Relativizing operational set theory

Gerhard Jäger*

Abstract

We introduce a way of relativizing operational set theory that also takes care of application. After presenting the basic approach and proving some essential properties of this new form of relativization we turn to the notion of relativized regularity and to the system $\mathsf{OST}(\mathsf{LR})$ that extends OST by a limit axiom claiming that any set is element of a relativized regular set. Finally we show that $\mathsf{OST}(\mathsf{LR})$ is proof-theoretically equivalent to the well-known theory KPi for a recursively inaccessible universe.

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1 Introduction

Feferman's original motivation for operational set theory was to provide a setting for the operational formulation of large cardinal statements directly over set theory in a way that seemed to him to be more natural mathematically than the metamathematical formulations using reflection and indescribability principles, etc. He saw operational set theory as a natural extension of the von Neumann approach to axiomatizing set theory.

The system OST has been introduced in Feferman [7] and further studied in Feferman [8] and Jäger [12, 13, 14, 15, 17, 18]. For a first discussion of operational set theory and some general motivation we refer to these articles, in particular to [8]. In addition, Cantini and Crosilla [4, 5] and Cantini [3] study the interplay between some constructive variants of operational set theory and constructive set theory.

A further principal motivation of Feferman [7, 8] was to relate formulations of classical large cardinal statements to their analogues in admissible set

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theory. However, in view of Jäger and Zumbrunnen [17] this aim of OST has to be analyzed further. It is shown in [17] that a direct relativization of operational reflection leads to theories that are significantly stronger than theories formalizing the admissible analogues of classical large cardinal axioms. The main reason is that simply restricting quantifiers to specific sets and operations to operations from and to specific sets does not affect the global application relation and thus substantial strength may be imported – so to say – through the back door.

In this paper we take care of this problem by introducing a new way of relativizing operational set theory such that also application is relativized. We first present the basic approach and prove some essential properties of this new form of relativization. Then we turn to relativized regularity and to the system $\mathsf{OST}(\mathsf{LR})$ that extends OST by a limit axiom claiming that any set is element of a relativized regular set. Finally we show that $\mathsf{OST}(\mathsf{LR})$ is proof-theoretically equivalent to the well-known theory KPi for a recursively inaccessible universe. This solves a problem that has been open for many years.

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2 The theory OST

Now we introduce the theory OST, though not in its original form but in a slightly modified but essentially equivalent way similar to that in Zumbrunnen [22]. In presenting the syntax of OST we follow Jäger and Zumbrunnen [18]. To begin with, let $\mathcal{L}$ be a typical language of first order set theory with the binary symbols $\in$ and $=$ as its only relation symbols, with countably many set variables $a, b, c, d, e, f, g, u, v, w, x, y, z, \ldots$ (possibly with subscripts), as well as with the logical symbols $\neg$, $\vee$, and $\exists$. We further assume that $\mathcal{L}$ has a constant ω for the collection of all finite von Neumann ordinals. The formulas of $\mathcal{L}$ are defined as usual.

The language $\mathcal{L}^\circ$ of operational set theory extends $\mathcal{L}$ by the binary function symbol $\circ$ for partial term application, the unary relation symbol $\downarrow$ for definedness, the binary relation symbol Reg, and a series of constants: (i) the combinators k and s, (ii) $\top$, $\bot$, $\mathbf{el}$, $\mathbf{reg}$, $\mathbf{non}$, $\mathbf{dis}$, and $\mathbf{e}$ for logical operations, (iii) $\mathbb{D}$, $\mathbb{U}$, $\mathbb{S}$, $\mathbb{R}$, and $\mathbb{C}$ for set-theoretic operations. The meaning of these symbols will be specified by the axioms below.

The *terms* $(r, s, t, r_1, s_1, t_1, \ldots)$ of $\mathcal{L}^\circ$ are built up from the variables and constants by means of our function symbol $\circ$ for application to form expressions $(r \circ s)$. In the following $(r \circ s)$ is often written as (rs) or (if no confusion arises) simply as rs. We adopt the convention of association to the left so that $r_1 r_2 \dots r_n$ stands for $(\dots(r_1 r_2)\dots r_n)$. In addition, we frequently write $r(s_1, \dots, s_n)$ for $rs_1 \dots s_n$ if this seems more intuitive. Self-application is possible but not necessarily total, and there may be terms which do not denote an object. We make use of the definedness predicate $\downarrow$ to single out those which do, and $(r\downarrow)$ is read "r is defined" or "r has a value".

The formulas $(A, B, C, D, A_1, B_1, C_1, D_1, \dots)$ of $\mathcal{L}^\circ$ are inductively generated as follows:

1. All expressions of the form $(r \in s)$, $(r = s)$, $(r\downarrow)$, and $\mathsf{Reg}(r, s)$ are formulas of $\mathcal{L}^\circ$, the so-called *atomic* formulas.

2. If A and B are formulas of $\mathcal{L}^\circ$, then so are $\neg A$ and $(A \vee B)$.

3. If A is a formula of $\mathcal{L}^\circ$ and if r is a term of $\mathcal{L}^\circ$ which does not contain x, then $(\exists x \in r)A$ and $\exists x A$ are formulas of $\mathcal{L}^\circ$.

We shall write $(A \wedge B)$ for $\neg(\neg A \vee \neg B)$, $(A \to B)$ for $(\neg A \vee B)$, $(A \leftrightarrow B)$ for $((A \to B) \wedge (B \to A))$, $(\forall x \in t)A$ for $\neg(\exists x \in t)\neg A$, and $\forall x A$ for $\neg \exists x \neg A$. We often omit parentheses and brackets whenever there is no danger of confusion and make use of the vector notation $\vec{r}$ as shorthand for a finite string $r_1, \dots, r_n$ of $\mathcal{L}^\circ$ terms whose length is either not important or evident from the context. If $\vec{u}$ is the sequence of pairwise different variables $u_1, \dots, u_n$ and $\vec{r} = r_1 \dots, r_n$, then $A[\vec{r}/\vec{u}]$ is the formula of $\mathcal{L}^\circ$ that is obtained from A by simultaneously replacing all free occurrences of the variables $\vec{u}$ by the $\mathcal{L}^\circ$ terms $\vec{r}$; in order to avoid collision of variables, a renaming of bound variables may be necessary. In case the $\mathcal{L}^\circ$ formula A is written as $B[\vec{u}]$, we often simply write $B[\vec{r}]$ instead of $B[\vec{r}/\vec{u}]$. Further variants of this notation will be obvious.

The Δ_0 formulas of $\mathcal{L}^\circ$ are those $\mathcal{L}^\circ$ formulas that do not contain the function symbol $\circ$, the relation symbol $\downarrow$ or unbounded quantifiers. Starting off from the Δ_0 formulas of $\mathcal{L}^\circ$, the Σ_1, Π_1, Σ, and Π formulas of $\mathcal{L}^\circ$ are defined as usual.[1]

To increase readability we freely use standard set-theoretic terminology; for example, $a \subseteq b$, $\{a_1, a_2\} = b$, $\cup a = b$, $\langle a_1, \dots, a_n \rangle = b$, $a^n = b$, $Tran[a]$, $Ord[a]$, and $Limit[a]$ express that a is a subset of b, b is the unordered pair of a_1 and a_2, b is the union of a, b is the (Kuratowski) n-tuple formed from the sets $a_1, \dots, a_n$, b is the n-times Cartesian product of a, a is transitive, a is an ordinal, and a is a limit ordinal, respectively. All these predicates have Δ_0

[1] Hence the Δ_0, Σ_1, Π_1, Σ, and Π formulas of $\mathcal{L}^\circ$ are the usual Δ_0, Σ_1, Π_1, Σ, and Π formulas of $\mathcal{L}$ extended by the relation symbol Reg, possibly containing additional constants.

definitions; see, e.g., Barwise [1]. Furthermore, we let the lower case Greek letters $\alpha, \beta, \gamma, \dots$ (possibly with subscripts) range over the ordinals.

Given an $\mathcal{L}^\circ$ formula $A[u]$, we write $\{x : A[x]\}$ to denote the collection of all sets x satisfying $A[x]$, and $u \in \{x : A[x]\}$ means $A[u]$. The collection $\{x : A[x]\}$ may be (extensionally equal to) a set, but this is not necessarily so. Special cases are

$$\mathbf{V} := \{x : x\downarrow\}, \quad \emptyset := \{x : x \neq x\}, \quad \mathbf{B} := \{x : x = \top \vee x = \bot\}$$

so that $\mathbf{V}$ denotes the collection of all sets (it is not a set itself), $\emptyset$ stands for the empty collection, and $\mathbf{B}$ for the unordered pair consisting of the truth values $\top$ and $\bot$ (it will turn out that $\emptyset$ and $\mathbf{B}$ are sets in OST). The following shorthand notation, for n an arbitrary natural number greater than 0,

$$(f : a^n \to b) \;:=\; (\forall x_1, \dots, x_n \in a)(f(x_1, \dots, x_n) \in b)$$

expresses that f is an n-ary operation from a to b. It does not say, however, that f is an n-ary function from a to b in the set-theoretic sense. In this definition the set variables a and b may be replaced by $\mathbf{V}$ and $\mathbf{B}$. So, for example, $(f : a \to \mathbf{V})$ means that f is total on a, $(f : \mathbf{V} \to b)$ means that f is an operation assigning an element of b to any set, and $(f : a \to \mathbf{B})$ means that f is an operation assigning a truth value to any element of a.

The logic of OST is the classical *logic of partial terms* (cf. Beeson [2] or Troestra and van Dalen [21]), including the common equality axioms. Partial equality of terms is introduced by

$$(r \simeq s) \;:=\; (r\downarrow \vee s\downarrow \;\to\; r = s)$$

and says that if either r or s denotes anything, then they both denote the same object.

The non-logical axioms of OST are divided into four groups and state that the universe is a partial combinatory algebra, formulate some basic set-theoretic properties, allow the representation of elementary logical connectives as operations, and provide for some operational set existence.

I. Applicative axioms.

(A1) $\mathsf{k}xy = x$,

(A2) $\mathsf{s}xy\downarrow \wedge \mathsf{s}xyz \simeq (xz)(yz)$.

II. Basic set-theoretic axioms. They comprise: (i) the usual extensionality axiom; (ii) the infinity axiom

(Inf) $$Limit[\omega] \;\wedge\; (\forall \xi \in \omega)\neg Limit[\xi];$$

(iii) $\in$-induction for arbitrary formulas $A[u]$ of $\mathcal{L}^\circ$,

$$\forall x((\forall y \in x)A[y] \to A[x]) \;\to\; \forall x A[x].$$

III. Logical operations axioms.

(L1) $\top \neq \bot$,

(L2) $(\mathbf{el} : \mathbf{V}^2 \to \mathbf{B}) \;\wedge\; \forall x \forall y(\mathbf{el}(x, y) = \top \;\leftrightarrow\; x \in y)$,

(L3) $(\mathbf{reg} : \mathbf{V}^2 \to \mathbf{B}) \;\wedge\; \forall x \forall y(\mathbf{reg}(x, y) = \top \;\leftrightarrow\; \mathsf{Reg}(x, y))$,

(L4) $(\mathbf{non} : \mathbf{B} \to \mathbf{B}) \;\wedge\; (\forall x \in \mathbf{B})(\mathbf{non}(x) = \top \;\leftrightarrow\; x = \bot)$,

(L5) $(\mathbf{dis} : \mathbf{B}^2 \to \mathbf{B}) \;\wedge\; (\forall x, y \in \mathbf{B})(\mathbf{dis}(x, y) = \top \;\leftrightarrow\; (x = \top \,\vee\, y = \top))$,

(L6) $(f : a \to \mathbf{B}) \;\to\; (\mathbf{e}(f, a) \in \mathbf{B} \;\wedge\; (\mathbf{e}(f, a) = \top \;\leftrightarrow\; (\exists x \in a)(fx = \top)))$.

IV. Set-theoretic operations axioms.

(S1) Unordered pair:

$$\mathbb{D}(a, b)\!\downarrow \;\wedge\; \forall x(x \in \mathbb{D}(a, b) \;\leftrightarrow\; x = a \,\vee\, x = b).$$

(S2) Union:

$$\mathbb{U}(a)\!\downarrow \;\wedge\; \forall x(x \in \mathbb{U}(a) \;\leftrightarrow\; (\exists y \in a)(x \in y)).$$

(S3) Separation for definite operations:

$$(f : a \to \mathbf{B}) \;\to\; (\mathbb{S}(f, a)\!\downarrow \,\wedge\, \forall x(x \in \mathbb{S}(f, a) \;\leftrightarrow\; (x \in a \,\wedge\, fx = \top))).$$

(S4) Replacement:

$$(f : a \to \mathbf{V}) \;\to\; (\mathbb{R}(f, a)\!\downarrow \,\wedge\, \forall x(x \in \mathbb{R}(f, a) \;\leftrightarrow\; (\exists y \in a)(x = fy))).$$

(S5) Choice:

$$\exists x(fx = \top) \;\to\; (\mathbb{C}f\!\downarrow \;\wedge\; f(\mathbb{C}f) = \top).$$

This finishes our description of the system OST. Because of the applicative axioms the universe is a partial combinatory algebra, and thus we have λ-abstraction: For each $\mathcal{L}^\circ$ term t we can introduce an $\mathcal{L}^\circ$ term $(\lambda x.t)$ whose variables are those of t other than x and is such that

$$(\lambda x.t)\!\downarrow \,\wedge\, (\lambda x.t)y \simeq t[y/x].$$

Clearly, λ-abstraction can be generalized to several arguments by simply iterating abstraction for one argument, and we set for all $\mathcal{L}^\circ$ terms t and all variables $x_1, \ldots, x_n$,

$$(\lambda x_1 \ldots x_n.t) \;:=\; (\lambda x_1.(\ldots(\lambda x_n.t)\ldots)).$$

Often the term $(\lambda x_1 \ldots x_n.t)$ is simply written as $\lambda x_1 \ldots x_n.t$. Furthermore, there exists a closed $\mathcal{L}^\circ$ term fix – a so-called fixed point operator – with

$$\mathsf{fix}(f){\downarrow} \;\wedge\; (\mathsf{fix}(f) = g \;\rightarrow\; gx \simeq f(g, x)).$$

Because of the logical operations axioms OST provides a term representation for every Δ_0 formula of $\mathcal{L}^\circ$ in the sense of the following lemma.

Lemma 1. *Let $\vec{u}$ be the sequence of variables $u_1, \ldots, u_n$. For every Δ_0 formula $A[\vec{u}]$ of $\mathcal{L}^\circ$ with at most the variables $\vec{u}$ free there exists a closed $\mathcal{L}^\circ$ term t such that OST proves*

$$t{\downarrow} \;\wedge\; (t : \mathbf{V}^n \rightarrow \mathbf{B}) \;\wedge\; \forall \vec{x}(A[\vec{x}] \;\leftrightarrow\; t(\vec{x}) = \top).$$

For a proof of this lemma see Feferman [7, 8]. For later purposes we need the following extension of this result.

Theorem 2. *Let $\vec{u}$ be the sequence of variables $u_1, \ldots, u_n$. For every pair of Σ_1 formulas $A[\vec{u}]$ and Π_1 formulas $B[\vec{u}]$ with at most the variables $\vec{u}$ free there exists a closed $\mathcal{L}^\circ$ term t such that OST proves*

$$\forall \vec{x}(A[\vec{x}] \leftrightarrow B[\vec{x}]) \;\rightarrow\; (t{\downarrow} \wedge (t : \mathbf{V}^n \rightarrow \mathbf{B}) \wedge \forall \vec{x}(A[\vec{x}] \;\leftrightarrow\; t(\vec{x}) = \top)).$$

Proof. By assumption, $A[\vec{u}]$ is of the form $\exists x C[\vec{u}, x]$ and $B[\vec{u}]$ of the form $\forall x D[\vec{u}, x]$, where $C[\vec{u}, v]$ and $D[\vec{u}, v]$ are Δ_0. Now we work in OST and know that

$$r_0{\downarrow} \;\wedge\; (r_0 : \mathbf{V}^{n+1} \rightarrow \mathbf{B}) \;\wedge\; \forall \vec{x}, y(C[\vec{x}, y] \;\leftrightarrow\; r_0(\vec{x}, y) = \top),$$
$$r_1{\downarrow} \;\wedge\; (r_1 : \mathbf{V}^{n+1} \rightarrow \mathbf{B}) \;\wedge\; \forall \vec{x}, y((C[\vec{x}, y] \vee \neg D[\vec{x}, y]) \;\leftrightarrow\; r_1(\vec{x}, y) = \top)$$

for closed $\mathcal{L}^\circ$ terms r_0 and r_1 chosen according to the previous lemma. Thus, if E abbreviates $\forall \vec{x}(A[\vec{x}] \leftrightarrow B[\vec{x}])$, we have

$$E \;\rightarrow\; \forall \vec{x} \exists y(r_1(\vec{x}, y) = \top).$$

Now we set $s := \lambda \vec{x}.\mathbb{C}(\lambda y.r_1(\vec{x}, y))$ and $t := \lambda \vec{x}.r_0(\vec{x}, s(\vec{x}))$. Then E implies that s and t are defined and that s is an operation from $\mathbf{V}^n$ to $\mathbf{V}$ and t one from $\mathbf{V}^n$ to $\mathbf{B}$. In addition, it is easy to check that

$$E \;\rightarrow\; \forall \vec{x}(A[\vec{x}] \;\leftrightarrow\; t(\vec{x}) = \top).$$

Hence t is the required closed term. $\square$

An $\mathcal{L}^\circ$ formula A is called Δ_1 with respect to a theory T iff there exist a Σ_1 formula B and a Π_1 formula C, both with the same free variables as A, such that T proves $(A \leftrightarrow B)$ and $(A \leftrightarrow C)$. If T is a theory containing OST, then by combining the previous theorem with (the proof of) Lemma 1 it is routine work to check that the term representation can be lifted to all formulas that are Δ_0 in one or several Δ_1 formulas with respect to T.

Corollary 3. *Assume that T is a theory containing OST and let $\vec{u}$ be the sequence of variables $u_1, \ldots, u_n$ and $A[\vec{u}]$ an $\mathcal{L}^\circ$ formula with at most the variables $\vec{u}$ free that is Δ_0 in Δ_1 with respect to T. Then there exists a closed $\mathcal{L}^\circ$ term t such that T proves*

$$t{\downarrow} \;\wedge\; (t : \mathbf{V}^n \to \mathbf{B}) \;\wedge\; \forall \vec{x}(A[\vec{x}] \;\leftrightarrow\; t(\vec{x}) = \top).$$

3 Relativized application

As standard in set theory we write A^p for the result of replacing all unbounded quantifiers $\exists x(\ldots)$ and $\forall x(\ldots)$ in A by $(\exists x \in p)(\ldots)$ and $(\forall x \in p)(\ldots)$, respectively. However, in contrast to this usual way of relativizing formulas with respect to a given set p, we now relativize our $\mathcal{L}^\circ$ formulas A with respect to a set p and a set $q \subseteq p^3$ to formulas $A^{(p,q)}$; then p is the new universe and q takes care of application in the sense described below.

Definition 4. *For all $\mathcal{L}^\circ$ terms r and q we define the formula $(r \,\partial\, q)$ by induction on the complexity of r as follows:*

1. *If r is a variable or a constant of $\mathcal{L}^\circ$, then $(r \,\partial\, q) := (r = r)$.*
2. *If r is the $\mathcal{L}^\circ$ term $r_1 r_2$, then*

$$(r \,\partial\, q) \;:=\; (r_1 \,\partial\, q) \wedge (r_2 \,\partial\, q) \wedge \exists x(\langle r_1, r_2, x\rangle \in q)$$

for some variable x not appearing in r_1, r_2, q.

Think of q as a ternary relation; then $(r \,\partial\, q)$ formalizes that the $\mathcal{L}^\circ$ term r is defined if application within r is treated according to q. For us only such relations are interesting that are compatible with the real term application. To single those out, we set

$$Comp[q] \;:=\; \forall x \forall y \forall z(\langle x, y, z\rangle \in q \;\to\; xy = z).$$

Furthermore, given an $\mathcal{L}^\circ$ term or a $\mathcal{L}^\circ$ formula $\mathcal{E}$, we write $VarCon_{\mathcal{E}}[p]$ for the $\mathcal{L}^\circ$ formula that states that all variables and constants appearing in $\mathcal{E}$ are elements of p. The following observation is a straightforward consequence of the previous definitions and abbreviations.

Lemma 5. OST *proves for all $\mathcal{L}^\circ$ terms r, p, and q:*

1. $(Comp[q] \wedge (r \,\partial\, q)) \rightarrow r{\downarrow}$.
2. $(VarCon_r[p] \wedge q \subseteq p^3 \wedge Comp[q] \wedge (r \,\partial\, q)) \rightarrow r \in p$.

However, observe that in general we may have $Comp[q]$ and $r{\downarrow}$, but not $(r \,\partial\, q)$; so it is possible that term r has a value without being defined in the sense of q.

In a next step this form of relativizing application via q is combined with restricting the universe of discourse to p and formulated for arbitrary formulas of the language $\mathcal{L}^\circ$. Given $\mathcal{L}^\circ$ terms p, q and an $\mathcal{L}^\circ$ formula A, we call p, q *suitable for relativizing* A iff the variables appearing somewhere in p or q are different from those appearing in A.

Definition 6. *For all $\mathcal{L}^\circ$ formulas A and all $\mathcal{L}^\circ$ terms p, q that are suitable for relativizing A we define the formula $A^{(p,q)}$ by induction on the complexity of A.*

1. *For atomic formulas we set:*

$$\begin{aligned}
(r = s)^{(p,q)} &:= (r \,\partial\, q) \wedge (s \,\partial\, q) \wedge r = s, \\
(r \in s)^{(p,q)} &:= (r \,\partial\, q) \wedge (s \,\partial\, q) \wedge r \in s, \\
(r{\downarrow})^{(p,q)} &:= (r \,\partial\, q) \wedge r \in p, \\
\mathsf{Reg}(r, s)^{(p,q)} &:= (r \,\partial\, q) \wedge (s \,\partial\, q) \wedge \mathsf{Reg}(r, s).
\end{aligned}$$

2. *If A is the formula $\neg B$ we set $A^{(p,q)} := \neg B^{(p,q)}$.*
3. *If A is the formula $(B \vee C)$ we set $A^{(p,q)} := (B^{(p,q)} \vee C^{(p,q)})$.*
4. *If A is the formula $(\exists x \in r)B$ we set*

$$A^{(p,q)} := (r \,\partial\, q) \wedge (\exists x \in r)B^{(p,q)}.$$

5. *If A is the formula $\exists x B$ we set $A^{(p,q)} := (\exists x \in p)B^{(p,q)}$.*

Whenever we write $A^{(p,q)}$ we tacitly assume that p, q are suitable for relativizing A. This form of relativizing formulas of $\mathcal{L}^\circ$ has a useful substitution property.

Lemma 7. *Let x be a variable that does not occur in the $\mathcal{L}^\circ$ terms p and q. Then* OST *proves for all $\mathcal{L}^\circ$ terms r and s as well as for all $\mathcal{L}^\circ$ formulas A:*

1. $(Comp[q] \wedge (r \,\partial\, q)) \;\rightarrow\; ((s[r/x] \,\partial\, q) \leftrightarrow (s \,\partial\, q)[r/x])$.

2. $(Comp[q] \wedge (r \,\partial\, q)) \;\rightarrow\; (A[r/x]^{(p,q)} \leftrightarrow A^{(p,q)}[r/x])$.

The proof of the first assertion is by straightforward induction on the complexity of the term s. The second assertion is established by induction on the complexity of A, employing the first assertion.

In general, the relativizations A^p and $A^{(p,q)}$ have different meanings. However, there is a special case for which they agree.

Lemma 8. *If A is an $\mathcal{L}^\circ$ formula that does not contain terms of the form st, then* OST *proves for all $\mathcal{L}^\circ$ terms p, q that*

$$VarCon_A[p] \;\rightarrow\; (A^{(p,q)} \leftrightarrow A^p).$$

Now the relation Reg comes into play. $\mathsf{Reg}(p,q)$ says that set p is regular with respect to q and has the following intuitive interpretation: (i) p is a transitive set containing all constants of $\mathcal{L}^\circ$ as elements, and q is a ternary relation on p compatible with the general application; (ii) if application is interpreted in the sense of q, then p satisfies the axioms of OST; (iii) we claim a linear ordering of those pairs $\langle p, q\rangle$ for which $\mathsf{Reg}(p,q)$ holds. To make this precise, we add to OST additional so-called Reg-axioms. Here $TranCon[p]$ is short for the $\mathcal{L}^\circ$ formula stating that p is transitive and contains all constants of $\mathcal{L}^\circ$.

V. Axioms for Reg.

(Reg1) $\mathsf{Reg}(d,e) \;\rightarrow\; (TranCon[d] \wedge e \subseteq d^3 \wedge Comp[e])$.

(Reg2) If A is an applicative axiom, logical operations axiom, or set-theoretic operations axiom with at most the variables $\vec{x}$ free such that neither the variables d, e appear in the list $\vec{x}$, then

$$\mathsf{Reg}(d,e) \;\rightarrow\; (\forall \vec{x} \in d)A^{(d,e)}.$$

(Reg3) $\mathsf{Reg}(d_1,e_1) \wedge \mathsf{Reg}(d_2,e_2) \;\rightarrow\; d_1 \in d_2 \vee d_1 = d_2 \vee d_2 \in d_1$.

(Reg4) $\mathsf{Reg}(d_1,e_1) \wedge \mathsf{Reg}(d_2,e_2) \wedge d_1 \in d_2 \;\rightarrow\; e_1 \in d_2 \wedge e_1 \subseteq e_2$.

OST+(Reg) is defined to be OST + (Reg1) + … + (Reg4). Considering the previous lemmas and the Reg-axioms, it is easy to see the (p,q)-relativizations of all axioms of the logic of partial terms and of all non-logical axioms of OST can be proved in OST + (Reg). Therefore, the following theorem follows by induction on the derivations in OST.

Theorem 9. *Let A be an $\mathcal{L}^\circ$ formula with at most $\vec{x}$ free such that neither the variables of p nor those of q appear in the list $\vec{x}$. Then we have that*

$$\mathsf{OST} \vdash A \quad \Longrightarrow \quad \mathsf{OST} + (\mathsf{Reg}) \vdash \mathsf{Reg}(p,q) \;\rightarrow\; (\forall \vec{x} \in p) A^{(p,q)}.$$

Observe that the Reg-axioms only state properties of sets satisfying $\mathsf{Reg}(p,q)$; however, they do not claim that there exist p and q for which we have $\mathsf{Reg}(p,q)$. This can be achieved, for example, by the following axiom, claiming that the universe of sets is a limit of regulars.

VI. Limit of regulars.

$$(\mathsf{Lim\text{-}Reg}) \qquad \forall x \exists y \exists z (x \in y \;\wedge\; \mathsf{Reg}(y,z)).$$

In the following we write $\mathsf{OST(LR)}$ for the extension of OST by the axioms (Reg1)–(Reg4) and ($\mathsf{Lim\text{-}Reg}$). As we will see later, $\mathsf{OST(LR)}$ is proof-theoretically equivalent to the theory KPi, which describes a recursively inaccessible universe.

In setting up this equivalence, the notion "d is admissible" in KPi will be translated into "there exists an x such that (d,x) is regular" in $\mathsf{OST(LR)}$. Accordingly, we define

$$Ad^\circ[d] \;:=\; \exists x \mathsf{Reg}(d,x).$$

However, before turning to the proof-theoretic analysis of $\mathsf{OST(LR)}$ in the following sections, we consider a property of $Ad^\circ[d]$, which will play an important role later.

Lemma 10. *In* $\mathsf{OST(LR)}$ *we can prove that*

$$Ad^\circ[d] \;\leftrightarrow\; \forall y \forall z (\mathsf{Reg}(y,z) \;\wedge\; d \in y \;\rightarrow\; (\exists x \in y)\mathsf{Reg}(d,x)).$$

Proof. The direction from left to right follows from (Reg4), the direction from right to left is a consequence of ($\mathsf{Lim\text{-}Reg}$). $\square$

This lemma implies in particular that the formula $Ad^\circ[u]$ is Δ_1 with respect to $\mathsf{OST(LR)}$. In view of Corollary 3 we thus have term representations of all $\mathcal{L}^\circ$ formulas that are Δ_0 in Ad°.

Theorem 11. *Let $\vec{u}$ be the sequence of variables $u_1, \dots, u_n$ and $A[\vec{u}]$ an $\mathcal{L}^\circ$ formula with at most the variables $\vec{u}$ free that is Δ_0 in Ad°. Then there exists a closed $\mathcal{L}^\circ$ term t such that* $\mathsf{OST(LR)}$ *proves*

$$t\!\downarrow \;\wedge\; (t : \mathbf{V}^n \to \mathbf{B}) \;\wedge\; \forall \vec{x}(A[\vec{x}] \;\leftrightarrow\; t(\vec{x}) = \top).$$

4 The theories KP, KPi, and related systems

The theory KPi is a well-known theory of iterated admissible sets that extends Kripke-Platek set theory with infinity by the additional assertion that every set is contained in an admissible set.[2] The least standard model of KPi is L_{ι_0} with ι_0 being the first recursively inaccessible ordinal, and we have the proof-theoretic equivalences

$$\mathsf{KPi} \;\equiv\; \Delta^1_2\text{-}\mathsf{CA} + (\mathsf{BI}) \;\equiv\; \mathsf{T}_0,$$

where Δ^1_2-CA + (BI) the usual system of second order arithmetic with the axiom of Δ^1_2-comprehension plus bar induction and T_0 is a central system of explicit mathematics; cf. Feferman [6], Jäger [10], and Jäger and Pohlers [16].

The theory KPi can be conveniently formulated in the language $\mathcal{L}^* = \mathcal{L}(\mathsf{Ad})$ that extends our language $\mathcal{L}$ of first order set theory by a unary relation symbol Ad for admissible sets. The terms of $\mathcal{L}^*$ are the variables of $\mathcal{L}$ plus the constant ω, and the formulas of $\mathcal{L}^*$ are defined in the usual way, with $\mathsf{Ad}(u)$ considered to be a Δ_0 formula of $\mathcal{L}^*$ as well.

The underlying logic of KPi is the classical first order logic with equality. The non-logical axioms of KPi comprise the Kripke-Platek axioms, the Ad-axioms, and the limit axiom for admissibles.

I. Kripke-Platek axioms. They consist of: (i) the extensionality axiom; (ii) pairing and union; (iii) the infinity axiom (Inf); (iv) $\in$-induction for arbitrary $\mathcal{L}^*$ formulas; (v) Δ_0 separation and Δ_0 collection, i.e. for all Δ_0 formulas $A[u]$ and $B[u,v]$ of $\mathcal{L}^*$,

$(\Delta_0\text{-}\mathsf{Sep})$ $$\exists x \forall y (y \in x \;\leftrightarrow\; y \in a \,\land\, A[y]),$$

$(\Delta_0\text{-}\mathsf{Col})$ $$(\forall x \in a)\exists y B[x,y] \;\to\; \exists z (\forall x \in a)(\exists y \in z) B[x,y].$$

II. Axioms for Ad.

(Ad1) $\mathsf{Ad}(d) \;\to\; \mathit{Tran}[d] \land \omega \in d.$

(Ad2) $\mathsf{Ad}(d) \;\to\; (\forall x, y \in d)(\{x,y\} \in d \,\land\, \cup x \in d).$

(Ad3) If A is an instance of $(\Delta_0\text{-}\mathsf{Sep})$ or $(\Delta_0\text{-}\mathsf{Col})$ with at most the variables $\vec{x}$ free, then
$$\mathsf{Ad}(d) \;\to\; (\forall \vec{x} \in d) A^d.$$

[2]The theory KPi has been introduced in Jäger [9] and is formulated there as a system above the natural numbers as urelements. This has some advantages in studying subsystems of KPi. However, in the presence of full $\in$-induction, it is obviously equivalent to our formulation below.

(Ad4) $\mathsf{Ad}(d_1) \wedge \mathsf{Ad}(d_2) \;\rightarrow\; d_1 \in d_2 \vee d_1 = d_2 \vee d_2 \in d_1$.

III. Limit of admissibles.

$$(\mathsf{Lim\text{-}Ad}) \qquad\qquad \forall x \exists y (x \in y \wedge \mathsf{Ad}(y)).$$

Kripke-Platek set theory KP is the subsystem of KPi obtained from KPi by deleting the axiom ($\mathsf{Lim\text{-}Ad}$). Clearly, the axioms for Ad imply that every set satisfying Ad is transitive, contains ω and is closed under pairing, union, Δ_0 separation and Δ_0 collection. Together with the axiom ($\mathsf{Lim\text{-}Ad}$) we thus know that every model of KPi is an admissible limit of admissibles.

By $\overline{\mathsf{KPi}}$ we denote the subsystem of KPi that is obtained from KPi by restricting the axioms (Ad3) to formulas of $\mathcal{L}$, i.e. to formulas not containing the relation symbol Ad. It is easy to see that $\overline{\mathsf{KPi}}$ is of the same proof-theoretic strength as KPi. For example, the embedding of Δ^1_2-CA + (BI) into KPi, as presented in Jäger [11], also works for $\overline{\mathsf{KPi}}$.

Following standard terminology, we call a formula A of $\mathcal{L}^*$ a $\Delta(\mathsf{KP})$ formula iff there exist a Σ formula B and a Π formula C of $\mathcal{L}^*$, both with the same free variables as A, such that

$$\mathsf{KP} \vdash (A \leftrightarrow B) \wedge (A \leftrightarrow C).$$

The constructible hierarchy provides for important examples of $\Delta(\mathsf{KP})$ formulas. We cannot introduce it here but refer for all relevant details to, for example, Barwise [1] or Kunen [19]. All we need is that $(a \in L_\alpha)$ states that the set a is an element of the α-th level L_α of the constructible hierarchy and $(a \in \mathbf{L})$ is short for $\exists \alpha (a \in L_\alpha)$; besides that $(a <_{\mathbf{L}} b)$ means that a is smaller than b according to the well-ordering $<_{\mathbf{L}}$ on the constructible universe $\mathbf{L}$. The axiom of constructibility is the statement $(\mathbf{V} = \mathbf{L})$, i.e. $\forall x (x \in \mathbf{L})$. It is well-known that the assertions $(a \in L_\alpha)$ and $(a <_{\mathbf{L}} b)$ are $\Delta(\mathsf{KP})$ formulas. In addition, the theories $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ and $\mathsf{KPi} + (\mathbf{V}{=}\mathbf{L})$ are of the same consistency strength as KP and KPi, respectively.

In the following we write $d \models \ulcorner\mathsf{KP}\urcorner$ to state that d is a transitive standard model of KP and refer to Barwise [1] and Probst [20] for details. Then we set

$$\mathsf{Ad}_{\mathbf{L}}[d] \;:=\; \exists \xi (\mathit{Limit}[\xi] \wedge d = L_\xi) \;\wedge\; d \models \ulcorner\mathsf{KP}\urcorner.$$

If d satisfies $\mathsf{Ad}_{\mathbf{L}}[d]$, we call it an $\mathbf{L}$-*admissible* set. Since KP proves the equivalence of the assertion $\exists \xi (\mathit{Limit}[\xi] \wedge d = L_\xi)$ with

$$\emptyset \in d \;\wedge\; \mathit{Tran}[d] \;\wedge\; (\forall x \in d)(\exists \xi \in d)(x \in L_\xi \wedge L_\xi \in d),$$

we conclude that $\mathsf{Ad}_{\mathbf{L}}[d]$ is a $\Delta(\mathsf{KP})$ formula. The first assertion of the following lemma is immediate from the definition of $\mathsf{Ad}_{\mathbf{L}}$, the second follows from the first and some obvious persistency arguments, and the third is by an inner model construction; see again [1, 20].

Lemma 12.

1. *If A is a closed $\mathcal{L}^*$ formula that is provable in* KP*, then we have that* KP *also proves*
$$\mathsf{Ad}_{\mathbf{L}}[d] \;\rightarrow\; A^d.$$
2. *If $A[\vec{u}]$ is a $\Delta(\mathsf{KP})$ formula with at most $\vec{u}$ free, then* KP *proves*
$$\mathsf{Ad}_{\mathbf{L}}(d) \wedge \vec{a} \in d \;\rightarrow\; (A[\vec{a}] \leftrightarrow A^d[\vec{a}]).$$
3. $\mathsf{KPi} + (\mathbf{V}{=}\mathbf{L}) \vdash \forall x \exists y (x \in y \wedge \mathsf{Ad}_{\mathbf{L}}[y])$.

Now let P be a fresh n-ary relation symbol and write $\mathcal{L}^*(\mathsf{P})$ for the extension of $\mathcal{L}^*$ by P. We call an $\mathcal{L}^*(\mathsf{P})$ formula that contains at most $u_0, \ldots, u_n$ free and that is Δ with respect to KP an *n-ary* $\Delta(\mathsf{KP})$ *operator form* and let $\mathfrak{A}[\mathsf{P}, u_0, \ldots, u_n]$ range over such forms. Given another formula $B[v_1, \ldots, v_n]$ with distinguished variables $v_1, \ldots, v_n$ we write $\mathfrak{A}[B[.], r_0, \ldots, r_n]$ for the result of substituting $B[s_1, \ldots, s_n]$ for each occurrence of $\mathsf{P}(s_1, \ldots, s_n)$ in $\mathfrak{A}[\mathsf{P}, r_0, \ldots, r_n]$.

For modeling $\mathsf{OST(LR)}$ in KPi we will later work with a specific $\Delta(\mathsf{KP})$ operator form. But first we turn to a central recursion theorem, available for arbitrary $\Delta(\mathsf{KP})$ operator forms.

Theorem 13. *Let $\mathfrak{A}[\mathsf{P}, u_0, \ldots, u_n]$ be an n-ary $\Delta(\mathsf{KP})$ operator form. Then there exists a Σ formula $F_{\mathfrak{A}}[u_0, \ldots, u_n]$ of $\mathcal{L}^*$ with at most $u_0, \ldots, u_n$ free such that $F_{\mathfrak{A}}[u_0, \ldots, u_n]$ is $\Delta(\mathsf{KP})$ and* KP *proves*
$$F_{\mathfrak{A}}[\alpha, \vec{a}] \;\leftrightarrow\; (\vec{a} \in L_{\alpha+\omega} \wedge \mathfrak{A}[(\exists \xi < \alpha) F_{\mathfrak{A}}[\xi, .], \alpha, \vec{a}])$$
for all ordinals α and sets $\vec{a} = a_1, \ldots, a_n$.

Proof. In Jäger and Zumbrunnen [17] it is shown that for every $\Delta(\mathsf{KP})$ operator form $\mathfrak{A}[\mathsf{P}, u_0, \ldots, u_n]$ there exists a Σ formula $F_{\mathfrak{A}}[u_0, \ldots, u_n]$ that satisfies the equivalence stated in our theorem. It is easy to check that this formula is $\Delta(\mathsf{KP})$. $\square$

In view of Lemma 12 we can relativize the $\Delta(\mathsf{KP})$ formula $F_{\mathfrak{A}}[u_0, \ldots, u_n]$ that comes with the $\Delta(\mathsf{KP})$ operator form $\mathfrak{A}[\mathsf{P}, u_0, \ldots, u_n]$ to all $\mathbf{L}$-admissible sets.

Corollary 14. *Assume that $\mathfrak{A}[\mathsf{P}, \vec{u}]$ is an n-ary $\Delta(\mathsf{KP})$ operator form and that $F_{\mathfrak{A}}[\vec{u}]$ is associated with $\mathfrak{A}[\mathsf{P}, \vec{u}]$ according to the previous theorem. Then* KP *proves for all α and $\vec{a} = a_1, \ldots, a_n$ that*
$$\mathsf{Ad}_{\mathbf{L}}[d] \wedge \alpha, \vec{a} \in d \;\rightarrow\; (F^d_{\mathfrak{A}}[\alpha, \vec{a}] \;\leftrightarrow\; F_{\mathfrak{A}}[\alpha, \vec{a}]).$$

5 The proof-theoretic strength of $\mathsf{OST}(\mathsf{LR})$

In this section we establish the proof-theoretic equivalence of the theories $\mathsf{OST}(\mathsf{LR})$ and KPi by showing that: (i) $\overline{\mathsf{KPi}}$ can be embedded into $\mathsf{OST}(\mathsf{LR})$, and (ii) $\mathsf{OST}(\mathsf{LR})$ is interpretable in $\mathsf{KPi} + (\mathbf{V}{=}\mathbf{L})$.

The first part is easy. Given an $\mathcal{L}^*$ formula A, we write A° for the result of substituting the $\mathcal{L}^\circ$ formula $Ad^\circ[s]$ for each occurrence of $\mathsf{Ad}(s)$ in A, thus translating A into an $\mathcal{L}^\circ$ formula. Then we have the following embedding result.

Theorem 15. *For all $\mathcal{L}^*$ formulas A we have that*

$$\overline{\mathsf{KPi}} \vdash A \quad \Longrightarrow \quad \mathsf{OST}(\mathsf{LR}) \vdash A^\circ.$$

Proof. Because of Because of (Reg1)–(Reg3), and ($\mathsf{Lim\text{-}Reg}$) it is clear that the translations of (Ad1), (Ad2), (Ad4), and ($\mathsf{Lim\text{-}Ad}$) are provable in $\mathsf{OST}(\mathsf{LR})$. In addition, if $A[\vec{u}]$ is a formula of $\mathcal{L}$ with at most $\vec{u}$ free and an axiom of KP, then we know from Feferman [8] and Jäger [12] that

$$\mathsf{OST} \vdash A[\vec{a}]$$

for all $\vec{a}$. Therefore, by Theorem 9 we also have

$$\mathsf{OST}(\mathsf{LR}) \vdash \mathsf{Reg}(d,e) \rightarrow (\forall \vec{x} \in d) A^{(d,e)}[\vec{x}]$$

and in view of Lemma 8 even

$$\mathsf{OST}(\mathsf{LR}) \vdash Ad^\circ[d] \rightarrow (\forall \vec{x} \in d) A^{d}[\vec{x}]$$

for these $A[\vec{u}]$. This implies that the translations of the axioms (Ad3) restricted to $\mathcal{L}$ formulas are provable in $\mathsf{OST}(\mathsf{LR})$. Of course, also the translations of $\in$-induction for arbitrary $\mathcal{L}^*$ formulas, pairing, union and the role of ω create no problems.

It remains to deal with Δ_0 separation and Δ_0 collection. Here we have to keep in mind that the Δ_0 formulas of $\mathcal{L}^*$ may contain the relation symbol Ad, and (sub)formulas of the form $\mathsf{Ad}(u)$ are translated into the Σ_1 formula $Ad^\circ[u]$. Thus if A is a Δ_0 formula of $\mathcal{L}^*$, then A° is Δ_0 in Ad°, and it follows from Theorem 11 that A° can be represented by a closed term. Taking this into account, we can validate Δ_0 separation and Δ_0 collection of $\overline{\mathsf{KPi}}$ as in the embedding of Kripke-Platek set theory into OST, which is presented – as mentioned above – in Feferman [8] and Jäger [12].

So $\mathsf{OST}(\mathsf{LR})$ proves A° for all axioms of $\overline{\mathsf{KPi}}$. From that our assertion follows by straightforward induction on the proof of A in $\overline{\mathsf{KPi}}$. $\square$

For interpreting $\mathsf{OST}(\mathsf{LR})$ in $\mathsf{KPi} + (\mathbf{V}{=}\mathbf{L})$ we can follow Jäger [12] and Jäger and Zumbrunnen [17] to a large part. As there, we begin with some notational preliminaries:

- For any natural number n greater than 0 and any natural number i we select Δ_0 formulas $Tup_n[u]$ and $(u)_i = v$ formalizing that u is an ordered n-tuple and v the projection of u on its i-th component; hence $Tup_n(\langle u_0, \ldots, u_{n-1}\rangle)$ and $(\langle u_0, \ldots, u_{n-1}\rangle)_i = u_i$ for $0 \le i \le n-1$.
- Then we fix pairwise different elements $\widehat{\mathsf{k}}$, $\widehat{\mathsf{s}}$, $\widehat{\top}$, $\widehat{\bot}$, $\widehat{\mathbf{el}}$, $\widehat{\mathbf{reg}}$, $\widehat{\mathbf{non}}$, $\widehat{\mathbf{dis}}$, $\widehat{\mathbf{e}}$, $\widehat{\mathbb{D}}$, $\widehat{\mathbb{U}}$, $\widehat{\mathbb{S}}$, $\widehat{\mathbb{R}}$, and $\widehat{\mathbb{C}}$ of ω, making sure that they all do not belong to the collection of ordered pairs and triples; they will later act as the codes of the corresponding constants of $\mathcal{L}^\circ$.

The $\mathcal{L}^\circ$ terms $\mathsf{k}x$, $\mathsf{s}x$, $\mathsf{s}xy$, ... will be coded by the ordered tuples $\langle\widehat{\mathsf{k}}, x\rangle$, $\langle\widehat{\mathsf{s}}, x\rangle$, $\langle\widehat{\mathsf{s}}, x, y\rangle$, ... of the corresponding form. For example, to satisfy $\mathsf{k}xy = x$ we interpret $\mathsf{k}x$ as $\langle\widehat{\mathsf{k}}, x\rangle$, and make sure that the translation of application is so that "$\langle\widehat{\mathsf{k}}, x\rangle$ applied to y" yields x.

Next let P be a fresh 3-place relation symbol and extend $\mathcal{L}^*$ to the language $\mathcal{L}^*(\mathsf{P})$ as above. The following definition introduces the $\Delta(\mathsf{KP})$ operator form $\mathfrak{A}[\mathsf{P}, \alpha, u, v, w]$ which will afterwards lead to the interpretation of the application relation $(uv = w)$.

Definition 16. *We choose $\mathfrak{A}[\mathsf{P}, \alpha, u, v, w]$ to be the $\mathcal{L}^*(\mathsf{P})$ formula defined as the disjunction of the following formulas (1)–(29):*

(1) $u = \widehat{\mathsf{k}} \ \wedge\ w = \langle\widehat{\mathsf{k}}, v\rangle$,

(2) $Tup_2[u] \ \wedge\ (u)_0 = \widehat{\mathsf{k}} \ \wedge\ w = (u)_1$,

(3) $u = \widehat{\mathsf{s}} \ \wedge\ w = \langle\widehat{\mathsf{s}}, v\rangle$,

(4) $Tup_2[u] \ \wedge\ (u)_0 = \widehat{\mathsf{s}} \ \wedge\ w = \langle\widehat{\mathsf{s}}, (u)_1, v\rangle$,

(5) $Tup_3[u] \ \wedge\ (u)_0 = \widehat{\mathsf{s}} \ \wedge$
$(\exists x, y \in L_\alpha)(\mathsf{P}((u)_1, v, x) \ \wedge\ \mathsf{P}((u)_2, v, y) \ \wedge\ \mathsf{P}(x, y, w))$,

(6) $u = \widehat{\mathbf{el}} \ \wedge\ w = \langle\widehat{\mathbf{el}}, v\rangle$,

(7) $Tup_2[u] \ \wedge\ (u)_0 = \widehat{\mathbf{el}} \ \wedge\ (u)_1 \in v \ \wedge\ w = \widehat{\top}$,

(8) $Tup_2[u] \ \wedge\ (u)_0 = \widehat{\mathbf{el}} \ \wedge\ (u)_1 \notin v \ \wedge\ w = \widehat{\bot}$,

(9) $u = \widehat{\mathbf{reg}} \ \wedge\ w = \langle\widehat{\mathbf{reg}}, v\rangle$,

(10) $Tup_2[u] \;\wedge\; (u)_0 = \widehat{\mathbf{reg}} \;\wedge\; (u)_1 \in L_{\alpha+1} \setminus L_\alpha \;\wedge\; v \in L_{\alpha+\omega} \;\wedge$
$(u)_1 = L_\alpha \;\wedge\; \mathsf{Ad}_{\mathbf{L}}[L_\alpha] \;\wedge\; v = \{\langle x,y,z\rangle \in (u)_1 : \mathsf{P}(x,y,z)\} \;\wedge\; w = \widehat{\top}$,

(11) $Tup_2[u] \;\wedge\; (u)_0 = \widehat{\mathbf{reg}} \;\wedge\; (u)_1 \in L_{\alpha+1} \setminus L_\alpha \;\wedge\; v \in L_{\alpha+\omega} \;\wedge$
$((u)_1 \neq L_\alpha \;\vee\; \neg\mathsf{Ad}_{\mathbf{L}}[L_\alpha] \;\vee\; v \neq \{\langle x,y,z\rangle \in (u)_1 : \mathsf{P}(x,y,z)\}) \;\wedge\; w = \widehat{\bot}$,

(12) $Tup_2[u] \;\wedge\; (u)_0 = \widehat{\mathbf{reg}} \;\wedge\; (\exists \xi < \alpha)((u)_1 \in L_{\xi+1} \setminus L_\xi \;\wedge\; v \notin L_{\xi+\omega}) \;\wedge$
$w = \widehat{\bot}$,

(13) $u = \widehat{\mathbf{non}} \;\wedge\; v = \widehat{\top} \;\wedge\; w = \widehat{\bot}$,

(14) $u = \widehat{\mathbf{non}} \;\wedge\; v = \widehat{\bot} \;\wedge\; w = \widehat{\top}$,

(15) $u = \widehat{\mathbf{dis}} \;\wedge\; w = \langle \widehat{\mathbf{dis}}, v\rangle$,

(16) $Tup_2[u] \;\wedge\; (u)_0 = \widehat{\mathbf{dis}} \;\wedge\; (u)_1 = \widehat{\top} \;\wedge\; w = \widehat{\top}$,

(17) $Tup_2[u] \;\wedge\; (u)_0 = \widehat{\mathbf{dis}} \;\wedge\; (u)_1 = \widehat{\bot} \;\wedge\; v = \widehat{\top} \;\wedge\; w = \widehat{\top}$,

(18) $Tup_2[u] \;\wedge\; (u)_0 = \widehat{\mathbf{dis}} \;\wedge\; (u)_1 = \widehat{\bot} \;\wedge\; v = \widehat{\bot} \;\wedge\; w = \widehat{\bot}$,

(19) $u = \widehat{\mathbf{e}} \;\wedge\; w = \langle \widehat{\mathbf{e}}, v\rangle$,

(20) $Tup_2[u] \;\wedge\; (u)_0 = \widehat{\mathbf{e}} \;\wedge\; (\exists x \in v)\mathsf{P}((u)_1, x, \widehat{\top}) \;\wedge\; w = \widehat{\top}$,

(21) $Tup_2[u] \;\wedge\; (u)_0 = \widehat{\mathbf{e}} \;\wedge\; (\forall x \in v)\mathsf{P}((u)_1, x, \widehat{\bot}) \;\wedge\; w = \widehat{\bot}$,

(22) $u = \widehat{\mathbb{D}} \;\wedge\; w = \langle \widehat{\mathbb{D}}, v\rangle$,

(23) $Tup_2[u] \;\wedge\; (u)_0 = \widehat{\mathbb{D}} \;\wedge\; w = \{(u)_1, v\}$,

(24) $u = \widehat{\mathbb{U}} \;\wedge\; w = \cup v$,

(25) $u = \widehat{\mathbb{S}} \;\wedge\; w = \langle \widehat{\mathbb{S}}, v\rangle$,

(26) $Tup_2[u] \;\wedge\; (u)_0 = \widehat{\mathbb{S}} \;\wedge\; (\forall x \in v)(\mathsf{P}((u)_1, x, \widehat{\top}) \vee \mathsf{P}((u)_1, x, \widehat{\bot})) \;\wedge$
$(\forall x \in w)(x \in v \;\wedge\; \mathsf{P}((u)_1, x, \widehat{\top})) \;\wedge\; (\forall x \in v)(\mathsf{P}((u)_1, x, \widehat{\top}) \;\to\; x \in w)$,

(27) $u = \widehat{\mathbb{R}} \;\wedge\; w = \langle \widehat{\mathbb{R}}, v\rangle$,

(28) $Tup_2[u] \;\wedge\; (u)_0 = \widehat{\mathbb{R}} \;\wedge\; (\forall x \in v)(\exists y \in w)\mathsf{P}((u)_1, x, y) \;\wedge$
$(\forall y \in w)(\exists x \in v)\mathsf{P}((u)_1, x, y)$,

(29) $u = \widehat{\mathbb{C}} \wedge \mathsf{P}(v, w, \widehat{\top}) \wedge (\forall x \in L_\alpha)(x <_{\mathbf{L}} w \to \neg \mathsf{P}(v, x, \widehat{\top})) \wedge$
$(\forall x \in L_\alpha) \neg \mathsf{P}(\widehat{\mathbb{C}}, v, x)$.

It is easy to see that $\mathfrak{A}[\mathsf{P}, \alpha, u, v, w]$ is a 4-ary $\Delta(\mathsf{KP})$ operator form and deterministic in the following sense: from $\mathfrak{A}[\mathsf{P}, \alpha, u, v, w]$ we can conclude that exactly one of the clauses (1)–(29) of the previous definition is satisfied for these α, u, v, and w. Now we recall Theorem 13 and associate with the operator form $\mathfrak{A}[\mathsf{P}, \alpha, u, v, w]$ a Σ formula $F_{\mathfrak{A}}[\alpha, u, v, w]$, which is $\Delta(\mathsf{KP})$, such that KP – and thus also $\mathsf{KPi} + (\mathbf{V}{=}\mathbf{L})$ – proves

$$(*) \qquad F_{\mathfrak{A}}[\alpha, a, b, c] \leftrightarrow (a, b, c \in L_{\alpha+\omega} \wedge \mathfrak{A}[(\exists \xi < \alpha) F_{\mathfrak{A}}[\xi, .], \alpha, a, b, c])$$

for all ordinals α and all sets a, b, c. Definition 16 is similar to a definition in Jäger and Zumbrunnen [17], but clauses (10)–(12) are new. They entail the following properties of the formula $F_{\mathfrak{A}}[\alpha, u, v, w]$.

Lemma 17. *For all a, b, c, and α we can prove in* KP*:*

1. $F_{\mathfrak{A}}[\alpha, \langle \widehat{\mathbf{reg}}, a\rangle, b, c] \to (c = \widehat{\top} \vee c = \widehat{\bot})$.
2. $(a \in L_{\alpha+1} \setminus L_\alpha \wedge b \in L_{\alpha+\omega}) \to$
   $(F_{\mathfrak{A}}[\alpha, \langle \widehat{\mathbf{reg}}, a\rangle, b, \widehat{\top}] \vee F_{\mathfrak{A}}[\alpha, \langle \widehat{\mathbf{reg}}, a\rangle, b, \widehat{\bot}])$.
3. $F_{\mathfrak{A}}[\alpha, \langle \widehat{\mathbf{reg}}, a\rangle, b, \widehat{\top}] \to (a \in L_{\alpha+1} \setminus L_\alpha \wedge b \in L_{\alpha+\omega})$.
4. $F_{\mathfrak{A}}[\alpha, \langle \widehat{\mathbf{reg}}, a\rangle, b, \widehat{\top}] \to$
   $(a = L_\alpha \wedge \mathsf{Ad}_{\mathbf{L}}[L_\alpha] \wedge b = \{\langle x, y, z\rangle \in a : (\exists \xi < \alpha) F_{\mathfrak{A}}[\xi, x, y, z]\})$.
5. $(a \in L_{\alpha+1} \setminus L_\alpha \wedge b \in L_\beta \setminus L_{\alpha+\omega}) \to$
   $(F_{\mathfrak{A}}[\beta, \langle \widehat{\mathbf{reg}}, a\rangle, b, \widehat{\bot}] \wedge \neg \exists \xi F_{\mathfrak{A}}[\xi, \langle \widehat{\mathbf{reg}}, a\rangle, b, \widehat{\top}])$.

Proof. Because of (*) the first four assertions follow immediately from Definition 16; observe that only the clauses (10)–(12) of this definition can apply. If $a \in L_{\alpha+1} \setminus L_\alpha$ and $b \in L_\beta \setminus L_{\alpha+\omega}$, then $\alpha < \beta$ and $F_{\mathfrak{A}}[\beta, \langle \widehat{\mathbf{reg}}, a\rangle, b, \widehat{\bot}]$ follows because of (*) and clause (12). Moreover, from the third assertion we conclude that there exists no ξ such that $F_{\mathfrak{A}}[\xi, \langle \widehat{\mathbf{reg}}, a\rangle, b, \widehat{\top}]$. Thus we also have the fifth assertion. $\Box$

To interpret the application $(uv = w)$ of $\mathsf{OST(LR)}$ we finally set

$$Ap[u, v, w] := \exists \xi F_{\mathfrak{A}}[\xi, u, v, w].$$

Clearly, an application relation has to be functional in its third argument, but following Jäger and Zumbrunnen [17] we can easily verify that

$$\mathsf{KP} \vdash \forall x \forall y \forall z_1 \forall z_2 (Ap[x,y,z_1] \wedge Ap[x,y,z_2] \;\rightarrow\; z_1 = z_2),$$

and thus this important property is satisfied in KP and $\mathsf{KPi} + (\mathbf{V}{=}\mathbf{L})$. Since $F_{\mathfrak{A}}[\alpha,u,v,w]$ is a Σ formula of $\mathcal{L}^*$, $Ap[u,v,w]$ is upward persistent, and since $F_{\mathfrak{A}}[\alpha,u,v,w]$ is also $\Delta(\mathsf{KP})$, Corollary 14 implies an important property with respect to relativization to $\mathbf{L}$-admissible sets.

Lemma 18. *In* KP *we can prove:*

1. $(d_1 \subseteq d_2 \;\wedge\; Ap^{d_1}[a,b,c]) \;\rightarrow\; Ap^{d_2}[a,b,c]$.
2. $Ap^{d}[a,b,c] \;\rightarrow\; Ap[a,b,c]$.
3. $(\mathsf{Ad}_{\mathbf{L}}[d] \;\wedge\; \alpha,a,b,c \in d) \;\rightarrow\; (F_{\mathfrak{A}}[\alpha,a,b,c] \leftrightarrow F^{d}_{\mathfrak{A}}[\alpha,a,b,c])$.

In interpreting $\mathsf{OST(LR)}$ in $\mathsf{KPi} + (\mathbf{V}{=}\mathbf{L})$ we also have to handle assertions $\mathsf{Reg}(d,e)$ stating that the set d is regular with respect to application in the sense of e. We do this within $\mathsf{KPi} + (\mathbf{V}{=}\mathbf{L})$ by claiming d to be an admissible set L_α and by collecting in e those triples that satisfy Ap relativized to d,

$$Ad^*[d,e] \;:=\; \mathsf{Ad}_{\mathbf{L}}[d] \;\wedge\; e = \{\langle x,y,z\rangle \in d : Ap^{d}[x,y,z]\}.$$

The clauses (10)–(12) of Definition 16 take care of the constant $\mathbf{reg}$, and the following lemma tells us that Ad^* and $\widehat{\mathbf{reg}}$ appropriately reflect the relation symbol Reg and the constant $\mathbf{reg}$, respectively, if application is treated in the sense of Ap.

Lemma 19. *For all d, e, and α we can prove in* $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$*:*

1. $Ap[\langle \widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}] \vee Ap[\langle \widehat{\mathbf{reg}}, d\rangle, e, \widehat{\bot}]$.
2. $(\mathsf{Ad}_{\mathbf{L}}[L_\alpha] \;\wedge\; d,e \in L_\alpha) \;\rightarrow\; (Ap^{L_\alpha}[\langle \widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}] \;\vee\; Ap^{L_\alpha}[\langle \widehat{\mathbf{reg}}, d\rangle, e, \widehat{\bot}])$.
3. $Ad^*[L_\alpha, e] \;\rightarrow\; F_{\mathfrak{A}}[\alpha, \langle \widehat{\mathbf{reg}}, L_\alpha\rangle, e, \widehat{\top}]$.
4. $F_{\mathfrak{A}}[\alpha, \langle \widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}] \;\rightarrow\; (d = L_\alpha \;\wedge\; Ad^*[d,e])$.
5. $Ap[\langle \widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}] \;\leftrightarrow\; Ad^*[d,e]$.
6. $(\mathsf{Ad}_{\mathbf{L}}[L_\alpha] \;\wedge\; d,e \in L_\alpha) \;\rightarrow\; (Ap[\langle \widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}] \;\leftrightarrow\; Ap^{L_\alpha}[\langle \widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}]$.
7. $(\mathsf{Ad}_{\mathbf{L}}[L_\alpha] \;\wedge\; d,e \in L_\alpha) \;\rightarrow\; (Ap^{L_\alpha}[\langle \widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}] \;\leftrightarrow\; Ad^*[d,e])$.

Proof. Because of (**V**=**L**) we know that there exist ordinals α and β such that $d \in L_{\alpha+1} \setminus L_\alpha$ and $e \in L_\beta$. Hence the first assertion follows from Lemma 17.

To prove the second assertion, assume $\mathsf{Ad}_{\mathbf{L}}[L_\alpha]$ and $d, e \in L_\alpha$. Then there exists an ordinal $\gamma < \alpha$ for which $d \in L_{\gamma+1} \setminus L_\gamma$, and we distinguish the following two cases:

(i) $e \in L_{\gamma+\omega}$. By Lemma 17 we then have

$$F_{\mathfrak{A}}[\gamma, \langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}] \vee F_{\mathfrak{A}}[\gamma, \langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\bot}]$$

and thus Lemma 18 yields

$$F_{\mathfrak{A}}^{L_\alpha}[\gamma, \langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}] \vee F_{\mathfrak{A}}^{L_\alpha}[\gamma, \langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\bot}].$$

Consequently, we have

$$Ap^{L_\alpha}[\langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}] \vee Ap^{L_\alpha}[\langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\bot}].$$

(ii) $e \notin L_{\gamma+\omega}$. Now we choose an ordinal $\delta < \alpha$ for which $e \in L_\delta$. Lemma 17 now implies

$$F_{\mathfrak{A}}[\delta, \langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\bot}].$$

Hence an application of Lemma 18 gives us

$$F_{\mathfrak{A}}^{L_\alpha}[\delta, \langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\bot}],$$

and $Ap^{L_\alpha}[\langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\bot}]$ is an immediate consequence.

In both cases (i) and (ii) we have what we want, and the second assertion is proved.

To show the third assertion we assume $Ad^*[L_\alpha, e]$. Then we have $\mathsf{Ad}_{\mathbf{L}}[L_\alpha]$ and

$$e = \{\langle x, y, z\rangle \in L_\alpha : (\exists \xi < \alpha) F_{\mathfrak{A}}^{L_\alpha}[\xi, x, y, z]\}.$$

Obviously, $e \in L_{\alpha+\omega}$. Furthermore, by applying Lemma 18 we can conclude that

$$e = \{\langle x, y, z\rangle \in L_\alpha : (\exists \xi < \alpha) F_{\mathfrak{A}}[\xi, x, y, z]\}.$$

In view of clause (10) of Definition 16 and equivalence (*) above this implies $F_{\mathfrak{A}}[\alpha, \langle\widehat{\mathbf{reg}}, L_\alpha\rangle, e, \widehat{\top}]$.

Now we turn to the fourth assertion and assume $F_{\mathfrak{A}}[\alpha, \langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}]$. Because of equivalence (*) this implies

$$\langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top} \in L_{\alpha+\omega} \wedge \mathfrak{A}[(\exists \xi < \alpha) F_{\mathfrak{A}}[\xi, .], \alpha, \langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}].$$

Only clause (10) of Definition 16 applies, yielding that $d = L_\alpha$, $\mathsf{Ad}_{\mathbf{L}}[d]$, and

$$e = \{\langle x, y, z\rangle \in d : (\exists \xi < \alpha) F_{\mathfrak{A}}[\xi, x, y, z]\}.$$

As in the proof of the previous assertion we make use of Lemma 18 and conclude that

$$e = \{\langle x, y, z\rangle \in d : (\exists \xi < \alpha) F^d_{\mathfrak{A}}[\xi, x, y, z]\}.$$

Thus we have $Ad^*[d, e]$. This completes the proof of the fourth assertion. The fifth assertion is an immediate consequence of the third and the fourth.

For proving the sixth assertion we assume $\mathsf{Ad}_{\mathbf{L}}[L_\alpha]$ and $d, e \in L_\alpha$. Then $Ap^{L_\alpha}[\langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}]$ implies $Ap[\langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}]$ according to Lemma 18. For the converse direction we make use of the fact that there exists a $\beta < \alpha$ such that $d \in L_{\beta+1} \setminus L_\beta$. From $Ap[\langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}]$, the equivalence (*), and Definition 16 we thus obtain $F_{\mathfrak{A}}[\beta, \langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}]$, hence $F^{L_\alpha}_{\mathfrak{A}}[\beta, \langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}]$ according to Lemma 18. This means that we also have $Ap^{L_\alpha}[\langle\widehat{\mathbf{reg}}, d\rangle, e, \widehat{\top}]$, finishing the proof of assertion six. Assertions five and six imply the seventh assertion. □

The next lemma provides further indication that Ad^* is the adequate analogue in the context of admissible sets of the $\mathsf{OST(LR)}$ notion of relativized regularity.

Lemma 20. *In* KP *we can prove:*

1. $Ad^*[d, e] \;\rightarrow\; Tran[d] \wedge e \subseteq d^3$.
2. $Ad^*[d_1, e_1] \wedge Ad^*[d_2, e_2] \;\rightarrow\; d_1 \in d_2 \vee d_1 = d_2 \vee d_2 \in d_1$.
3. $Ad^*[d_1, e_1] \wedge Ad^*[d_2, e_2] \wedge d_1 \in d_2 \;\rightarrow\; e_1 \in d_2 \wedge e_1 \subseteq e_2$.
4. $Ad^*[d, e] \wedge \langle a, b, c\rangle \in e \;\rightarrow\; Ap[a, b, c]$.

Proof. The first and the second assertion follow immediately from the properties of admissible sets. Since admissibles satisfy Δ_0 separation and because of Lemma 18 we have the third assertion. Lemma 18 also implies the fourth assertion. □

Now we give the translation of the formulas of $\mathcal{L}^\circ$ into formulas of $\mathcal{L}^*$. As in [17] we first introduce for each $\mathcal{L}^\circ$ term t an $\mathcal{L}^*$ formula $Val_t[u]$, formalizing that t has the value u if the application of $\mathsf{OST(LR)}$ is interpreted by Ap. In addition, we also consider a relativized version $Val_t[d, u]$ in order to state that term t has the value u provided that application is interpreted by Ap^d; in all relevant cases d will be admissible.

Definition 21. *For each $\mathcal{L}^\circ$ term r and variables u and d not occurring in r we introduce $\mathcal{L}^*$ formulas $Val_r[u]$ and $Val_r[d,u]$ that are inductively defined as follows:*

1. *If r is a variable or the constant ω, then $Val_r[u]$ and $Val_r[d,u]$ are the formula $(r = u)$.*

2. *If r is another constant, then $Val_r[u]$ and $Val_r[d,u]$ are the formula $(\widehat{r} = u)$.*

3. *If r is the term (st), then we set (for x and y chosen so that they do not occur in r)*

$$\begin{aligned} Val_r[u] &:= \exists x \exists y (Val_s[x] \wedge Val_t[y] \wedge Ap[x,y,u]), \\ Val_r[d,u] &:= (\exists x, y \in d)(Val_s[d,x] \wedge Val_t[d,y] \wedge Ap^d[x,y,u]). \end{aligned}$$

Notice that for every term r of $\mathcal{L}^\circ$ its translation formula $Val_r[u]$ is a Σ formula of $\mathcal{L}^*$; in general, it is not $\Delta(\mathsf{KP})$. The translation formula $Val_r[d,u]$ is the restriction of $Var_r[u]$ to d and thus a Δ_0 formula of $\mathcal{L}^*$. The following observation is proved by induction on the buildup of r and an immediate consequence of the functionality of $Ap[u,v,w]$ in its third argument and of Lemma 18.

Lemma 22. KP *proves for all $\mathcal{L}^\circ$ terms r and all variables d:*

1. *$\forall x \forall y (Val_r[x] \wedge Val_r[y] \;\rightarrow\; x = y)$.*

2. *$\forall x (Val_r[d,x] \rightarrow Val_r[x])$.*

3. *$\forall x \forall y (Val_r[d,x] \wedge Val_r[d,y] \;\rightarrow\; x = y)$.*

Clearly, the values of terms also satisfy the following substitution property. Again, its proof is by induction on the buildup of r.

Lemma 23. *If all variables of the $\mathcal{L}^\circ$ term r come from the list $u_1, \ldots, u_n$ and if s is the $\mathcal{L}^\circ$ term $r[t_1, \ldots, t_n / u_1, \ldots, u_n]$, then* KP *proves*

$$\bigwedge_{i=1}^{n} Val_{t_i}[u_i] \;\rightarrow\; \forall x (Val_r[x] \;\leftrightarrow\; Val_s[x]).$$

The above treatments of the application of $\mathsf{OST}(\mathsf{LR})$ determine canonical translations of the formulas of $\mathcal{L}^\circ$ into formulas of $\mathcal{L}^*$.

Definition 24. *The translations of an $\mathcal{L}^\circ$ formula A into the $\mathcal{L}^*$ formula A^* and its relativized version $A^{[d]}$ are inductively defined as follows.*

1. *If A is the atomic formula $(r = s)$ we set:*

$$\begin{aligned} A^* &:= \exists x(\mathit{Val}_r[x] \wedge \mathit{Val}_s[x]), \\ A^{[d]} &:= (\exists x \in d)(\mathit{Val}_r[d, x] \wedge \mathit{Val}_s[d, x]). \end{aligned}$$

2. *If A is the atomic formula $(r \in s)$ we set:*

$$\begin{aligned} A^* &:= \exists x \exists y(\mathit{Val}_r[x] \wedge \mathit{Val}_s[y] \wedge x \in y), \\ A^{[d]} &:= (\exists x, y \in d)(\mathit{Val}_r[d, x] \wedge \mathit{Val}_s[d, y] \wedge x \in y). \end{aligned}$$

3. *If A is the atomic formula $(r\downarrow)$ we set:*

$$A^* := \exists x\, \mathit{Val}_r[x] \quad \text{and} \quad A^{[d]} := (\exists x \in d)\, \mathit{Val}_r[d, x].$$

4. *If A is the atomic formula $\mathsf{Reg}(r, s)$ we set:*

$$\begin{aligned} A^* &:= \exists x \exists y(\mathit{Val}_r[x] \wedge \mathit{Val}_s[y] \wedge \mathit{Ad}^*[x, y]), \\ A^{[d]} &:= (\exists x, y \in d)(\mathit{Val}_r[d, x] \wedge \mathit{Val}_s[d, y] \wedge \mathit{Ad}^*[x, y]). \end{aligned}$$

5. *If A is the formula $\neg B$ we set:*

$$A^* := \neg B^* \quad \text{and} \quad A^{[d]} := \neg B^{[d]}.$$

6. *If A is the formula $(B \vee C)$ we set:*

$$A^* := (A^* \vee B^*), \quad \text{and} \quad A^{[d]} := (A^{[d]} \vee B^{[d]}).$$

7. *If A is the formula $(\exists x \in r)B$ we set:*

$$\begin{aligned} A^* &:= \exists y(\mathit{Val}_r[y] \wedge (\exists x \in y)B^*), \\ A^{[d]} &:= (\exists y \in d)(\mathit{Val}_r[d, y] \wedge (\exists x \in y)B^{[d]}). \end{aligned}$$

8. *If A is the formula $\exists x B$ we set:*

$$A^* := \exists x B^*, \quad \text{and} \quad A^{[d]} := (\exists x \in d)B^{[d]}.$$

If d is admissible, then $A^{[d]}$ is equivalent to the restriction of A^* to d. Before turning to the proof that this $*$-translation provides an embedding of $\mathsf{OST}(\mathsf{LR})$ into $\mathsf{KPi}+(\mathbf{V}{=}\mathbf{L})$, we compile some useful properties concerning substitutions of terms in $*$-translation and the relationship between $*$-translations and $[d]$-translations.

Lemma 25. *Let A be a formula of $\mathcal{L}^\circ$ with at most the variables $u_1, \dots, u_n$ free, let $t_1, \dots, t_n$ be a list of $\mathcal{L}^\circ$ terms, and set $B := A[t_1, \dots, t_n/u_1, \dots, u_n]$. Then* KP *proves that*

$$\bigwedge_{i=1}^{n} Val_{t_i}[u_i] \;\rightarrow\; (A^* \leftrightarrow B^*).$$

Furthermore, if A is a formula of $\mathcal{L}$, then KP *even proves*

$$\bigwedge_{i=1}^{n} Val_{t_i}[u_i] \;\rightarrow\; (A \leftrightarrow B^*).$$

These assertions are established by induction on A, using Lemma 23 in the case of atomic formulas. The following lemma is proved by straightforward induction on the complexity of the terms r and the formulas A, respectively. The previous lemma is useful for handling the first assertion.

Lemma 26.

1. *Let r be an $\mathcal{L}^\circ$ term whose variables are from the list $\vec{u}$ and let d, e be variables different from $\vec{u}$. Then* KP *proves that*

$$(Ad^*[d,e] \wedge \vec{u} \in d) \;\rightarrow\; ((r \,\partial\, e)^* \;\leftrightarrow\; (\exists x \in d)\, Val_r[d,x]).$$

2. *Let A be an $\mathcal{L}^\circ$ formula with at most $\vec{u}$ free and let d, e be variables different from $\vec{u}$. If we set $B := A^{(d,e)}$, then* KP *proves that*

$$Ad^*[d,e] \wedge \vec{u} \in d \;\rightarrow\; (B^* \;\leftrightarrow\; A^{[d]}).$$

In Jäger and Zumbrunnen [17] we have interpreted an operational set theory into a theory of admissible sets. There we have been working with an inductive definition for translating application very similar to Definition 16. What is new here are the relation symbol Reg and the constant **reg** with their corresponding axioms plus the axiom ($\mathsf{Lim\text{-}Reg}$).

Theorem 27. *If A is an applicative axiom, a basic set-theoretic axiom, a logical operations axiom, or a set-theoretic operations axiom of* OST *with at most the variables $\vec{u}$ free, then* $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ *proves for all variables d, e different from $\vec{u}$ that*

$$Ad^*[d,e] \wedge \vec{u} \in d \ \rightarrow \ A^{[d]}.$$

Proof. For the treatment of the logical operations axiom (L3) see Lemma 19. In all other cases we only have to follow [12, 17]. □

Corollary 28. *If A is any axiom (Reg1)–(Reg4), then* $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ *proves its translation A^*.*

Proof. Clearly, for any variables d and e, $\mathsf{Reg}(d,e)^*$ is (logically equivalent to) the formula $Ad^*[d,e]$. Hence Lemma 20 yields our assertion for the axioms (Reg1), (Reg3), and (Reg4). To prove our claim for (Reg2), let A be an applicative axiom, a logical operations axiom or a set-theoretic operations axiom with at most $\vec{u}$ free. In view of the previous theorem and Lemma 26 we know that $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ proves

$$Ad^*[d,e] \wedge \vec{u} \in d \ \rightarrow \ B^*$$

for all d, e not from $\vec{u}$, where B stands for the $\mathcal{L}^\circ$ formula $A^{(d,e)}$. From this it follows immediately that $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L})$ proves A^* for all instances A of (Reg2). □

Theorem 29. *If A is any axiom of* $\mathsf{OST}(\mathsf{LR})$*, then* $\mathsf{KPi} + (\mathbf{V}{=}\mathbf{L})$ *proves its translation A^*.*

Proof. As in the proof of Theorem 27 we observe that with exception of the logical operations axiom (L3) the translations of all applicative axioms, basic set-theoretic axioms, logical operations axioms, and set-theoretic operations axioms can be proved in $\mathsf{KPi} + (\mathbf{V}{=}\mathbf{L})$ as in [12, 17] and that the provability of the translation of the logical operations axiom (L3) follows from Lemma 19. For the translations of the axioms (Reg1) – (Reg4) see the previous corollary.

Finally, if A is the axiom (Lim-Reg), then A^* is equivalent to the formula

$$\forall x \exists y \exists z (x \in y \wedge Ad^*[y,z]).$$

So given an arbitrary set x, Lemma 12 implies the existence of an $\mathbf{L}$-admissible d such that $x \in d$. Furthermore, by Δ_0 separation there also exists the set

$$z \ = \ \{\langle u,v,w\rangle \in d : Ap^d[u,v,w]\},$$

and thus we have $Ad^*[d,z]$. Hence also the translation of the axiom (Lim-Reg) is provable in $\mathsf{KPi} + (\mathbf{V}{=}\mathbf{L})$. □

From this theorem we conclude that the system $\mathsf{OST}(\mathsf{LR})$ is interpretable in the theory $\mathsf{KPi} + (\mathbf{V}{=}\mathbf{L})$. Moreover, $\mathsf{KPi} + (\mathbf{V}{=}\mathbf{L})$ is conservative over KPi for formulas which are absolute with respect to KP. Together with Theorem 15 we thus obtain the following final result.

Corollary 30. *The two theories* $\mathsf{OST}(\mathsf{LR})$ *and* KPi *are proof-theoretically equivalent.*

In this paper a new form of relativizing operational set theory has been introduced and, based on that, a natural operational set theory of the same proof-theoretic strength as the theory KPi has been formulated and analyzed. The heart of the matter in interpretating $\mathsf{OST}(\mathsf{LR})$ into KPi is giving an inductive definition of the application relation. By restricting this application relation to suitable sets we then can deal with relativized regularity.

This is just one specific application of this new way of relativizing operational set theory. A uniform version of the limit axiom ($\mathsf{Lim\text{-}Reg}$) will be discussed elsewhere.

In future work various large cardinal notions will be reexamined under the perspective this new form of relativizing operational set theory, for example by adding power set and unbounded existential quantification.

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Address
Gerhard Jäger, Institut für Informatik, Universität Bern, Neubrückstrasse 10, CH-3012 Bern, Switzerland, jaeger@inf.unibe.ch