A remark on positive sojourn times of symmetric processes

Christophe Profeta

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Christophe Profeta[1]

Abstract : We show that under some slight assumptions the positive sojourn time of a product of symmetric processes converges towards one half as the number of processes increases. Monotony properties are then exhibited in the case of symmetric stable processes, and used, via a recurrence relation, to get upper and lower bounds on the moments of the occupation time (in the first and third quadrants) for 2-dimensional Brownian motion. Explicit values are also given for the second and third moments in the n-dimensional Brownian case.

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1 Introduction

In this note, we are interested in the study of the random variables

$$\mathcal{A}_n = \int_0^1 1_{\{\prod_{i=1}^n X_u^{(i)} \geq 0\}} du$$

where $X^{(1)}, \dots, X^{(n)}$ are independent and identically distributed symmetric processes. The random variable $\mathcal{A}_n$ may be interpreted as the time spent by an n-dimensional process (with independent components) in some symmetric orthants.

When $n = 1$, the random variable $\mathcal{A}_1$ has been widely studied for several families of processes. The most celebrated example is the case of symmetric Lévy processes $X^{(1)} = L^{(1)}$ such that $\mathbb{P}(L_1^{(1)} = 0) = 0$, for which it is known that $\mathcal{A}_1$ follows the classic arcsine law (see Getoor and Sharpe [8]) :

$$\mathbb{P}(\mathcal{A}_1 \in dz) = \frac{1}{\pi\sqrt{z(1-z)}}\, dz, \qquad z \in (0,1).$$

When $n = 2$, the random variable $\mathcal{A}_2$ corresponds to the time spent by a planar symmetric process $X^{(1)} + iX^{(2)}$ in the first and third quadrant of the complex plane. In the special case of the planar Brownian motion $B^{(1)} + iB^{(2)}$, a first attempt to find the law of $\mathcal{A}_2$ has been undertaken by Ernst and Shepp [6] in which the authors try to compute the double Laplace transform of $\mathcal{A}_2$. More generally, the study of the sojourn times of planar Brownian motion in a cone has already attracted much attention. In particular it was proven by Mountford [12] that if C is a closed convex cone of magnitude θ with vertex at 0, then there exist two constants κ_1 and κ_2 such that

$$\kappa_1\, t^{1/\xi} \leq \mathbb{P}\left(\int_0^1 1_{\{(B_u^{(1)}, B_u^{(2)}) \in C\}} du \leq t\right) \leq \kappa_2\, t^{1/\xi}, \qquad t \in [0,1], \tag{1}$$

with $\xi = \frac{2}{\pi}(2\pi - \theta)$. The first moments of this random variable were computed by Desbois [4], see also Bingham and Doney [2] in the special case $\theta = \frac{\pi}{2}$. Analogues of (1) for n-dimensional Brownian motion were obtained by Meyre and Werner [13] and Nakayama [14], in which the exponent ξ is related to the first eigenvalue of the Laplacian operator $-\Delta/2$. In addition to these bounds, the strong arcsine law (see

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[1]) gives the asymptotics of the sojourn time of a n-dimensional Brownian motion $(B^{(i)}, 1 \leq i \leq n)$ in the positive orthant :

$$\frac{1}{\ln(t)} \int_1^t \prod_{i=1}^n 1_{\{B_u^{(i)} \geq 0\}} \frac{du}{u} \xrightarrow[t \to +\infty]{a.s.} \frac{1}{2^n}.$$

Observe that the summation here is logarithmic : we refer to [3] for a general discussion between summability methods and limits of occupation times.
In this note, we shall first study the limit of the variables $\mathcal{A}_n$ as the dimension n goes to ∞.

Theorem 1. *Let $(X^{(i)}, i \geq 1)$ be independent and identically distributed symmetric processes.*

1. *The strong law of large numbers holds for the sequence $(\mathcal{A}_n, n \geq 1)$:*

$$\frac{1}{k} \sum_{n=1}^k \mathcal{A}_n \xrightarrow[k \to +\infty]{a.s.} \frac{1}{2}.$$

2. *Assume that for a.e. $u \in (0, 1)$, the random variables $X_u^{(1)}$ have no atoms at 0 and that*

$$\text{for a.e. } 0 < u < s < 1, \qquad 0 < \mathbb{P}\left(X_u^{(1)} \geq 0,\, X_s^{(1)} \geq 0\right) < \frac{1}{2}. \tag{2}$$

Then, for any $p > 0$,

$$\mathcal{A}_n \xrightarrow[n \to +\infty]{L^p} \frac{1}{2}.$$

3. *Assume furthermore that*

$$\int_0^1 \int_0^1 \frac{1}{\mathbb{P}\left(X_u^{(1)} \leq 0,\, X_s^{(1)} \geq 0\right)} du\, ds < +\infty. \tag{3}$$

Then,

$$\mathcal{A}_n \xrightarrow[n \to +\infty]{a.s.} \frac{1}{2}.$$

When thinking of symmetric Lévy processes, an interpretation of this result is as follows : the usual arcsine law essentially explains that, although $L^{(1)}$ is centered, there is a high probability that it spends more time on one side of the axis than on the other one. As the number of Lévy processes increases, so do the changes of sign of the product, hence the resulting process spends a more balanced time on each side of the abscissa axis.

Remark 2. Note that an assumption such as (2) is necessary to obtain the L^p-convergence. Indeed, let for instance $(X_i,\, i \geq 1)$ be a family of i.i.d. symmetric random variables admitting a density. Define the processes :

$$X_t^{(i)} = tX_i \qquad (t \geq 0)$$

which do not satisfy the assumption $\mathbb{P}(X_u^{(1)} \geq 0,\, X_s^{(1)} \geq 0) < \frac{1}{2}$. In this case, the random variables $\mathcal{A}_n$ all have the same law :

$$\mathcal{A}_n \overset{\text{(law)}}{=} \frac{1}{2}(\delta_0 + \delta_1)$$

and the L^p-convergence of Theorem 1 cannot hold.

Example 3. Assumption (3) is for instance satisfied by symmetric α-stable Lévy processes L with $\alpha > 1$. Indeed, for $0 < u < s$, using the symmetry, independent increments and scaling properties, we first deduce that :

$$\mathbb{P}(L_s \leq 0,\, L_u \geq 0) = \mathbb{P}\left(\left(\frac{s}{u} - 1\right)^{1/\alpha} L_1 \geq Z_1,\, Z_1 \geq 0\right)$$

where Z_1 is a symmetric α-stable r.v. independent from L_1. Next, for $\nu > 0$ small enough, applying Fubini's theorem :

$$\begin{aligned}\int_0^{+\infty} t^{-\nu-1}\mathbb{P}(t^{1/\alpha}L_1 \geq Z_1,\, Z_1 \geq 0)dt &= \frac{1}{\nu}\mathbb{E}\left[L_1^{\nu\alpha}1_{\{L_1\geq 0\}}\right]\mathbb{E}\left[Z_1^{-\nu\alpha}1_{\{Z_1\geq 0\}}\right]\\ &= \frac{1}{4\nu}\mathbb{E}\left[|L_1|^{\nu\alpha}\right]\mathbb{E}\left[|Z_1|^{-\nu\alpha}\right]\\ &= \frac{1}{4\alpha^2\nu}\frac{\Gamma(\nu)}{\Gamma(\nu\alpha)\cos(\nu\alpha\pi/2)}\frac{\Gamma(-\nu)}{\Gamma(-\nu\alpha)\cos(\nu\alpha\pi/2)}\\ &= \frac{1}{2\alpha\nu}\frac{\sin(\nu\alpha\pi/2)}{\sin(\nu\pi)\cos(\nu\alpha\pi/2)}.\end{aligned}$$

Therefore, using the inverse mapping for the Mellin transform (see for instance Janson [9]), we obtain the asymptotics

$$F(t) := \mathbb{P}\left(t^{1/\alpha}L_1 \geq Z_1,\, Z_1 \geq 0\right) \underset{t\to 0^+}{\sim} \frac{1}{\pi\alpha\sin(\pi/\alpha)}t^{1/\alpha}$$

hence, by a change of variable

$$\int_0^1\left(\int_0^1 \frac{1}{\mathbb{P}(L_u\leq 0,\, L_s\geq 0)}du\right)ds = 2\int_0^1\left(\int_0^s \frac{1}{F(s/u-1)}du\right)ds = \int_0^{+\infty}\frac{1}{F(t)(t+1)^2}dt < +\infty$$

which is Assumption (3).

The remainder of the paper is organized as follows : we prove Theorem 1 in Section 2, then study some monotony properties of $\mathcal{A}_n$ when dealing with stable processes in Section 3, and finally compute the first moments of $\mathcal{A}_n$ and give some bounds on $\mathcal{A}_2$ for Brownian motion in Section 4.

2 Proof of Theorem 1

Proof. We start with the law of large numbers of Point 1. Let us first define the centered random variables

$$\mathcal{A}_n^* = \mathcal{A}_n - \frac{1}{2}$$

and observe that these random variables are uncorrelated. Indeed, decomposing $\mathcal{A}_{k+n}$ and using the tower property of conditional expectations for $k \geq 1$

$$\begin{aligned}\mathbb{E}[\mathcal{A}_n^*\mathcal{A}_{n+k}^*] &= \mathbb{E}[\mathcal{A}_n\mathcal{A}_{n+k}] - \frac{1}{4}\\ &= \mathbb{E}\left[\mathcal{A}_n\int_0^1\left(1_{\{\prod_{i=1}^n X_u^{(i)}\geq 0\}}1_{\{\prod_{i=n+1}^{n+k} X_u^{(i)}\geq 0\}} + 1_{\{\prod_{i=1}^n X_u^{(i)}\leq 0\}}1_{\{\prod_{i=n+1}^{n+k} X_u^{(i)}\leq 0\}}\right)du\right] - \frac{1}{4}\\ &= \mathbb{E}\left[\mathcal{A}_n\int_0^1\left(1_{\{\prod_{i=1}^n X_u^{(i)}\geq 0\}}\frac{1}{2} + 1_{\{\prod_{i=1}^n X_u^{(i)}\leq 0\}}\frac{1}{2}\right)du\right] - \frac{1}{4}\\ &= \frac{1}{2}\mathbb{E}[\mathcal{A}_n] - \frac{1}{4} = 0.\end{aligned}$$

Now, since the random variables $(\mathcal{A}_n^*, n \geq 1)$ are uniformly bounded by 1, the result will follow from Theorem 1 in [10] after having checked that

$$\sum_{k\geq 1}\frac{1}{k}\mathbb{E}\left[\left(\frac{1}{k}\sum_{n=1}^k \mathcal{A}_n^*\right)^2\right] < \infty.$$

But this is immediate since developing the square and applying Fubini's theorem :

$$\sum_{k\geq 1}\frac{1}{k}\mathbb{E}\left[\left(\frac{1}{k}\sum_{n=1}^k \mathcal{A}_n^*\right)^2\right] = \sum_{k\geq 1}\frac{1}{k^3}\sum_{n=1}^k\mathbb{E}\left[(\mathcal{A}_n^*)^2\right] \leq \sum_{k\geq 1}\frac{1}{k^2} < +\infty,$$

hence we conclude that

$$\frac{1}{k}\sum_{n=1}^{k}\mathcal{A}_n^* \xrightarrow[k\to+\infty]{\text{a.s.}} 0,$$

which is Point 1 of Theorem 1.

To prove the L^p-convergence, let us consider, for $n \geq 1$, the function $F_n : [0,1]^2 \to [0,1]$ defined by

$$F_n(u,s) = \mathbb{P}\left(\prod_{i=1}^n X_u^{(i)} \geq 0,\, \prod_{i=1}^n X_s^{(i)} \geq 0\right).$$

By symmetry and since there are no atoms at 0, we may decompose F_{n+1} as

$$\begin{aligned} F_{n+1}(u,s) = 2\,\mathbb{P}\left(\prod_{i=1}^n X_u^{(i)} \geq 0,\, \prod_{i=1}^n X_s^{(i)} \geq 0\right)\mathbb{P}\left(X_u^{(n+1)} \geq 0,\, X_s^{(n+1)} \geq 0\right) \\ + 2\,\mathbb{P}\left(\prod_{i=1}^n X_u^{(i)} \leq 0,\, \prod_{i=1}^n X_s^{(i)} \geq 0\right)\mathbb{P}\left(X_u^{(n+1)} \leq 0,\, X_s^{(n+1)} \geq 0\right) \end{aligned} \tag{4}$$

and rewrite this under the form

$$\begin{aligned} F_{n+1}(u,s) &= 2F_n(u,s)F_1(u,s) + 2\left(\frac{1}{2} - F_n(u,s)\right)\left(\frac{1}{2} - F_1(u,s)\right) \\ &= 4\left(F_n(u,s) - \frac{1}{4}\right)\left(F_1(u,s) - \frac{1}{4}\right) + \frac{1}{4}. \end{aligned}$$

In particular, we deduce by iteration that

$$F_{n+1}(u,s) - \frac{1}{4} = \left(F_n(u,s) - \frac{1}{4}\right)(4F_1(u,s) - 1) = \frac{1}{4}\left(4F_1(u,s) - 1\right)^{n+1}. \tag{5}$$

Now, for a.e. $u \neq s$, we have by assumption $-1 < 4F_1(u,s) - 1 < 1$, so we may let $n \to +\infty$ to obtain

$$F_n(u,s) \xrightarrow[n\to+\infty]{} \frac{1}{4}.$$

Finally, applying the dominated convergence theorem

$$\mathbb{E}\left[\left(\mathcal{A}_n - \frac{1}{2}\right)^2\right] = \mathbb{E}\left[\mathcal{A}_n^2\right] - \frac{1}{4} = \int_0^1\int_0^1 F_n(u,s)\,duds - \frac{1}{4} \xrightarrow[n\to+\infty]{} 0$$

which proves the L^2-convergence of Theorem 1, hence the L^p-convergence for any $0 < p \leq 2$ by Hölder's inequality. But, since for any $n \in \mathbb{N}$, $|\mathcal{A}_n - \frac{1}{2}| \leq 1$, we further obtain that for $p \geq 2$

$$\mathbb{E}\left[\left|\mathcal{A}_n - \frac{1}{2}\right|^p\right] \leq \mathbb{E}\left[\left(\mathcal{A}_n - \frac{1}{2}\right)^2\right] \xrightarrow[n\to+\infty]{} 0$$

which ends the proof of Point 2.

Finally, to get the a.s. convergence of Point 3, we apply Fubini's theorem to obtain the bound, thanks to (5) :

$$\begin{aligned} \sum_{n=1}^{+\infty}\mathbb{E}\left[\left(\mathcal{A}_n - \frac{1}{2}\right)^2\right] &= \sum_{n=1}^{+\infty}\int_0^1\int_0^1\left(F_n(u,s) - \frac{1}{4}\right)duds \\ &= \frac{1}{4}\int_0^1\int_0^1 \frac{4F_1(u,s)-1}{2-4F_1(u,s)}\,du\,ds \leq \frac{1}{16}\int_0^1\int_0^1 \frac{1}{\mathbb{P}(X_u^{(1)} \leq 0,\, X_s^{(1)} \geq 0)}\,du\,ds < +\infty. \end{aligned}$$

The a.s. convergence then follows from the usual application of the Bienaymé-Tchebychev inequality and the Borel-Cantelli lemma. ∎

3 Monotonicity for stable processes

We now assume that $(X^{(i)} = L^{(i)})_{i\geq 1}$ are independent symmetric α-stable Lévy processes with $\alpha \in (0,2]$ defined on a probability space $(\Omega, \mathcal{F}_\infty, \mathbb{P})$. From Point 2 of Theorem 1, we deduce that for any $p > 0$:

$$\mathbb{E}[\mathcal{A}_n^p] \xrightarrow[n\to+\infty]{} \left(\frac{1}{2}\right)^p.$$

When dealing with stable processes, it turns out that the sequence $(\mathbb{E}[\mathcal{A}_n^p],\, n \geq 1)$ is monotone, according to the value of p (i.e. to the convexity of the function $x \mapsto x^p$).

Proposition 4. *Let $p > 0$ be fixed. The sequence*

$$(\mathbb{E}[\mathcal{A}_n^p],\, n \geq 1) \text{ is } \begin{cases} \text{decreasing if } p \in \mathbb{N}^* \\ \text{increasing if } 0 < p < 1. \end{cases}$$

As a consequence, for any $\lambda \in \mathbb{R}$ and $n \geq 1$:

$$\mathbb{E}\left[e^{\lambda \mathcal{A}_{n+1}}\right] \leq \mathbb{E}\left[e^{\lambda \mathcal{A}_n}\right].$$

For symmetric stable Lévy processes, the r.v.'s $(\mathcal{A}_n, n \geq 1)$ are thus ordered via moment-generating functions or Laplace transforms.

Proof. We start the proof with a simple lemma.

Lemma 5. *Let $n \geq 1$ and $(X_i)_{i\leq n}$ be i.i.d. symmetric random variables with common density f and let $(A_i)_{i\leq n}$ be random variables, independent from the $(X_i)_{i\leq n}$ and such that $\mathbb{P}\left(\prod_{i=1}^n A_i > 0\right) = 1$. Then, the function*

$$t \mapsto \mathbb{P}\left(\prod_{i=1}^n (X_i + tA_i) \geq 0\right) \qquad \text{is increasing from 1/2 to 1.}$$

Proof. Observe first that by conditioning on the distribution of the sequence $(A_i)_{i\leq n}$

$$\mathbb{P}\left(\prod_{i=1}^n (X_i + tA_i) \geq 0\right) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathbb{P}\left(\prod_{i=1}^n (X_i + ta_i) \geq 0\right) \mathbb{P}(A_1 \in da_1, \dots, A_n \in da_n)$$

we only need to prove that the function

$$\Psi_n(t) = \mathbb{P}\left(\prod_{i=1}^n (X_i + ta_i) \geq 0\right)$$

is increasing, under the assumption $\prod_{i=1}^n a_i > 0$. Next, for any $n \geq 1$, we have $\Psi_n(0) = 1/2$ and $\lim_{t\to+\infty} \Psi_n(t) = 1$. We shall prove that Ψ_n is increasing by induction on n. For $n = 1$, the result is clear since in this case $a_1 > 0$. Assume now that $n \geq 2$ and that Ψ_{n-1} is increasing from $1/2$ to 1. Since the (X_i) are independent, we may decompose

$$\Psi_n(t) = \mathbb{P}(X_n + ta_n \geq 0)\mathbb{P}\left(\prod_{i=1}^{n-1} (X_i + ta_i) \geq 0\right) + \mathbb{P}(X_n + ta_n \leq 0)\mathbb{P}\left(\prod_{i=1}^{n-1} (X_i + ta_i) \leq 0\right).$$

We now separate two cases.

1. Assume first that $a_n > 0$. Then $\prod_{i=1}^{n-1} a_i > 0$ and differentiating

$$\begin{aligned}
\Psi_n'(t) &= a_n f(-ta_n)\Psi_{n-1}(t) + \mathbb{P}(X_n + ta_n \geq 0)\Psi_{n-1}'(t) \\
&\qquad - a_n f(-ta_n)(1 - \Psi_{n-1}(t)) - \mathbb{P}(X_n + ta_n \leq 0)\Psi_{n-1}'(t) \\
&= a_n f(-ta_n)\left(2\Psi_{n-1}(t) - 1\right) + \Psi_{n-1}'(t)\left(\mathbb{P}(X_n + ta_n \geq 0) - \mathbb{P}(X_n + ta_n \leq 0)\right).
\end{aligned}$$

   Since $\Psi_{n-1}(t) \geq 1/2$ and $\mathbb{P}(X_n + ta_n \geq 0) > \mathbb{P}(X_n + ta_n \leq 0)$, we deduce from the recursion hypothesis that $\Psi_n'(t) > 0$.

2. Assume now that $a_n < 0$. Then $\prod_{i=1}^{n-1} a_i < 0$ and we deduce from the symmetry of X_1 and X_n that[2]

$$\Psi_n(t) = \mathbb{P}(X_n + t(-a_n) \leq 0)\mathbb{P}\left((X_1 + t(-a_1))\prod_{i=2}^{n-1}(X_i + ta_i) \leq 0\right) \\ + \mathbb{P}(X_n + t(-a_n) \geq 0)\mathbb{P}\left((X_1 + t(-a_1))\prod_{i=2}^{n-1}(X_i + ta_i) \geq 0\right).$$

The result then follows from the first case, since $-a_n > 0$ and $-\prod_{i=1}^{n-1} a_i > 0$.

□

We now come back to the proof of Proposition 4. To simplify the notation, we set :

$$P_u^{(n)} = \prod_{i=1}^{n} L_u^{(i)}.$$

Let us consider the function $F : \mathbb{R}^+ \to [0, 1]$ defined by :

$$F(x) = \mathbb{E}\left[\left(\int_0^1 1_{\{(x+Z_u)P_u^{(n)} \geq 0\}} du\right)^p\right]$$

where Z is another α-stable Lévy process independent from the $(L^{(i)})$. We shall prove that F is increasing on $[0, +\infty)$. By the change of variable $u = x^\alpha s$ and scaling, we have

$$F(x) = x^{\alpha p}\,\mathbb{E}\left[\left(\int_0^{1/x^\alpha} 1_{\{(1+Z_s)P_s^{(n)} \geq 0\}} ds\right)^p\right].$$

Differentiating with respect to x and going back to the original variable, we obtain

$$F'(x) = \frac{\alpha p}{x}\mathbb{E}\left[\left(\int_0^1 1_{\{(x+Z_u)P_u^{(n)} \geq 0\}} du\right)^p - \left(\int_0^1 1_{\{(x+Z_u)P_u^{(n)} \geq 0\}} du\right)^{p-1} 1_{\{(x+Z_1)P_1^{(n)} \geq 0\}}\right].$$

Applying Fubini's theorem, we need to prove that

$$\int_0^1 \dots \int_0^1 \mathbb{P}\left(\bigcap_{i=1}^{p}\left\{(x + Z_{u_i})P_{u_i}^{(n)} \geq 0\right\}\right) du_1 \dots du_p \\ \geq \int_0^1 \dots \int_0^1 \mathbb{P}\left(\bigcap_{i=1}^{p-1}\left\{(x + Z_{u_i})P_{u_i}^{(n)} \geq 0\right\} \cap \left\{(x + Z_1)P_1^{(n)} \geq 0\right\}\right) du_1 \dots du_p.$$

We shall in fact simply prove that the inequality holds on the integrands :

$$\mathbb{P}\left(\bigcap_{i=1}^{p-1}\left\{(x + Z_{u_i})P_{u_i}^{(n)} \geq 0\right\} \cap \left\{(x + Z_{u_p})P_{u_p}^{(n)} \geq 0\right\}\right) \\ \geq \mathbb{P}\left(\bigcap_{i=1}^{p-1}\left\{(x + Z_{u_i})P_{u_i}^{(n)} \geq 0\right\} \cap \left\{(x + Z_1)P_1^{(n)} \geq 0\right\}\right). \quad (6)$$

where we may assume, up to renaming the variables, that $0 \leq u_1 \leq u_2 \leq \ldots \leq u_p \leq 1$. To simplify the notation, let us introduce the measure $\mathbb{Q}$ defined, for $\Lambda \in \mathcal{F}_\infty$, by :

$$\mathbb{Q}(\Lambda) = \mathbb{P}\left(\Lambda \Big| \bigcap_{i=1}^{p-1}\left\{(x + Z_{u_i})P_{u_i}^{(n)} \geq 0\right\}\right).$$

[2] with the usual convention that empty products are 1

Dividing both sides of (6) by $\mathbb{P}\left(\bigcap_{i=1}^{p-1}\left\{(x+Z_{u_i})P_{u_i}^{(n)}\geq 0\right\}\right)$, we are thus led to prove that the function

$$t\to\mathbb{Q}\left((x+Z_{t+u_{p-1}})P_{t+u_{p-1}}^{(n)}\geq 0\right)\quad\text{ is decreasing on } [0,1-u_{p-1}].$$

Applying the Markov property, we may decompose

$$\mathbb{Q}\left((x+Z_{t+u_{p-1}})P_{t+u_{p-1}}^{(n)}\geq 0\right)=\mathbb{Q}\left(\left(x+Z_{u_{p-1}}+t^{1/\alpha}\widehat{Z}_1\right)\prod_{i=1}^{n}\left(L_{u_{p-1}}^{(i)}+t^{1/\alpha}\widehat{L}_1^{(i)}\right)\geq 0\right)$$

where $\widehat{Z_1}$ and $(\widehat{L}_1^{(i)})$ are independent symmetric α-stable random variables, independent from Z and the $(L^{(i)})$. Observe furthermore that by definition of $\mathbb{Q}$:

$$\mathbb{Q}\left((x+Z_{u_{p-1}})\prod_{i=1}^{n}L_{u_{p-1}}^{(i)}>0\right)=\mathbb{Q}\left((x+Z_{u_{p-1}})P_{u_{p-1}}^{(n)}>0\right)=1.$$

Therefore, applying Lemma 5 with the sequences

$$(X_i,1\leq i\leq n)=(\widehat{L}_1^{(i)},1\leq i\leq n),\qquad X_{n+1}=\widehat{Z}_1$$

and

$$(A_i,1\leq i\leq n)=(L_{u_{p-1}}^{(i)},1\leq i\leq n),\qquad A_{n+1}=x+Z_{u_{p-1}},$$

we deduce by composition that the function $t\to\mathbb{Q}\left((x+Z_{t+u_{p-1}})P_{t+u_{p-1}}^{(n)}\geq 0\right)$ is decreasing on $[0,1-u_{p-1}]$ (in fact on $[0,+\infty)$), hence the function F is increasing on $[0,+\infty)$. It remains then to apply the dominated convergence theorem, upon noticing that

$$\mathbb{E}[\mathcal{A}_{n+1}^p]=F(0)\leq\lim_{x\to+\infty}F(x)=\mathbb{E}[\mathcal{A}_n^p]$$

which yields the proof for integer values.

Summing the different moments, we deduce that for $\lambda\geq 0$:

$$\mathbb{E}\left[e^{\lambda\mathcal{A}_{n+1}}\right]\leq\mathbb{E}\left[e^{\lambda\mathcal{A}_n}\right]$$

Next, by symmetry

$$\mathbb{E}\left[e^{\lambda(1-\mathcal{A}_{n+1})}\right]\leq\mathbb{E}\left[e^{\lambda(1-\mathcal{A}_n)}\right]\quad\Longleftrightarrow\quad\mathbb{E}\left[e^{-\lambda\mathcal{A}_{n+1}}\right]\leq\mathbb{E}\left[e^{-\lambda\mathcal{A}_n}\right]$$

hence, for $0<p<1$:

$$\int_0^{+\infty}\lambda^{-p-1}\left(1-\mathbb{E}\left[e^{-\lambda\mathcal{A}_n}\right]\right)d\lambda\leq\int_0^{+\infty}\lambda^{-p-1}\left(1-\mathbb{E}\left[e^{-\lambda\mathcal{A}_{n+1}}\right]\right)d\lambda$$

which is exactly

$$\mathbb{E}[\mathcal{A}_n^p]\leq\mathbb{E}[\mathcal{A}_{n+1}^p]\qquad\qquad(0<p<1).$$

□

Remark 6. We give below an example of a process satisfying Assumption (2), but for which the sequence $(\mathbb{E}[\mathcal{A}_n^2],n\geq 1)$ is not decreasing. Take for instance

$$X_t^{(1)}=B_t1_{\{t\leq 1/2\}}-B_{t-1/2}1_{\{t>1/2\}}\qquad\text{and}\qquad X_t^{(2)}=W_t1_{\{t\leq 1/2\}}-W_{t-1/2}1_{\{t>1/2\}}$$

where B and W are two independent Brownian motions started from 0. Then :

$$\mathcal{A}_1=\frac{1}{2}\qquad\text{and}\qquad\mathcal{A}_2=2\int_0^{1/2}1_{\{B_tW_t\geq 0\}}dt\overset{\text{(law)}}{=}\int_0^1 1_{\{B_uW_u\geq 0\}}du$$

hence (see next Section for the value of $\mathbb{E}[\mathcal{A}_2^2]$)

$$\mathbb{E}\left[\mathcal{A}_1^2\right]=\frac{1}{4}<\mathbb{E}\left[\mathcal{A}_2^2\right]=\frac{3}{8}-\frac{1}{2\pi^2}.$$

4 A study of moments in the Brownian case

4.1 Second and third moments

The first three moments are easy to compute in the Brownian case. Indeed, from Formula (5), the second moment of $\mathcal{A}_n$ equals:

$$\mathbb{E}[\mathcal{A}_n^2] = \frac{1}{4} + \frac{1}{2}\int_0^1 \left(\int_0^s (4F_1(u,s)-1)^n \, du\right) ds$$

where, from Bingham and Doney [2], the quadrant probability is given for $0 \leq u \leq s$ by :

$$F_1(u,s) = \frac{1}{4} + \frac{1}{2\pi}\arcsin\left(\sqrt{\frac{u}{s}}\right).$$

After some changes of variables and successive integrations by parts, we deduce that

$$\mathbb{E}[\mathcal{A}_n^2] = \frac{1}{4} + \frac{1}{8\pi^n}\int_0^\pi t^n \sin(t)dt = \frac{1}{4} + (-1)^{\lfloor n/2\rfloor+1}\frac{n!(n-2\lfloor n/2\rfloor-1)}{8\pi^n} + \frac{1}{8}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{n!}{(n-2k)!}\pi^{-2k}.$$

By symmetry, since $\mathbb{E}[(1-\mathcal{A}_n)^3] = \mathbb{E}[\mathcal{A}_n^3]$, we further obtain :

$$\mathbb{E}\left[\mathcal{A}_n^3\right] = \frac{3}{2}\mathbb{E}\left[\mathcal{A}_n^2\right] - \frac{1}{4}.$$

We give below the first values of the second and third moments, in which the decreasing property may be observed.

n	$\mathbb{E}[\mathcal{A}_n^2]$	$\mathbb{E}[\mathcal{A}_n^3]$
1	$\frac{3}{8} \simeq 0,375$	$\frac{5}{16} \simeq 0,3125$
2	$\frac{3}{8} - \frac{1}{2\pi^2} \simeq 0,3243$	$\frac{5}{16} - \frac{3}{4\pi^2} \simeq 0,2365$
3	$\frac{3}{8} - \frac{3}{4\pi^2} \simeq 0,299$	$\frac{5}{16} - \frac{9}{8\pi^2} \simeq 0,1985$
4	$\frac{3}{8} - \frac{3}{2\pi^2} + \frac{6}{\pi^4} \simeq 0,2846$	$\frac{5}{16} - \frac{9}{4\pi^2} + \frac{9}{\pi^4} \simeq 0,1769$
5	$\frac{3}{8} - \frac{5}{2\pi^2} + \frac{15}{\pi^4} \simeq 0,2757$	$\frac{5}{16} - \frac{15}{4\pi^2} + \frac{45}{2\pi^4} \simeq 0,1635$
∞	$0,25$	$0,125$

4.2 Higher moments for two Brownian motions

Getting the explicit values of higher moments seems a hard task as outlined in several papers [2, 4, 6]. We propose here a method to get lower and upper bounds on these moments. Recall the moments of the arcsine distribution :

$$\mathbb{E}[\mathcal{A}_1^p] = \frac{1}{2^{2p}}\binom{p}{2p} = \frac{(2p)!}{2^{2p}(p!)^2} = \frac{\Gamma(p+1/2)}{\sqrt{\pi}\Gamma(p+1)} \underset{p\to+\infty}{\sim} \frac{1}{\sqrt{\pi p}}$$

Proposition 7. *For any $p \geq 1$, we have*

$$\mathbb{E}[\mathcal{A}_2^p] \leq \frac{1}{2p+1}\frac{8}{\pi^2}{}_3F_2\left[\begin{matrix}1/2 \ 1/2 \ 1\\ p+3/2 \ 3/2\end{matrix};1\right] + \frac{1}{\pi^2}\sum_{k=0}^{p-1}\frac{2}{(p-k)^2}\mathbb{E}[\mathcal{A}_1^k]$$

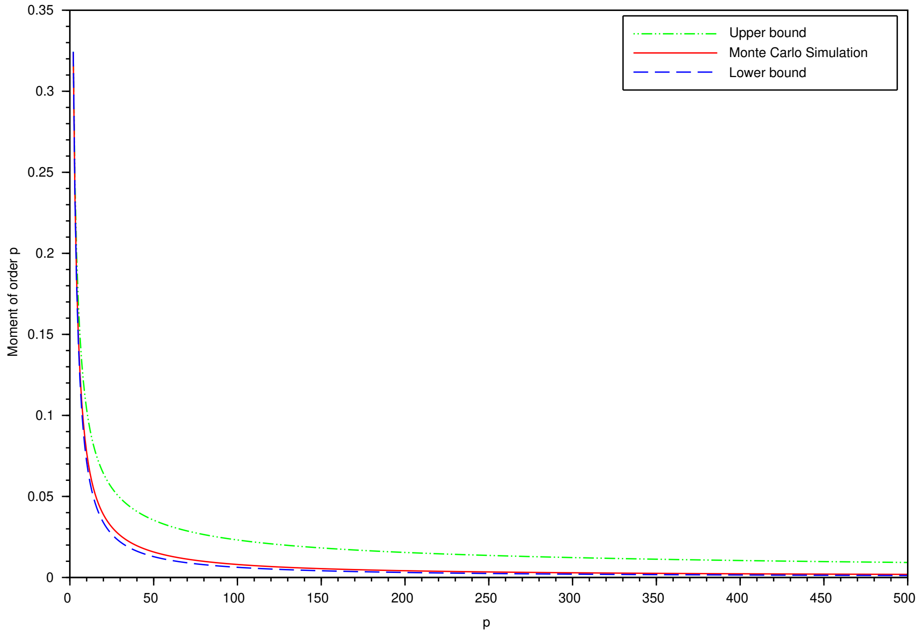

Figure 1: Monte Carlo simulation of $\mathbb{E}[\mathcal{A}_2^p]$ for $1 \le p \le 500$.

and

$$\mathbb{E}[\mathcal{A}_2^p] \ge \frac{1}{2p+1}\frac{8}{\pi^2}{}_3F_2\left[\begin{matrix}1/2 & 1/2 & 1\\ p+3/2 & 3/2\end{matrix};1\right] + \frac{1}{\pi^2}\sum_{k=0}^{p-1}\frac{2}{(p-k)^2}\mathbb{E}[\mathcal{A}_2^k]$$

where ${}_3F_2$ denotes the usual generalized hypergeometric function, see [7, Section 9.1]. Note that both bounds are the same when p equals 1 and 2. Asymptotically, we further obtain that

$$\frac{6}{\pi^2 p} \le \mathbb{E}[\mathcal{A}_2^p] \le \frac{1}{3\sqrt{\pi p}} \qquad\qquad (p \to +\infty).$$

In particular, this implies that the r.v. $\mathcal{A}_2$ cannot follow a Beta distribution (hence neither a generalized arcsine distribution). Indeed, otherwise the Beta distribution would be $\beta\left(\frac{1}{2}+\frac{4}{\pi^2-4}, \frac{1}{2}+\frac{4}{\pi^2-4}\right)$, since then

$$\mathbb{E}[\beta] = \frac{1}{2}, \qquad \mathbb{E}[\beta^2] = \frac{3}{8} - \frac{1}{2\pi^2} \qquad \text{and} \qquad \mathbb{E}[\beta^3] = \frac{5}{16} - \frac{3}{4\pi^2}.$$

But, as $p \to +\infty$, we would have

$$\mathbb{E}[\beta^p] = \mathcal{O}\left(\frac{1}{p^{\frac{1}{2}+\frac{4}{\pi^2-4}}}\right)$$

which would contradict the lower bound since $\frac{1}{2}+\frac{4}{\pi^2-4} > 1$. Numerical computations are shown in Figure 1, in which it is seen that the lower bound is clearly the better one.

Proof. Let B and W be two independent Brownian motions and define

$$M_p(x) = \int_0^{+\infty} e^{-t/2}\mathbb{E}\left[\left(\int_0^t \mathbf{1}_{\{(x+B_u)W_u>0\}}du\right)^p\right]dt$$

so that

$$\mathbb{E}[\mathcal{A}_2^p] = \frac{M_p(0)}{2^{p+1}p!}.$$

Applying first the Markov property at the stopping time $T_x = \inf\{u \geq 0,\, x + B_u = 0\}$ and Fubini's theorem, we deduce that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \mathbf{1}_{\{(x+B_u)W_u>0\}}du\right)^p\right] \\
&= \mathbb{E}\left[\left(\int_0^t \mathbf{1}_{\{W_u>0\}}du\right)^p 1_{\{T_x>t\}}\right] + \mathbb{E}\left[\left(\int_0^{T_x} \mathbf{1}_{\{W_u>0\}}du + \int_{T_x}^t \mathbf{1}_{\{(x+B_u)W_u>0\}}du\right)^p 1_{\{T_x\leq t\}}\right] \\
&= t^p\mathbb{E}\left[\mathcal{A}_1^p\right]\mathbb{P}(T_x>t) + \sum_{k=0}^p \binom{p}{k}\mathbb{E}\left[\left(\int_0^{T_x}\mathbf{1}_{\{W_u>0\}}du\right)^{p-k}\left(\int_0^{t-T_x}\mathbf{1}_{\{\widehat{B}_s(W_{T_x}+\widehat{W}_s)>0\}}du\right)^k 1_{\{T_x\leq t\}}\right]
\end{aligned}$$

where $\widehat{B}$ and $\widehat{W}$ are two independent Brownian motions, independent from B and W. We now take the Laplace transform of both sides. Applying the Fubini-Tonelli theorem and a change of variable, we obtain

$$\begin{aligned}
M_p(x) &= \mathbb{E}\left[\mathcal{A}_1^p\right]\int_0^{+\infty} e^{-t/2}t^p\mathbb{P}(T_x>t)dt + \sum_{k=0}^p\binom{p}{k}\mathbb{E}\left[e^{-T_x/2}\left(\int_0^{T_x}\mathbf{1}_{\{W_u>0\}}du\right)^{p-k} M_k(W_{T_x})\right] \\
&= R_{p-1}(x) + \mathbb{E}\left[e^{-T_x/2}M_p(W_{T_x})\right]
\end{aligned}$$

where, by scaling, R_{p-1} is defined by

$$R_{p-1}(x) = \mathbb{E}[\mathcal{A}_1^p]\int_0^{+\infty} e^{-t/2}t^p\mathbb{P}(T_x>t)dt + \sum_{k=0}^{p-1}\binom{p}{k}\mathbb{E}\left[e^{-T_x/2}\,T_x^{p-k}\left(\int_0^1 \mathbf{1}_{\{W_u>0\}}du\right)^{p-k} M_k(\sqrt{T_x}W_1)\right].$$

We thus obtain the relation, since $\mathbb{E}\left[e^{-T_x/2}\right] = e^{-|x|}$,

$$M_p(x) - M_p(0) = R_{p-1}(x) - (1-e^{-|x|})M_p(0) + \mathbb{E}\left[e^{-T_x/2}(M_p(W_{T_x}) - M_p(0))\right]. \tag{7}$$

The expectation on the right-hand side may be computed to give

$$\begin{aligned}
\mathbb{E}\left[e^{-T_x/2}(M_p(W_{T_x}) - M_n(0))\right] &= 2\int_0^{+\infty}\frac{x}{\sqrt{2\pi t^3}}e^{-\frac{x^2}{2t}-t/2}dt\int_0^{+\infty}\frac{1}{\sqrt{2\pi t}}e^{-\frac{z^2}{2t}}(M_p(z)-M_p(0))dz \\
&= 2\int_0^{+\infty}\frac{x}{\pi}\frac{K_1(\sqrt{x^2+z^2})}{\sqrt{x^2+z^2}}(M_p(z)-M_p(0))dz
\end{aligned}$$

where K_ν denotes the modified Bessel function of the second kind of order ν. Integrating (7) with respect to $K_0(x)\frac{dx}{x}$, we deduce from the formulae [5, p.377 $\mathrm{n}^0(33)$] and [7, p.695 $\mathrm{n}^0(9)$]

$$\int_0^{+\infty}\frac{K_1(\sqrt{x^2+z^2})}{\sqrt{x^2+z^2}}K_0(x)dx = \frac{\pi}{2z}K_0(z) \qquad \text{and} \qquad \int_0^{+\infty}(1-e^{-x})K_0(x)\frac{dx}{x} = \frac{\pi^2}{8}$$

that

$$\int_0^{+\infty}(M_p(x)-M_p(0))K_0(x)\frac{dx}{x} = \int_0^{+\infty}(R_{p-1}(x)-(1-e^{-|x|})M_p(0))K_0(x)\frac{dx}{x} + \int_0^{+\infty}K_0(z)(M_p(z)-M_p(0))\frac{dz}{z}$$

hence

$$M_p(0) = \frac{8}{\pi^2}\int_0^{+\infty}R_{p-1}(x)K_0(x)\frac{dx}{x}. \tag{8}$$

Remark 8. Note that we might have used the Kontorovitch-Lebedev transform (see for instance [11]) to get a recurrence relation between $M_p(x)$ and $R_{p-1}(x)$:

$$M_p(x) = \frac{2}{\pi^2}\int_0^{+\infty}\frac{\cosh(\pi\gamma/2)}{\cosh(\pi\gamma/2)-1}K_{i\gamma}(x)\left(\int_0^{+\infty}K_{i\gamma}(z)R_{p-1}(z)\frac{dz}{z}\right)\gamma\sinh(\pi\gamma)d\gamma$$

but this leads to quite complicated calculations, even for $p=3$.

We shall rather find bounds on R_{p-1}. Plugging the expression of R_{p-1} in (8), we first need to compute :

$$\begin{aligned}
&\int_0^{+\infty}\left(\int_0^{+\infty} e^{-t/2}t^p\mathbb{P}(T_x>t)dt\right)K_0(x)\frac{dx}{x}\\
&\qquad= \int_0^{+\infty} e^{-t/2}t^p dt\int_t^{+\infty}\frac{1}{4s}e^{s/4}K_0(s/4)ds\\
&\qquad= \int_1^{+\infty}\frac{du}{4u}\int_0^{+\infty} t^p e^{-t(2-u)/4}K_0(ut/4)dt\\
&\qquad= 2^{p-1}\sqrt{\pi}\frac{(\Gamma(p+1))^2}{\Gamma(p+3/2)}\int_1^{+\infty} {}_2F_1\left[\begin{matrix}p+1,1/2\\p+3/2\end{matrix};1-u\right]\frac{du}{u} \qquad \text{(see [7, p.700])}\\
&\qquad= 2^{p-1}\sqrt{\pi}\frac{(\Gamma(p+1))^2}{\Gamma(p+3/2)}\int_0^{1} {}_2F_1\left[\begin{matrix}p+1,1/2\\p+3/2\end{matrix};1-\frac{1}{x}\right]\frac{dx}{x}\\
&\qquad= 2^{p-1}\sqrt{\pi}\frac{(\Gamma(p+1))^2}{\Gamma(p+3/2)}\int_0^{1} {}_2F_1\left[\begin{matrix}1/2,1/2\\p+3/2\end{matrix};1-x\right]\frac{dx}{\sqrt{x}} \qquad \text{(using Pfaff's formula)}\\
&\qquad= 2^{p}\sqrt{\pi}\frac{(\Gamma(p+1))^2}{\Gamma(p+3/2)}{}_3F_2\left[\begin{matrix}1/2,1/2,1\\p+3/2,3/2\end{matrix};1\right] \qquad \text{(see [7, p.813])}.
\end{aligned}$$

Next from Section 3 and the proof of monotony of the moments, we deduce that

$$M_k(0)\le M_k(\sqrt{T_x}W_1)\le M_k(+\infty)$$

hence, going back to the expression of R_{p-1}, it remains to compute

$$\begin{aligned}
\int_0^{+\infty}\mathbb{E}\left[e^{-T_x/2}\,T_x^{p-k}\right]K_0(x)\frac{dx}{x} &= \int_0^{+\infty}K_0(x)\frac{dx}{x}\int_0^{+\infty}\frac{x}{\sqrt{2\pi t^3}}t^{p-k}e^{-t/2}e^{-\frac{x^2}{2t}}dt\\
&= 4^{p-k-1}\int_0^{+\infty}t^{p-k-1}e^{-t}K_0(t)dt\\
&= \frac{\sqrt{\pi}}{4}2^{p-k}\frac{(\Gamma(p-k))^2}{\Gamma(p-k+1/2)}.
\end{aligned}$$

Therefore, we get the lower bound

$$\begin{aligned}
\mathbb{E}[\mathcal{A}_2^p] &= \frac{1}{2^{p+1}p!}\frac{8}{\pi^2}\int_0^{+\infty}R_{p-1}(x)K_0(x)\frac{dx}{x}\\
&\ge \frac{\mathbb{E}[\mathcal{A}_1^p]}{2\,p!}\frac{8}{\pi^2}\sqrt{\pi}\frac{(\Gamma(p+1))^2}{\Gamma(p+3/2)}{}_3F_2\left[\begin{matrix}1/2,1/2,1\\p+3/2,3/2\end{matrix};1\right]+\sum_{k=0}^{p-1}\sqrt{\pi}2^{-k}\frac{(\Gamma(p-k))^2}{\Gamma(p-k+1/2)}\binom{p}{k}\mathbb{E}[\mathcal{A}_1^{p-k}]2^{k+1}k!\mathbb{E}[\mathcal{A}_2^k]
\end{aligned}$$

Then, using the explicit value of the moments of the arcsine distribution, we finally obtain after some simplifications

$$\mathbb{E}[\mathcal{A}_2^p]\ge\frac{1}{2p+1}\frac{8}{\pi^2}{}_3F_2\left[\begin{matrix}1/2\ 1/2\ 1\\p+3/2\ 3/2\end{matrix};1\right]+\frac{1}{\pi^2}\sum_{k=0}^{p-1}\frac{2}{(p-k)^2}\mathbb{E}[\mathcal{A}_2^k]$$

which is the announced result. The computations for the upper bound are similar. □

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