 := Y_0 \cup \{C^N\} \cup \{\sigma^N(\ulcorner B \urcorner) \mid B \in X_0\}.$$

We note that the $\sigma^N(\ulcorner B \urcorner)$ for B in X_0 are in U since $\mathcal{S}$ semi-numerates X. So $Y_1 \subseteq_{\mathsf{fin}} U$. Consider any $A \in Y_1$. In case $A \in Y_0 \cup \{C^N\}$, we have a proof of A from X_0, so certainly, $Y_1 \vdash (\Box_\sigma A)^N$, i.o.w. $Y_1 \vdash (\Box_{\mathsf{ax}} A)^{\mathcal{S}}$. Suppose $A = \sigma^N(\ulcorner B^* \urcorner)$, for some $B^* \in X_0$. Then, $Y_1 \vdash \sigma^N(\ulcorner B^* \urcorner)$. It follows from the fact that $C^N \in Y_1$ in combination with Σ^0_1-completeness in EA that $Y_1 \vdash (\Box_\sigma \sigma^N(\ulcorner B^* \urcorner))^N$, i.o.w., $Y_1 \vdash (\Box_{\mathsf{ax}} A)^{\mathcal{S}}$. ❑

We remind the reader that if we want to push down our result from EA to S^1_2 without solving any complexity theoretic problems, then we can replace Σ^0_1 by $\exists\Sigma^{\mathsf{b}}_1$.

There is an alternative proof Theorem 6.20 that runs as follows.

SECOND PROOF OF THEOREM 6.20. Suppose $U \vdash (\Diamond_{\mathsf{ax}} \top)^{\mathcal{S}}$. It follows, by Theorem 4.14, that $U \vdash (\Diamond_{\mathsf{ax}} \top)^{\mathsf{Cr}(\mathcal{S})}$. By, Theorem 4.15(a), we see that U is closed under the necessitation rule for $(\Box_{\mathsf{ax}}(\cdot))^{\mathsf{Cr}(\mathcal{S})}$. Inspection of the definition of $\mathsf{Cr}(\cdot)$ shows that $\mathsf{ax}^{\mathsf{Cr}(\mathcal{S})}$ is elementary. Hence, $(\Box_{\mathsf{ax}})^{\mathsf{Cr}(\mathcal{S})}$ satisfies the Löb conditions. It follows that $U \vdash \bot$. *Quod non.* ❑

REMARK 6.21. The attentive reader may wonder whether the alternative proof is not more successful than I make it here. Does not the argument establish the theorem with S^1_2 substituted for EA? I do not think so. The point is that $\Box^{\mathsf{Cr}(\mathcal{S})}_{\mathsf{ax}}$ is still Σ^0_1 rather than $\exists\Sigma^{\mathsf{b}}_1$. There are tricks to improve Craigification to yield a Δ^{b}_0-set of axioms, but I suspect that these tricks manage to obstruct the proof of Theorem 4.15(a). Whatever is the case, there is more to explore here. ❍

§7. Extensions of Peano Arithmetic: examples. In this section we provide various examples of applications of our main theorem for theories extending Peano Arithmetic.

7.1. Feferman provability. The contraposition of Theorem 5.1 tells us that if U is consistent and $M : U \rhd (\mathsf{S}^1_2 + \Diamond_{\mathsf{ax}} \top)$, then $\mathcal{A} := \tau_M$ is not a uniform semi-numeration for U. The interesting case is of course, when $\mathcal{A}$ is a semi-numeration.

We consider the theory PA. Let π be a usual arithmetization of the set of axioms of PA. We define π_y by $\pi_y(x) \leftrightarrow \pi(x) \wedge x \leq y$. Feferman defines a predicate $\digamma$ by $\digamma(x) := \pi(x) \wedge \Diamond_{\pi_x} \top$. The Feferman Presentation is $\mathfrak{F} := \mathsf{Id}_{\mathsf{Ar}}[\mathsf{ax} := \digamma]$. Since PA is reflexive, it follows that $\mathfrak{F}$ is, indeed, a presentation of PA. On the other hand, trivially, we have $\mathsf{PA} \vdash (\Diamond_{\mathsf{ax}} \top)^{\mathfrak{F}}$. Thus, $\mathfrak{F}$ is not uniform.

Feferman provability $\Box_{\digamma}$ has been studied intensively by provability logicians. See [13], [22] and [17]. Shavrukov in [17] characterizes the bimodal provability logic of $\Box_\pi$ and $\Box_{\digamma}$.[12]

Applications of Feferman provability are based on the use of Feferman consistency as basis for a Henkin-style interpretations existence result. See [25] and [28]. The full generality of Feferman's method is only realized if we admit also restricted provability where we restrict our proofs to proofs only involving formulas below a given complexity. In a sense, we leave the Feferman framework there, since we employ other notions of proof than the standard one.

[12] For some purposes, logicians have worked with a different definition of π_y since alternative definitions add some good properties to Feferman provability. The most salient example is to take as elements of π_x the axioms of the theories $\mathrm{I}\Sigma_y + \mathsf{Exp}$ for $y \leq x$. See, e.g., [19], [22], [17], [16].

7.2. Oracle provability. We consider the theory U axiomatized by the set X consisting of the usual PA-axioms plus all true Π^0_n-sentences. We define $\pi^{[n]}(x) := \pi(x) \vee \mathsf{true}_{\Pi^0_n}(x)$.

It is easy to see that $\mathfrak{P}^{[n]} := \mathsf{Id}_{\mathsf{Ar}}[\mathsf{ax} := \pi^{[n]}(x)]$ uniformly numerates X in U. Hence, we have the Second Incompleteness Theorem for $\mathfrak{P}^{[n]}$. We can also verify this fact by showing that $(\Box_{\mathsf{ax}}(\cdot))^{\mathfrak{P}^{[n]}}$ satisfies the Löb Conditions.

The provability logic of the predicates $\mathfrak{P}^{[n]}$ is Japaridze's Logic. It has been studied in great detail by provability logicians. See, e.g., [7] and [2].

7.3. Failure of the formalized Second Incompleteness Theorem over Peano Arithmetic. Recently Taishi Kurahashi characterized the provability logics of some Σ^0_2-provability predicates. See [11] and [12]. One of his results is that there is a Σ_2-numeration of the axioms of PA such that the provability logic for that numeration is precisely K. Thus, Kurahashi provides an example of a Σ^0_2-axiomatization for which we do have the Second Incompleteness Theorem, but which does not satisfy the Löb-conditions. We provide a quick example of the same phenomenon here. Our aim is far more modest than Kurahashi's; we just provide failure of the Second Incompleteness Theorem and not a characterization of a provability logic.

We (locally) write: $\vdash$ for provability in PA. Suppose the usual representation of the axioms of PA is π. As before, let $\pi_y(x) := (\pi(x) \wedge x \leq y)$. We write PA_k for the theory axiomatized by π_k. We write $\Box_x$ for $\Box_{\pi_x}$. We define:

- $\mathfrak{P} := \mathsf{Id}_{\mathsf{Ar}}[\mathsf{ax} := \pi]$, $\Box := \Box_\pi$.
- $\digamma(x) := \pi(x) \wedge \Diamond_{\pi_x} \top$, $\mathfrak{F} := \mathsf{Id}_{\mathsf{Ar}}[\mathsf{ax} := \digamma]$, $\vartriangle_0 := \Box_\digamma$.
- $\pi^*(x) := \pi(x) \wedge \forall y \leq x \, \neg\, \mathsf{proof}_\pi(y, \Box_\pi \bot)$, $\mathfrak{P}^* := \mathsf{Id}_{\mathsf{Ar}}[\mathsf{ax} := \pi^*]$, $\vartriangle_1 := \Box_{\pi^*}$.
- $\widetilde{\pi} := (\digamma \vee \pi^*)(x) := (\digamma(x) \vee \pi^*(x))$, $\widetilde{\mathfrak{P}} := \mathsf{Id}_{\mathsf{Ar}}[\mathsf{ax} := \widetilde{\pi}]$, $\vartriangle := \Box_{\widetilde{\pi}}$.

We follow the usual practice of writing $\triangledown$ for $\neg\vartriangle\neg$.

It is easy to see that the axioms of PA are uniformly $\widetilde{\mathfrak{P}}$-complete, since $\mathfrak{P}^*$ already uniformly semi-numerates the axioms of PA in PA. We have the stronger: there is a fixed k such that if A is an axiom of PA, then $\mathsf{PA}_k \vdash \widetilde{\pi}(A)$. We simply take k to be sufficiently large such that $\mathsf{PA}_k \vdash \mathsf{EA}$. Thus, $\mathsf{PA} \nvdash \triangledown \top$. We note that, since $\digamma$ and π^* are PA-provably subsets of π, we even have that $\widetilde{\mathfrak{P}}$ uniformly numerates the axioms of PA in PA.

We show that PA does not prove the arithmetized Second Incompleteness Theorem for $\vartriangle$. Thus, *a fortiori*, we do not have the Löb conditions for $\vartriangle$.

We enumerate some salient facts.

a. $\vdash \vartriangle A \leftrightarrow (\vartriangle_0 A \vee \vartriangle_1 A)$.
b. $\vdash \vartriangle A \to \Box A$.
c. $\vdash \Diamond \top \to (\vartriangle A \leftrightarrow \Box A)$.
d. $\vdash \vartriangle \bot \leftrightarrow \vartriangle_1 \bot$.
e. $\vdash \Box\Box\bot \to \Box(\vartriangle A \leftrightarrow \vartriangle_0 A)$.
f. $\vdash \Box\Box\bot \to \Box \triangledown \top$.
g. $\vdash \neg\, \Box\Box\bot \to \neg\, \Box \triangledown \top$.
h. $\vdash \Box \triangledown \top \leftrightarrow \Box\Box\bot$.

The items (a,b,c,d) are trivial.

Ad (e): We reason in PA. Suppose $\Box\Box\bot$. Let x be the smallest witness of $\Box\Box\bot$. We reason inside $\Box$. We find that $\triangle A$ is equivalent to $\triangle_0 A$ or $\Box_{x-1} A$. Since, by the reflexiveness of PA, $\Diamond_{x-1}\top$, we see that $\Box_{x-1} A$ implies $\triangle_0 A$, and we are done.

(f) is an immediate consequence of (e).

(g) is by formalizing the reasoning for $\nvdash \triangledown\top$.

(h) follows by combining (f) and (g).

Using the above facts, we can show that $\nvdash \triangle\triangledown\top \to \triangle\bot$. Suppose (†) $\vdash \triangle\triangledown\top \to \triangle\bot$. We have:

$$\begin{aligned} \Box\Box\bot \wedge \Diamond\top &\vdash_{(\mathrm{f})} \Box\triangledown\top \\ &\vdash_{(\mathrm{c})} \triangle\triangledown\top \\ &\vdash_{(\dagger)} \triangle\bot \\ &\vdash_{(\mathrm{b})} \Box\bot \end{aligned}$$

It follows, by propositional logic, that $\Box\Box\bot \vdash \Box\bot$, and hence $\vdash \Box\bot$. *Quod non.* Thus, PA does not verify the Second Incompleteness Theorem for $\triangle$. It follows that $\triangle \to \triangle\triangle$ also fails over PA.

§8. Beyond reduction to finite. We have provided a reasonably general version of the Second Incompleteness Theorem. Is this general version, the last word? *No.* As a matter of principle, there is no last word, there is nothing like the most general version of a theorem.

In the first place there are always new insights. Secondly, salient versions of a theorem are often dependent on further interests. In our case, we already know some limitations of what we do in the present paper.

We have already seen, in the alternative proof of Theorem 6.20, that the pattern of proving the Second Incompleteness Theorem for a presentation by proving the same theorem for a 'weaker' presentation extends beyond the reduction to finite axiomatizability. Thus, our scheme can be extended.

In fact, the most important use of this kind of methodology does not fully fall under our scheme: Pudlák's proof of the Second Incompleteness Theorem for recursively enumerable extensions of Q.[13] Here is an outline of the proof.

THEOREM 8.1. *Let U be axiomatized by $X := \{C \in \mathsf{sent}_\Theta \mid \mathbb{N} \models \sigma(\ulcorner C \urcorner)\}$, where $\sigma(x)$ is a Σ^0_1-formula. Suppose $U \rhd (\mathsf{Q} + \Diamond_\sigma \top)$. Then, U is inconsistent.*

PROOF. Suppose $N : U \rhd (\mathsf{Q} + \Diamond_\sigma \top)$. One can construct a definable cut N_0 of N such that $N_0 : U \rhd \mathsf{S}^1_2$. (See, e.g., [6].) Since N_0 is a cut and $\Diamond_\sigma \top$ is $\Pi_{1,1}$, i.e., is of the form $\forall x\, \exists y < t(x)\, \forall z\, S_0(x, y, z)$, where S_0 is Δ_0, we find $U \vdash (\Diamond_\sigma \top)^{N_0}$. We may now apply Theorem 6.1. ❏

[13] Pudlák's proof is an adaptation of a proof due to Mycielski of the Second Incompleteness Theorem for his finitistic theory FIN in an unpublished manuscript "Finitistic intuitions supporting the consistency of ZF and $\mathsf{ZF}+\mathsf{AD}$".

We note that Pudlák's argument transcends our framework since it involves the interplay between two interpretations of a weak arithmetic. In his paper [15], Pudlák discusses a bimodal logic where two modalities work together to prove the Second Incompleteness Theorem for one of them. In fact, Pudlák addresses a problem that we did not touch upon in this paper: how to deal with inefficient Gödel numberings. As far as I know, this proposal in its application to inefficient Gödel numberings was never seriously further explored.

One thing suggested by Pudlák's argument is to develop an ordering of presentations that also covers cases where the restriction to the arithmetical repertoire is not constant. This would involve definable initial mappings between such representations.

The Fefermanian restriction to a fixed proofpredicate excludes a lot of interesting contexts where consistency statements occur. There is a lot to say about cut-free, Herbrand, and versions of restricted provability. For example, finitely axiomatized sequential theories prove their own cut-free consistency on a definable cut. There is the question whether, for example, S^1_2 proves its own cut-free consistency. The proper generalization of Feferman provability involves restricted consistency statements. Pavel Pudlák studied finitistic versions of consistency proofs. Etcetera.

As the reader can see, much work is left be done, and Gödel's legacy has not nearly been exhausted.

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Appendix A. Translations and interpretations. We present the notion of *m-dimensional interpretation without parameters*. There are two extensions of this notion: we can consider piecewise interpretations and we can add parameters. We just treat the ordinary m-dimensional case without parameters or pieces here.

Consider two signatures Ξ and Θ. An m-dimensional translation $\tau : \Xi \to \Theta$ is a quadruple $\langle \Xi, \delta, \mathcal{F}, \Theta \rangle$, where $\delta(v_0, \dots, v_{m-1})$ is a Θ-formula and where for any n-ary predicate P of Ξ, $\mathcal{F}(P)$ is a formula $A(\vec{v}_0, \dots, \vec{v}_{n-1})$ in the language of signature Θ, where $\vec{v}_i = v_{i0}, \dots, v_{i(m-1)}$. Both in the case of δ and A all free variables are among the variables shown. Moreover, if $i \neq j$ or $k \neq \ell$, then v_{ik} is syntactically different from $v_{j\ell}$.

We require that we have $\vdash \mathcal{F}(P)(\vec{v}_0, \dots, \vec{v}_{n-1}) \to \bigwedge_{i<n} \delta(\vec{v}_i)$. Here $\vdash$ is provability in predicate logic. This demand is inessential, but it is convenient to have.

We allow identity to be translated to a formula that is not identity.

We define B^τ as follows:

- $(P(x_0, \ldots, x_{n-1}))^\tau := \mathcal{F}(P)(\vec{x}_0, \ldots, \vec{x}_{n-1})$.
- $(\cdot)^\tau$ commutes with the propositional connectives.
- $(\forall x\, A)^\tau := \forall \vec{x}\, (\delta(\vec{x}\,) \to A^\tau)$.
- $(\exists x\, A)^\tau := \exists \vec{x}\, (\delta(\vec{x}\,) \wedge A^\tau)$.

There are two worries about this definition. First, what variables $\vec{x}_i$ on the side of the translation A^τ correspond with x_i in the original formula A? The second worry is that substitution of variables in δ and $\mathcal{F}(P)$ may cause variable clashes. These worries are never important in practice: we choose 'suitable' sequences $\vec{x}$ to correspond to variables x, and we avoid clashes by α-conversions. However, if we want to give precise definitions of translations and, for example, of composition of translations these problems come into play. The problems are clearly solvable, but a worked out solution is beyond the scope of this paper.

Instead of introducing τ explicitly as being $\langle \Xi, \delta, \mathcal{F}, \Theta\rangle$, we will write, e.g., δ_τ for the δ of τ, and $P_\tau := \mathcal{F}_\tau(P)$.

We specify the identity translation and composition of translations.

- id_Ξ is the identity translation. We take $\delta_{\mathsf{id}_\Xi}(v) := (v = v)$ and $\mathcal{F}(P) := P(\vec{v}\,)$.
- We can compose translations. Suppose $\tau : \Xi \to \Theta$ and $\nu : \Theta \to \Lambda$. Then $\nu \circ \tau$ or $\tau\nu$ is a translation from Ξ to Λ. We define:
  - $\delta_{\tau\nu}(\vec{v}_0, \ldots, \vec{v}_{m_\tau - 1}) := \bigwedge_{i<m_\tau} \delta_\nu(\vec{v}_i) \wedge (\delta_\tau(v_0, \ldots, v_{m_\tau-1}))^\nu$.
  - $P_{\tau\nu}(\vec{v}_{0,0}, \ldots, \vec{v}_{0,m_\tau-1}, \ldots, \vec{v}_{n-1,0}, \ldots, \vec{v}_{n-1,m_\tau-1}) :=$ $\bigwedge_{i<n, j<m_\tau} \delta_\nu(\vec{v}_{i,j}) \wedge (P(v_0, \ldots, v_{n-1})^\tau)^\nu$.

A translation relates signatures; an interpretation relates theories. An interpretation $K : U \to V$ is a triple $\langle U, \tau, V\rangle$, where U and V are theories and $\tau : \Xi_U \to \Xi_V$. We require: for U-sentences A, if $U \vdash A$, then $V \vdash A^\tau$.

We can define the identity interpretation and composition of interpretations as follows.

- $\mathsf{ID}_U : U \to U$ is the interpretation $\langle U, \mathsf{id}_{\Xi_U}, U\rangle$.
- Suppose $K : U \to V$ and $M : V \to W$. Then, $KM := M \circ K : U \to W$ is $\langle U, \tau_M \circ \tau_K, W\rangle$.

Appendix B. Alternative proofs for Theorem 6.1. Here is Theorem 6.1 again.

Theorem 6.1. *Suppose $\sigma(x)$ is a Σ^0_1-formula that numerates the axioms of U in the standard model. Then, if $U \rhd (\mathsf{S}^1_2 + \Diamond_\sigma \top)$, we have that U is inconsistent.*

Alternative Proof 1. Suppose $N : U \rhd (\mathsf{S}^1_2 + \Diamond_\sigma \top)$. Then, $\mathsf{S}^1_2 \vdash \Box_\sigma(\bigwedge \mathsf{S}^1_2 \wedge \Diamond_\sigma \top)^N$. Let $\gamma(x) := (x = \ulcorner \bigwedge \mathsf{S}^1_2 \wedge \Diamond_\sigma \top \urcorner)$. We find: $\mathsf{S}^1_2 \vdash \Box_\gamma \bot \to \Box_\sigma \bot$, in other words, $\mathsf{S}^1_2 + \Diamond_\sigma \top \vdash \Diamond_\gamma \top$. Since we have the Löb conditions for $\Box_\gamma$ over $\mathsf{S}^1_2 + \Diamond_\sigma \top$ we obtain $\mathsf{S}^1_2 + \Diamond_\sigma \top \vdash \bot$, by the Second Incompleteness Theorem. Hence, $\mathsf{S}^1_2 \vdash \Box_\sigma \bot$. But then, $U \vdash \bot$. ❑

ALTERNATIVE PROOF 2. Suppose $N : U \rhd (\mathsf{S}^1_2 + \Diamond_\sigma \top)$. Let N_0 be a cut in N on which we have $\mathsf{S}^1_2 + \mathrm{B}\Sigma_1$ with the additional property that $U \vdash \forall x \in \delta_{N_0}\, 2^x \in \delta_N$. We claim that:

$$U \vdash \forall A \in \delta_{N_0}\, ((\Box_\sigma A)^{N_0} \to (\Box_\sigma (\Box_\sigma A)^{N_0})^N).$$

We briefly sketch the idea for the verification of this last equation. First, inside N_0 we can transform $\Box_\sigma A$ from a $\Sigma^0_{1,1}$-formula to a Σ^0_1-formula, say $(\Box_\sigma A)^\star$. We can construct $(\Box_\sigma A)^\star$ in such a way that $\mathsf{S}^1_2 \vdash (\Box_\sigma A)^\star \to \Box_\sigma A$. Second we can estimate the transformation of a witness x of a Σ^0_1-formula S to a witness of $\Box_\sigma S^{N_0}$ as of order $2^{k\cdot x\cdot|S|}$, for standard k. Note that this uses the fact that we have a standardly finite verification of $(\mathsf{S}^1_2)^{\tau_{N_0}}$ in U. So, if we start with a witness p in N_0 of $(\Box_\sigma A)^\star$, we have a witness $p^\star$ of $\Box_\sigma(\Box_\sigma A)^{\star N_0}$ in N. We can transform $p^\star$ easily to a witness $\widetilde{p}$ of $\Box_\sigma(\Box_\sigma A)^{N_0}$ in N.

By the Gödel Fixed-Point Lemma, we find G such that $\mathsf{S}^1_2 \vdash G \leftrightarrow \neg \Box_\sigma G^{N_0}$.

We now reason in U as follows. Suppose $(\Box_\sigma G^{N_0})^{N_0}$. Then, we have both $(\Box_\sigma G^{N_0})^N$ and $(\Box_\sigma(\Box_\sigma G^{N_0})^{N_0})^N$. By the fixed-point equation, it follows that $(\Box_\sigma \neg G^{N_0})^N$, and, hence, $(\Box_\sigma \bot)^N$.

On the other hand, we have $(\Diamond_\sigma \top)^N$. So, we may conclude $\bot$. Eliminating our assumption, we obtain $\neg (\Box_\sigma G^{N_0})^{N_0}$, or, in other words, $(\neg \Box_\sigma G^{N_0})^{N_0}$. The fixed-point equation gives us: G^{N_0}.

We leave U. We have shown $U \vdash G^{N_0}$. Since $\mathcal{T} := \tau_{N_0}[\mathsf{ax} := \sigma^{\tau_{N_0}}]$ semi-numerates X in U, we find $U \vdash (\Box_\sigma G^{N_0})^{N_0}$ and, so, $U \vdash (\neg G)^{N_0}$. So $U \vdash \bot$. ❑

ALTERNATIVE PROOF 3. Suppose $N : U \rhd (\mathsf{S}^1_2 + \Diamond_\sigma \top)$. Suppose $\sigma(x)$ is written in the form $\exists u\, \sigma_0(u, x)$, where σ_0 is Δ_0. Without loss of generality we may assume that $x < u$ is implied by $\sigma_0(u, x)$. We define:

$$\alpha(x) := \exists w < x\, \exists u < |x|\, \exists y < |x|\, (w = \mathsf{tally}(y) \wedge \sigma_0(u, y) \wedge x = \mathsf{conj}(\mathsf{id}(w, w), y)).$$

Here tally computes the standard *non*-efficient numeral. We assume that our coding is such that $2^y \leq \mathsf{tally}(y)$. We note that all terms in $\sigma_0(u, y)$ can be bounded by polynomials in $|x|$. So, α is Σ^{b}_1. By the Σ^{b}_1-collection principle, we find that $\mathsf{proof}_\alpha(y)$ is Σ^{b}_1 (modulo S^1_2-provable equivalence). See [4], p. 53. It follows that $\Box_\alpha$ is $\exists\Sigma^{\mathsf{b}}_1$.

We set $\mathcal{S} := \tau_N[\mathsf{ax} := \sigma(x)^{\tau_N}]$ and $\mathcal{T} := \tau_N[\mathsf{ax} := \alpha(x)^{\tau_N}]$. We find that $\mathcal{T} \preceq_{U,\tau_K} \mathcal{S}$. Hence, $U \vdash (\Diamond_{\mathsf{ax}} \top)^{\mathcal{T}}$.

Since σ represents the axioms of U in the standard model, it follows that $\mathsf{Th}(\mathcal{T})$ semi-numerates U. The reason is that S^1_2 verifies the existence of $\mathsf{tally}(\underline{n})$, for standard n. Thus, $(\Box_{\mathsf{ax}})^{\mathcal{T}}$ satisfies the Löb conditions. It follows that $U \vdash \bot$. ❑

ALTERNATIVE PROOF 4. Suppose $N : U \rhd (\mathsf{S}^1_2 + \Diamond_\sigma \top)$.

In [26], we show that the theory Peano Basso is locally cut-interpretable in PA^-. We use the following consequence of this fact: the theory

$$W := \mathsf{S}^1_2 + \mathrm{B}\Sigma^0_1 + \{S \to S^I \mid I \text{ is an } \mathsf{S}^1_2\text{-definable cut and } S \text{ is a } \Sigma^0_1\text{-sentence}\}$$

is locally cut-interpretable in S^1_2. It follows, by the downwards preservation of $\Pi_{1,1}$-sentences, that $T := W + \Diamond_\sigma \top$ is locally cut-interpretable in $\mathsf{S}^1_2 + \Diamond_\sigma \top$.

Let θ be a Δ_0^{b} standard representation of the axioms of T. We define $\theta_n(x) := \theta(x) \wedge x \leq n$. We have that U as axiomatized by σ interprets W_n axiomatized by θ_n. Let us, *ad hoc*, write this as $\mathsf{int}[\sigma, \theta_n]$. Now, by $\Sigma^0_{1,1}$-completeness (in the metalanguage), we have $U \vdash (\mathsf{int}[\sigma, \theta_n])^N$. It follows that $U \vdash (\Diamond_{\theta_n} \top)^N$.

By the Interpretation Existence Lemma (see [28]), we obtain an interpretation $M : U \rhd T$. It is easily seen that, in U, we have the Löb Conditions for $(\Box_\sigma(\cdot))^M$, which yields a contradiction. ❑

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