# OPTIMAL CONSUMPTION OF MULTIPLE GOODS IN INCOMPLETE MARKETS 

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#### Abstract

We consider the problem of optimal consumption of multiple goods in incomplete semimartingale markets. We formulate the dual problem and identify conditions that allow for existence and uniqueness of the solution and give a characterization of the optimal consumption strategy in terms of the dual optimizer. We illustrate our results with examples in both complete and incomplete models. In particular, we construct closed-form solutions in some incomplete models.


## 1. Introduction

The problem of optimal consumption of multiple goods has been investigated in Fis75, Bre79. For a single consumption good in continuous-time settings, it was first formulated in Mer69. Since then, this problem was analyzed in a large number of papers in both complete and incomplete settings with a range of techniques based on Hamilton-Jacobi-Bellman equations, backward stochastic differential equations, and convex duality being used for its analysis.

In the present paper, we formulate a problem of optimal consumption of multiple goods in a general incomplete semimartingale model of a financial market. We construct the dual problem and characterize optimal consumption policies in terms of the solution to the dual problem. We also identify mathematical conditions, that allow for existence and uniqueness of the solution and a dual characterization. We illustrate our results by examples, where in particular we obtain closed-form solutions in incomplete markets. Our proofs rely on certain results on weakly measurable correspondences for Carathéodory functions, multidimensional convex-analytic techniques, and some recent advances in stochastic analysis in mathematical finance, in particular, the characterization of the "no unbounded profit with bounded risk" condition in terms of non-emptiness of the set of equivalent local martingale deflators from CCFM17, KKS16] and sharp conditions for solvability of the expected utility maximization problem in a single good setting from Mos15.

[^0]The remainder of this paper is organized as follows: in Section 2 we specify the model setting, formulate the problem, and state main results (in Theorem 2.4). In Section 3 we discuss various specific cases. In particular, we present there the structure of the solution in complete models and the additive utility case as well as closed-form solutions in some incomplete models (with and without an additive structure of the utility). We conclude the paper with Section 4, which contains proofs.

## 2. SEtting And main Results

2.1. Setting. Let $\widetilde{S}=\left(\widetilde{S}_{t}\right)_{t \geq 0}$ an $\mathbb{R}^{d}$-valued semimartingale, representing the discounted price 1 of $d$ risky assets on a complete stochastic basis $\left(\Omega, \mathcal{F},\left(\mathcal{F}_{t}\right)_{t \in[0, \infty)}, \mathbb{P}\right)$, with $\mathcal{F}_{0}$ being the trivial $\sigma$-algebra. We fix a stochastic clock $\kappa=\left(\kappa_{t}\right)_{t \geq 0}$, which is a nondecreasing, càdlàg, adapted process, such that

$$
\begin{equation*}
\kappa_{0}=0, \quad \mathbb{P}\left(\kappa_{\infty}>0\right)>0 \quad \text { and } \quad \kappa_{\infty} \leq \bar{A} \tag{2.1}
\end{equation*}
$$

where $\bar{A}$ is a positive constant. The stochastic clock $\kappa$ specifies times when consumption is assumed to occur. Various optimal investment-consumption problems can be recovered from the present general setting by suitably specifying the clock process $\kappa$. For example, the problem of maximizing expected utility of terminal wealth at some finite investment horizon $T<\infty$ can be recovered by simply letting $\kappa \triangleq \mathbb{I}_{[T, \infty \llbracket}$. Likewise, maximization of expected utility from consumption only up to a finite horizon $T<\infty$ can be obtained by letting $\kappa_{t} \triangleq \min (t, T)$, for $t \geq 0$. Other specifications include maximization of utility form lifetime consumption, from consumption at a finite set of stopping times, and from terminal wealth at a random horizon, see e.g., Mos15, Examples 2.5-2.9] for a description of possible standard choices of the clock process $\kappa$.

We suppose that there are $m$ different consumption goods, where $S_{t}^{k}$ denotes the discounted price of commodity $k$ at time $t$. We assume that for each $k \in\{1, \ldots, m\}, S^{k}=\left(S_{t}^{k}\right)_{t \geq 0}$ is a strictly positive optional processes on $\left(\Omega, \mathcal{F},\left(\mathcal{F}_{t}\right)_{t \in[0, \infty)}, \mathbb{P}\right)$.

A portfolio is defined by a triplet $\Pi=(x, H, c)$, where $x \in \mathbb{R}$ represents an initial capital, $H=\left(H_{t}\right)_{t \geq 0}$ is a $d$-dimensional $\widetilde{S}$-integrable process, $H_{t}^{j}$ represents the holdings in the $j$-th risky asset at time $t, j \in\{1, \ldots, d\}, t \geq 0, c$ is an $m$-dimensional consumption process, whose every component $\left(c_{t}^{k}\right)_{t \geq 0}$ is a nonnegative optional process representing the consumption rate of commodity $k, k=\{1, \ldots, m\}$. The wealth process $X=\left(X_{t}\right)_{t \geq 0}$ of a portfolio $\Pi=(x, H, c)$ is defined as

$$
\begin{equation*}
X_{t} \triangleq x+\int_{0}^{t} H_{u} \mathrm{~d} \widetilde{S}_{u}-\sum_{k=1}^{m} \int_{0}^{t} c_{u}^{k} S_{u}^{k} \mathrm{~d} \kappa_{u}, \quad t \geq 0 \tag{2.2}
\end{equation*}
$$

[^1]2.2. Absence of arbitrage. The main objective of this part is to specify the no-arbitrage type condition (NUPBR) below. As it is commonly done in the literature (see for example KS99), we begin defining $\mathcal{X}$ to be the collection of all nonnegative wealth processes associated to portfolios of the form $\Pi=(1, H, 0)$, i.e.,
$$
\mathcal{X} \triangleq\left\{X \geq 0: X_{t}=1+\int_{0}^{t} H_{u} \mathrm{~d} \widetilde{S}_{u}, \quad t \geq 0\right\} .
$$

In this paper, we suppose the following no-arbitrage-type condition:
(NUPBR) the set $\mathcal{X}_{T} \triangleq\left\{X_{T}: X \in \mathcal{X}\right\}$ is bounded in probability, for every $T \in \mathbb{R}_{+}$,
where (NUPBR) stands for no unbounded profit with bounded risk. This condition was originally introduced in KK07. It is proven in Kar10, Proposition 1], that (NUPBR) is equivalent to another (weak) no-arbitrage condition, namely absence of arbitrages of the first kind on $[0, T]$, see Kar14, Definition 1].

A useful characterization of (NUPBR) is given via the set of equivalent local martingale deflators (ELMD) that is defined as follows:

$$
\begin{array}{ll}
\mathcal{Z} \triangleq\{Z>0: & Z \text { is a càdlàg local martingale such that } Z_{0}=1 \text { and }  \tag{2.3}\\
& \left.Z X=\left(Z_{t} X_{t}\right)_{t \geq 0} \text { is a local martingale for every } X \in \mathcal{X}\right\} .
\end{array}
$$

It is proven in [CCFM17, Proposition 2.1] (see also [KKS16) that condition (NUPBR) holds if and only if $\mathcal{Z} \neq \emptyset$. This result was previously established in the one-dimensional case in the finite time horizon in Kar12, Theorem 2.1]. Also, TS14, Theorem 2.6] contains a closely related result (in a finite time horizon) in terms of strict $\sigma$-martingale densities, see [TS14 for the corresponding definition and details.

Remark 2.1. Condition (NUPBR) is weaker than the existence of an equivalent martingale measure (see for example [DS94, p. 463] for the definition an equivalent martingale measure), another classical no-arbitrage type assumption, which in the infinite time horizon is even stronger than

$$
\begin{equation*}
\{Z \in \mathcal{Z}: Z \text { is a martingale }\} \neq \emptyset . \tag{2.4}
\end{equation*}
$$

Note that in the finite time horizon setting, (2.4) is equivalent to the existence of an equivalent martingale measure. Besides, (2.4) is apparently stronger than (NUPBR) (by comparison of (2.3) and (2.4) combined with [CCFM17, Proposition 2.1]). We also would like to point out that (2.4) holds in every original formulation of Mer69], where the problem of optimal consumption from investment (in a single consumption good setting) was introduced, including the infinitetime horizon case. In general, (2.4) can be stronger than (NUPBR). A classical example, where (NUPBR) holds but (2.4) fails, corresponds to the three-dimensional Bessel process driving the stock price, see e.g., KK07, Example 4.6].
2.3. Admissible consumptions. For a given initial capital $x>0$, an $m$-dimensional optional consumption process $c$ is said to be $x$-admissible if there exists an $\mathbb{R}^{d}$-valued predictable $\widetilde{S}$ integrable process $H$ such that the wealth process $X$ in (2.2), corresponding to the portfolio $\Pi=(x, H, c)$ is nonnegative; the set of $x$-admissible consumption processes corresponding to a stochastic clock $\kappa$ is denoted by $\mathcal{A}(x)$. For brevity, we denote $\mathcal{A} \triangleq \mathcal{A}(1)$.
2.4. Preferences of a rational economic agent. Building from the formulation of Mer09, we assume that preferences of a rational economic agent are represented by a optional utilityvalued process (or simply a utility process) $U=U(t, \omega, x):[0, \infty) \times \Omega \times[0, \infty)^{m} \rightarrow \mathbb{R} \cup\{-\infty\}$, where for every $(t, \omega) \in[0, \infty) \times \Omega, U(t, \omega, \cdot)$ is an Inada-type utility function, i.e., $U(t, \omega, \cdot)$ satisfies the following (technical) assumption.

Assumption 2.2. For every $(t, \omega) \in[0, \infty) \times \Omega$, the function

$$
\mathbb{R}_{+}^{m} \ni x \mapsto U(t, \omega, x) \in \mathbb{R} \cup\{-\infty\}
$$

is strictly concave, strictly increasing in every component, finite-valued and continuously differentiable in the interior of the positive orthant, and satisfies the Inada conditions

$$
\lim _{x_{i} \downarrow 0} \partial_{x_{i}} U(t, \omega, x)=\infty \quad \text { and } \quad \lim _{x_{i} \uparrow \infty} \partial_{x_{i}} U(t, \omega, x)=0, \quad i=1, \ldots, m,
$$

where $\partial_{x_{i}} U(t, \omega, \cdot): \mathbb{R}_{++}^{m} \mapsto \mathbb{R}$ is the partial derivative of $U(t, \omega, \cdot)$ with respect to the $i$-th spatial variable ${ }^{2}$. On the boundary of the first orthant, by upper semicontinuity, we suppose that $U(t, \omega, x)=\limsup _{x^{\prime} \rightarrow x} U\left(t, \omega, x^{\prime}\right)$ (note that some of these values may be $-\infty$ and that $U(t, \omega, x)=$ $\lim _{t \downarrow 0} U\left(t, \omega, x+t\left(x^{\prime}-x\right)\right)$, where $x^{\prime}$ is an arbitrary element in the interior of the first orthant, see HUL04, Proposition B.1.2.5]). Finally, for every $x \in \mathbb{R}_{+}^{m}$, we assume that the stochastic process $U(\cdot, \cdot, x)$ is optional.

Remark 2.3. The Inada conditions in Assumption [2.2 were introduced in [na63]. These are technical assumptions that have natural economic interpretations and that allow for a deeper tractability of the problem (as e.g., in KS99]). Likewise, the semicontinuity of $U$ is imposed for regularity purposes. It also used in e.g., Sio15, Sio16.

In particular, modeling preferences via utility process allows to take into account utility maximization problems under a change of numéraire (see e.g., [Mos17, Example 4.2]). This is the primary reason why we suppose that the prices of the traded stocks are discounted, as this allows to simplify notations without any loss of generality. Note also that Assumption 2.2 does not make any requirement on the asymptotic elasticity of $U$, introduced in KS99].

To a utility process $U$ satisfying Assumption [2.2, we associate the primal value function, defined as

$$
\begin{equation*}
u(x) \triangleq \sup _{\mathbf{c}=\left(c^{1}, \ldots, c^{m}\right) \in \mathcal{A}(x)} \mathbb{E}\left[\int_{0}^{\infty} U\left(t, \omega, \mathbf{c}_{t}\right) \mathrm{d} \kappa_{t}\right], \quad x>0 \tag{2.5}
\end{equation*}
$$

[^2]To ensure that the integral above is well-defined, we use the convention

$$
\begin{equation*}
\mathbb{E}\left[\int_{0}^{\infty} U\left(t, \omega, \mathbf{c}_{t}\right) \mathrm{d} \kappa_{t}\right] \triangleq-\infty \quad \text { if } \quad \mathbb{E}\left[\int_{0}^{\infty} U^{-}\left(t, \omega, \mathbf{c}_{t}\right) \mathrm{d} \kappa_{t}\right]=\infty \tag{2.6}
\end{equation*}
$$

where $U^{-}(t, \omega, \cdot)$ is the negative part of $U(t, \omega, \cdot)$. Note that formulation (2.5) is a generalization of the formulation in Mer09, p. 205], in the form (2.5) we allow for stochastic preferences and include several standard formulations as particular cases.
2.5. Dual problem. In order to specify model assumptions that ensure existence and uniqueness of solutions to (2.5) and to give a characterization of this solution, we need to formulate an appropriate dual problem. Let us define

$$
\begin{equation*}
U^{*}(t, \omega, x) \triangleq \sup _{\substack{\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{R}_{+}^{m}:}} U\left(t, \omega, x^{1}, \ldots, x^{m}\right), \quad(t, \omega, x) \in[0, \infty) \times \Omega \times[0, \infty) \tag{2.7}
\end{equation*}
$$

Let us set a family of transformations $A:[0, \infty) \times \Omega \times \mathbb{R}^{m} \mapsto \mathbb{R}$, as

$$
A\left(t, \omega, x_{1}, \ldots, x_{m}\right) \triangleq S_{t}^{1}(\omega) x_{1}+\cdots+S_{t}^{m}(\omega) x_{m}, \quad\left(t, \omega, x_{1}, \ldots, x_{m}\right) \in[0, \infty) \times \Omega \times[0, \infty)^{m}
$$

Note that for every $(t, \omega) \in[0, \infty) \times \Omega, A(t, \omega, \cdot)$ is a linear transformation from $\mathbb{R}^{m}$ to $\mathbb{R}$ and $U^{*}(t, \omega, \cdot)$ is the image of $U(t, \omega, \cdot)$ under $A(t, \omega, \cdot)$ (see e.g., [HUL04, p. 96] for the definition and properties of the image of a function under a linear mapping ${ }_{3}^{3}$ ). We define a stochastic field $V^{*}$ as the pointwise conjugate of $U^{*}$ (equivalently, as the pointwise conjugate of the image function of $U$ under $A$ ) in the sense that

$$
V^{*}(t, \omega, y) \triangleq \sup _{x>0}\left(U^{*}(t, \omega, x)-x y\right), \quad(t, \omega, y) \in[0, \infty) \times \Omega \times[0, \infty)
$$

where $\sup _{x>0}$ and $\sup _{x \geq 0}$ coincide thanks to continuity of $U^{*}$ established in Lemma 4.1. We also introduce the following set of dual processes:

$$
\begin{aligned}
& \mathcal{Y}(y) \triangleq \operatorname{cl}\{Y: Y \text { is càdlàg adapted and } \\
& \qquad 0 \leq Y \leq y Z(\mathrm{~d} \kappa \times \mathbb{P}) \text {-a.e. for some } Z \in \mathcal{Z}\}
\end{aligned}
$$

where the closure is taken in the topology of convergence in measure $(\mathrm{d} \kappa \times \mathbb{P})$ on the measure space of real-valued optional processes $(\Omega \times[0, \infty), \mathcal{O}, \mathrm{d} \kappa \times \mathbb{P})$, where $\mathcal{O}$ is the optional sigmafield. We write $\mathcal{Y} \triangleq \mathcal{Y}(1)$ for brevity. Note that $\mathcal{Y}$ is closely related to - but different from the set with the same name in KS99. The value function of the dual optimization problem, or equivalently, the dual value function, is then defined as

$$
\begin{equation*}
v(y) \triangleq \inf _{Y \in \mathcal{Y}(y)} \mathbb{E}\left[\int_{0}^{\infty} V^{*}\left(t, \omega, Y_{t}\right) \mathrm{d} \kappa_{t}\right], \quad y>0 \tag{2.8}
\end{equation*}
$$

with the convention $\mathbb{E}\left[\int_{0}^{\infty} V^{*}\left(t, \omega, Y_{t}\right) \mathrm{d} \kappa_{t}\right] \triangleq \infty$ if $\mathbb{E}\left[\int_{0}^{\infty} V^{*+}\left(t, \omega, Y_{t}\right) \mathrm{d} \kappa_{t}\right]=\infty$, where $V^{*+}(t, \omega, \cdot)$ is the positive part of $V^{*}(t, \omega, \cdot)$. We are now in a position to state the following theorem, which is the main result of this paper.

[^3]Theorem 2.4. Assume that conditions (2.1) and (NUPBR) hold true and let $U$ satisfies Assumption 2.2. Let us also suppose that

$$
\begin{equation*}
v(y)<\infty \quad \text { for every } y>0 \quad \text { and } \quad u(x)>-\infty \quad \text { for every } x>0 . \tag{2.9}
\end{equation*}
$$

Then we have
(i) $u(x)<\infty$, for every $x>0$, and $v(y)>-\infty$, for every $y>0$, i.e., the value functions are finite-valued.
(ii) The functions $u$ and $-v$ are continuously differentiable on $(0, \infty)$, strictly concave, strictly increasing and satisfy the Inada conditions

$$
\begin{align*}
\lim _{x \downarrow 0} u^{\prime}(x) & =\infty, & \lim _{y \downarrow 0}-v^{\prime}(y) & =\infty,  \tag{2.10}\\
\lim _{x \rightarrow \infty} u^{\prime}(x) & =0, & \lim _{y \rightarrow \infty}-v^{\prime}(y) & =0 .
\end{align*}
$$

(iii) For every $x>0$ and $y>0$, the solutions $\widehat{c}(x)=\left(\widehat{c}^{1}(x), \ldots, \widehat{c}^{m}(x)\right)$ to (2.5) and $\widehat{Y}(y)$ to (2.8) exist and are unique and, if $y=u^{\prime}(x)$, we have the optimality characterizations

$$
\begin{equation*}
\widehat{Y}_{t}(y)(\omega)=\frac{\partial_{x_{i}} U\left(t, \omega, \widehat{c}_{t}^{1}(x)(\omega), \ldots, \widehat{c}_{t}^{m}(x)(\omega)\right)}{S_{t}^{i}(x)(\omega)}, \quad(\mathrm{d} \kappa \times \mathbb{P}) \text {-a.e. }, \quad i=1, \ldots, m \tag{2.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{Y}_{t}(y)(\omega)=U_{x}^{*}\left(t, \omega, \sum_{i=1}^{m} \hat{c}_{t}^{i}(x)(\omega) S_{t}^{i}(\omega)\right), \quad(\mathrm{d} \kappa \times \mathbb{P})-\text { a.e. }, \tag{2.12}
\end{equation*}
$$

with $U_{x}^{*}$ denoting the partial derivative of $U^{*}$ with respect to its third argument.
(iv) For every $x>0$, the constraint $x$ is binding in the sense that

$$
\begin{equation*}
\mathbb{E}\left[\int_{0}^{\infty} \sum_{i=1}^{m} \widehat{c}_{t}^{i}(x) S_{t}^{i} \frac{\widehat{Y}_{t}(y)}{y} \mathrm{~d} \kappa_{t}\right]=x, \quad \text { where } y=u^{\prime}(x) . \tag{2.13}
\end{equation*}
$$

(v) The functions $u$ and $v$ are Legendre conjugate, i.e.,

$$
\begin{equation*}
v(y)=\sup _{x>0}(u(x)-x y), \quad y>0, \quad u(x)=\inf _{y>0}(v(y)+x y), \quad x>0 . \tag{2.14}
\end{equation*}
$$

(vi) The dual value function $v$ can be represented as

$$
\begin{equation*}
v(y)=\inf _{Z \in \mathcal{Z}} \mathbb{E}\left[\int_{0}^{\infty} V\left(t, \omega, y Z_{t}(\omega)\right) \mathrm{d} \kappa_{t}(\omega)\right], \quad y>0 . \tag{2.15}
\end{equation*}
$$

Remark 2.5 (On sufficient conditions for the validity of (2.9)). Condition (2.9) holds if there exists one primal element $c \in \mathcal{A}$ and one dual element $Y \in \mathcal{Y}$ such that

$$
\mathbb{E}\left[\int_{0}^{\infty} U\left(t, \omega, z c_{t}^{1}, \ldots, z c_{t}^{m}\right) \mathrm{d} \kappa_{t}\right]>-\infty \quad \text { and } \quad \mathbb{E}\left[\int_{0}^{\infty} V^{*}\left(t, \omega, z Y_{t}\right) \mathrm{d} \kappa_{t}\right]<\infty, \quad z>0 .
$$

In particular, for every $x>0$, as an $m$-dimensional optional process with constant values $\left(\frac{x}{A m}, \ldots, \frac{x}{A m}\right)$ belongs to $\mathcal{A}(x)$, a sufficient condition in (2.9) for the finiteness of $u$ is

$$
\mathbb{E}\left[\int_{0}^{\infty} U\left(t, \omega, \frac{x}{A m}, \ldots, \frac{x}{A m}\right) \mathrm{d} \kappa_{t}\right]>-\infty, \quad x>0,
$$

which typically holds if $U$ is nonrandom. Likewise, as $\mathcal{Z} \neq \emptyset$ (by (NUPBR) and CCFM17, Proposition 2.1]), finiteness of $v$ holds if for one equivalent local martingale deflator $Z$, we have

$$
\mathbb{E}\left[\int_{0}^{\infty} V^{*}\left(t, \omega, y Z_{t}\right) \mathrm{d} \kappa_{t}\right]<\infty, \quad y>0
$$

## 3. Examples

## Complete market solution and dual characterization

If the model is complete, the dual characterization of the optimal consumption policies has a particularly nice form, as $\mathcal{Z}$ contains a unique element, $Z$. The solutions corresponding to different $y$ 's in the dual problem (2.8) are $y Z, y>0$. Therefore, in (2.12) and (2.11) we have $\widehat{Y}(y)=y Z, y>0$.

## Special case: Additive utility

An important example of $U^{*}$ corresponds to $U$ having an additive form with respect to its spatial components, i.e., when

$$
U\left(t, \omega, c_{1}, \ldots, c_{m}\right)=U^{1}\left(t, \omega, c_{1}\right)+\cdots+U^{m}\left(t, \omega, c_{m}\right), \quad(t, \omega) \in[0, \infty) \times \Omega
$$

where for every $k=1, \ldots, m, U^{k}$ is a utility process in the sense of Mos15, Assumption 2.1] and a utility process in sense of the Assumption 2.2 with $m=1$. In this case, for every $(t, \omega) \in[0, \infty) \times \Omega, U^{*}(t, \omega, \cdot)$ is given by the infimal convolution of $U^{k}(t, \omega, \cdot)$ 's, see the definition in e.g., Roc70, p. 34]. Let $V^{i}(t, \omega, \cdot)$ denote the convex conjugate of $U^{i}(t, \omega, \cdot), i=1, \ldots, m$. Then the convex conjugate of $U^{*}(t, \omega, \cdot)$ is $V^{*}(t, \omega, \cdot)$ given by

$$
V^{*}(t, \omega, \cdot)=V^{1}(t, \omega, \cdot)+\cdots+V^{m}(t, \omega, \cdot) .
$$

This result was established e.g., in Roc70, Theorem 16.4, p. 145]. In this case, the optimal $\widehat{c}(x)=\left(\widehat{c}^{1}(x), \ldots, \widehat{c}^{m}(x)\right)$ has a more explicit characterization via $I_{i}(t, \omega, \cdot) \triangleq\left(U_{x}^{i}\right)^{-1}(t, \omega, \cdot)$, the the pointwise inverse of the partial derivative of $U^{i}(t, \omega, \cdot)$ with respect to the third argument, as (2.11) can be solved for $\widehat{c}^{i}(x), i=1, \ldots, m$, as follows

$$
\begin{equation*}
\widehat{c}_{t}^{i}(x)(\omega)=I_{i}\left(t, \omega, \widehat{Y}_{t}(y)(\omega) S_{t}^{i}(\omega)\right), \quad(\mathrm{d} \kappa \times \mathbb{P}) \text {-a.e. }, \quad i=1, \ldots, m . \tag{3.1}
\end{equation*}
$$

Using (2.12), we can restate (3.1) as

$$
\widehat{c}_{t}^{i}(x)(\omega)=I_{i}\left(t, \omega, U_{x}^{*}\left(t, \omega, \widehat{c}_{t}^{*}(x)(\omega)\right) S_{t}^{i}(\omega)\right), \quad(\mathrm{d} \kappa \times \mathbb{P}) \text {-a.e., } \quad i=1, \ldots, m,
$$

where $\widehat{c}^{*}(x)$ is the optimizer to the auxiliary problem (4.2) corresponding to the initial wealth $x>0$.

Remark 3.1. In the following three examples we consider some incomplete models that admit closed-form solutions for one good and show how these results apply to multiple good settings.

## Example of a closed form solution in an incomplete model with additive LOGARITHMIC UTILITY

Let us suppose that $d$ traded discounted assets are modeled with Ito processes of the form

$$
\begin{equation*}
d \widetilde{S}_{t}^{i}=\widetilde{S}_{t}^{i} b_{t}^{i} d t+\widetilde{S}_{t}^{i} \sum_{j=1}^{n} \sigma_{t}^{i j} d W_{t}^{j}, \quad i=1, \ldots, d, \quad \widetilde{S}_{0} \in \mathbb{R}^{d} \tag{3.2}
\end{equation*}
$$

where $W$ is an $\mathbb{R}^{n}$-valued standard Brownian motion and $b^{i}, \sigma^{i j}, i=1, \ldots, d, j=1, \ldots, n$, are predictable processes, such that the unique strong solution to (3.2) exists, see e.g., KS98. Let us suppose that there are $m$ consumption goods and that the value function of a rational economic agent is given by

$$
\sup _{c \in \mathcal{A}(x)} \mathbb{E}\left[\int_{0}^{T} e^{-\nu t} \log \left(c_{1} \ldots c_{m}\right) d t\right], \quad x>0
$$

(with the same convention as the one specified after (2.5)), where an impatience rate $\nu$ and a time horizon $T$ are positive constants. Note that in this case $\kappa_{t}=\frac{1-e^{-\nu t}}{\nu}, t \in[0, T]$, i.e., $\kappa$ is deterministic. Let us also suppose that there exists an $\mathbb{R}^{d}$-valued process $\gamma$, such that

$$
b_{t}-\sigma_{t} \sigma_{t}^{T} \gamma_{t}=0 \quad(\mathrm{~d} \kappa \times \mathbb{P})-a . e
$$

Let $\mathcal{E}$ denotes the Doléans-Dade exponential. Then, using GK00, Theorem 3.1 and Example 4.2] and Theorem [2.4, we get

$$
\begin{gathered}
\widehat{c}_{t}^{*}(x)=\frac{x \nu}{1-e^{-\nu T}} \mathcal{E}\left(\int_{0} \gamma_{s}^{T} d \widetilde{S}_{s}\right)_{t}, \quad x>0 \\
\widehat{c}_{t}^{i}(x)=\frac{\widehat{c}_{t}^{*}(x)}{S_{t}^{i} M}, \quad i=1, \ldots, m, \quad x>0 \\
\widehat{Y}_{t}(y)=\frac{y}{\mathcal{E}\left(\int_{0}^{*} \gamma_{s}^{T} d \widetilde{S}_{s}\right)_{t}}, \quad y>0, \quad t \in[0, T]
\end{gathered}
$$

## Example of a closed-Form solution and dual characterization in an incomplete

 ADDITIVE CASELet us fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\left(\mathcal{F}_{t}\right)_{t \geq 0}$ is the augmentation of the filtration generated by a two-dimensional Brownian motion $\left(W^{1}, W^{2}\right)$. Let us suppose that there are two traded securities: a risk-free asset $B$, such that

$$
B_{t}=e^{r t}, \quad t>0
$$

where $r$ is a nonnegative constant, and a risky stock $\widetilde{S}$ with the dynamics

$$
d \widetilde{S}_{t}=\widetilde{S}_{t} \mu_{t} d t+\widetilde{S}_{t} \sigma_{t} d W_{t}^{1}, \quad t \geq 0, \quad \widetilde{S}_{0} \in \mathbb{R}_{+}
$$

where processes $\mu$ and $\sigma$ are such that $\theta_{t}=\frac{\mu_{t}-r}{\sigma_{t}}, t \geq 0$, the market price of risk process, follows the Ornstein-Uhlenbeck process

$$
d \theta_{t}=-\lambda_{\theta}\left(\theta_{t}-\bar{\theta}\right) d t+\sigma_{\theta}\left(\rho d W_{t}^{1}+\sqrt{1-\rho^{2}} d W_{t}^{2}\right), \quad t \geq 0, \quad \theta_{0} \in \mathbb{R}_{+}
$$

where $\lambda_{\theta}, \sigma_{\theta}$, and $\bar{\theta}$ are positive constants, $\rho \in(-1,1)$. Let us also assume that $\kappa$ corresponds to the expected utility maximization from terminal wealth, i.e., $\kappa=\mathbb{I}_{[T, \infty[ }, T \in \mathbb{R}_{+}$, that there are $m$ consumption goods, where $S^{i}, i=1, \ldots, m$, are deterministic, and

$$
U\left(T, \omega, c_{1}, \ldots, c_{m}\right)=\frac{c_{1}^{p}}{p}+\cdots+\frac{c_{m}^{p}}{p}, \quad\left(c_{1}, \ldots, c_{m}\right) \in \mathbb{R}_{+}^{m}, \quad \omega \in \Omega,
$$

where $p<0$. Let us set

$$
q \triangleq \frac{p}{1-p}, \quad A \triangleq \sum_{i=1}^{m}\left(S_{T}^{i}\right)^{-q}, \quad \text { and } \quad B \triangleq A^{1-p} .
$$

Then, by direct computations, we get

$$
U^{*}(T, \omega, x)=\frac{x^{p}}{p} B, \quad x>0 .
$$

Using the argument in KO96], one can express the optimal trading strategy is $\widehat{H}(x)$ in a closed form in terms of a solution to a system of (nonlinear) ordinary differential equations (see K096, p. 147]), where $\widehat{H}_{t}(x)$ is the number of shares of the risky asset in the portfolio at time $t$, $t \in[0, T]$. With $\widehat{X}(x)$ such that

$$
d \widehat{X}_{t}(x)=\widehat{H}_{t}(x) d \widetilde{S}_{t}+\left(\widehat{X}_{t}(x)-\widehat{H}_{t}(x) \widetilde{S}_{t}\right) r d t, \quad \widehat{X}_{0}(x)=x
$$

using Theorem [2.4, we get

$$
\begin{gathered}
\widehat{c}_{T}^{*}(x)=\widehat{X}_{T}(x), \quad x>0, \\
\widehat{Y}_{T}(y)=\frac{y}{\mathbb{E}\left[\left(\widehat{c}_{T}^{*}(1)\right)^{p}\right]}\left(\widehat{c}_{T}^{*}(1)\right)^{p-1}, \quad y>0, \\
\widehat{c}_{T}^{i}(x)=\frac{\widehat{c}_{T}^{*}(x)}{A}\left(S_{T}^{i}\right)^{-(1+q)}, \quad x>0 .
\end{gathered}
$$

## Example of a closed-form solution and dual characterization in an incomplete NON-ADDITIVE CASE

Here we will suppose that $\kappa=\mathbb{I}_{[T, \infty[ }$, where $T \in \mathbb{R}_{+}$, and let

$$
U\left(t, \omega, c_{1}, c_{2}\right)=-\frac{c_{1}^{p_{1}}}{p_{1}} \frac{c_{2}^{p_{2}}}{p_{2}}, \quad p_{1}<0, p_{2}<0
$$

i.e., there are two consumption goods. One can see that $U(t, \omega, \cdot)$ is jointly concave, since the Hessian of $-U(t, \omega, \cdot)$ is positive definite on $\mathbb{R}_{++}^{2}$. We also extend $U(t, \omega, \cdot)$ to the boundary of $\mathbb{R}_{+}^{2}$ by $-\infty$. Then, with $p \triangleq p_{1}+p_{2}<0, U^{*}$ is given by

$$
U^{*}(t, \omega, x)=\frac{x^{p}}{p} \frac{\left(-p_{1}\right)^{p_{1}-1}\left(-p_{2}\right)^{p_{2}-1}}{(-p)^{p-1}}\left(S_{t}^{1}\right)^{-p_{1}}\left(S_{t}^{2}\right)^{-p_{2}}, \quad x>0 .
$$

Let us define $G \triangleq \frac{\left(-p_{1}\right)^{p_{1}-1}\left(-p_{2}\right)^{p_{2}-1}}{(-p)^{p-1}}\left(S_{T}^{1}\right)^{-p_{1}}\left(S_{T}^{2}\right)^{-p_{2}}$. Then $U(T, \omega, x)=\frac{x^{p}}{p} G(\omega), x>0$. Let us suppose that $W^{1}$ and $W^{2}$ are two Brownian motions with a fixed correlation $\rho$ such that $0<|\rho|<1$. Let $\left(\mathcal{F}_{t}\right)_{t \geq 0}$ be the usual augmentation of the filtration generated by $W^{1}$ and $W^{2}$ and $\left(\mathcal{G}_{t}\right)_{t \geq 0}$ be the usual augmentation of the filtration generated by $W^{2}$. We also assume that there is a bond $B$ and a stock $\widetilde{S}$ on the market. Their dynamics are given by

$$
d \widetilde{S}_{t}=\widetilde{S}_{t}\left(\mu_{t} d t+\sigma_{t} d W_{t}^{1}\right), \quad \widetilde{S}_{0} \in \mathbb{R}
$$

$$
d B_{t}=B_{t} r_{t} d t, \quad B_{0}=1
$$

where the drift $\mu$, volatility $\sigma$, and sport interest rate $r$ are bounded, progressively measurable processes with respect to $\left(\mathcal{G}_{t}\right)_{t} \in[0, T]$, and $\sigma$ is strictly positive.

Let us suppose that $S_{T}^{1}$ and $S_{T}^{2}$ are $\mathcal{G}_{T}$-measurable random variables with moments of all orders. Then $G$ is also $\mathcal{G}_{T}$-measurable random variable with moments of all orders (by Hölder's inequality) and the auxiliary value function $u^{*}$ defined in (4.2) satisfies the settings of Teh04]. Also, as $u^{*}(x) \geq \frac{x^{p}}{p} \mathbb{E}[G]>-\infty$ and since $V(T, \omega, \cdot)$ is negative-valued (thus, $v(y) \leq 0$ ), the assumption (2.9) holds.

Let us set

$$
\begin{gathered}
\lambda_{t} \triangleq \frac{\mu_{t}-r_{t}}{\sigma_{t}}, \quad \delta \triangleq \frac{1-p}{1-p+\rho^{2} p}, \quad \frac{d \mathbb{Q}}{d \mathbb{P}} \triangleq \exp \left(-\frac{\rho^{2} p^{2}}{2(1-p)^{2}} \int_{0}^{T} \lambda_{s}^{2} d s+\frac{\rho p}{1-p} \int_{0}^{T} \lambda_{s} d W_{s}^{2}\right), \\
K_{t} \triangleq \frac{p}{(1-p)}\left(\lambda_{t}+\rho \delta \frac{\beta_{t}}{\mathbb{E} \mathbb{Q}\left[\exp \left(\int_{0}^{T}\left(r_{s} / \delta\right) d s\right) \mid \mathcal{F}_{t}\right]}\right), \quad t \in[0, T] .
\end{gathered}
$$

Then, using [Teh04, Proposition 3.4] and Theorem [2.4, we deduce that

$$
\begin{gathered}
\widehat{c}_{T}^{*}(x)=x \exp \left(\int_{0}^{T}\left(r+K_{s} \lambda_{s}-\frac{1}{2} K_{s}^{2}\right) d s+\int_{0}^{T} K_{s} d W_{s}^{1}\right), \quad x>0 \\
\widehat{Y}_{T}(y)=\frac{y}{\mathbb{E}\left[\left(\widehat{c}_{T}^{*}(1)\right)^{p}\right]} \exp \left(\int_{0}^{T}(p-1)\left(r+K_{s} \lambda_{s}-\frac{1}{2} K_{s}^{2}\right) d s+\int_{0}^{T}(p-1) K_{s} d W_{s}^{1}\right), \quad y>0 \\
\widehat{c}_{T}^{i}=\frac{\widehat{c}_{T}^{*}(x) p_{i}}{p S_{T}^{i}}, \quad i=1,2, \quad x>0
\end{gathered}
$$

are the optimizers to (2.8), (4.2), and (2.5), respectively. From Theorem [2.4, we conclude that for every $x>0, \widehat{c}_{T}^{i}(x), i=1,2$, and $\widehat{Y}_{T}\left(u^{\prime}(x)\right)$ are related via (2.11) and (2.12).

## 4. Proofs

We begin from a characterization of the utility process $U^{*}$ defined in (2.7).
Lemma 4.1. Let $U$ satisfies Assumption 2.2 and $U^{*}$ be defined in (2.7). Then, $U^{*}$ is an Inada-type utility process for $m=1$ in the sense of Assumption 2.2, i.e., $U^{*}$ satisfies:
(1) For every $(t, \omega) \in[0, \infty) \times \Omega$, the function $x \mapsto U^{*}(t, \omega, x)$ is finite-valued on $(0, \infty)$, strictly concave, and strictly increasing.
(2) For every $(t, \omega) \in[0, \infty) \times \Omega$, the function $x \mapsto U^{*}(t, \omega, x)$ is continuously differentiable on $(0, \infty)$ and satisfies the Inada conditions

$$
\lim _{z \downarrow 0} U_{x}^{*}(t, \omega, z)=\infty \quad \text { and } \quad \lim _{z \uparrow \infty} U_{x}^{*}(t, \omega, z)=0
$$

(3) For every $(t, \omega) \in[0, \infty) \times \Omega$, at $z=0$, we have

$$
U^{*}(t, \omega, 0)=\lim _{z \downarrow 0} U^{*}(t, \omega, z)
$$

(note that this value may be $-\infty$ ).
(4) For every $z \geq 0$, the stochastic process $U^{*}(\cdot, \cdot, z)$ is optional.

Proof. For every $(t, \omega) \in[0, \infty) \times \Omega$, as $U^{*}(t, \omega, \cdot)$ is an image function under an appropriate linear transformation of a concave function $U(t, \omega, \cdot)$, therefore using e.g., HUL04, Theorem B.2.4.2], one can show that $U^{*}(t, \omega, \cdot)$ is concave. In order to show strict concavity of $U^{*}(t, \omega, \cdot)$, one can proceed as follows. First, for some positive numbers $x_{1} \neq x_{2}$, let $\mathbf{c}^{i}=\left(c^{i, 1}, \ldots, c^{i, m}\right)$ be such that

$$
\begin{align*}
\sum_{k=1}^{m} S_{t}^{k} c^{i, k} & \leq x_{i}, \quad \text { and }  \tag{4.1}\\
U^{*}\left(t, \omega, x_{i}\right) & =U\left(t, \omega, c^{i, 1}, \ldots, c^{i, m}\right), \quad i=1,2 .
\end{align*}
$$

The existence of such $\mathbf{c}^{i}$,s follows from compactness of the domain of the optimization problem in the definition of $U^{*}(t, \omega, x)$ (for every $x>0$ ) and upper semicontinuity of $U(t, \omega, \cdot)$. Since in (4.1), $\mathbf{c}^{i}$ necessarily satisfies inequality $\sum_{k=1}^{m} S_{t}^{k} c^{i, k} \leq x_{i}$ with equality, $i=1,2$, from the strict monotonicity of $U(t, \omega, \cdot)$ in every spatial component and $x_{1} \neq x_{2}$, we deduce that $\mathbf{c}^{1} \neq \mathbf{c}^{2}$. Consequently, from strict concavity of $U(t, \omega, \cdot)$, we get

$$
\begin{aligned}
U^{*}\left(t, \omega, \frac{x_{1}+x_{2}}{2}\right)= & \sup _{\left(c_{1}, \ldots, c_{m}\right) \in \mathbb{R}_{+}^{m} ;} U\left(t, \omega, c_{1}, \ldots, c_{m}\right) \\
& \sum_{k=1}^{m} c_{k} S_{t}^{k}(\omega) \leq \frac{x_{1}+x_{2}}{2} \\
\geq & U\left(t, \omega, \frac{c^{1,1}+c^{2,1}}{2}, \ldots, \frac{c^{1, m}+c^{2}, m}{2}\right) \\
> & \frac{1}{2} U\left(t, \omega, c^{1,1}, \ldots, c^{1, m}\right)+\frac{1}{2} U\left(t, \omega, c^{2,1}, \ldots, c^{2, m}\right) \\
= & \frac{1}{2} U^{*}\left(t, \omega, x_{1}\right)+\frac{1}{2} U^{*}\left(t, \omega, x_{2}\right) .
\end{aligned}
$$

Therefore, $U^{*}(t, \omega, \cdot)$ is strictly concave. As $U^{*}(t, \omega, \cdot)$ is increasing and strictly concave, it is strictly increasing.

For every $(t, \omega) \in[0, \infty) \times \Omega$ and $x>0$, using the Inada conditions for $U(t, \omega, \cdot)$ one can show that there exists $\left(c_{1}, \ldots, c_{m}\right)$ in the interior of the first orthant, such that $\sum_{i=1}^{m} c_{i} S_{t}^{i}(\omega)=x$ and $U^{*}(t, \omega, x)=U\left(t, \omega, c_{1}, \ldots, c_{m}\right)$. As a result, differentiability of $U^{*}(t, \omega, \cdot)$ (in the third argument) follows from differentiability of $U(t, \omega, \cdot)$ and general properties of the subgradient of the image function, see e.g., HUL04, Corollary D.4.5.2]. As $U^{*}(t, \omega, \cdot)$ is concave and differentiable, we deduce that $U^{*}(t, \omega, \cdot)$ is continuously differentiable in the interior of its domain, see HUL04, Theorem D.6.2.4]. The Inada conditions for $U^{*}(t, \omega, \cdot)$ follow from the (version of the) Inada conditions for $U(t, \omega, \cdot)$ and HUL04, Theorem D.4.5.1, p.192].

For every $(t, \omega) \in[0, \infty) \times \Omega$, as $U(t, \omega, \cdot)$ is a closed concave function, using e.g., Roc70, Theorem 9.2, p. 75], we deduce that $U^{*}(t, \omega, \cdot)$ is also a closed concave function. In particular, we get

$$
U^{*}(t, \omega, 0)=\lim _{z \downarrow 0} U^{*}(t, \omega, z), \quad(t, \omega) \in[0, \infty) \times \Omega .
$$

Finally, for every $x \geq 0, U^{*}(\cdot, \cdot, x)$ is optional as a supremum of countably many optional processes (where from continuity of $U(t, \omega, \cdot)$ in the relative interior of its effective domain, it is

[^4]enough to take the supremum (in the definition of $U^{*}(t, \omega, \cdot)$ ) over the $m$-dimensional vectors, whose components take only rational values).

Remark 4.2. Lemma 4.1 asserts that $U^{*}$ satisfies Assumption 2.1 in Mos15.
For every $x>0$, we denote by $\mathcal{A}^{*}(x)$ the set of 1-dimensional optional processes $c^{*}$, for which there exists an $\mathbb{R}^{d}$-valued predictable $\widetilde{S}$-integrable process $H$, such that

$$
X_{t} \triangleq x+\int_{0}^{t} H_{u} \mathrm{~d} \widetilde{S}_{u}-\int_{0}^{t} c_{u}^{*} \mathrm{~d} \kappa_{u}, \quad t \geq 0
$$

is nonnegative, $\mathbb{P}$-a.s. We also define

$$
\begin{equation*}
u^{*}(x) \triangleq \sup _{c^{*} \in \mathcal{A}^{*}(x)} \mathbb{E}\left[\int_{0}^{\infty} U\left(t, \omega, c_{t}^{*}(\omega)\right) \mathrm{d} \kappa_{t}(\omega)\right], \quad x>0 \tag{4.2}
\end{equation*}
$$

with the convention analogous to (2.6):

$$
\mathbb{E}\left[\int_{0}^{\infty} U^{*}\left(t, \omega, c_{t}^{*}(\omega)\right) \mathrm{d} \kappa_{t}(\omega)\right] \triangleq-\infty, \quad \text { if } \quad \mathbb{E}\left[\int_{0}^{\infty} U^{*-}\left(t, \omega, c_{t}^{*}(\omega)\right) \mathrm{d} \kappa_{t}(\omega)\right]=\infty
$$

Proof of Theorem 2.4. Let $x>0$ be fixed and $c \in \mathcal{A}(x)$. Then $c_{t}^{*} \triangleq \sum_{k=1}^{m} c_{t}^{k} S_{t}^{k}, t \geq 0$, is an optional process such that $c^{*} \in \mathcal{A}^{*}(x)$. Therefore,

$$
\begin{equation*}
u^{*}(x) \geq u(x)>-\infty, \quad x>0 \tag{4.3}
\end{equation*}
$$

Since $U^{*}$ satisfies the assertions of Lemma 4.1, standard techniques in convex analysis show that $-V^{*}$ has the same properties as $U^{*}$. Therefore, optimization problems (4.2) and (2.8) satisfy the assumptions of Mos15, Theorem 3.2]. Consequently, Mos15, Theorem 3.2] applies, which in particular asserts that $u^{*}$ and $v$ are finite-valued and that for every $x>0$, the exists a strictly positive optional process $\widehat{c^{*}}(x)$, the unique maximizer to (4.2).

Let us consider

$$
\begin{align*}
& \sup _{\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{R}_{+}^{m}:} U\left(t, \omega, x^{1}, \ldots, x^{m}\right), \quad(t, \omega) \in[0, \infty) \times \Omega,  \tag{4.4}\\
& \sum_{k=1}^{m} x_{k} S_{t}^{k}(\omega) \leq \widehat{c_{t}^{*}}(x)(\omega)
\end{align*}
$$

and define a correspondence $\varphi:[0, \infty) \times \Omega \rightarrow \mathbb{R}^{m}$ as follows

$$
\varphi(t, \omega) \triangleq\left\{\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{R}_{+}^{m}: \sum_{k=1}^{m} x_{k} S_{t}^{k}(\omega) \leq \widehat{c_{t}^{*}}(x)(\omega)\right\}
$$

From strict positivity of the $S^{k}$ 's and positivity and $(\mathrm{d} \kappa \times \mathbb{P})$-a.e. finiteness of $\widehat{c^{*}}(x)$ (by Mos15, Theorem 3.2]), we deduce that $\varphi$ has nonempty ${ }^{5}$ compact values $(\mathrm{d} \kappa \times \mathbb{P})$-a.e. Let us consider the lower inverse of $\varphi^{l}$ defined by

$$
\varphi^{l}(G) \triangleq\{(t, \omega) \in[0, \infty) \times \Omega: \varphi(t, \omega) \cap G \neq \emptyset\}, \quad G \subset \mathbb{R}^{m}
$$

Let us also consider a subset of $\mathbb{R}^{m}$ of the form $A \triangleq\left[a_{1}, b_{1}\right] \times \cdots \times\left[a_{m}, b_{m}\right]$, where $a_{i}$ 's and $b_{i}$ 's are real numbers. In view of the weak measurability of $\varphi$ (see AB06, Definition 18.1, p. 592])

[^5]that we are planning to show, it is enough to consider $b_{i} \geq 0, i=1, \ldots, m$. In addition, let us set $\bar{a}_{i}=\max \left(0, a_{i}\right)$. One can see that for such a set $A$, as
$$
\varphi^{l}(A)=\varphi^{l}\left(\left[\bar{a}_{1}, b_{1}\right] \times \cdots \times\left[\bar{a}_{m}, b_{m}\right]\right),
$$
we have
$$
\varphi^{l}(A)=\left\{(t, \omega): \sum_{i=1}^{m} \bar{a}_{i} S_{t}^{i}(\omega) \leq \widehat{c_{t}^{*}}(x)(\omega)\right\} .
$$

As $\widehat{c^{*}}(x)$ and $S^{i}$,s are optional processes and since $\varphi^{l}\left(\bigcup_{n \in \mathbb{N}} A_{n}\right)=\bigcup_{n \in \mathbb{N}} \varphi^{l}\left(A_{n}\right)$ (see AB06, Section 17.1], where $A_{n}$ 's are subsets of $\mathbb{R}^{m}$, we deduce that $\varphi^{l}(G) \in \mathcal{O}$ for every open subset $G$ of $\mathbb{R}^{m}$, i.e., $\varphi$ is weakly measurable. As $U$ is a Carathéodory function (see AB06, Definition 4.50, p. 153]), we conclude from [AB06, Theorem 18.19 , p. 605] that there exists an optional $\mathbb{R}^{m_{-}}$ valued process $\widehat{c}_{t}(x), t \in[0, T]$, the maximizer of (4.4) for $(\mathrm{d} \kappa \times \mathbb{P})$-a.e. $(t, \omega) \in[0, \infty) \times \Omega$. The uniqueness of such a maximizer follows from strict concavity of $U(t, \omega, \cdot)$ (for every $(t, \omega) \in$ $[0, \infty) \times \Omega \sqrt{6}$. As $\widehat{c^{*}}(x) \in \mathcal{A}^{*}(x)$, we deduce that $\widehat{c}(x) \in \mathcal{A}(x)$. Combining this with (4.3), we conclude that $\widehat{c}(x)$ is the unique (up to an equivalence class) maximizer to (2.5).

For $x>0$, let $\widehat{c}_{t}^{i}(x), i=1, \ldots, m$, denote the components of $\widehat{c}_{t}(x)$. As $\sum_{i=1}^{m} \widehat{c}_{t}^{i}(x)(\omega) S_{t}^{i}(\omega)=$ $\widehat{c}_{t}^{*}(\omega),(\mathrm{d} \kappa \times \mathbb{P})$-a.e., (where the argument here is similar to the discussion after (4.1)) relations (2.10), (2.12), (2.13), and (2.14) follow from Mos15, Theorem 3.2], whereas (2.15) results from [Mos15, Theorem 3.3] (equivalently, from CCFM17, Theorem 2.4]). In turn, combining (2.12) with HUL04, Theorem D.4.5.1], we get

$$
\begin{aligned}
\widehat{Y}_{t}(\omega)= & U_{x}^{*}\left(t, \omega, \widehat{c_{t}^{*}}(x)(\omega)\right) \\
= & \left\{s(t, \omega) \in \mathbb{R}: S_{t}^{i}(\omega) s(t, \omega)=\partial_{x_{i}} U\left(t, \omega, \widehat{c}^{1}(x)(\omega), \ldots, \widehat{c}^{m}(x)(\omega)\right), \quad i=1, \ldots, m\right\} \\
& (\mathrm{d} \kappa \times \mathbb{P})-a . e .
\end{aligned}
$$

i.e., (2.11) holds.

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[^0]:    Date: March 8, 2022.
    2010 Mathematics Subject Classification. 91G10, 93E20. JEL Classification: C61, G11.
    Key words and phrases. optimal consumption, multiple goods, utility maximization, no unbounded profit with bounded risk, arbitrage of the first kind, local martingale deflator, duality theory, semimartingale, incomplete market, optimal investment.

    The author would like to thank Robert C. Merton for suggesting this problem and for a discussion on the subject of the paper. The author is also thankful to an anonymous referee for valuable comments. The author's research is supported by NSF Grant DMS-1600307. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect those of the National Science Foundation

[^1]:    ${ }^{1}$ Since we allow preferences to be stochastic (see the definition below), there is no loss of generality in assuming that asset prices are discounted, see Mos15, Remark 2.2] for a more detailed explanation of this observation.

[^2]:    ${ }^{2}$ For the results below, we only need to specify the gradient of $U(t, \omega, \cdot)$ in the interior of the first orthant, i.e., at the points $x \in \mathbb{R}^{m}$, where $U(t, \omega, x)$ is (finite-valued and) differentiable.

[^3]:    ${ }^{3}$ Equivalently, see Roc70 Theorem 5.2], where $U^{*}(t, \omega, \cdot)$ is named the image of $U(t, \omega, \cdot)$ under the linear transformation $A(t, \omega, \cdot),(t, \omega) \in[0, \infty) \times \Omega$.

[^4]:    ${ }^{4}$ Note that in general, the image of a closed convex or concave function under a linear transformation need not be closed, see a discussion in HUL04 p.97].

[^5]:    ${ }^{5}$ Note that the origin in $\mathbb{R}^{m}$ is in $\varphi(t, \omega)$ for every $(t, \omega) \in[0, \infty) \times \Omega$.

[^6]:    6 AB06, Theorem 18.19 , p. 605] gives a maximizer, which is a measurable multifunction, and from the uniqueness of the maximizer it is a single valued multifunction, for which the concept of measurability coincides with measurability for functions.

