# ON RANDOMLY SPACED OBSERVATIONS AND CONTINUOUS TIME RANDOM WALKS 

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#### Abstract

We consider random variables observed at arrival times of a renewal process, which possibly depends on those observations and has regularly varying steps with infinite mean. Due to the dependence and heavy tailed steps, the limiting behavior of extreme observations until a given time $t$ tends to be rather involved. We describe this asymptotics and generalize several partial results which appeared in this setting. In contrast to the earlier studies, our analysis is based in the point processes theory. The theory is applied to determine the asymptotic distribution of maximal excursions and sojourn times for continuous time random walks. Keywords: extreme value theory; point process; renewal process; continuous time random walk; excursion; sojourn time 2010 Mathematics Subject Classification: Primary 60G70


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## 1. Introduction

A sequence of observations collected at arrival times of a renewal process represents a popular modeling framework in applied probability. It underlies many standard stochastic models, ranging from risk theory and engineering to theoretical physics. The asymptotic distribution of the maximum of such observations until a given time $t$

[^0]is well understood in the case of the renewal process with finite mean interarrival times, see Basrak and Špoljarić [3] and references therein. Shanthikumar and Sumita in [26] and Anderson in [1] considered the problem in the infinite mean case, allowing certain degree of dependence between the observations and the interarrival times. In such a setting, the asymptotic theory for partial maxima tends to be more complicated.

It is often essential to understand the distribution of all extreme observations and not merely the partial maximum until a given time. Therefore, we aim to characterize the limiting behavior of all upper order statistics in a sequence $\left(X_{n}\right)$ up to a time $\tau(t)$, where $(\tau(t))$ represents the renewal process generated by a sequence of nonnegative (and nontrivial) random variables $\left(Y_{n}\right)$, i.e.

$$
\begin{equation*}
\tau(t)=\inf \left\{k: Y_{1}+\cdots+Y_{k}>t\right\}, \quad \text { for } t \geq 0 \tag{1.1}
\end{equation*}
$$

Throughout we shall assume that $\left(X_{n}, Y_{n}\right), n \geq 1$, form iid pairs and that the distribution of $X_{1}$ belongs to the maximum domain of attraction (MDA for short) of one of the three extreme value distributions, denoted by $G$. Moreover, the interarrival times are assumed to be regularly varying with index $\alpha \in(0,1)$ (as in [1, 16, 19). In particular, $Y_{i}$ 's have infinite mean and belong to the MDA of a Fréchet distribution themselves.

To explain the limiting behavior of all large values in the sequence $\left(X_{n}\right)$, which arrive before time $t$, we rely on the theory of point processes. Such an approach seems to be new in this context. It does not only yield more general results, but we believe, it provides a very good insight into why all the previously established results actually hold. Moreover, we relax restrictions used in the literature concerning dependence between the observations and interarrival times.

We apply our results to study the continuous time random walk (CTRW), introduced by Montroll and Weiss [18. It is essentially a random walk subordinated to a renewal process. It has numerous applications in physics and has been used to model various phenomena in finance, hydrology, quantum transport theory and seismology. For an overview of the literature on the theory and various applications of CTRW we refer
to [24]. Under mild assumptions, excursions of such a walk are regularly varying with index $1 / 2$. Hence, our theory applies, and one can determine the limiting distribution of extremely long excursions and sojourn times at level zero of a CTRW.

The paper is organized as follows: notation and auxiliary results are introduced in section 2 In section 3 we present the limiting distribution of all extreme observations until a given time, discussing asymptotic tail independence and asymptotic full tail dependence between the observations and interarrival times in detail. Our main result extends previously mentioned results in this context, as well as some more recent results in Meerschaert and Stoev [17] and Pancheva et al. [19] for instance. In the asymptotic full tail dependence case. our result could be applied to study the longest intervals of the renewal process $\tau$ itself, which is the subject of recent papers by Goldrèche et al. [12] and Basrak [2]. In section [ 4 we apply the main result to study the longest sojourn times and the longest excursions of a continuous time random walk. In particular, this section extends the analysis of asymptotic distribution for the ranked lengths of excursions in the simple symmetric random walk given by Csáki and Hu [5.

## 2. Preliminaries

As already mentioned in the introduction, we assume that the observations $X_{i}$ belong to MDA of some extreme value distributions, denoted by $G$. Because of the correspondence between MDA's of Fréchet and Weibull distributions, we discuss only observations in the Gumbel (which is denoted by $G=\Lambda$ ) and Fréchet ( $G=\Phi_{\beta}$, for $\beta>0$ ) MDA's in detail (see subsection 3.3.2 in Embrechts et al. (9). In particular, there exist functions $a(t)$ and $b(t)$ such that

$$
\begin{equation*}
t P\left(X_{1}>a(t) x+b(t)\right) \rightarrow-\log G(x) \tag{2.1}
\end{equation*}
$$

as $t \rightarrow \infty$ (cf. Resnick [22]).
Recall next that we assumed that the renewal steps $Y$ have regularly varying
distribution of infinite mean with index $\alpha \in(0,1)$ (we denote this by $Y \sim \operatorname{Reg} \operatorname{Var}(\alpha)$ ). In such a case, it is well known (see Feller [11) that there exists a strictly positive sequence $\left(d_{n}\right)$ such that

$$
d_{n}^{-1}\left(Y_{1}+\cdots+Y_{n}\right) \xrightarrow{d} S_{\alpha}
$$

where random variable $S_{\alpha}$ has the stable law with the index $\alpha$, scale parameter $\sigma=1$, skewness parameter $\beta=1$ and shift parameter $\mu=0$. In particular, $S_{\alpha}$ is strictly positive a.s. The sequence $\left(d_{n}\right)$ can be chosen such that

$$
\begin{equation*}
n\left(1-F_{Y}\left(d_{n}\right)\right) \rightarrow 1 \tag{2.2}
\end{equation*}
$$

as $n \rightarrow \infty$, where $F_{Y}$ denotes cdf of $Y_{1}$. Denote $d(t)=d_{\lfloor t\rfloor}$, for $t \geq 0$, with $d_{0}=0$ and observe that the function $d$ is regularly varying with index $1 / \alpha$. Consider the partial sum process

$$
\begin{equation*}
T(t)=\sum_{i=1}^{\lfloor t\rfloor} Y_{i} \tag{2.3}
\end{equation*}
$$

with $T(t)=0$ for $0 \leq t<1$. It is well known that

$$
\begin{equation*}
\left(\frac{T(t c)}{d(t)}\right)_{c \geq 0} \quad \stackrel{d}{\longrightarrow}\left(S_{\alpha}(c)\right)_{c \geq 0} \tag{2.4}
\end{equation*}
$$

as $t \rightarrow \infty$, in a space of càdlàg functions $D[0, \infty)$ endowed with Skorohod $J_{1}$ topology (see Skorohod [28] or Resnick [23]). The limiting process $\left(S_{\alpha}(c)\right)_{c \geq 0}$ is an $\alpha$-stable process with strictly increasing sample paths.

Recall that for a function $z \in D([0, \infty),[0, \infty))$, the right-continuous generalized inverse is defined by the relation

$$
z^{\leftarrow}(u)=\inf \{s \in[0, \infty): z(s)>u\}, \quad u \geq 0
$$

The following lemma shows that under certain conditions, the convergence of functions $z_{t}$ to $z$ in $J_{1}$ topology implies the convergence of corresponding generalized
inverses in the same topology. The content of the lemma seems to be known (cf. Resnick [23, p. 266] and Theorem 7.2 in Whitt [29]), for convenience we add a short proof.

Lemma 1. Suppose that $z_{t} \in D([0, \infty),[0, \infty))$ are non-decreasing functions for all $t \geq 0$. Further, let $z \in D([0, \infty),[0, \infty))$ be strictly increasing to infinity. If $z_{t} \xrightarrow{J_{1}} z$, as $t \rightarrow \infty$, then $z_{t}^{\leftarrow} \xrightarrow{J_{1}} z^{\leftarrow}$, as $t \rightarrow \infty$, in $D([0, \infty),[0, \infty))$ as well.

Proof. Since $z$ is strictly increasing, $z^{\leftarrow}$ is continuous. According to Theorem 2.15 in Jacod and Shiryaev [13, Chapter VI], it sufficies to show $z_{t}^{\leftarrow}(u) \rightarrow z^{\leftarrow}(u)$, for all $u \geq 0$. One can prove this by showing that, for an arbitrary fixed $u \geq 0$, the function $y \mapsto y^{\leftarrow}(u)$ is continuous at $z$. However, this follows at once from Proposition 2.11 in [13, Chapter VI]. Note that this proposition is actually proved using the leftcontinuous generalized inverse, but under our assumptions, the proof can be easily adapted to the right-continuous case.

According to Seneta [25] (see also [16, 23]), there exists a function $\widetilde{d}$ which is an asymptotic inverse of $d$, that is

$$
\begin{equation*}
d(\widetilde{d}(t)) \sim \tilde{d}(d(t)) \sim t \tag{2.5}
\end{equation*}
$$

as $t \rightarrow \infty$. Moreover, $\widetilde{d}$ is known to be a regularly varying function with index $\alpha$.
Denote by

$$
\begin{equation*}
W_{\alpha}(c)=\inf \left\{x: S_{\alpha}(x)>c\right\}=S_{\alpha}^{\leftarrow}(c), \quad c \geq 0 \tag{2.6}
\end{equation*}
$$

the first hitting-time process of the $\alpha$-stable subordinator $\left(S_{\alpha}(t)\right)_{t \geq 0}$. As we shall see in the sequel (see (3.7)), Lemma 1 together with (2.4) and (2.5) implies

$$
\begin{equation*}
\left(\frac{\tau(t c)}{\widetilde{d}(t)}\right)_{c \geq 0} \xrightarrow{d}\left(W_{\alpha}(c)\right)_{c \geq 0} \tag{2.7}
\end{equation*}
$$

in $D([0, \infty),[0, \infty))$ endowed with $J_{1}$ topology. For an $\alpha$-stable process $S_{\alpha}$ and fixed $c \geq 0$, the hitting-time $W_{\alpha}(c)$ has the Mittag-Leffler distribution.

## 3. Extremes of randomly spaced observations

With observations $\left(X_{n}\right)$ and interarrival times $\left(Y_{n}\right)$ dependent, it can be difficult to describe explicitly limiting behavior of observations $X_{i}$ with index $i \leq \tau(t)$. Here, we show that this can be done using the convergence of suitably chosen point processes based on iid random vectors $\left(X_{n}, Y_{n}\right)$. Convergence of this type is well understood in the extreme value theory, see for instance Resnick [22].

Again, we assume that $X_{1} \in \operatorname{MDA}(G)$ and $Y_{1} \sim \operatorname{Reg} \operatorname{Var}(\alpha), 0<\alpha<1$. For $t \geq 0$ we define point process $N_{t}$ as

$$
\begin{equation*}
N_{t}=\sum_{i \geq 1} \delta_{\left(\frac{i}{d(t)}, \widetilde{X}_{t, i}, \widetilde{Y}_{t, i}\right)} \tag{3.1}
\end{equation*}
$$

with $N_{0}=0$, where

$$
\begin{equation*}
\widetilde{Y}_{t, i}=\frac{Y_{i}}{t} \tag{3.2}
\end{equation*}
$$

and $\widetilde{X}_{t, i}$ is defined by

$$
\begin{equation*}
\widetilde{X}_{t, i}=\frac{X_{i}-\widetilde{b}(t)}{\widetilde{a}(t)} \tag{3.3}
\end{equation*}
$$

with $\widetilde{a}(t):=a(\widetilde{d}(t)), \widetilde{b}(t):=b(\widetilde{d}(t))$ where $a(t)$ and $b(t)$ satisfy (2.1) and $\widetilde{d}(t)$ is defined in (2.5).

The state space of $N_{t}$ depends on the MDA of the observations (see (2.1)), it can be written as $[0, \infty) \times \mathbb{E}$ where

$$
\mathbb{E}= \begin{cases}{[-\infty, \infty] \times[0, \infty] \backslash\{(-\infty, 0)\},} & X_{1} \in \operatorname{MDA}(\Lambda) \\ {[0, \infty] \times[0, \infty] \backslash\{(0,0)\},} & X_{1} \in \operatorname{MDA}\left(\Phi_{\beta}\right)\end{cases}
$$

Throughout we use the standard vague topology on the space of point measures $M_{p}([0, \infty) \times \mathbb{E})$, see Resnick [22].

Recall that

$$
\begin{equation*}
\widetilde{d}(t) P\left(\left(\tilde{X}_{t, i}, \tilde{Y}_{t, i}\right) \in \cdot\right) \xrightarrow{v} \mu_{0}(\cdot), \quad \text { as } t \rightarrow \infty \tag{3.4}
\end{equation*}
$$

is necessary and sufficient for

$$
N_{t} \xrightarrow{d} N,
$$

as $t \rightarrow \infty$, where $N$ is Poisson random measure with mean measure $\lambda \times \mu_{0}$, denoted by $\operatorname{PRM}\left(\lambda \times \mu_{0}\right)$ (see Proposition 3.21 in Resnick [22] or Theorem 1.1.6 in de Haan and Ferreira (6]). It turns out that (3.4) is immediate when one considers the observations independent of interarrival times. The same holds in the important special case when they are exactly equal. In that case, one can prove an interesting invariance principle concerning extremely long steps of a renewal process, see Basrak [2] and references therein.

Theorem 3.1. Assume that (3.4) holds, then

$$
\begin{equation*}
\left(N_{t}, \frac{T(\widetilde{d}(t) \cdot)}{t}, \frac{\tau(t)}{\widetilde{d}(t)}\right) \stackrel{d}{\longrightarrow}\left(N, S_{\alpha}(\cdot), W_{\alpha}\right), \tag{3.5}
\end{equation*}
$$

as $t \rightarrow \infty$, in the product space $M_{p} \times D[0, \infty) \times \mathbb{R}$ and the corresponding product topology (of vague, $J_{1}$ and Euclidean topologies), where $N$ is a $\operatorname{PRM}\left(\lambda \times \mu_{0}\right)$, while $S_{\alpha}$ and $W_{\alpha}=W_{\alpha}(1)$ denote the $\alpha$-stable subordinator and the first passage time from (2.4) and (2.7) respectively.

Remark 1. If one denotes the limiting point process above as $N=\sum_{i} \delta_{\left(T_{i}, P_{i}, Q_{i}\right)}$, then $S_{\alpha}$ has representation

$$
S_{\alpha}(t)=\sum_{T_{i} \leq t} Q_{i}
$$

while $W_{\alpha}(c)$ is the inverse of the increasing process $S_{\alpha}$ as in (2.6). Actually, due to lemma 11 we prove a stronger result than the theorem above claims. Namely, under the same assumption, (3.5) holds jointly with (2.7).

Proof. The first part of the proof is standard, and essentially follows the lines of the
proof of Theorem 7.1 in Resnick [23. Observe that for $s, t \geq 0$ and $T(\cdot)$ in (2.3)

$$
\frac{T(\widetilde{d}(t) s)}{t}=\sum_{i=1}^{\lfloor\widetilde{d}(t) s\rfloor} \widetilde{Y}_{t, i}
$$

From (2.4) we know

$$
\begin{equation*}
\frac{T(\widetilde{d}(t) \cdot)}{t} \stackrel{d}{\longrightarrow} S_{\alpha}(\cdot), \text { as } t \rightarrow \infty \tag{3.6}
\end{equation*}
$$

in $D([0, \infty),[0, \infty))$ with $J_{1}$ topology. The limiting process $S_{\alpha}(\cdot)$ has the same distribution as in (2.4). Notice that the generalized inverse of $T(\tilde{d}(t) \cdot) / t$ equals

$$
\begin{aligned}
\left(\frac{T(\widetilde{d}(t) \cdot)}{t}\right)^{\leftarrow}(u) & =\inf \left\{s: \sum_{i=1}^{\lfloor\tilde{d}(t) s\rfloor} Y_{i}>t u\right\}=\text { substituting } k=\lfloor\widetilde{d}(t) s\rfloor \\
& =\frac{\inf \left\{k \in \mathbb{N}: \sum_{i=1}^{k} Y_{i}>t u\right\}}{\widetilde{d}(t)}=\frac{\tau(t u)}{\widetilde{d}(t)}
\end{aligned}
$$

for every $u \geq 0$, where $\tau(\cdot)$ is defined in (1.1). Using (3.6) and Lemma 1, we apply continuous mapping theorem to obtain

$$
\begin{equation*}
\frac{\tau(t \cdot)}{\widetilde{d}(t)} \stackrel{d}{\longrightarrow} W_{\alpha}(\cdot), \tag{3.7}
\end{equation*}
$$

in $D([0, \infty),[0, \infty))$ with $J_{1}$ topology, where $W_{\alpha} \stackrel{d}{=} S_{\alpha}^{\leftarrow}$. Finally, the continuous mapping argument yields the joint convergence in (3.5).

Abusing the notation somewhat, for any time period $A \subseteq[0, \infty)$ and an arbitrary point measure $n \in M_{p}([0, \infty) \times \mathbb{E})$, we introduce restricted point measure

$$
\begin{equation*}
\left.n\right|_{A} \text { as }\left.n\right|_{A \times \mathbb{E}} \tag{3.8}
\end{equation*}
$$

Since the distribution of point processes $N_{t}$ contains the information about all upper order statistics in the sequence $\left(X_{n}\right)$, it is useful to study the limit of point processes $N_{t}$ restricted to time intervals determined by the renewal process $(\tau(t))$. For an
illustration, denote by $M^{\tau}(t), t \geq 0$, the maximum of observations $\left\{X_{1}, \ldots, X_{\tau(t)}\right\}$.
Note that

$$
\frac{M^{\tau}(t)-\widetilde{b}(t)}{\widetilde{a}(t)} \leq x \quad \text { if and only if }\left.\quad N_{t}\right|_{\left[0, \frac{\tau(t)}{d(t)}\right] \times(x, \infty] \times[0, \infty]}=0,
$$

for all suitably chosen $x$. Using similar arguments, the following theorem yields the limiting distribution of all extremes until a given time $t$.

Theorem 3.2. Assume that (3.4) holds, then

$$
\begin{equation*}
\left.\left.N_{t}\right|_{\left[0, \frac{\tau(t)}{d(t)}\right]} \xrightarrow{d} N\right|_{\left[0, W_{\alpha}\right]} \quad \text { and }\left.\left.\quad N_{t}\right|_{\left[0, \frac{\tau(t)}{d(t)}\right)} \xrightarrow{d} N\right|_{\left[0, W_{\alpha}\right)}, \tag{3.9}
\end{equation*}
$$

as $t \rightarrow \infty$, where $N$ is a $\operatorname{PRM}\left(\lambda \times \mu_{0}\right)$, and $W_{\alpha}=W_{\alpha}(1)$ denotes the fist passage time from (2.7).

Proof. Observe that, the limiting point process $N$ in general has a point exactly at the right end point of the interval $\left[0, W_{\alpha}\right]$. This forbids direct application of the continuous mapping argument. Fortunately, one can adapt the argument from the proof of Theorem 4.1 in [2].

Denote by $n, n_{t}, t>0$, arbitrary Radon point measures in $M_{p}([0, \infty) \times \mathbb{E})$. One can always write

$$
n_{t}=\sum_{i} \delta_{\left(v_{i}^{t}, x_{i}^{t}, y_{i}^{t}\right)}, n=\sum_{i} \delta_{\left(v_{i}, x_{i}, y_{i}\right)}
$$

for some sequences $\left(v_{i}^{t}, x_{i}^{t}, y_{i}^{t}\right)$ and $\left(v_{i}, x_{i}, y_{i}\right)$ with values in $[0, \infty) \times \mathbb{E}$. Denote further their corresponding cumulative sum functions, by

$$
s_{t}(u)=\int_{[0, u] \times \mathbb{E}} y n_{t}(d v, d x, d y), u \geq 0,
$$

and

$$
s(u)=\int_{[0, u] \times \mathbb{E}} y n(d v, d x, d y), u \geq 0,
$$

Suppose that they take finite value for each $u>0$, but tend to infinity as $u \rightarrow \infty$.

This makes $s_{t}$ and $s$ well defined, unbounded, nondecreasing elements of the space of càdlàg functions $D[0, \infty$ ). Their right-continuous generalized inverses (or hitting time functions) we denote by $s^{\leftarrow}$ and $s_{t}^{\leftarrow}$.

Assume that

$$
\begin{equation*}
\left(n_{t}, s_{t}\right) \rightarrow(n, s), \tag{3.10}
\end{equation*}
$$

in the product topology (of vague and $J_{1}$ topologies) as $t \rightarrow \infty$. If

$$
\begin{equation*}
0<s\left(s^{\leftarrow}(1)-\right)<1<s\left(s^{\leftarrow}(1)\right) \text { and } n(\{v\} \times(0, \infty) \times \mathbb{E}) \leq 1 \tag{3.11}
\end{equation*}
$$

for all $v \geq 0$, by straightforward adaptation of Theorem $4.1[2]$ to three-dimensional case

$$
\begin{equation*}
\left.\left.n_{t}\right|_{\left[0, s_{t}^{\leftarrow}(1)\right]} \xrightarrow{v} n\right|_{[0, s \leftarrow(1)]} \quad \text { and }\left.\left.\quad n_{t}\right|_{\left[0, s_{t}^{\leftarrow}(1)\right)} \xrightarrow{v} n\right|_{[0, s \leftarrow(1))} . \tag{3.12}
\end{equation*}
$$

We can apply continuous mapping theorem now. Observe now that by Theorem 3.1

$$
\left(N_{t}, \frac{T(\widetilde{d}(t) \cdot)}{t}\right) \stackrel{d}{\longrightarrow}\left(N, S_{\alpha}(\cdot)\right)
$$

On the other hand, the limiting Poisson process $N$ and $\alpha$-stable subordinator $S_{\alpha}$ satisfy regularity assumptions (3.11) with probability one, and therefore (3.9) holds.

The limiting distribution of all upper order statistics until a given time $t$, determined by the theorem above, can be quite complicated depending on the joint distribution of $N$ and $W_{\alpha}$ in (3.5). Hence, we examine two particular types of dependence between $X_{n}^{\prime} \mathrm{s}$ and $Y_{n}^{\prime} \mathrm{s}$ in detail. The first of them is called the asymptotic tail independence (see e.g. de Haan and Ferreira [6] and Resnick [22]). Rougly speaking, it requires that when $Y_{n}$ is large, there is negligible probability of $X_{n}$ being large. The second type of dependence we consider in detail is called the asymptotic full tail dependence. Intuitively, it implies that the $X_{n}$ and $Y_{n}$ are highly tail dependent in the sense that if one of them is large, then the other one is also large, asymptotically with probability 1 (see Sibuya [27], de Haan and Resnick [7] or Resnick [22, pp.296-298]).

### 3.1. Asymptotic tail independence

Recall that $\left(\left(X_{n}, Y_{n}\right)\right)$ is an iid sequence of random vectors such that $X_{1} \in \operatorname{MDA}\left(G_{1}\right)$ where $G_{1}=\Lambda$ or $\Phi_{\beta}$ with $\beta>0$ and $Y_{1} \sim \operatorname{Reg} \operatorname{Var}(\alpha)$ for $\alpha \in(0,1)$. Hence, $Y_{1} \in$ $\operatorname{MDA}\left(G_{2}\right)$ where $G_{2}=\Phi_{\alpha}$. By $F_{X, Y}$ denote the joint cdf of $\left(X_{1}, Y_{1}\right)$, and set $U_{X}=$ $1 /\left(1-F_{X}\right), U_{Y}=1 /\left(1-F_{Y}\right)$. Note, $U_{X}(X)$ and $U_{Y}(Y)$ are RegVar $(1)$ at infinity.

It is known, cf. de Haan and Resnick [7, that for the so-called tail independent $X_{1}$ and $Y_{1}$, the measure $\mu_{0}$ in (3.4) is concentrated on the axes. In particular, for $X_{1} \in \operatorname{MDA}(\Lambda)$ and $(x, y) \in[-\infty, \infty] \times[0, \infty] \backslash\{(-\infty, 0)\}$, we have

$$
\begin{equation*}
\mu_{0}\left(([-\infty, x] \times[0, y])^{c}\right)=-\log G_{1}(x)-\log G_{2}(y)=e^{-x}+y^{-\alpha} \tag{3.13}
\end{equation*}
$$

If $X_{1} \in \operatorname{MDA}\left(\Phi_{\beta}\right)$ and $(x, y) \in[0, \infty]^{2} \backslash\{(0,0)\}$, then

$$
\begin{equation*}
\mu_{0}\left(([0, x] \times[0, y])^{c}\right)=-\log G_{1}(x)-\log G_{2}(y)=x^{-\beta}+y^{-\alpha} \tag{3.14}
\end{equation*}
$$

Let $\left\{N_{t}: t \geq 0\right\}$ be point processes from (3.1). It is known that $N_{t} \xrightarrow{d} N$, where $N$ is $\operatorname{PRM}\left(\lambda \times \mu_{0}\right)$ is equivalent to $F_{X, Y} \in \operatorname{MDA}(G)$ (we refer to Resnick [22, Section 5.4] for a definition of multivariate MDA) with

$$
G(x, y)= \begin{cases}\exp \left\{-\mu_{0}\left(([-\infty, x] \times[0, y])^{c}\right)\right\}, & X_{1} \in \operatorname{MDA}(\Lambda)  \tag{3.15}\\ \exp \left\{-\mu_{0}\left(([0, x] \times[0, y])^{c}\right)\right\}, & X_{1} \in \operatorname{MDA}\left(\Phi_{\beta}\right)\end{cases}
$$

and $\mu_{0}$ defined in (3.13) and (3.14). The measure $\mu_{0}$ is often called the exponent measure.

Under our assumptions, necessary and sufficient condition for (3.4) can be inferred from the literature (cf. Theorem 6.2.3 in de Haan and Ferreira [6]). Next theorem summarizes this for completeness.

Theorem 3.3. For measure $\mu_{0}$ described in (3.13) and (3.14), (3.4) is equivalent to

$$
\begin{equation*}
\lim _{x \rightarrow \infty} P\left(X_{1}>U_{X}^{\overleftarrow{ }}(x) \mid Y_{1}>U_{Y}^{\leftarrow}(x)\right)=0 \tag{3.16}
\end{equation*}
$$

Remark 2. If (3.16) holds, $X_{1}$ and $Y_{1}$ are called asymptotically tail independent. Under this condition, the limiting Poisson process $N$ in (3.5) can be decomposed into two independent parts, e.g. for $X_{1} \in \operatorname{MDA}(\Lambda)$

$$
N=\sum_{i} \delta_{\left(T_{i}, P_{i}, 0\right)}+\sum_{i} \delta_{\left(T_{i}^{\prime}, 0, Q_{i}\right)}
$$

This makes the restriction of $N$ to the first two coordinates

$$
N^{(2)}=\sum_{i \geq 1} \delta_{\left(T_{i}, P_{i}\right)}
$$

independent of $S_{\alpha}=\sum_{T_{i}^{\prime} \leq t} Q_{i}$, and $W_{\alpha}$ for the same reason.

Proof. Under our assumptions, (3.4) is known to be equivalent to $F_{X, Y} \in \operatorname{MDA}(G)$ for $G$ given by (3.15). Since $U_{X}(X)$ and $U_{Y}(Y)$ are Reg $\operatorname{Var}(1)$ at infinity, by Sibuya's theorem (see Theorem 5 in de Haan and Resnick [7), $F_{X, Y} \in \operatorname{MDA}(G)$ is further equivalent to

$$
\begin{equation*}
\lim _{x \rightarrow \infty} P\left(U_{X}\left(X_{1}\right)>x \mid U_{Y}\left(Y_{1}\right)>x\right)=0 \tag{3.17}
\end{equation*}
$$

It remains to prove the equivalence between (3.16) and (3.17). From Theorem 1.7.13 in Leadbetter et al. [15] we conclude that

$$
\frac{P\left(X=U_{\overleftarrow{X}}^{\overleftarrow{ }}(x)\right)}{P\left(X \geq U_{\overleftarrow{X}}^{\overleftarrow{ }}(x)\right)} \rightarrow 0
$$

as $x \rightarrow \infty$. By the proof of Proposition 5.15 in Resnick [22], on the other hand, $x P\left(X>U_{X}^{\overleftarrow{X}}(x)\right) \rightarrow 1$, as $x \rightarrow \infty$. Now, observing that

$$
\left\{U_{X}(X)>x\right\} \subseteq\left\{X \geq U_{X}^{\overleftarrow{ }}(x)\right\} \text { and }\left\{X>U_{X}^{\overleftarrow{K}}(x)\right\} \subseteq\left\{U_{X}(X)>x\right\}
$$

one can show that (3.17) is equivalent to (3.16).

Condition (3.16) does not seem easy to verify directly in general. In some examples one can verify a simpler sufficient condition introduced in the following lemma (cf. condition A introduced in Anderson [1]).

Lemma 2. If

$$
\begin{equation*}
\lim _{x \rightarrow \infty} \limsup _{y \rightarrow \infty} P\left(X_{1}>x \mid Y_{1}>y\right)=0 \tag{3.18}
\end{equation*}
$$

then

$$
\lim _{x \rightarrow \infty} P\left(X_{1}>U_{X}^{\overleftarrow{X}}(x) \mid Y_{1}>U_{Y}^{\overleftarrow{K}}(x)\right)=0
$$

Proof. Assuming (3.18), for every $\varepsilon>0$ there exists $x_{1} \in \mathbb{R}$ such that for all $x \geq x_{1}$ we have

$$
\limsup _{y \rightarrow \infty} P\left(X_{1}>x \mid Y_{1}>y\right)<\varepsilon
$$

In particular,

$$
\limsup _{y \rightarrow \infty} P\left(X_{1}>x_{1} \mid Y_{1}>y\right)<\varepsilon
$$

Moreover, there exists $y_{0}>0$ such that for all $y>y_{0}$ we have

$$
P\left(X_{1}>x_{1} \mid Y_{1}>y_{0}\right)<\varepsilon
$$

If we denote $x_{0}=\inf \left\{x>0: U_{X}^{\overleftarrow{ }}(x)>x_{1}, U_{Y}^{\overleftarrow{ }}(x)>y_{0}\right\}$, then for all $x>x_{0}$

$$
P\left(X_{1}>U_{X}^{\overleftarrow{ }}(x) \mid Y_{1}>U_{Y}^{\overleftarrow{ }}(x)\right) \leq P\left(X>x_{1} \mid Y_{1}>U_{Y}^{\overleftarrow{ }}(x)\right)<\varepsilon
$$

An application of Theorem 3.3, yields the asymptotic behavior of the $k$-th upper order statistics in a sample indexed by the renewal process $(\tau(t))$.

Example 1. Let $M_{k}^{\tau}(t), t \geq 0$ represent the $k$-th upper order statistics in the sample $\left\{X_{1} \ldots, X_{\tau(t)}\right\}$. Under the assumptions of Theorem 3.3 we can find the limiting distribution for suitable normalized random variables $M_{k}^{\tau}(t)$. Clearly, from (3.9) we
obtain

$$
\begin{aligned}
& P\left(M_{k}^{\tau}(t) \leq \widetilde{a}(t) x+\widetilde{b}(t)\right)=P\left(N_{t}\left([0, \tau(t) / \widetilde{d}(t)] \times(x, \infty] \times \mathbb{R}_{+}\right) \leq k-1\right) \\
& \quad \rightarrow \quad P\left(N\left(\left[0, W_{\alpha}\right] \times(x, \infty] \times \mathbb{R}_{+}\right) \leq k-1\right)=E\left(\frac{\Gamma_{k}\left(W_{\alpha} \mu_{G}\left((x, \infty] \times \mathbb{R}_{+}\right)\right)}{\Gamma(k)}\right),
\end{aligned}
$$

as $t \rightarrow \infty$, where $H_{\alpha}(x)=P\left(W_{\alpha} \leq x\right)$ represents the cdf of the random variable $W_{\alpha}$ and $\Gamma_{k}(x)$ is an incomplete gamma function. For $k=1$, i.e. for the partial maxima of the first $\tau(t)$ observations, the result first appears in Berman [4]. For linearly growing $\tau(t)$ independent of the observations, a similar result can be found in Theorem 4.3.2 of Embrechts et al. [9].

### 3.2. Asymptotic full tail dependence

In the case when observations and interarrival times are exactly equal, the limiting behavior of the maximum has been found already by Lamperti [14. We will show here that one can extend his results to study all the upper order statistics in a more general setting. We keep the assumptions and notation from subsection 3.1. The main difference is that the limiting measure $\mu_{0}$ in (3.4) will be concentrated on a line, i.e. on the set

$$
C= \begin{cases}\left\{(u, v) \in(-\infty, \infty) \times(0, \infty): e^{-u}=v^{-\alpha}\right\}, & \text { if } G_{1}=\Lambda \\ \left\{(u, v) \in(0, \infty) \times(0, \infty): u^{-\beta}=v^{-\alpha}\right\}, & \text { if } G_{1}=\Phi_{\beta}\end{cases}
$$

More precisely, for $y>0$ and $C_{(y, \infty)}=\{(u, v) \in C: v>y\}$, the measure $\mu_{0}$ is determined by

$$
\mu_{0}\left(C_{(y, \infty)}\right)=y^{-\alpha}
$$

Under these assumptions, necessary and sufficient condition for (3.4) is the full tail dependence condition known from the literature (cf. de Haan and Resnick [7]). We again summarize this for completeness.

Theorem 3.4. For measure $\mu_{0}$ given above, (3.4) is equivalent to

$$
\begin{equation*}
\lim _{x \rightarrow \infty} P\left(X>U_{X}^{\overleftarrow{K}}(x) \mid Y>U_{Y}^{\leftarrow}(x)\right)=1 \tag{3.19}
\end{equation*}
$$

Proof. The proof follows the lines of the proof of Theorem 3.3, except that, instead of Theorem 5, we use Theorem 6 in de Haan and Resnick [7]).

For an application of Theorem 3.4 we refer to section 4.2 where the problem of the longest excursion of the continuous time random walk is considered.

## 4. Excursions and sojourn times of continuous time random walk

In this section we use results from section 3 to obtain the limiting distribution of extremely long sojourn times at level zero of a CTRW. We consider a CTRW which is a simple symmetric random walk subordinated to a certain renewal process. Hence, jumps are always of magnitude one, while waiting times between jumps correspond to interarrival times of the subordinating renewal process.

Let $\left(E_{n}\right)_{n \geq 1}$ be an iid sequence of non-negative random variables with finite expectation. Suppose $E_{1} \in \operatorname{MDA}(G)$ where $G=\Lambda$ or $G=\left(\Phi_{\beta}\right)$, for $\beta>1$. Denote the partial sum of the sequence $\left(E_{n}\right)$ by $T(n)=\sum_{i=1}^{n} E_{i}$ and set $T(0)=0$. Additionally, let $(N(t))_{t \geq 0}$ denote a renewal process generated by the sequence $\left(E_{n}\right)$, that is

$$
N(t)=\max \left\{k \geq 0: T(k)=\sum_{i=1}^{k} E_{i} \leq t\right\}
$$

The sequence $\left(E_{n}\right)$ models the waiting times between jumps, whereas the renewal process $(N(t))$ counts the number of jumps up to a time $t$. Furthermore, let $\left(\varepsilon_{n}\right)_{n \geq 1}$ be the iid sequence of Rademacher random variables, that is $P\left(\varepsilon_{1}=1\right)=P\left(\varepsilon_{1}=\right.$ $-1)=1 / 2$. Let $S_{n}=\sum_{i=1}^{n} \varepsilon_{i}$ denote the partial sums of $\left(\varepsilon_{n}\right)$. Set $S_{0}=0$ and define
continuous time random walk as the process

$$
Z(t)=S_{N(t)}=\sum_{i=1}^{N(t)} \varepsilon_{i}, \quad t \geq 0
$$

Our goal is to determine the limiting distribution of the longest time interval during which CTRW $Z(\cdot)$ remains at level zero before time $t$, including the last possibly incomplete part. With that in mind, we define some auxiliary random variables.

### 4.1. The longest sojourn time at level zero

Let $X_{1}$ denote the time spent during the first visit to the origin. Clearly, $X_{1}=E_{1}$. Since $Z(t)=0$ for all $t \in\left[0, E_{1}\right)$, and $Z\left(E_{1}\right) \neq 0$, the time of the first return to the origin is defined as

$$
A_{1}=\inf \left\{t \geq X_{1}: Z(t)=0\right\}
$$

Further, if we set $Y_{1}=A_{1}$, then $Y_{1}$ represents the sum of the first sojourn time at the origin and the time spent in the first excursion away from the origin. Clearly, duration of the first excursion, denoted by $R_{1}$, satisfies $R_{1}=Y_{1}-X_{1}$. If we set $A_{0}=0$, then for $i \geq 1$, the time spent during the $i$-th visit at the origin or the time spent on the $i$-th excursion can be defined recursively as

$$
\begin{aligned}
X_{i} & =\inf \left\{t \geq A_{i-1}: Z(t) \neq 0\right\}-A_{i-1}=E_{N\left(A_{i-1}\right)+1} \\
A_{i} & =\inf \left\{t \geq A_{i-1}+X_{i}: Z(t)=0\right\} \\
Y_{i} & =A_{i}-A_{i-1} \\
R_{i} & =Y_{i}-X_{i} .
\end{aligned}
$$

An illustration of these random variables is given in Figure 4.1

Clearly, the sequence $\left(X_{n}\right)_{n \geq 1}$ defined above is iid. Moreover, $X_{1}$ belongs to the same MDA as $E_{1}$. Since, $\left(Y_{n}\right)_{n \geq 1}$ is iid sequence of non-negative random variables,


Figure 1: A sketch of CTRW and associated random variables $X_{i}, A_{i}, Y_{i}, R_{i}$.
the renewal process

$$
\begin{equation*}
\tau(t)=\inf \left\{k \geq 1: \sum_{i=1}^{k} Y_{i}>t\right\}, t \geq 0 \tag{4.1}
\end{equation*}
$$

is well defined.

To apply Theorem 3.2 we need to determine the joint tail behavior of random variables $Y_{i}$ and $R_{i}$. Note that $\left(R_{n}\right)_{n \geq 1}$ is an iid sequence of non-negative random variables, and for $u>0$, it satisfies

$$
P\left(R_{1}>u\right)=P\left(\sum_{i=2}^{K} E_{i}>u\right)
$$

where $K=\inf \left\{m \geq 1: S_{m}=0\right\}$, and $S_{n}=\sum_{i=1}^{n} \varepsilon_{i}$. It is well known (see e.g. Durrett [8, Section 4.3]) that $K \sim \operatorname{Reg} \operatorname{Var}(1 / 2)$. Therefore, for instance, by Proposition 4.3. in Faÿ et al. 10

$$
P\left(R_{1}>u\right) \sim E\left(\sqrt{E_{1}}\right) P(K>u),
$$

as $u \rightarrow \infty$. In particular, $R_{1} \sim \operatorname{Reg} \operatorname{Var}(1 / 2)$, and, consequently, $Y_{1} \sim \operatorname{Reg} \operatorname{Var}(1 / 2)$.

To verify condition (3.18), observe that $Y_{1}=R_{1}+X_{1}$ with $R_{1}$ and $X_{1}$ independent. Therefore

$$
\frac{P\left(Y_{1}>y, X_{1}>x\right)}{P\left(Y_{1}>y\right)} \leq \frac{P\left(R_{1}>y-y^{2 / 3}\right) P\left(X_{1}>x\right)}{P\left(Y_{1}>y\right)}+\frac{P\left(X_{1}>y^{2 / 3}\right)}{P\left(Y_{1}>y\right)}
$$

The second term on the right tends to 0 as $y \rightarrow \infty$ because $X_{1}$ has finite mean and $Y_{1} \sim \operatorname{Reg} \operatorname{Var}(1 / 2)$, while the first term is bounded by $P\left(X_{1}>x\right)$, which clearly tends to 0 , if we let first $y$, then $x \rightarrow \infty$.

If we construct point processes $N_{t}, t \geq 0$ as in (3.1), observe that functions $d, \widetilde{d}$ are regularly varying with indices 2 and $1 / 2$, respectively. Now, using Theorems 3.1 and 3.3 together with Lemma 2 we obtain

$$
\begin{equation*}
\left(N_{t}, \frac{\tau(t)}{\widetilde{d}(t)}\right) \xrightarrow{d}\left(N, W_{1 / 2}\right) \tag{4.2}
\end{equation*}
$$

as $t \rightarrow \infty$, where $N \sim \operatorname{PRM}\left(\lambda \times \mu_{0}\right)$ is such that $N^{(2)}$ (see Remark 2) is independent of the random variable $W_{1 / 2}$.

The following theorem describes the asymptotic distribution of the longest sojourn time of CTRW at level zero. Denote the longest sojourn time at level 0 up to time $t$ by $Q(t)$. First observe that, if

$$
\sum_{i=1}^{\tau(t)-1} Y_{i}+X_{\tau(t)}<t
$$

$Q(t)$ simply equals $M^{\tau}(t)=\max \left\{X_{1}, \ldots, X_{\tau(t)}\right\}$. On the other hand, if

$$
\begin{gathered}
t \leq \sum_{i=1}^{\tau(t)-1} Y_{i}+X_{\tau(t)} \\
Q(t)=\max \left\{M^{\tau-1}(t), t-\sum_{i=1}^{\tau(t)-1} Y_{i}\right\} .
\end{gathered}
$$

In either case

$$
\begin{equation*}
M^{\tau-1}(t) \leq Q(t) \leq M^{\tau}(t) \tag{4.3}
\end{equation*}
$$

Observe further that the random events

$$
\left\{\frac{M^{\tau}(t)-\widetilde{b}(t)}{\widetilde{a}(t)} \leq x\right\} \quad \text { and } \quad\left\{\frac{M^{\tau-1}(t)-\widetilde{b}(t)}{\widetilde{a}(t)} \leq x\right\}
$$

correspond to

$$
\left\{\left.N_{t}\right|_{\left[0, \frac{\tau(t)}{\bar{d}(t)}\right] \times(x, \infty] \times[0, \infty]}=0\right\} \quad \text { and } \quad\left\{\left.N_{t}\right|_{\left[0, \frac{\tau(t)}{\bar{d}(t)}\right) \times(x, \infty] \times[0, \infty]}=0\right\}
$$

Theorem 4.1. Under the assumptions above

$$
P\left(\frac{Q(t)-\widetilde{b}(t)}{\widetilde{a}(t)} \leq x\right) \rightarrow E\left(G(x)^{W_{1 / 2}}\right)
$$

as $t \rightarrow \infty$.

Proof. Using (4.2) and Theorem 3.2 by the discussion before the theorem for an arbitrary $x \in \mathbb{R}$, we obtain

$$
P\left(\frac{M^{\tau}(t)-\widetilde{b}(t)}{\widetilde{a}(t)} \leq x\right) \rightarrow P\left(\left.N\right|_{\left[0, W_{1 / 2}\right] \times(x, \infty] \times[0, \infty]}=0\right)
$$

as $t \rightarrow \infty$, and similarly

$$
P\left(\frac{M^{\tau-1}(t)-\widetilde{b}(t)}{\widetilde{a}(t)} \leq x\right) \rightarrow P\left(\left.N\right|_{\left[0, W_{1 / 2}\right) \times(x, \infty] \times[0, \infty]}=0\right)
$$

By the independence between $N^{(2)}$ and $W_{1 / 2}$, see Remark 2, both limiting probabilities above equal

$$
\begin{aligned}
& P\left(\left.N^{(2)}\right|_{\left[0, W_{1 / 2}\right] \times(x, \infty]}=0\right) \\
&=\int_{0}^{+\infty} e^{-s \mu_{0}\left(([-\infty, x] \times[0, \infty])^{c}\right)} d F_{W_{1 / 2}}(s)=\int_{0}^{+\infty} G(x)^{s} d F_{W_{1 / 2}}(s) \\
& \quad=E\left(G(x)^{W_{1 / 2}}\right)
\end{aligned}
$$

By (4.3)

$$
P\left(\frac{Q(t)-\widetilde{b}(t)}{\widetilde{a}(t)} \leq x\right) \rightarrow E\left(G(x)^{W_{1 / 2}}\right)
$$

as $t \rightarrow \infty$.

Similarly, one can use (4.2) to obtain the joint distribution of the two longest sojourn times of CTRW. Denote by $Q(t)$ and by $Q^{\prime}(t)$ the longest and the second longest sojourn times until time $t$. Fix levels $u_{1}>u_{2}>0$. Observe that

$$
\begin{aligned}
& P\left(\frac{Q(t)-\widetilde{b}(t)}{\widetilde{a}(t)} \leq u_{1}, \frac{Q^{\prime}(t)-\widetilde{b}(t)}{\widetilde{a}(t)} \leq u_{2}\right) \\
& \quad=P\left(N_{t}\left(\left[0, \frac{\tau(t)}{\widetilde{d}(t)}\right] \times\left(u_{1}, \infty\right] \times[0, \infty]\right)=0, N_{t}\left(\left[0, \frac{\tau(t)}{\widetilde{d}(t)}\right] \times\left(u_{2}, \infty\right] \times[0, \infty]\right) \leq 1\right)
\end{aligned}
$$

Now one can apply (4.2) and Theorem 3.2 to show that the last expression converges to

$$
E\left(G\left(u_{2}\right)^{W_{1 / 2}}\right)+\mu_{G}\left(u_{2}, u_{1}\right] E\left(W_{1 / 2} G\left(u_{2}\right)^{W_{1 / 2}}\right)
$$

### 4.2. The longest excursion

Our next goal is to determine the limiting distribution of the length of the longest excursion completed until time $t$, i.e. the longest time interval during which CTRW $Z(t)$ is not equal to zero, completed until time $t$. In contrast to subsection 4.1, here we only assume that $E_{i}$ 's have finite expectation. As in subsection 4.1 we denote by $R_{i}$ the time spent on the $i$-th excursion, and by $Y_{i}$ the total time spent on the $i$-th stay at zero and the $i$-th excursion. Now, we are interested in determining the limiting distribution of

$$
\begin{equation*}
M^{\tau-1}(t)=\sup \left\{R_{i}: i \leq \tau(t)-1\right\} \tag{4.4}
\end{equation*}
$$

with $\tau(t)$ given in (4.1), which corresponds to the length of the longest completed excursion.

In the present model, the point processes $N_{t}$ of Theorem 3.4 are constructed using sequences $\left(R_{n}\right)$ (instead of $\left.\left(X_{n}\right)\right)$ and $\left(Y_{n}\right)$. Recall that $U_{R}=1 /\left(1-F_{R}\right)$, and observe that $U_{R}^{\leftarrow}(x) \leq U_{Y}^{\leftarrow}(x)$ for all $x$, since $Y_{1}=R_{1}+X_{1}$. To show (3.19), observe that

$$
x P\left(Y_{1}>U_{Y}^{\leftarrow}(x)\right) \rightarrow 1
$$

and, since $Y_{1}$ is $\operatorname{Reg} \operatorname{Var}(1 / 2)$, it is well know that the same holds for $R_{1}=Y_{1}-X_{1}$ because $E X_{1}<\infty$. Moreover, one can show that

$$
x P\left(R_{1}>U_{Y}^{\overleftarrow{ }}(x)\right)=x P\left(Y_{1}-X_{1}>U_{Y}^{\overleftarrow{ }}(x)\right) \rightarrow 1
$$

Clearly,

$$
\begin{aligned}
& P\left(R_{1}>U_{R}^{\leftarrow}(x) \mid Y_{1}>U_{Y}^{\leftarrow}(x)\right) \\
& \quad=\frac{P\left(R_{1}>U_{R}^{\leftarrow}(x), R_{1}+X_{1}>U_{Y}^{\leftarrow}(x)\right)}{P\left(Y_{1}>U_{Y}^{\leftarrow}(x)\right)} \geq \frac{P\left(R_{1}>U_{Y}^{\leftarrow}(x)\right)}{P\left(Y_{1}>U_{Y}^{\overleftarrow{K}}(x)\right)}
\end{aligned}
$$

Since the last ratio above tends to 1 as $x \rightarrow \infty$, (3.19) is proved.
From Theorems 3.2 and 3.4, we obtain

$$
\begin{equation*}
\left.\left.N_{t}\right|_{\left[0, \frac{\tau(t)}{d(t)}\right)} \xrightarrow{d} N\right|_{\left[0, W_{1 / 2}\right)} \tag{4.5}
\end{equation*}
$$

as $t \rightarrow \infty$. In this case,

$$
N=\sum_{i \geq 1} \delta_{\left(T_{i}, P_{i}, P_{i}\right)}
$$

while $W_{1 / 2}=W_{1 / 2}(1)=S_{1 / 2}^{\leftarrow}(1)$ with $S_{1 / 2}$ equal to the $1 / 2$-stable subordinator

$$
S_{1 / 2}(t)=\sum_{T_{i} \leq t} P_{i}
$$

which makes $W_{1 / 2}$ completely dependent on $N^{(2)}=\sum_{i \geq 1} \delta_{\left(T_{i}, P_{i}\right)}$, cf. Remark 2.
Observe that (4.5) identifies the limiting distribution of all upper order statistics in the sequence of excursions. It follows that rescaled excursions behave asymptotically as completed jumps of the subordinator $S_{1 / 2}$ until the passage of the level 1. Distribution of those jumps is well understood, see Perman [20] or Pitman and Yor [21]. Alternatively, one could include the last, possibly incomplete excursion in the analysis. It turns out that the convergence result still holds, but one needs to add the last a.s. incomplete jump of the subordinator $S_{1 / 2}$ to the limiting distribution. All those jumps
have the distribution of the Pitman-Yor point processes, see [20, 21.
To illustrate our claims, the next theorem gives the limiting distribution of $M^{\tau-1}(t)$.

Theorem 4.2. Under the assumptions above, $M^{\tau-1}(t)$ in (4.4) satisfies

$$
\frac{M^{\tau-1}(t)}{t} \xrightarrow{d} V
$$

as $t \rightarrow \infty$, where $V$ has the distribution of the largest jump of $1 / 2-$ stable suboordinator completed before the passage of the level 1.

Proof. Observe that for arbitrary $x>0$

$$
P\left(\frac{M^{\tau-1}(t)}{\widetilde{a}(t)} \leq x\right)=P\left(\left.N_{t}\right|_{\left[0, \frac{\tau(t)}{\tilde{d}(t)}\right) \times(x, \infty] \times[0, \infty]}=0\right)
$$

Theorem 3.2 yields (4.5), and therefore

$$
P\left(\left.N_{t}\right|_{\left[0, \frac{\tau(t)}{d(t)}\right) \times(x, \infty] \times[0, \infty]}=0\right) \rightarrow P\left(\left.N\right|_{\left[0, W_{1 / 2}\right) \times(x, \infty] \times[0, \infty]}=0\right) .
$$

According to the discussion following (4.5),

$$
\left\{\left.N\right|_{\left[0, W_{1 / 2}\right) \times(x, \infty] \times[0, \infty]}=0\right\}=\left\{\sup _{T_{i}<W_{1 / 2}} P_{i} \leq x\right\}
$$

which is exactly the probability that the largest jump of $1 / 2$-stable subordinator, before it hits $[1, \infty)$, is less than or equal to $x$.

The limiting random variable $V$ in Theorem 4.2 has a continuous distribution with support on $(0,1)$, see Perman [20]. This is also the distribution of the longest completed excursion of the standard Brownian motion during time interval $[0,1]$. Its density is given in [20, Corollary 9]. Interestingly, the closed form expression for the density is known only on the interval $(1 / 3,1)$.

In the special case when all waiting times of CTRW have unit length, the model above boils down to the simple symmetric random walk on integers. In particular, from
(4.5) one can deduce the asymptotic distribution of all upper order statistics for the length of excursions of the simple symmetric random walk given in Csáki and Hu [5].

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