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Stein's method for negatively associated random variables with applications to second order stationary random fields

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Abstract

Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m)$ be a negatively associated mean zero random vector with components that obey the bound $|\xi_i| \le B, i = 1, \dots, m$, and whose sum $W = \sum_{i=1}^m \xi_i$ has variance 1, the bound

$$d_1\big(\mathcal{L}(W), \mathcal{L}(Z)\big) \le 5B - 5.2 \sum_{i \ne j} \sigma_{ij}.$$

is obtained where Z has the standard normal distribution and $d_1(\cdot, \cdot)$ is the L^1 metric. The result is extended to the multidimensional case with the L^1 metric replaced by a smooth functions metric. Applications to second order stationary random fields with exponential decreasing covariance are also presented.

1 Introduction

This work is extended from the recent paper [GW18] that focuses only on positive association. Particularly, in this work, we provide non-asymptotic L^1 bounds to the normal for negatively associated random fields in $\mathbb{Z}^d$ using the same technique developed in [GW18]. We recall that the L^1, or Wasserstein, distance between the distributions $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ of real valued random variables X and Y is given by

$$d_1\big(\mathcal{L}(X), \mathcal{L}(Y)\big) = \int_{-\infty}^{\infty} |P(X \le t) - P(Y \le t)| dt. \tag{1}$$

A random vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ is said to be *positively associated* whenever

$$\mathrm{Cov}\big(f(\boldsymbol{\xi}), g(\boldsymbol{\xi})\big) \ge 0$$

for all real valued coordinate-wise nondecreasing functions f and g on $\mathbb{R}^m$ such that $f(\boldsymbol{\xi})$ and $g(\boldsymbol{\xi})$ have finite second moments. A random vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ is said to be *negatively associated* if for any disjoint subsets A, B of $[m] := \{1, 2, \dots, m\}$,

$$\mathrm{Cov}\big(f(\xi_i, i \in A), g(\xi_j, j \in B)\big) \le 0 \tag{2}$$

for all coordinate-wise increasing functions $f : \mathbb{R}^{|A|} \to \mathbb{R}$ and $g : \mathbb{R}^{|B|} \to \mathbb{R}$ such that $f(\xi_i, i \in A)$ and $g(\xi_j, j \in B)$ have finite second moments. In general, a collection $\{\xi_\alpha : \alpha \in I\}$ of real valued random variables indexed by a set I is said to be positively associated (resp. negatively associated) if all

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finite subcollections are positively associated (resp. negatively associated). Positive association was introduced in [EPW67] and has been found frequently in probabilistic models in several areas, especially statistical physics. In some literature positive association is termed the 'FKG-inequality' or simply 'association' (see [New80] and [CG84] for examples). Negative association was later introduced in [JP83] and has well-known applications related to permutation distributions.

Over the last few decades, many researchers established central limit theorems and rates of convergence for sums of positively associated random variables ([New80],[Bir88],[Bul95]) and negatively associated random variables ([LW08],[CW09]) under different assumptions. Recently, the work [GW18] developed an L^1 version of Stein's method adapted to sums of positively associated random variables with applications to statistical physics. Stein's method, introduced by [Ste72], is nowadays one of the most powerful methods to prove convergence in distribution as it has main advantages that it provides non-asymptotic bounds on the distance between distributions, and that it can handle various situations involving dependence. Thus far, many applications in several areas such as statistics, statistical physics and applied sciences have been developed using this method. For more detail about the method in general, see the text [CGS11] and the introductory notes [Ros11].

The one dimensional result of [GW18] is stated below.

Theorem 1.1 ([GW18]) *Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m)$ be a positively associated mean zero random vector with components obeying the bound $|\xi_i| \le B$ for some $B > 0$, and whose sum $W = \sum_{i=1}^m \xi_i$ has variance 1. Let Z be a standard normal random variable. Then*

$$d_1\big(\mathcal{L}(W), \mathcal{L}(Z)\big) \le 5B + \sqrt{\frac{8}{\pi}} \sum_{i \ne j} \sigma_{ij} \quad \textit{where} \quad \sigma_{ij} = \mathbb{E}[\xi_i \xi_j].$$

The multidimensional result was also obtained in [GW18] with the L^1 metric replaced by a smooth functions metric, following the development of Chapter 12 of [CGS11].

For

$$\mathbf{x} \in \mathbb{R}^p \quad \text{let} \quad |\mathbf{x}|_1 = \sum_{i=1}^p |x_i|, \quad \text{the } L^1 \text{ vector norm,}$$

and for a real valued function $\varphi(u)$ defined on the domain $\mathcal{D}$, let $|\varphi|_\infty = \sup_{x \in \mathcal{D}} |\varphi(x)|$. We include in this definition the $|\cdot|_\infty$ norm of vectors and matrices, for instance, by considering them as real valued functions of their indices. Also, from this point, we denote $\mathbb{N}_k = [k, \infty) \cap \mathbb{Z}$ for $k \in \mathbb{Z}$.

For $m \in \mathbb{N}_0$, let $L_m^\infty(\mathbb{R}^p)$ be the collection of all functions $h : \mathbb{R}^p \to \mathbb{R}$ such that for all $\mathbf{k} = (k_1, \dots, k_p) \in \mathbb{N}_0^p$ with $|\mathbf{k}|_1 \le m$, the partial derivative

$$h^{(\mathbf{k})}(\mathbf{x}) = \frac{\partial^{|\mathbf{k}|_1} h}{\partial^{k_1} x_1 \cdots \partial^{k_p} x_p}$$

exists, and

$$|h|_{L_m^\infty(\mathbb{R}^p)} := \max_{0 \le |\mathbf{k}|_1 \le m} |h^{(\mathbf{k})}|_\infty \quad \text{is finite.}$$

For $f \in L_m^\infty(\mathbb{R}^p)$ let

$$\mathcal{H}_{m,\infty,p} = \left\{ h \in L_m^\infty(\mathbb{R}^p) : |h|_{L_m^\infty(\mathbb{R}^p)} \le 1 \right\},$$

and for random vectors $\mathbf{X}$ and $\mathbf{Y}$ in $\mathbb{R}^p$, define the smooth functions metric

$$d_{\mathcal{H}_{m,\infty,p}}\big(\mathcal{L}(\mathbf{X}), \mathcal{L}(\mathbf{Y})\big) = \sup_{h \in \mathcal{H}_{m,\infty,p}} |\mathbb{E}h(\mathbf{X}) - \mathbb{E}h(\mathbf{Y})|. \tag{3}$$

For a positive semidefinite matrix H, we let $H^{1/2}$ denote the unique positive semidefinite square root of H. When H is positive definite, we write $H^{-1/2} = (H^{1/2})^{-1}$. The following theorem states the multidimensional result of [GW18].

Theorem 1.2 ([GW18]) *With $m, p \in \mathbb{N}_1$, let $\{\xi_{i,j} : i \in [m], j \in [p]\}$ be positively associated mean zero random variables bounded in absolute value by some positive constant B. Let $\mathbf{S} = (S_1, S_2, \ldots, S_p)$ where $S_j = \sum_{1\le i\le m} \xi_{i,j}$ for $j \in [p]$ and assume that $\Sigma = \operatorname{Var}\big(\mathbf{S}\big)$ is positive definite. Then*

$$\begin{aligned}
& d_{\mathcal{H}_{3,\infty,p}}\big(\mathcal{L}(\Sigma^{-1/2}\left(\mathbf{S} - \mathbb{E}\mathbf{S}\right)), \mathcal{L}(\mathbf{Z})\big) \\
&\le \left(\frac{1}{6} + 2\sqrt{2}\right) p^3 B |\Sigma^{-1/2}|_\infty^3 \sum_{j=1}^{p} \Sigma_{j,j} \\
&\quad + \left(\frac{3}{\sqrt{2}} + \frac{1}{2}\right) p^2 |\Sigma^{-1/2}|_\infty^2 \sum_{j=1}^{p} \sum_{i,k\in[m], i\ne k} \operatorname{Cov}\left(\xi_{i,j}, \xi_{k,j}\right) \\
&\quad + \left(2\sqrt{2} p^3 B |\Sigma^{-1/2}|_\infty^3 + \left(\frac{3}{\sqrt{2}} + \frac{1}{2}\right) p^2 |\Sigma^{-1/2}|_\infty^2\right) \sum_{j,l\in[p], j\ne l} \Sigma_{j,l},
\end{aligned}$$

where $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, I_p)$, a standard normal vector in $\mathbb{R}^p$.

The results above were applied to second order stationary random fields assuming exponential decreasing covariance and to four models in statistical physics; Ising and voter models, bond percolation and contact process. In the present work, we prove similar results adapted to negative association.

Stein's method has been used previously for negative association and some related concepts. In [LW08], Stein's method was used in normal approximation for sums of pairwise negative quadrant dependent random variables which allows one to derive a CLT for pairwise negative quadrant dependent random variables with Lindeberg's condition. We note that when $m = 2$ in (2) negative association and negative quadrant dependent are equivalent (see [JP83]). In [Dal13], Stein's method was used to obtain the total variation distance between compound Poisson distribution and sums of positively associated or negatively associated random variables.

Our main results in the one dimensional and multidimensional cases are stated in Theorems 1.3 and 1.4, respectively, as follows.

Theorem 1.3 *Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)$ be a negatively associated mean zero random vector with components obeying the bound $|\xi_i| \le B$ for some $B > 0$, and whose sum $W = \sum_{i=1}^m \xi_i$ has variance 1. Let Z be a standard normal random variable. Then, with $\sigma_{ij} = \mathbb{E}[\xi_i \xi_j]$,*

$$d_1\big(\mathcal{L}(W), \mathcal{L}(Z)\big) \le 5B - 5.2 \sum_{i\ne j} \sigma_{ij}. \tag{4}$$

Theorem 1.4 *With $m, p \in \mathbb{N}_1$, let $\{\xi_{i,j} : i \in [m], j \in [p]\}$ be negatively associated mean zero random variables satisfying $|\xi_{i,j}| \le B$ for some $B > 0$. Let $\mathbf{S} = (S_1, S_2, \ldots, S_p)$ where $S_j =$*

$\sum_{1\le i\le m}\xi_{i,j}$ *for* $j\in[p]$ *and assume that* $\Sigma=\mathrm{Var}\big(\mathbf{S}\big)$ *is positive definite. Then*

$$\begin{aligned}
&d_{\mathcal{H}_{3,\infty,p}}\big(\mathcal{L}(\Sigma^{-1/2}\left(\mathbf{S}-\mathbb{E}\mathbf{S}\right)),\mathcal{L}(\mathbf{Z})\big)\\
&\quad\le \frac{5}{6}p^3B|\Sigma^{-1/2}|_\infty^3\sum_{j=1}^p\Sigma_{j,j}\\
&\qquad-\left(\frac{3}{2}p^3B|\Sigma^{-1/2}|_\infty^3+p^2|\Sigma^{-1/2}|_\infty^2\right)\sum_{j=1}^p\sum_{i,k\in[m],i\neq k}\mathrm{Cov}\left(\xi_{i,j},\xi_{k,j}\right)\\
&\qquad-\left(\frac{2}{3}p^3B|\Sigma^{-1/2}|_\infty^3+p^2|\Sigma^{-1/2}|_\infty^2\right)\sum_{j,l\in[p],j\neq l}\Sigma_{j,l},
\end{aligned}\tag{5}$$

where $\mathbf{Z}\sim\mathcal{N}(\mathbf{0},I_p)$*, a standard normal random vector in* $\mathbb{R}^p$*.*

Remark 1.5 *We note that the differences between our results in Theorems 1.3 and 1.4 for negative association and those in Theorems 1.1 and 1.2 of [GW18] for positive association, are that the signs of the covariance terms are reverse and that the constants are different. Nevertheless, in Section 3 we have an example where these changes do not contribute rates of convergence. We also note that the bounds in the four theorems above are particularly useful when the variables one handles are bounded and (positively or negatively) associated. However let us compare the one dimensional results in Theorems 1.1 and 1.3 with the classical result for independent and identically distributed variables* X_i*,* $i\in[n]$ *with* $\mathbb{E}X_i=0$*,* $\mathrm{Var}X_i=\sigma^2>0$ *and* $|X_i|\le K$ *for some* $K>0$*. With* $W=\left(\sum_i X_i\right)/(\sigma\sqrt{n})$*, both Theorems 1.1 and 1.3 give that* $d_1\big(\mathcal{L}(W),\mathcal{L}(Z)\big)\le 5K\sigma^{-1}n^{-1/2}$*. The classical Berry Esseen theorem provides the bound* $\sup_{x\in\mathbb{R}}|P(W\le x)-P(Z\le x)|\le CK^3\sigma^{-3}n^{-1/2}$ *with the smallest constant* $C=0.4748$ *obtained recently by [She11]. Though the rates of convergence in* n *are the same, the constants are different and actually the two distances are not comparable as it is known that* $\sup_{x\in\mathbb{R}}|P(X\le x)-P(Y\le x)|\le\sqrt{2cd_1\big(\mathcal{L}(X),\mathcal{L}(Y)\big)}$ *for some* $c>0$ *but not conversely (See Proposition 1.2 of [Ros11]).*

The remainder of this work is organized as follows. In the next section, we use Stein's method to prove the two main results, Theorems 1.3 and 1.4. We state our results for negatively associated random fields whose covariance decays exponentially in Section 3 . One similar advantage of the four theorems stated in this section is that, unlike many results based on Stein's method, they may be applied without the need for coupling constructions.

2 Proofs of main theorems

In this section we prove our main results, Theorems 1.3 and 1.4, using similar techniques as in [GW18]. For this purpose, we first state the following two lemmas proved in [JP83] and [CW09], respectively. The more general version of the first lemma was originally proved in [BS98] in Russian and the English version can be found in the book [BS07]. Lemma 2.1 states that two disjoint sums of negatively associated random variables are negatively associated which will be used throughout the remainder of this work. Lemma 2.2 allows us to bound $|\mathrm{Cov}(f(\xi_i:i\in A),g(\xi_j:j\in B))|$ by a linear combination of $-\mathrm{Cov}(\xi_i,\xi_j)$ with $i\in A,j\in B$ when A and B are disjoint.

Lemma 2.1 ([JP83]) *Increasing functions defined on disjoint subsets of a set of negatively associated random variables are negatively associated.*

Lemma 2.2 ([BS98], [CW09]) *Let A and B be disjoint finite sets, and let ξ_j, $j \in A \cup B$, be negatively associated random variables. If $f : \mathbb{R}^{|A|} \to \mathbb{R}$, $g : \mathbb{R}^{|B|} \to \mathbb{R}$ are partially differentiable with bounded partial derivatives, then*

$$|\mathrm{Cov}(f(\xi_i : i \in A), g(\xi_j : j \in B))| \le -\sum_{i \in A} \sum_{j \in B} \left| \frac{\partial f}{\partial \xi_i} \right|_\infty \left| \frac{\partial g}{\partial \xi_j} \right|_\infty \mathrm{Cov}(\xi_i, \xi_j).$$

We note that the difference between the proofs below and the ones in [GW18] results from that Lemma 2.2 requires A and B to be disjoint unlike the one for positive association. Therefore we add a few more steps in the proofs to handle this situation. In the proof that follows, we use the alternate form of the L^1, or Wasserstein distance (see e.g. [Rac84]);

$$d_1\big(\mathcal{L}(X), \mathcal{L}(Y)\big) = \sup_{h \in \mathbb{L}} |\mathbb{E}h(X) - \mathbb{E}h(Y)| \quad \text{where} \quad \mathbb{L} = \{h : |h(y) - h(x)| \le |y - x|\}. \tag{6}$$

Proof of Theorem 1.3 For given $h \in \mathbb{L}$ let f be the unique bounded solution to the Stein equation

$$f'(w) - wf(w) = h(w) - Nh \quad \text{where} \quad Nh = \mathbb{E}h(Z), \tag{7}$$

with $\mathcal{L}(Z)$ the standard normal distribution. Then, (see e.g. Lemma 2.4 of [CGS11]),

$$|f'|_\infty \le \sqrt{\frac{2}{\pi}} \quad \text{and} \quad |f''|_\infty \le 2. \tag{8}$$

Recall that in the proof below we use the notations $\sigma_{ij} = \mathbb{E}[\xi_i \xi_j]$ and $\sigma_i^2 = \mathrm{Var}(\xi_i)$ for $i \neq j \in [n]$. As $\mathrm{Var}(W) = \sum_{i=1}^n \sigma_i^2 + \sum_{i \neq j} \sigma_{ij} = 1$, we obtain

$$\begin{aligned} \mathbb{E}[f'(W)] &= \mathbb{E}\left(\sum_{i=1}^m \sigma_i^2 f'(W) + \sum_{i \neq j} \sigma_{ij} f'(W) \right) \\ &= \mathbb{E}\left(\sum_{i=1}^m \xi_i^2 f'(W) + \sum_{i \neq j} \sigma_{ij} f'(W) + \sum_{i=1}^m (\sigma_i^2 - \xi_i^2) f'(W) \right). \end{aligned}$$

Now letting $W^i = W - \xi_i$, write

$$\mathbb{E}[Wf(W)] = \mathbb{E}\sum_{i=1}^m \xi_i f(W) = \mathbb{E}\sum_{i=1}^m \xi_i f(W^i + \xi_i) = \mathbb{E}\sum_{i=1}^m \left[\xi_i f(W^i) + \xi_i^2 \int_0^1 f'(W^i + u\xi_i) du \right].$$

Recalling the Stein equation (7) and subtracting two equations above, we obtain

$$\begin{aligned} &\mathbb{E}[h(W) - Nh] = \mathbb{E}[f'(W) - Wf(W)] \\ &\qquad = \mathbb{E}\left(\sum_{i=1}^m \xi_i^2 \left(\int_0^1 \big(f'(W) - f'(W^i + u\xi_i) \big) du \right) + \sum_{i=1}^m (\sigma_i^2 - \xi_i^2) f'(W) \right. \\ &\qquad\qquad \left. + \sum_{i \neq j} \sigma_{ij} f'(W) - \sum_{i=1}^m \xi_i f(W^i) \right). \end{aligned} \tag{9}$$

Using the second inequality in (8), we bound the first term in (9) by

$$\left|\mathbb{E}\sum_{i=1}^{m}\xi_i^2\int_0^1\big(f'(W)-f'(W^i+u\xi_i)\big)\,du\right|$$
$$=\left|\mathbb{E}\sum_{i=1}^{m}\xi_i^2\int_0^1\int_{u\xi_i}^{\xi_i}f''(W^i+t)dtdu\right|\le 2\mathbb{E}\sum_{i=1}^{m}\xi_i^2\left(\int_0^1\int_{u|\xi_i|}^{|\xi_i|}dtdu\right)$$
$$=\mathbb{E}\sum_{i=1}^{m}|\xi_i|^3\le B\mathbb{E}\sum_{i=1}^{m}\xi_i^2=B\left(1-\sum_{i\neq j}\sigma_{ij}\right). \tag{10}$$

To handle the second term in (9), using the triangle inequality, we first bound it by the three terms denoted by I_1, I_2, I_3, respectively,

$$\begin{aligned}
\left|\mathbb{E}\sum_{i=1}^{m}f'(W)(\sigma_i^2-\xi_i^2)\right| &\le \left|\mathbb{E}\sum_{i=1}^{m}f'(W^i)(\sigma_i^2-\xi_i^2)\right|+\left|\mathbb{E}\sum_{i=1}^{m}(f'(W)-f'(W^i))(\sigma_i^2-\xi_i^2)\right|\\
&\le \left|\mathbb{E}\sum_{i=1}^{m}f'(W^i)(\sigma_i^2-\xi_i^2)\right|+\mathbb{E}\sum_{i=1}^{m}\left|f'(W)-f'(W^i)\right|\left|\sigma_i^2-\xi_i^2\right|\\
&\le \left|\mathbb{E}\sum_{i=1}^{m}f'(W^i)(\sigma_i^2-\xi_i^2)\right|+\mathbb{E}\sum_{i=1}^{m}\left|f'(W)-f'(W^i)\right|\sigma_i^2\\
&\quad+\mathbb{E}\sum_{i=1}^{m}\left|f'(W)-f'(W^i)\right|\xi_i^2 := I_1+I_2+I_3.
\end{aligned}$$

Note that W^i and ξ_i are coordinate-wise increasing functions defined on disjoint subsets of $\boldsymbol{\xi}$, and hence negatively associated by Lemma 2.1. Now for I_1, applying Lemma 2.2 with

$$g(x)=\begin{cases} x^2 & |x|\le B\\ B^2 & |x|>B\end{cases}$$

and using the second inequality in (8), we have

$$I_1=\left|\sum_{i=1}^{m}\mathrm{Cov}\big(f'(W^i),g(\xi_i)\big)\right|\le -4B\sum_{i=1}^{m}\mathrm{Cov}\big(W^i,\xi_i\big)=-4B\sum_{i\neq j}\sigma_{ij}. \tag{11}$$

For I_2 and I_3, applying again the second inequality in (8), we obtain

$$I_2\le |f''|_\infty\sum_{i=1}^{m}\sigma_i^2\mathbb{E}|W-W^i|\le 2B\sum_{i=1}^{m}\sigma_i^2=2B\left(1-\sum_{i\neq j}\sigma_{ij}\right), \tag{12}$$

and

$$I_3\le |f''|_\infty\sum_{i=1}^{m}\mathbb{E}\xi_i^2|W-W^i|\le 2B\sum_{i=1}^{m}\mathbb{E}\xi_i^2=2B\left(1-\sum_{i\neq j}\sigma_{ij}\right). \tag{13}$$

For the third term in (9), using the negativity of the covariances σ_{ij}, $i\neq j$, and the first inequality in (8) we obtain

$$\left|\mathbb{E}\sum_{i\neq j}\sigma_{ij}f'(W)\right|\le -|\mathbb{E}f'(W)|\sum_{i\neq j}\sigma_{ij}\le -\sqrt{\frac{2}{\pi}}\sum_{i\neq j}\sigma_{ij}. \tag{14}$$

For the final term in (9), using again the fact that the pair (W^i, ξ_i) is negatively associated and applying Lemma 2.2 and the first inequality in (8) now yields

$$\left|\mathbb{E}\sum_{i=1}^{m} \xi_i f(W^i)\right| = \left|\sum_{i=1}^{m} \mathrm{Cov}\big(\xi_i, f(W^i)\big)\right| \le -\sqrt{\frac{2}{\pi}}\sum_{i=1}^{m} \mathrm{Cov}\big(\xi_i, W^i\big) = -\sqrt{\frac{2}{\pi}}\sum_{i\neq j} \sigma_{ij}. \tag{15}$$

Summing the bounds (10)-(15), taking supremum over $h \in \mathbb{L}$ and using the form of the L^1 distance given in (6), we obtain

$$d_1\big(\mathcal{L}(W), \mathcal{L}(Z)\big) \le 5B - 9B\sum_{i\neq j}\sigma_{ij} - \sqrt{\frac{8}{\pi}}\sum_{i\neq j}\sigma_{ij}.$$

Using the fact that $d_1(\cdot,\cdot) \le 2$ and that $\sigma_{i,j}$ are negative, we can assume that $B \le 0.4$ and thus the last expression is bounded by the right hand side of (4).

□

Next we use the following result which is slightly different from Lemma 2.6 of [CGS11] due to [Bar90] to prove Theorem 1.4. Let $\mathbf{Z}$ be a standard normal random vector in $\mathbb{R}^p$. For $h : \mathbb{R}^p \to \mathbb{R}$ let $Nh = \mathbb{E}h(\mathbf{Z})$ and for $u \ge 0$ define

$$(T_u h)(\mathbf{s}) = \mathbb{E}h(\mathbf{s}e^{-u} + \sqrt{1 - e^{-2u}}\mathbf{Z}).$$

We write $D^2 h$ for the Hessian matrix of h when it exists.

Lemma 2.3 *For $m \ge 3$ and $h \in L_m^\infty(\mathbb{R}^p)$ the function*

$$g(\mathbf{s}) = -\int_0^\infty [T_u h(\mathbf{s}) - Nh]du$$

solves

$$\mathrm{tr}D^2 g(\mathbf{s}) - \mathbf{s}\cdot\nabla g(\mathbf{s}) = h(\mathbf{s}) - Nh,$$

and for any $0 \le |\mathbf{k}|_1 \le m$

$$|g^{(\mathbf{k})}|_\infty \le \frac{1}{|\mathbf{k}|_1}|h^{(\mathbf{k})}|_\infty.$$

Furthermore, for any $\boldsymbol{\lambda} \in \mathbb{R}^p$ and positive definite $p \times p$ matrix Σ, f defined by the change of variable

$$f(\mathbf{s}) = g(\Sigma^{-1/2}(\mathbf{s} - \boldsymbol{\lambda})) \tag{16}$$

solves

$$tr\Sigma D^2 f(\mathbf{s}) - (\mathbf{s} - \boldsymbol{\lambda})\cdot\nabla f(\mathbf{s}) = h(\Sigma^{-1/2}(\mathbf{s} - \boldsymbol{\lambda})) - Nh, \tag{17}$$

and satisfies

$$|f^{(\mathbf{k})}|_\infty \le \frac{p^{|\mathbf{k}|_1}}{|\mathbf{k}|_1}|\Sigma^{-1/2}|_\infty^{|\mathbf{k}|_1}|h^{(\mathbf{k})}|_\infty.$$

In particular, if $h \in \mathcal{H}_{m,\infty,p}$ then

$$|f^{(\mathbf{k})}|_\infty \le \frac{p^{|\mathbf{k}|_1}}{|\mathbf{k}|_1}|\Sigma^{-1/2}|_\infty^{|\mathbf{k}|_1} \quad \textit{for all} \;\; 0 \le |\mathbf{k}|_1 \le m. \tag{18}$$

We apply the same technique as in the univariate case, along with Lemmas 2.2 and 2.3, to prove our main multivariate theorem below.

Proof of Theorem 1.4 Given $h \in \mathcal{H}_{3,\infty,p}$, let f be the solution of (17) given by (16) with $\boldsymbol{\lambda} = \mathbf{0}$. Writing out the expressions in (17) yields

$$\begin{aligned}
\mathbb{E}\left[h(\Sigma^{-1/2}\mathbf{S}) - Nh\right] &= \mathbb{E}\left[\sum_{j=1}^{p}\sum_{l=1}^{p}\Sigma_{j,l}\frac{\partial^2}{\partial s_j \partial s_l}f(\mathbf{S}) - \sum_{j=1}^{p} S_j \frac{\partial}{\partial s_j}f(\mathbf{S})\right] \\
&= \mathbb{E}\sum_{j=1}^{p}\Sigma_{j,j}\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}) + \mathbb{E}\sum_{j,l\in[p],j\neq l}\Sigma_{j,l}\frac{\partial^2}{\partial s_j \partial s_l}f(\mathbf{S}) - \mathbb{E}\sum_{j=1}^{p} S_j\frac{\partial}{\partial s_j}f(\mathbf{S}). \quad (19)
\end{aligned}$$

We consider the first term of (19) and deal with each term under the sum separately for $j = 1, \ldots, p$. Letting $\sigma_{i,j}^2 := \mathrm{Var}\big(\xi_{i,j}\big)$ and $\sigma_{i,j;k,l} := \mathrm{Cov}\big(\xi_{i,j}, \xi_{k,l}\big)$, we have

$$\begin{aligned}
\Sigma_{j,j}\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}) &= \sum_{i=1}^{m}\sigma_{i,j}^2\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}) + \sum_{i,k\in[m],i\neq k}\sigma_{i,j;k,j}\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}) \\
&= \sum_{i=1}^{m}\xi_{i,j}^2\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}) + \sum_{i,k\in[m],i\neq k}\sigma_{i,j;k,j}\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}) + \sum_{i=1}^{m}(\sigma_{i,j}^2 - \xi_{i,j}^2)\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}). \quad (20)
\end{aligned}$$

Now, with $S_{j*i} := S_j - \xi_{i,j}$ we write the summands of the third term on the right hand side of (19) as

$$\begin{aligned}
S_j\frac{\partial}{\partial s_j}f(\mathbf{S}) &= \sum_{i=1}^{m}\xi_{i,j}\frac{\partial}{\partial s_j}f(\mathbf{S}) \\
&= \sum_{i=1}^{m}\xi_{i,j}\frac{\partial}{\partial s_j}f(S_1, \ldots, S_{j*i}, \ldots, S_p) \\
&\quad + \sum_{i=1}^{m}\xi_{i,j}^2\int_0^1\frac{\partial^2}{\partial s_j^2}f(S_1, \ldots, S_{j*i} + u\xi_{i,j}, \ldots, S_p)du. \quad (21)
\end{aligned}$$

Substituting (20) and (21) into (19) and letting $\mathbf{S}^{j*i} =: (S_1, \ldots, S_{j-1}, S_{j*i}, S_{j+1}, \ldots, S_p)$, we obtain

$$\begin{aligned}
\mathbb{E}\left[h(\Sigma^{-1/2}\mathbf{S}) - Nh\right] &= \mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}\xi_{i,j}^2\int_0^1\left(\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}) - \frac{\partial^2}{\partial s_j^2}f(S_1, \ldots, S_{j*i} + u\xi_{i,j}, \ldots, S_p)\right)du \\
&\quad + \mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}(\sigma_{i,j}^2 - \xi_{i,j}^2)\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}) - \mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}\xi_{i,j}\frac{\partial}{\partial s_j}f(\mathbf{S}^{j*i}) \\
&\quad + \mathbb{E}\sum_{j=1}^{p}\sum_{i,k\in[m],i\neq k}\sigma_{i,j;k,j}\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}) + \mathbb{E}\sum_{j,l\in[p],j\neq l}\Sigma_{j,l}\frac{\partial^2}{\partial s_j \partial s_l}f(\mathbf{S}). \quad (22)
\end{aligned}$$

Now we handle these five terms in (22) separately. For the first term, using (18) we have

$$\begin{aligned}
&\left|\mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m} \quad \xi_{i,j}^2 \quad \int_0^1 \left(\frac{\partial^2}{\partial s_j^2} f(\mathbf{S}) - \frac{\partial^2}{\partial s_j^2} f(S_1,\ldots,S_{j*i}+u\xi_{i,j},\ldots,S_p)\right) du\right| \\
&\qquad = \left|\mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}\xi_{i,j}^2\int_0^1\int_{u\xi_{i,j}}^{\xi_{i,j}}\frac{\partial^3}{\partial s_j^3} f(S_1,\ldots,S_{j*i}+t,\ldots,S_p)dtdu\right| \\
&\qquad \le \frac{p^3}{3}|\Sigma^{-1/2}|_\infty^3 \mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}\xi_{i,j}^2\int_0^1\int_{u|\xi_{i,j}|}^{|\xi_{i,j}|} dtdu \\
&\qquad = \frac{p^3}{6}|\Sigma^{-1/2}|_\infty^3 \sum_{j=1}^{p}\sum_{i=1}^{m}\mathbb{E}|\xi_{i,j}|^3 \\
&\qquad \le \frac{p^3}{6}|\Sigma^{-1/2}|_\infty^3 B\sum_{j=1}^{p}\sum_{i=1}^{m}\mathbb{E}\xi_{i,j}^2 \\
&\qquad = \frac{p^3}{6}|\Sigma^{-1/2}|_\infty^3 B\sum_{j=1}^{p}\left(\Sigma_{j,j} - \sum_{i,k\in[m],i\neq k}\sigma_{i,j;k,j}\right), \qquad (23)
\end{aligned}$$

where we have used the almost sure bound on the variables $\xi_{i,j}$, and that their sum S_j over i from 1 to m has mean zero in the last two inequalities, respectively.

For the second term in (22), we first bound it by the three terms denoted by I_1, I_2, I_3, respectively,

$$\begin{aligned}
&\left|\mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}(\sigma_{i,j}^2-\xi_{i,j}^2)\frac{\partial^2}{\partial s_j^2}f(\mathbf{S})\right| \\
&\quad \le \left|\mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}(\sigma_{i,j}^2-\xi_{i,j}^2)\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}^{j*i})\right| + \left|\mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}(\sigma_{i,j}^2-\xi_{i,j}^2)\left(\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}^{j*i})-\frac{\partial^2}{\partial s_j^2}f(\mathbf{S})\right)\right| \\
&\quad \le \left|\sum_{j=1}^{p}\sum_{i=1}^{m}\mathrm{Cov}\left(\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}^{j*i}),\xi_{i,j}^2\right)\right| + \mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}\sigma_{i,j}^2\left|\left(\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}^{j*i})-\frac{\partial^2}{\partial s_j^2}f(\mathbf{S})\right)\right| \\
&\qquad +\mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}\xi_{i,j}^2\left|\left(\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}^{j*i})-\frac{\partial^2}{\partial s_j^2}f(\mathbf{S})\right)\right| := I_1+I_2+I_3.
\end{aligned}$$

Then we write I_1 as

$$I_1 = \left|\sum_{j=1}^{p}\sum_{i=1}^{m}\mathrm{Cov}\left(\frac{\partial^2}{\partial s_j^2}f(\mathbf{S}^{j*i}),g(\xi_{i,j})\right)\right|,$$

where

$$g(x) = \begin{cases} x^2 & |x|\le B \\ B^2 & |x|>B, \end{cases}$$

As $\mathbf{S}^{j*i}$ and $\xi_{i,j}$ are increasing functions defined on disjoint subsets of $\boldsymbol{\xi}$, by Lemma 2.1, $(\mathbf{S}^{j*i}, \xi_{i,j})$

are negatively associated for all i, j. Applying Lemma 2.2 and using the bound (18), we obtain

$$\begin{aligned}
I_1 &\le \left|\sum_{j=1}^{p}\sum_{i=1}^{m}\left(\sum_{l\in[p],l\neq j}\left|\frac{\partial^3}{\partial s_l \partial s_j^2}f\right|_\infty \left|\frac{\partial g}{\partial x}\right|_\infty \mathrm{Cov}\left(S_l, \xi_{i,j}\right) + \left|\frac{\partial^3}{\partial s_j^3}f\right|_\infty \left|\frac{\partial g}{\partial x}\right|_\infty \mathrm{Cov}\left(S_{j*i}, \xi_{i,j}\right)\right)\right| \\
&\le \frac{2}{3}p^3|\Sigma^{-1/2}|_\infty^3 B\left|\sum_{j=1}^{p}\sum_{i=1}^{m}\left(\sum_{l\in[p],l\neq j}\mathrm{Cov}\big(S_l, \xi_{i,j}\big) + \mathrm{Cov}\left(S_{j*i}, \xi_{i,j}\right)\right)\right| \\
&= \frac{2}{3}p^3|\Sigma^{-1/2}|_\infty^3 B\left|\sum_{j,l\in[p],j\neq l}\sum_{i,k=1}^{m}\sigma_{i,j;k,l} + \sum_{j=1}^{p}\sum_{i,k\in[m],i\neq k}\sigma_{i,j;k,j}\right| \\
&= -\frac{2}{3}p^3|\Sigma^{-1/2}|_\infty^3 B\left(\sum_{j,l\in[p],j\neq l}\Sigma_{j,l} + \sum_{j=1}^{p}\sum_{i,k\in[m],i\neq k}\sigma_{i,j;k,j}\right). \qquad (24)
\end{aligned}$$

Again using (18), we have

$$\begin{aligned}
I_2 &\le \sum_{j=1}^{p}\sum_{i=1}^{m}\sigma_{i,j}^2\left|\frac{\partial^3}{\partial s_j^3}f\right|_\infty \mathbb{E}\left|S_{j*i} - S_j\right| = \sum_{j=1}^{p}\sum_{i=1}^{m}\sigma_{i,j}^2\left|\frac{\partial^3}{\partial s_j^3}f\right|_\infty \mathbb{E}\left|\xi_{i,j}\right| \\
&\le \frac{p^3}{3}|\Sigma^{-1/2}|_\infty^3 B\sum_{j=1}^{p}\left(\Sigma_{j,j} - \sum_{i,k\in[m],i\neq k}\sigma_{i,j;k,j}\right), \qquad (25)
\end{aligned}$$

and

$$\begin{aligned}
I_3 &\le \sum_{j=1}^{p}\sum_{i=1}^{m}\left|\frac{\partial^3}{\partial s_j^3}f\right|_\infty \mathbb{E}\xi_{i,j}^2\left|S_{j*i} - S_j\right| \le \frac{p^3}{3}|\Sigma^{-1/2}|_\infty^3 B\sum_{j=1}^{p}\sum_{i=1}^{m}\mathbb{E}\xi_{i,j}^2 \\
&= \frac{p^3}{3}|\Sigma^{-1/2}|_\infty^3 B\sum_{j=1}^{p}\left(\Sigma_{j,j} - \sum_{i,k\in[m],i\neq k}\sigma_{i,j;k,j}\right). \qquad (26)
\end{aligned}$$

For the third term in (22), again applying Lemma 2.2 and arguing as for I_1 in (24), we have

$$\begin{aligned}
&\left|\mathbb{E}\sum_{j=1}^{p}\sum_{i=1}^{m}\xi_{i,j}\frac{\partial}{\partial s_j}f(\mathbf{S}^{j*i})\right| = \left|\sum_{j=1}^{p}\sum_{i=1}^{m}\mathrm{Cov}\left(\xi_{i,j}, \frac{\partial}{\partial s_j}f(\mathbf{S}^{j*i})\right)\right| \\
&\quad\le \left|\sum_{j=1}^{p}\sum_{i=1}^{m}\left(\sum_{l\in[p]\setminus\{j\}}\left|\frac{\partial^2}{\partial s_l \partial s_j}f\right|_\infty \mathrm{Cov}\left(\xi_{i,j}, S_l\right) + \left|\frac{\partial^2}{\partial s_j^2}f\right|_\infty \mathrm{Cov}\left(\xi_{i,j}, S_{j*i}\right)\right)\right| \\
&\quad\le -\frac{p^2}{2}|\Sigma^{-1/2}|_\infty^2\left(\sum_{j,l\in[p],j\neq l}\sum_{i=1}^{m}\mathrm{Cov}\left(\xi_{i,j}, S_l\right) + \sum_{j=1}^{p}\sum_{i=1}^{m}\mathrm{Cov}\left(\xi_{i,j}, S_{j*i}\right)\right) \\
&\quad= -\frac{p^2}{2}|\Sigma^{-1/2}|_\infty^2\left(\sum_{j,l\in[p],j\neq l}\sum_{i,k=1}^{m}\sigma_{i,j;k,l} + \sum_{j=1}^{p}\sum_{i,k\in[m],i\neq k}\sigma_{i,j;k,j}\right) \\
&\quad= -\frac{p^2}{2}|\Sigma^{-1/2}|_\infty^2\left(\sum_{j,l\in[p],j\neq l}\Sigma_{j,l} + \sum_{j=1}^{p}\sum_{i,k\in[m],i\neq k}\sigma_{i,j;k,j}\right). \qquad (27)
\end{aligned}$$

For the fourth and the fifth terms in (22), again using (18) we obtain

$$\left| \sum_{j=1}^{p} \sum_{i,k \in [m], i \neq k} \sigma_{i,j;k,j} \frac{\partial^2}{\partial s_j^2} f(\mathbf{S}) \right| \leq -\frac{p^2}{2} |\Sigma^{-1/2}|_\infty^2 \sum_{j=1}^{p} \sum_{i,k \in [m], i \neq k} \sigma_{i,j;k,j} \tag{28}$$

and

$$\left| \mathbb{E} \sum_{j,l \in [p], j \neq l} \Sigma_{j,l} \frac{\partial^2}{\partial s_j \partial s_l} f(\mathbf{S}) \right| \leq -\frac{p^2}{2} |\Sigma^{-1/2}|_\infty^2 \sum_{j,l \in [p], j \neq l} \Sigma_{j,l}. \tag{29}$$

Summing the bounds (23)-(29) we find that $\left| \mathbb{E} \left[h(\Sigma^{-1/2}\mathbf{S}) - Nh \right] \right|$ is bounded by the right hand side of (5). Taking supremum over $h \in \mathcal{H}_{3,\infty,p}$ and using the definition (3) of $d_{\mathcal{H}_{m,\infty,p}}$, yields the claim.

□

3 Applications

In this section, we follow the same structure as in Section 2.1 of [GW18] with positive association replaced by negative association. In particular, we apply our main theorems in the first section to second order stationary negatively associated random fields with exponential covariance decay.

First we introduce the definitions and notations used in [GW18] that will also be used here. Let $\{X_\mathbf{j} : \mathbf{j} \in \mathbb{Z}^d\}$ be a negatively associated random field on the d-dimensional integer lattice $\mathbb{Z}^d$ and assume that the field is second order stationary. We recall that a random field $\{X_\mathbf{j} : \mathbf{j} \in \mathbb{Z}^d\}$ is called *second order stationary* when $\mathbb{E}X_\mathbf{j}^2 < \infty$ for all $\mathbf{j} \in \mathbb{Z}^d$ and the covariance $\mathrm{Cov}\big(X_\mathbf{i}, X_\mathbf{j}\big) = R(\mathbf{j} - \mathbf{i})$ for all $\mathbf{i}, \mathbf{j} \in \mathbb{R}^d$, with $R(\cdot)$ given by

$$R(\mathbf{k}) = \mathrm{Cov}\big(X_\mathbf{0}, X_\mathbf{k}\big). \tag{30}$$

We let $\mathbf{1} \in \mathbb{Z}^d$ denotes the vector with all components 1, and write inequalities such as $\mathbf{a} < \mathbf{b}$ for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ when they hold componentwise. For $\mathbf{k} \in \mathbb{Z}^d, n \in \mathbb{N}_1$, define the 'block sum' variables, over a block with side length n, by

$$S_\mathbf{k}^n = \sum_{\mathbf{j} \in B_\mathbf{k}^n} X_\mathbf{j} \quad \text{where} \quad B_\mathbf{k}^n = \left\{ \mathbf{j} \in \mathbb{Z}^d : \mathbf{k} \leq \mathbf{j} < \mathbf{k} + n\mathbf{1} \right\}. \tag{31}$$

Note that $B_\mathbf{k}^n = B_\mathbf{0}^n + \mathbf{k}$.

For $R(\cdot)$ given in (30), we have

$$\mathrm{Var}\big(S_\mathbf{k}^n\big) = \sum_{\mathbf{i},\mathbf{j} \in B_\mathbf{k}^n} \mathrm{Cov}\big(X_\mathbf{i}, X_\mathbf{j}\big) = n^d A_n \quad \text{where} \quad A_n = \frac{1}{n^d} \sum_{\mathbf{i},\mathbf{j} \in B_\mathbf{1}^n} R(\mathbf{i} - \mathbf{j}). \tag{32}$$

Since the field is negatively associated, $R(\mathbf{k}) \leq 0$ for all $\mathbf{k} \neq \mathbf{0}$, which implies that $0 \leq A_n \leq R(\mathbf{0})$ for all $n \in \mathbb{N}_1$. For simplicity, in this work, we assume that $\inf_n A_n > 0$ which implies that A_n is of constant order. With this assumption, we may include A_n in our bounds without affecting the rate of convergence.

With $S_\mathbf{k}^n$ defined in (31), we consider the standardized variables

$$W_\mathbf{k}^n = \frac{S_\mathbf{k}^n - \mathbb{E}S_\mathbf{k}^n}{\sqrt{n^d A_n}}, \quad \mathbf{k} \in \mathbb{Z}^d, n \in \mathbb{N}_1, \tag{33}$$

that have mean zero and variance 1. The following theorem provides a bound of order $n^{-d/(2d+2)}$ with an explicit constant on the L^1 distance between the distribution of $W_{\mathbf{k}}^n$ and the normal under the assumption that the covariance function $R(\cdot)$ decays at exponential rate in the L^1 norm in $\mathbb{R}^d$. Since all norms in $\mathbb{R}^d$ are equivalent, we use the L^1 norm that makes our calculation simplest.

Theorem 3.1 *Let $d \in \mathbb{N}_1$ and $\{X_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^d\}$ be a negatively associated second order stationary random field with covariance function $R(\mathbf{k}) = \mathrm{Cov}(X_{\mathbf{j}}, X_{\mathbf{j}+\mathbf{k}})$ for all $\mathbf{j}, \mathbf{k} \in \mathbb{Z}^d$, and suppose that for some $K > 0$, $|X_{\mathbf{j}}| \le K$ a.s. for all $\mathbf{j} \in \mathbb{Z}^d$. Assume that there exist $\lambda > 0$ and $\kappa_0 > 0$ such that*

$$-R(\mathbf{k}) \le \kappa_0 e^{-\lambda|\mathbf{k}|_1} \quad \textit{for all } \mathbf{k} \in \mathbb{Z}^d/\{\mathbf{0}\} \tag{34}$$

and $\inf_n A_n > 0$ where A_n is given in (32). Let

$$\mu_\lambda = \frac{e^\lambda}{(e^\lambda - 1)^2}, \quad \nu_\lambda = \frac{e^{2\lambda}}{(e^\lambda - 1)^2} \quad \textit{and} \quad \gamma_{\lambda,d} = (4\mu_\lambda + 2\nu_\lambda)^d - (2\nu_\lambda)^d \tag{35}$$

and

$$C_{\lambda,\kappa_0,d} = \frac{10Kd\sqrt{A_n}}{5.2\kappa_0\gamma_{\lambda,d}} \tag{36}$$

Then, for any $\mathbf{k} \in \mathbb{Z}^d$, with $W_{\mathbf{k}}^n$ as given in (33) and Z a standard normal random variable,

$$d_1\big(\mathcal{L}(W_{\mathbf{k}}^n), \mathcal{L}(Z)\big) \le \frac{\kappa_1}{n^{d/(2d+2)}} \quad \textit{for all} \quad n \ge \max\left\{C_{\lambda,\kappa_0,d}^{2/d}, C_{\lambda,\kappa_0,d}^{-2/(d+2)}\right\}, \tag{37}$$

where

$$\kappa_1 = \left(\frac{10K(5.2\kappa_0\gamma_{\lambda,d})^d}{A_n^{d+1/2}}\right)^{1/(d+1)} \left(\frac{1}{d^{\frac{d}{d+1}}} + 2d^{\frac{1}{d+1}}\right).$$

The bound in (37) is of order $n^{-d/(2d+2)}$ recalling that A_n is bounded away from zero and infinity and hence does not contribute the rate. We also extend Theorem 3.1 to the multidimensional case. For any $p \in \mathbb{N}_1$ and indices $\mathbf{k}_1, \ldots, \mathbf{k}_p$ in $\mathbb{Z}^d$ such that $B_{\mathbf{k}_i}^n, i \in [p]$ are disjoint, Theorem 3.2 provides a bound in the metric $d_{\mathcal{H}_{3,\infty,p}}$ to the multivariate normal for $\mathbf{S}^n = (S_{\mathbf{k}_1}^n, \ldots, S_{\mathbf{k}_p}^n)$ under exponential decay of the covariance function. We note the difference between the assumption (38) below and the one in (30) of [GW18] that we require $B_{\mathbf{k}_i}^n, i \in [p]$ here to be disjoint, otherwise, there exists $\mathbf{j} \in \mathbb{Z}^d$ that belongs to both $B_{\mathbf{k}_i}^n$ and $B_{\mathbf{k}_j}^n$ for some i, j and $(X_{\mathbf{j}}, X_{\mathbf{j}})$ is positively associated to which Theorem 1.4 is not applicable. The same issue does not arise in the positively associated case as a pair of the same variable has positive covariance. In the following result and its proof, constants will not be tracked with precision, but will be indexed by the set of variables on which it depends.

Theorem 3.2 *For $d \in \mathbb{N}_1$, let $\{X_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^d\}$ be a negatively associated second order stationary random field with covariance function $R(\mathbf{k}) = \mathrm{Cov}(X_{\mathbf{j}}, X_{\mathbf{j}+\mathbf{k}})$ for all $\mathbf{j}, \mathbf{k} \in \mathbb{Z}^d$, and suppose that there exist constants $K > 0, \kappa_0 > 0$ and $\lambda > 0$ such that $|X_{\mathbf{j}}| \le K$ a.s. for all $\mathbf{j} \in \mathbb{Z}^d$,*

$$-R(\mathbf{k}) \le \kappa_0 e^{-\lambda|\mathbf{k}|_1} \quad \textit{for all} \;\; \mathbf{k} \in \mathbb{Z}^d/\{\mathbf{0}\},$$

and $\inf_n A_n > 0$ where A_n is given in (32). For $p \in \mathbb{N}_1$ let $\mathbf{k}_1, \ldots, \mathbf{k}_p \in \mathbb{Z}^d$ be such that

$$\min_{q,s \in [p], q \ne s} |\mathbf{k}_q - \mathbf{k}_s|_\infty \ge n. \tag{38}$$

Let $\mathbf{S}^n = (S^n_{\mathbf{k}_1}, \dots, S^n_{\mathbf{k}_p})$*, where* $S^n_{\mathbf{k}}$ *is defined as in* (31) *and* Σ *be the covariance matrix of* $\mathbf{S}^n$*. Then, for* $n > (p-1)\kappa_0 \nu_\lambda^d e^{-\lambda} A_n^{-1}$ *with* ν_λ *as in* (35)*,* Σ *is invertible and*

$$|\Sigma^{-1}|_\infty \le \frac{1}{n^{d-1}(nA_n - (p-1)\kappa_0 \nu_\lambda^d e^{-\lambda})}. \tag{39}$$

Furthermore, with

$$\psi_n = n^{d/2} |\Sigma^{-1/2}|_\infty \quad \text{for} \quad n > (p-1)\kappa_0 \nu_\lambda^d e^{-\lambda} A_n^{-1} \quad \text{and} \quad B_{n,d} = d\psi_n A_n$$

there exists a constant $C_{\lambda,\kappa_0,d,p,K}$ *such that, for*

$$n > \max\left\{ B_{n,d}^{2/d}, B_{n,d}^{-2/(d+2)}, (p-1)\kappa_0 \nu_\lambda^d e^{-\lambda} A_n^{-1} \right\}, \tag{40}$$

$$\begin{aligned} d_{\mathcal{H}_{3,\infty,p}}&\big(\mathcal{L}(\Sigma^{-1/2}(\mathbf{S}^n - \mathbb{E}\mathbf{S}^n)), \mathcal{L}(\mathbf{Z})\big) \\ &\le C_{\lambda,\kappa_0,d,p,K} \left(\frac{\psi_n^{(2d+4)/(d+1)}}{A_n^{(d-1)/(d+1)} n^{d/(d+1)}} + \frac{\psi_n^{(2d+3)/(d+1)}}{A_n^{d/(d+1)} n^{(3d+2)/(2d+2)}} \right. \\ &\qquad \left. + \frac{\psi_n^2}{n} + \frac{A_n^{1/(d+1)} \psi_n^{(2d+3)/(d+1)}}{n^{d/(2d+2)}} \right), \end{aligned} \tag{41}$$

where $\mathbf{Z}$ *is a standard normal random vector in* $\mathbb{R}^p$*.*

Since A_n is of constant order, $|\Sigma^{-1/2}|_\infty$ is of order at most $n^{-d/2}$ by (39). This implies that ψ_n is of at most constant order and thus so is $B_{n,d}$. Therefore the last term on the right hand side of (41) is the only one that contributes the rate of convergence of order $n^{-d/(2d+2)}$ as the other terms converge to zero at much faster rates. We note that the bounds in Theorems 3.1 and 3.2 have the same order as the ones in Theorems 2.1 and 2.2 of [GW18], respectively. However, comparing to the results of [GW18], the constant of the bound (37) is bigger and the bound (41) has the extra terms that do not contribute the rate.

To prove Theorems 3.1 and 3.2, we use the same technique as in [GW18] decomposing the sum $S^n_{\mathbf{k}}$ over the block $B^n_{\mathbf{k}}$ into sums over smaller, disjoint blocks whose side lengths are at most some integer l. That is, for $1 \le l \le n$, we uniquely write $n = (m-1)l + r$ with $m \ge 1$ and $1 \le r \le l$ and correspondingly decompose $B^n_{\mathbf{k}}$ into m^d disjoint blocks $D^l_{\mathbf{i},\mathbf{k}}, \mathbf{i} \in [m]^d$, where there are $(m-1)^d$ 'main' blocks having all sides of length l, and $m^d - (m-1)^d$ remainder blocks having all sides of length r or l, with at least one side of length r.

To be more precise, for $\mathbf{k} \in \mathbb{Z}^d$ and $\mathbf{i} \in [m]^d$ set $D^l_{\mathbf{i},\mathbf{k}} = D^l_{\mathbf{i}} + \mathbf{k} - \mathbf{1}$ where

$$\begin{aligned} D^l_{\mathbf{i}} \quad = \quad &\big\{ \mathbf{j} \in \mathbb{Z}^d : (i_s - 1)l + 1 \le j_s \le i_s l \text{ for } i_s \ne m, \\ &\qquad (m-1)l + 1 \le j_s \le (m-1)l + r \text{ for } i_s = m \big\}. \end{aligned}$$

It is easy to see that for $\mathbf{i} \in [m-1]^d$, the vectors indexing the 'main blocks', we have

$$D^l_{\mathbf{i}} = B^l_{(\mathbf{i}-\mathbf{1})l+\mathbf{1}} \quad \text{for} \quad \mathbf{i} \in [m-1]^d, \tag{42}$$

and if $r = l$ then $D^l_{\mathbf{i}}$ is given by (42) for all $i \in [m]^d$. Furthermore, it is straightforward to verify that the elements of the collection $\{D^l_{\mathbf{i},\mathbf{k}}, \mathbf{i} \in [m]^d\}$ is the partition of $B^n_{\mathbf{k}}$.

Letting

$$\xi^l_{\mathbf{i},\mathbf{k}} = \sum_{\mathbf{t} \in D^l_{\mathbf{i},\mathbf{k}}} (X_{\mathbf{t}} - \mathbb{E}X_{\mathbf{t}}) \quad \text{for} \quad \mathbf{i} \in [m]^d, \quad \text{and} \quad W^n_{\mathbf{k}} = \sum_{\mathbf{i} \in [m]^d} \frac{\xi^l_{\mathbf{i},\mathbf{k}}}{\sqrt{n^d A_n}}, \tag{43}$$

we see that $\xi^n_{\mathbf{i},\mathbf{k}}$ has mean zero, and $W^n_{\mathbf{k}}$ as in (43) agrees with its representation as given in (33), and has mean zero and variance one. For simplicity we will drop the index $\mathbf{k}$ in $\xi_{\mathbf{i},\mathbf{k}}$ when $\mathbf{k} = \mathbf{1}$, as we do also for $D_{\mathbf{i},\mathbf{k}}$, and also suppress n in $\xi^n_{\mathbf{i},\mathbf{k}}$.

As the elements of $\{\xi_{\mathbf{i},\mathbf{k}} : \mathbf{i} \in [m]^d\}$ are increasing functions of disjoint subsets of $\{X_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^d\}$, they are negatively associated by Lemma 2.1. We prove Theorems 3.1 and 3.2 with the help of the following three lemmas. The first, Lemma 3.3 bounds the sum of the covariances between $\xi^l_{\mathbf{i},\mathbf{k}}$ and $\xi^l_{\mathbf{j},\mathbf{k}}$, defined in (43), over $\mathbf{i}, \mathbf{j} \in [m]^d$. Next, we state Lemma 3.4, proved in [GW18], which is used in the proof of Lemma 3.5 that bounds the covariance between two non-overlapped block sums of size n^d.

Lemma 3.3 *Let $\{X_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^d\}$ be a second order, negatively associated stationary random field with covariance function $R(\mathbf{k}) = \mathrm{Cov}(X_{\mathbf{j}}, X_{\mathbf{j}+\mathbf{k}})$ for all $\mathbf{j} \in \mathbb{Z}^d$ and $\mathbf{k} \in \mathbb{Z}^d/\{\mathbf{0}\}$ where $R(\cdot)$ satisfies the exponential decay condition* (34)*. For $n \geq 2$ and $1 \leq l \leq n$, let $n = (m-1)l + r$ for integers $m \in \mathbb{N}_1$ and $1 \leq r \leq l$. Then for $\mathbf{i} \in [m]$ and $\mathbf{k} \in \mathbb{Z}^d$, with $\xi_{\mathbf{i},\mathbf{k}}$ given by* (43) *we have*

$$\sum_{\mathbf{i},\mathbf{j} \in [m]^d, \mathbf{i} \neq \mathbf{j}} -\mathbb{E}\left[\xi^l_{\mathbf{i},\mathbf{k}} \xi^l_{\mathbf{j},\mathbf{k}}\right] \leq \frac{\kappa_0 \gamma_{\lambda,d} n^d}{l},$$

where κ_0 is given in (34)*, and $\gamma_{\lambda,d}$ in* (35)*.*

Proof: We note that the difference between (34) of this work and (22) of [GW18] is only the sign on the left hand side of the inequality. Thus the proof follows immediately from the proof of Lemma 2.4 of [GW18] with $\mathbb{E}\left[\xi^l_{\mathbf{i},\mathbf{k}} \xi^l_{\mathbf{j},\mathbf{k}}\right]$ and $R(\mathbf{k})$ replaced by $-\mathbb{E}\left[\xi^l_{\mathbf{i},\mathbf{k}} \xi^l_{\mathbf{j},\mathbf{k}}\right]$ and $-R(\mathbf{k})$, respectively.

□

Lemma 3.4 ([GW18]) *For all $n \in \mathbb{N}_2$ and $\lambda > 0$,*

$$\sum_{a=-n+1}^{n-1} (n - |a|) e^{-\lambda |q+a|} \quad \textit{is decreasing as a function of } |q| \in \mathbb{N}_0.$$

In the following we will use the identities

$$\sum_{k=1}^{n-1} (n-k) w^k = \frac{w\left((n-1) - nw + w^n\right)}{(w-1)^2} \quad \text{for} \quad w \neq 1, \tag{44}$$

and

$$n + 2\sum_{b=1}^{n-1} (n-b) u^b = \frac{(1-u^2)n - 2u + 2u^{n+1}}{(u-1)^2} \quad \text{for} \quad u \neq 1. \tag{45}$$

Lemma 3.5 *Let $\{X_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^d\}$ be a second order stationary random field with covariance function $R(\mathbf{k}) = \mathrm{Cov}(X_{\mathbf{j}}, X_{\mathbf{j}+\mathbf{k}})$ for all $\mathbf{j} \in \mathbb{Z}^d$ and $\mathbf{k} \in \mathbb{Z}^d/\{\mathbf{0}\}$ where $R(\cdot)$ satisfies* (34)*. Let $n \in \mathbb{N}_2$, $\mathbf{k}_1$ and $\mathbf{k}_2$ be vectors in $\mathbb{Z}^d$ such that*

$$|\mathbf{k}_1 - \mathbf{k}_2|_\infty \geq n,$$

then with λ and κ_0 as in (34)*, and ν_λ as in* (35)*,*

$$-\mathrm{Cov}\left(S^n_{\mathbf{k}_1}, S^n_{\mathbf{k}_2}\right) \leq \kappa_0 \nu^d_\lambda e^{-\lambda} n^{d-1}.$$

Proof: Using the definition of $S^n_{\mathbf{k}}$ and that $X_{\mathbf{j}}, \mathbf{j} \in \mathbb{Z}^d$ are second order stationary, we have

$$\begin{aligned}
-\mathrm{Cov}\left(S^n_{\mathbf{k}_1}, S^n_{\mathbf{k}_2}\right) &= -\sum_{\substack{p_1,\ldots,p_d=0\\ q_1,\ldots,q_d=0}}^{n-1} R\left(\begin{bmatrix}(p_1+k_1^2)-(q_1+k_1^1)\\ \vdots \\ (p_d+k_d^2)-(q_d+k_d^1)\end{bmatrix}\right)\\
&\le \kappa_0 \sum_{a_1,\ldots,a_d=-n+1}^{n-1} (n-|a_1|)\ldots(n-|a_d|)\exp\left(-\lambda\left|\begin{bmatrix}a_1+(k_1^2-k_1^1)\\ \vdots \\ a_d+(k_d^2-k_d^1)\end{bmatrix}\right|_1\right)\\
&= \kappa_0 \prod_{i=1}^{d}\sum_{a_i=-n+1}^{n-1}(n-|a_i|)e^{-\lambda|(k_i^2-k_i^1)+a_i|}. \qquad (46)
\end{aligned}$$

Applying Lemma 3.4, we have

$$\sum_{a_i=-n+1}^{n-1}(n-|a_i|)e^{-\lambda|(k_i^2-k_i^1)+a_i|} \quad \text{is a decreasing function of } |k_i^1-k_i^2|. \qquad (47)$$

Hence the i^{th} sum appearing in the product (46) is maximized by its value when $k_i^1 = k_i^2$. As $|\mathbf{k}_1-\mathbf{k}_2|_\infty \ge n$, there must exist at least one i for which $|k_i^2-k_i^1| \ge n$, and whose corresponding sum is bounded by its value when $|k_i^2-k_i^1|$ is exactly n, using (47). The product of these sums, by (47) again, is maximized when there is just a single coordinate achieving n as its absolute difference, and where this difference in all other terms achieve equality to zero. Therefore, by symmetry (46) is bounded by the case where $k_i^1 = k_i^2$ for $i \in [d-1]$ and $k_d^2 - k_d^1 = n$ and thus

$$\begin{aligned}
-\mathrm{Cov}\left(S^n_{\mathbf{k}_1}, S^n_{\mathbf{k}_2}\right) &\le \kappa_0 \prod_{i=1}^{d-1}\sum_{a_i=-n+1}^{n-1}(n-|a_i|)e^{-\lambda|a_i|}\sum_{a_d=-n+1}^{n-1}(n-|a_d|)e^{-\lambda|a_d+n|}\\
&\le \kappa_0\left(n\nu_\lambda\right)^{d-1}\sum_{a_d=-n+1}^{n-1}(n-|a_d|)e^{-\lambda|a_d+n|}, \qquad (48)
\end{aligned}$$

where we have applied (45) in the final inequality and ν_λ is given in (35).

Now considering the sum in (48), we obtain

$$\begin{aligned}
\sum_{a=-n+1}^{n-1}(n-|a|)e^{-\lambda|a+n|} &= \sum_{a=-n+1}^{n-1}(n-|a|)e^{-\lambda(a+n)}\\
&= \sum_{a=1}^{n-1}(n-a)e^{-\lambda(-a+n)} + \sum_{a=0}^{n-1}(n-a)e^{-\lambda(a+n)}.
\end{aligned}$$

For each sum, making a change of variable and applying (44), we obtain

$$\begin{aligned}
\sum_{a=1}^{n-1}(n-a)e^{-\lambda(-a+n)} &= e^{-\lambda n}\sum_{a=1}^{n-1}(n-a)e^{\lambda a}\\
&= \frac{e^{-\lambda n}e^{\lambda}\left(n-1-ne^{\lambda}+e^{\lambda n}\right)}{\left(e^{\lambda}-1\right)^2}\\
&= \frac{e^{\lambda}+n\left(1-e^{\lambda}\right)e^{\lambda(1-n)}-e^{\lambda(1-n)}}{\left(e^{\lambda}-1\right)^2}
\end{aligned}$$

and

$$\begin{aligned}\sum_{a=0}^{n-1}(n-a)e^{-\lambda(a+n)} &= ne^{-\lambda n}+e^{-\lambda n}\sum_{a=1}^{n-1}(n-a)e^{-\lambda a} \\ &= ne^{-\lambda n}+\frac{e^{-\lambda n}e^{-\lambda}\left(n-1-ne^{-\lambda}+e^{-\lambda n}\right)}{\left(e^{-\lambda}-1\right)^2} \\ &= ne^{-\lambda n}+\frac{e^{-\lambda n}e^{\lambda}\left(n-1-ne^{-\lambda}+e^{-\lambda n}\right)}{\left(e^{\lambda}-1\right)^2} \\ &= \frac{e^{\lambda(1-2n)}-e^{\lambda(1-n)}+n(e^{\lambda}-1)e^{\lambda(1-n)}}{\left(e^{\lambda}-1\right)^2}.\end{aligned}$$

Summing these two terms yields

$$\sum_{a=-n+1}^{n-1}(n-|a|)e^{-\lambda|a+n|}=\frac{e^{\lambda(1-2n)}+e^{\lambda}-2e^{\lambda(1-n)}}{\left(e^{\lambda}-1\right)^2}\le\frac{e^{\lambda}}{\left(e^{\lambda}-1\right)^2}=\nu_\lambda e^{-\lambda}.$$

Plugging the last bound in (48) yields the claim.

□

Now we have all ingredients to prove Theorems 3.1 and 3.2. In the following, we use the same technique as in (44) of [GW18], that is, for any positive real numbers a and b the minimum of al^d+b/l over real numbers l is achieved at $l_0=(b/ad)^{1/(d+1)}$. Taking $l=\lfloor l_0\rfloor$ when $l_0\ge 1$ and using that $l_0/2\le l\le l_0$ yields

$$\min_{l\in\mathbb{N}_1}\left(al^d+\frac{b}{l}\right)\le a\left(\frac{b}{ad}\right)^{\frac{d}{d+1}}+2b\left(\frac{ad}{b}\right)^{\frac{1}{d+1}}=a^{\frac{1}{d+1}}b^{\frac{d}{d+1}}\left(\frac{1}{d^{\frac{d}{d+1}}}+2d^{\frac{1}{d+1}}\right). \tag{49}$$

Proof of Theorem 3.1: By second order stationarity, it suffices to prove the case $\mathbf{k}=\mathbf{1}$. Let $n\ge 2, B_{\mathbf{1}}^n$ the block of size n^d as given in (31), and $W_{\mathbf{1}}^n$ the standardized sum over that block, as in (33). For any $1\le l\le n$ write $n=(m-1)l+r, 1\le r\le l$, and decompose $W_{\mathbf{1}}^n$ as the sum of $\xi_{\mathbf{i}}/\sqrt{n^dA_n}$ over $\mathbf{i}\in[m]$, as in (43).

We apply Theorem 1.3, dealing with the two terms on the right hand side of (4). For the first term, using $|X_{\mathbf{j}}|\le K$, the definition (43) of $\xi_{\mathbf{i}}$, and the fact that the side lengths of all blocks $D_{\mathbf{i}}^l$ are at most l, we have

$$\left|\frac{\xi_{\mathbf{i}}}{\sqrt{n^dA_n}}\right|\le B\quad\text{with}\quad B=\frac{2Kl^d}{\sqrt{n^dA_n}}\quad\text{for all}\quad\mathbf{i}\in[m]^d.$$

Applying Lemma 3.3 for the last term and invoking Theorem 1.3 now yields

$$d_1\big(\mathcal{L}(W_{\mathbf{1}}),\mathcal{L}(Z)\big)\quad\le\quad\frac{10Kl^d}{n^{d/2}A_n^{1/2}}+\frac{5.2\kappa_0\gamma_{\lambda,d}}{lA_n}. \tag{50}$$

Applying the bound (49) to the last expression with $l_0=C_{\lambda,\kappa_0,d}^{-1/(d+1)}n^{d/(2d+2)}$ and $C_{\lambda,\kappa_0,d}$ as in (36) and plugging in $l=\lfloor l_0\rfloor$ back to the right hand side of (50) yields the result. It is easy to check that $1\le l_0\le n$ for $n\ge\max\left\{C_{\lambda,\kappa_0,d}^{2/d},C_{\lambda,\kappa_0,d}^{-2/(d+2)}\right\}$.

□

To prove Theorem 3.2, we apply Theorem 1.4 and use the same techniques as in Theorem 3.1. We remind the reader that for this result we do not explicitly compute the constants, but index them by the parameters on which they depend.

Proof of Theorem 3.2: First we prove the claims that, Σ is invertible and $|\Sigma^{-1}|_\infty$ is bounded by (39) when $n > (p-1)\kappa_0\nu_\lambda^d e^{-\lambda}/A_n$. Applying Lemma 3.5, for all $q \in [p]$ we have

$$\Sigma_{q,q} - \sum_{1\le s\le p, s\neq q} |\Sigma_{q,s}| \ge n^{d-1}(nA_n - (p-1)\kappa_0\nu_\lambda^d e^{-\lambda}) > 0 \;\text{ if }\; n > \frac{(p-1)\kappa_0\nu_\lambda^d e^{-\lambda}}{A_n}.$$

which implies that Σ is a strictly diagonally dominant matrix, and is therefore invertible by the Gershgorin circle theorem, see for instance Theorem 15.10 of [BR14]. Another claim in (39) follows from [AN63], where it is shown that the bound (39) holds for the norm $||C||_\infty = \max_i \sum_{j=1}^p |c_{ij}|$, which dominates $|C|_\infty$.

Next we proceed as in the one dimensional case. For $n \ge 2$, and $1 \le l \le n$ we write $n = (m-1)l + r$ with $m \ge 1$ and $1 \le r \le l$, and decompose $S_{\mathbf{k}_q}^n - \mathbb{E}S_{\mathbf{k}_q}^n$ for $q \in [p]$ as the sum over $\mathbf{i} \in [m]^d$ of the variables $\xi_{\mathbf{i},\mathbf{k}_q}$ given in (43).

Applying Theorem 1.4, we handle the three terms on the right hand side of (5). Using the definition (43) of $\xi_{\mathbf{i},\mathbf{k}_q}$ and that $|X_\mathbf{t}| \le K$, we have

$$\left|\xi_{\mathbf{i},\mathbf{k}_q}\right| \le B \;\text{ where }\; B = 2Kl^d \;\text{ for all }\; \mathbf{i} \in [m]^d, q \in [p]$$

and thus using $|\Sigma^{-1/2}|_\infty = n^{-d/2}\psi_n$ and $\Sigma_{jj} = n^d A_n$, we may bound the first term as

$$\frac{5}{6}p^3 B n^{-3d/2}\psi_n^3 \sum_{q=1}^p \Sigma_{q,q} \le \frac{C_{p,K} l^d A_n \psi_n^3}{n^{d/2}}. \tag{51}$$

For the second term, by Lemma (3.3) we have

$$\begin{aligned}
&-\left(\frac{3}{2}p^3 B n^{-3d/2}\psi_n^3 + p^2 n^{-d}\psi_n^2\right) \sum_{q=1}^p \sum_{\mathbf{i},\mathbf{j}\in[m]^d, \mathbf{i}\neq\mathbf{j}} \mathbb{E}\left(\xi_{\mathbf{i},\mathbf{k}_q}\xi_{\mathbf{j},\mathbf{k}_q}\right) \\
&= -\left(\frac{C_{p,K} l^d \psi_n^3}{n^{d/2}} + C_p\psi_n^2\right) \sum_{q=1}^p \sum_{\mathbf{i},\mathbf{j}\in[m]^d, \mathbf{i}\neq\mathbf{j}} \mathbb{E}\left(\frac{\xi_{\mathbf{i},\mathbf{k}_q}\xi_{\mathbf{j},\mathbf{k}_q}}{n^d}\right) \le \frac{C_{\lambda,\kappa_0,p,K,d} l^{d-1}\psi_n^3}{n^{d/2}} + \frac{C_{\lambda,\kappa_0,p,d}\psi_n^2}{l}.
\end{aligned} \tag{52}$$

Next, invoking Lemma 3.5 and assumption (38) we have

$$-\Sigma_{q,s} = -\mathrm{Cov}\left(S_{\mathbf{k}_q}^n, S_{\mathbf{k}_s}^n\right) \le \kappa_0\nu_\lambda^d e^{-\lambda} n^{d-1} \;\text{ for }\; q \neq s \in [p],$$

and hence we may bound the last term as

$$\begin{aligned}
-\left(\frac{2}{3}p^3 B n^{-3d/2}\psi_n^3 + p^2 n^{-d}\psi_n^2\right) \sum_{q,s\in[p], q\neq s} \Sigma_{q,s} \;&\le\; \left(\frac{C_{p,K} l^d \psi_n^3}{n^{3d/2}} + \frac{C_p\psi_n^2}{n^d}\right)\kappa_0\nu_\lambda^d e^{-\lambda} n^{d-1} \\
&\le\; \frac{C_{\lambda,\kappa_0 d,p,K} l^d \psi_n^3}{n^{(d+2)/2}} + \frac{C_{\lambda,\kappa_0,d,p}\psi_n^2}{n}.
\end{aligned} \tag{53}$$

By Theorem 1.4 and (51)-(53), we have

$$d_{\mathcal{H}_{3,\infty,p}}\left(\mathcal{L}(\Sigma^{-1/2}\mathbf{W}), \mathcal{L}(\mathbf{Z})\right) \;\le\; C_{\lambda,\kappa_0,K,p,d}\left(\frac{l^{d-1}\psi_n^3}{n^{d/2}} + \frac{l^d\psi_n^3}{n^{(d+2)/2}} + \frac{\psi_n^2}{n} + \frac{l^d A_n\psi_n^3}{n^{d/2}} + \frac{\psi_n^2}{l}\right).$$

Since the first three terms do not contribute the rate, applying (49) to the last two terms in the parentheses, we obtain

$$l_0 = \left(\frac{1}{d\psi_n A_n}\right)^{\frac{1}{d+1}} n^{\frac{d}{2d+2}},$$

which satisfies $1 \le l_0 \le n$ for the range of n given in (40), and pluging in $l = \lfloor l_0 \rfloor$ back to the bound yields the result.

□

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