ON RANDOM QUADRATIC FORMS: SUPPORTS OF POTENTIAL LOCAL MAXIMA

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ABSTRACT. The selection model in population genetics is a dynamic system on the set of the probability distributions $\mathbf{p} = (p_1, \ldots, p_n)$ of the alleles A_1, \ldots, A_n , with $p_i(t+1)$ proportional to $p_i(t)$ times $\sum_i f_{i,j} p_j(t)$, and $f_{i,j} = f_{j,i}$ interpreted as a fitness of the gene pair (A_i, A_j) . It is known that $\hat{\mathbf{p}}$ is a locally stable equilibrium iff $\hat{\mathbf{p}}$ is a strict local maximum of the quadratic form $\mathbf{p}^T \mathbf{f} \mathbf{p}$. Usually there are multiple local maxima and $\lim \mathbf{p}(t)$ depends on $\mathbf{p}(0)$. To address the question of a typical behavior of $\{\mathbf{p}(t)\}$, John Kingman considered the case when the $f_{i,j}$ are independent, [0, 1]-uniform. He proved that with high probability (w.h.p.) no local maximum may have more than $2.49n^{1/2}$ positive components, and reduced 2.49 to 2.14 for a non-biological case of exponentials on $[0,\infty)$. We show that the constant 2.14 serves a broad class of the smooth densities on [0, 1] with the increasing hazard rate. As for a lower bound, we prove that w.h.p. for all $k \leq 2n^{1/3}$ there are many k-element subsets of [n] that pass a partial test to be a support of a local maximum. Still it may well be that w.h.p. the actual supports are much smaller. In that direction we prove that w.h.p. (i) a support of a local maximum, that does not contain a support of a local equilibrium, is very unlikely to have size exceeding $(2/3) \log_2 n$, and (ii) for the uniform fitnesses, there are super-polynomially many potential supports free of local equilibriums, of size close to $(1/2) \log_2 n$.

1. INTRODUCTION AND MAIN RESULTS

The classic selection model in population genetics is a dynamic system on the set of the probability distributions $\mathbf{p} = (p_1, \ldots, p_n) \in \Delta_n := \{\mathbf{x} \ge \mathbf{0}, \sum_{i \in [n]} x_i = 1\}$ of the alleles A_1, \ldots, A_n at the single locus:

(1.1)
$$p_i(t+1) = p_i(t) \cdot \frac{\sum_j f_{i,j} p_j(t)}{\sum_{r,s} f_{r,s} p_r(t) p_s(t)}, \quad i \in [n].$$

Here each $f_{r,s} = f_{s,r} \in [0, 1]$ is interpreted as the fitness, i.e. the probability that the unordered gene pair (A_r, A_s) survives to an adult age. While the dynamic behavior of $\mathbf{p}(t)$ in this model certainly depends on the fitness

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matrix $\mathbf{f} = \{f_{r,s}\}$, it has long been known that the *average fitness* $V(\mathbf{p}(t)) := \sum_{r,s} f_{r,s} p_r(t) p_s(t)$ strictly increases with t unless $\mathbf{p}(t+1) = \mathbf{p}(t)$. Hofbauer and Sigmund [9] (i) characterized this property as a consequence of Fisher's Fundamental Theorem of Natural Selection [6], (ii) provided a full proof following Kingman [10], and (iii) sketched the different proofs given by Scheuer and Mandel [20], and Baum and Eagon [2].

Using the increase of the average fitness, it was later proven by various researchers that (i) $\mathbf{p}(\infty) = \lim_{t\to\infty} \mathbf{p}(t)$ exists for every initial gene distribution $\mathbf{p}(0)$, and (ii) $\mathbf{p} := \mathbf{p}(\infty)$ is a fixed point of the mapping $\mathbf{\Phi}(\cdot) : \Delta_n \to \Delta_n$ defined by (1.1), with a property: for a nonempty $I \subseteq [n]$,

(1.2)
$$p_i = 0, \ (i \notin I), \quad \sum_{j \in I} f_{i,j} p_j \equiv V(\mathbf{p}), \ (i \in I).$$

Remarkably, a fixed point \mathbf{p} is a *locally* stable equilibrium iff \mathbf{p} is a strict local maximum of $V(\cdot)$. There is no reason to expect that a local maximum is unique; so typically the limit $\mathbf{p}(\infty)$ depends on $\mathbf{p}(0)$.

For $\mathbf{p} \in \Delta_n$ to be a local maximum of $V(\cdot)$, three sets of conditions must be satisfied, Kingman [12]. If $I = I(\mathbf{p}) := \{i : p_i > 0\}$, then

(1.3)
$$\sum_{\substack{j \in I}} f_{i,j}p_j \equiv V(\mathbf{p}), \quad (i \in I),$$
$$\sum_{\substack{i,j \in I}} f_{i,j}x_ix_j \leq 0, \quad \forall \{x_i\}_{i \in I} \text{ with } \sum_{i \in I} x_i = 0,$$
$$\sum_{\substack{j \in I}} f_{i,j}p_j \leq V(\mathbf{p}), \quad (i \notin I).$$

The second necessary condition applied to \mathbf{x} such that $x_i = 1, x_j = -1$, with the remaining $x_k = 0$, easily yields

(1.4)
$$f_{i,j} \ge \frac{f_{i,i} + f_{j,j}}{2}, \quad i, j \in I, \, i \neq j.$$

To quote from [12]: "This condition uses internal stability alone, and takes no account of vulnerability to mutation".

The inequality (1.4) was earlier obtained by Lewontin, Ginzburg and Tuljapurkar as a corollary of a determinantal criterion applied to the system of (|I| - 1) linear equations for p_i , $i \in I \setminus \{i_0\}$, implicit in

$$\sum_{j \in I} f_{i,j} p_j \equiv V(\mathbf{P}), \ (i \in I), \quad \sum_{i \in I} p_i = 1.$$

It was also asserted in [17] that $f_{i,j} < \max_k(f_{i,k} + f_{k,j})$; the proof is valid under an additional condition $f_{i,j} > \max\{f_{i,i}, f_{j,j}\}$.

A subset I meeting the condition (1.4) is a candidate to be a support set of a local maximum of $\mathbf{p}^T \mathbf{f} \mathbf{p}$. (We will use a term *K*-set for such sets I.) Kingman [12] posed a problem of analyzing these potential supports in a

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typical case, i.e. when $f_{i,j}$ are *i.i.d.* random variables with range [0, 1]. For the case when $f_{i,j}$ are [0, 1]-uniform, he proved that with high probability (w.h.p.), i.e. with probability approaching one, max $|I| \leq 2.49n^{1/2}$: so "the largest stable polymorphism will contain at most of the order of $n^{1/2}$ alleles". The key tool was the bound $P(D_I) \leq \frac{1}{r!}$, r := |I|, where D_I is the event in (1.4). He found that, for a (non-biological) exponential distribution on $[0, \infty)$, $P(D_I) = \left(\frac{2}{r+1}\right)^r \ll \frac{1}{r!}$ and the constant 2.49 got reduced to 2.14.

Haigh [7], [8] established the counterparts of some of Kingman's results for the case of a non-symmetric payoff matrix. For instance, he proved that for the density $e^{-x}/\sqrt{\pi x}$, (x > 0), of χ_1^2 distribution, with high probability, no evolutionarily stable strategy has support of size exceeding $1.64n^{2/3}$. Kontogiannis and Spirakis [16] used the technique from Haig [8] to resolve the cases of uniform distribution and standard normal distribution left open there.

Recently, and independently of the work cited above, Chen and Peng [4] studied, in an operations research context of the random quadratic optimization problems, the probability of the events quite similar to, but different from D_I . The probability bounds include $\frac{2^r}{(r+1)!}$ (general continuous distribution), and $\frac{2^r}{(r+1)^r}$ (uniform distribution), $\frac{2^r}{(r-1)^r}$ (exponential distribution).

In [12] Kingman suggested that it should be interesting "to carry out a comparative analysis for other distributions of the $f_{i,j}$ ", and conjectured, in [14], that "for every continuous distribution F of f, there is a finite $\beta(F) = \lim_{r\to\infty} \{r!P(D_I)\}^{1/r}$ ". Whenever this limit exists, max $|I| \leq 2.49\beta(F)n^{1/2}$ w.h.p.; in general, max $|I| \leq 4.98n^{1/2}$ w.h.p.

In this paper we consider a relatively broad class of the distributions F, meeting the conditions: **(I)** F(x) has a differentiable positive density g(x), $x \in [0, 1]$, such that $g'(x) \leq 0$, and **(II)** the hazard ratio $\lambda(x) := \frac{g(x)}{1 - F(x)}$ is increasing with x. The non-increasing linear density $g_c(x) = \frac{1-cx}{1-c/2}, c \in [0, 1]$ $(g_0(x) \equiv 1)$ meets these constraints, and so does $g(x) = \frac{ce^{-cx}}{1-e^{-c}}$, the density of the negative exponential distribution conditioned on [0, 1].

For F meeting the conditions (I) and (II), we prove that

(1.5)
$$\left(\frac{2}{r+1}\right)^r \le \mathcal{P}(D_I) \le \frac{r^r}{\binom{r}{2}^{(r)}} \le \frac{e}{2} \left(\frac{2}{r}\right)^r$$

In combination with Kingman's analysis of the exponential distribution on $[0, \infty)$, it follows from (1.5) that for every F meeting the constraints above, we have max $|I| \leq 2.14n^{1/2}$ with high probability. We see also that, for every F in question,

$$\lim_{r \to \infty} \left\{ r! \operatorname{P}(D_I) \right\}^{1/r} =: \beta(F) = \frac{2}{e}$$

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proving not only that $\beta(F)$ exists, but also that $\beta(F)$ does not depend on F in this class. This lends a certain support to Kingman's conjecture, [14], that $\lim_{r\to\infty} \{r! P(D_I)\}^{1/r}$ exists for every continuous F.

Suppose we restrict our attention to the minimal K-sets I, i.e. such that there is no $J \subset I$, $(|J| \ge 2)$, which supports a local equilibrium $\mathbf{p} = \{p_i\}_{i \in J}$, meeting the top two conditions in (1.3). Let \mathcal{D}_I be the corresponding event. For the distributions F from the class described above, we prove that

(1.6)
$$P(\mathcal{D}_I) \le 2^{-r^2/2} \left(\frac{4e}{r}\right)^{r/2} \exp(\Theta(r^{1/3})), \quad r := |I|.$$

Continuing with $P(D_I)$, suppose that, in addition, $g^{(3)}(0)$ exists. Then

(1.7)
$$P(D_I) = \left(1 + O(r^{-\sigma})\right) \left(\frac{2}{r}\right)^r \exp\left(\frac{g'(0)}{g^2(0)}\right), \quad \forall \, \sigma < 1/3,$$

and if $|I_1| = |I_2| = r$, $|I_1 \cap I_2| = k$, then

(1.8)
$$P(D_{I_1} \cap D_{I_2}) = O(P(r,k)), \quad P(r,k) := r^{-6} \left(\frac{2}{r}\right)^{2(r-k-1)} \left(\frac{2}{2r-k}\right)^{k-1},$$

uniformly for $r \ge 2$ and $k \in [1, r - 1]$.

Let $X_{n,r}$ be the total number of K-sets of [n] of cardinality r. We already know that w.h.p. $X_{n,r} = 0$ for $r > 2.14n^{1/2}$, and that $\mathbb{E}[X_{n,r}] \to \infty$ for every $r < 2.14n^{1/2}$. We use the estimates (1.5), (1.7) and (1.8) to show that

$$\frac{\operatorname{Var}(X_{n,r})}{\operatorname{E}^{2}[X_{n,r}]} = O(n^{-2/3}), \quad 2 \le r \le r(n) := \lceil 2n^{1/3} \rceil.$$

It follows that

$$P\left(\bigcap_{\rho=2}^{r(n)} \left\{ \left| \frac{X_{n,\rho}}{\operatorname{E}[X_{n,\rho}]} - 1 \right| \le n^{-1/6+\varepsilon} \right\} \right) = 1 - O(n^{-2\varepsilon}), \quad \varepsilon < 1/6,$$

i.e. w.h.p. the counts of the K-sets of size r ranging from 2 to r(n) are uniformly asymptotic to their expected values. In particular, setting $L_n = \max\{\rho: X_{n,\rho} > 0\}$, we have $P(L_n > 2n^{1/3}) \to 1$, i.e. w.h.p. the size of the largest potential support of a local maximum is sandwiched between $2n^{1/3}$ and $2.14n^{1/2}$.

We cannot rule out a possibility that, with high probability, the actual supports of local maxima are considerably smaller. In fact, we use the bound (1.6) to show that, with probability $> 1 - n^{-a}$, $(\forall a > 0)$, there is no K-set of cardinality $> (2/3) \log_2 n$ that contains, properly, a non-trivial support of a local equilibrium. Complementing this claim, we show that, with high probability, the number of K-sets of size $< 0.5 \log_2 n$ that do not contain the size 2 supports of local equilibriums is super-polynomially large.

The already cited paper [4] was preceded by Chen, Peng and Zhang [3]; both papers studied the likely behavior of an *absolute* minimum of a random quadratic form $\mathbf{x}^T Q \mathbf{x}$ for $\mathbf{x} \in \Delta_n$. Under the condition that the elements of Q are i.i.d. random variables with a c.d.f. F concave on its support, the support size of the absolute minimum point was shown to be *bounded* in probability, with its distribution tail decaying exponentially fast. In particular, it followed that, for $f_{i,j}$ uniform or *positive*-exponential on [0, 1], the absolute *maximum* of $\mathbf{p}^T \mathbf{f} \mathbf{p}$ is attained at a point of Δ_n with N, the number of positive components, satisfying $P(N \ge k) = O(\rho^k), k > 0$, as $n \to \infty$.

In view of all this information, it is tempting to conjecture that—for $f_{i,j}$ meeting the conditions (I) and (II)—the size of the largest support of a *local* maximum of $\mathbf{p}^T \mathbf{f} \mathbf{p}$ is, with high probability, of (poly)logarithmic order.

2. Proofs

2.1. Estimate of $\mathbf{P}(D_I)$.

Theorem 2.1. Suppose that F(i) has a positive, non-increasing, differentiable density g, and (2) has a non-decreasing hazard ratio $\lambda(x) = \frac{g(x)}{1-F(x)}$. Then, with $a^{(b)} := a(a+1)\cdots(a+b-1)$, we have

(2.1)
$$\left(\frac{2}{r+1}\right)^r \leq P(D_I) \leq \frac{r^r}{\binom{r}{2}^{(r)}} \leq \frac{e}{2} \left(\frac{2}{r}\right)^r.$$

In the special case of the uniform density $g(x) \equiv 1$, this bound improves Kingman's bound $P(D_I) \leq \frac{1}{r!}$. It also shows that, for all F meeting the conditions (i) and (ii),

$$\lim_{r \to \infty} \left\{ r! \operatorname{P}(D_I) \right\}^{1/r} = \frac{2}{e}.$$

Proof. As in [12], the probability of D_I , conditioned on $\{f_{i,i} = x_i, i \in I\}$, is

$$\prod_{(i,j)} \mathbf{P}\left(f \ge \frac{x_i + x_j}{2}\right) = \prod_{(i,j)} \left(1 - F\left(\frac{x_i + x_j}{2}\right)\right),$$

where $i \neq j$. The function 1 - F(x) is log-concave, since

$$\frac{d}{dx}\log(1-F(x)) = -\frac{g(x)}{1-F(x)} = -\lambda(x)$$

is decreasing with x. (a) Lower bound. By Jensen inequality,

$$\prod_{(i,j)} \left(1 - F\left(\frac{x_i + x_j}{2}\right) \right) \ge \prod_{(i,j)} (1 - F(x_i))^{1/2} (1 - F(x_j))^{1/2}$$
$$= \prod_{i=1}^r (1 - F(x_i))^{(r-1)/2}.$$

Consequently

$$P(D_I) \ge \int_{\mathbf{x} \in [0,1]^r} \cdots \int_{i=1}^r \prod_{i=1}^r (1 - F(x_i))^{(r-1)/2} \prod_{i=1}^r g(x_i) \, dx_i,$$

and, switching to the variables $y_i = F(x_i)$,

$$P(D_I) \ge \int_{\mathbf{y} \in [0,1]^n} \cdots \int_{i=1}^r \prod_{i=1}^r (1-y_i)^{(r-1)/2} d\mathbf{y}$$
$$= \left(\int_0^1 (1-y)^{(r-1)/2} dy\right)^r = \left(\frac{2}{r+1}\right)^r$$

.

(b) Upper bound. Again by Jensen inequality, denoting $s = \sum_i x_i$ we have

(2.2)
$$\prod_{(i,j)} \left(1 - F\left(\frac{x_i + x_j}{2}\right)\right) = \exp\left[\binom{r}{2} \sum_{(i,j)} \frac{1}{\binom{r}{2}} \log\left(1 - F\left(\frac{x_i + x_j}{2}\right)\right)\right]$$
$$\leq \exp\left[\binom{r}{2} \log\left(1 - F\left(\frac{1}{r(r-1)} \sum_{(i,j)} (x_i + x_j)\right)\right)\right]$$
$$= \exp\left[\binom{r}{2} \log\left(1 - F\left(\frac{s}{r}\right)\right)\right] = \left(1 - F\left(\frac{s}{r}\right)\right)^{\binom{r}{2}}.$$

Consequently

(2.3)
$$P(D_I) \le \int_{\mathbf{x} \in [0,1]^r} \cdots \int_{\mathbf{x} \in [0,1]^r} \left(1 - F\left(\frac{s}{r}\right)\right)^{\binom{r}{2}} \prod_{i \in I} g(x_i) \, dx_i.$$

Again change the variables of integration, setting $y_i = F(x_i)$, so that $x_i = F^{-1}(y_i)$, and $s = \sum_{i \in I} F^{-1}(y_i)$. Now

$$\frac{d^2}{dy^2} F^{-1}(y) = -\frac{g'(x)}{g(x)^3} \ge 0,$$

implying that $F^{-1}(y)$ is convex. Therefore, for each $t \leq r$, we have

$$r^{-1}\min\left\{\sum_{i\in I}F^{-1}(y_i):\sum_{i\in I}y_i=t\right\}=F^{-1}\left(\frac{t}{r}\right).$$

Hence

$$\max\left\{1 - F\left(r^{-1}\sum_{i\in I} x_i\right) : \sum_{i\in I} y_i = t\right\}$$

= $1 - \min\left\{F\left(r^{-1}\sum_{i\in I} F^{-1}(y_i)\right) : \sum_{i\in I} y_i = t\right\}$
= $1 - F\left(r^{-1}\min\left\{\sum_{i\in I} F^{-1}(y_i) : \sum_{i\in I} y_i = t\right\}\right)$
= $1 - F\left(F^{-1}\left(\frac{t}{r}\right)\right) = 1 - \frac{t}{r}.$

So (2.3) yields

$$P(D_I) \leq \int \cdots \int \left(1 - \frac{t}{r}\right)^{\binom{r}{2}} \prod_{i \in I} dy_i.$$

Since

$$\int_{\sum_i y_i \le t} \cdots \int_{i \in I} \prod_{i \in I} dy_i = \frac{t^r}{r!},$$

we conclude that

$$P(D_I) \leq \int_0^r \left(1 - \frac{t}{r}\right)^{\binom{r}{2}} \frac{t^{r-1}}{(r-1)!} dt$$

= $\frac{r^r}{(r-1)!} \int_0^1 (1-\tau)^{\binom{r}{2}} \tau^{r-1} d\tau$
= $\frac{r^r}{(r-1)!} \cdot \frac{\binom{r}{2}!(r-1)!}{\binom{r}{2}+r)!} = \frac{r^r}{\binom{r}{2}}^{(r)}.$

Theorem 2.2. Suppose that, in addition to conditions (i), (ii), we have (iii): $g^{(3)}(0)$ exists. Then

$$\mathbf{P}(D_I) = \left(1 + O(r^{-\sigma})\right) \left(\frac{2}{r}\right)^r \exp\left(\frac{g'(0)}{g^2(0)}\right),$$

for every $\sigma < 1/3$.

To prove this claim, we shrink, in steps, the cube $[0,1]^n$ to a subset C^* in such a way that (a) the integral of the product of $1 - F\left(\frac{x_i+x_j}{2}\right)$ over C^* sharply approximates that over $[0,1]^n$, and (b) the product itself admits a manageable approximation on C^* .

Given $C \subset [0,1]^n$, denote

$$P_C(D_I) = \int_{\mathbf{x}\in C} \cdots \int \prod_{(i\neq j)} \left(1 - F\left(\frac{x_i + x_j}{2}\right)\right) d\mathbf{x}.$$

Lemma 2.3. Let

$$C_1 := \left\{ \mathbf{x} \in [0,1]^n : \left| \sum_{i=1}^r F(x_i) - 2 \right| \le r^{-1/3} \right\}.$$

Then

$$P(D_I) - P_{C_1}(D_I) \le \left(\frac{2}{r}\right)^r \cdot \exp\left(-\frac{r^{1/3}}{10}\right).$$

Proof. Let $\tau_{1,2} = \frac{2}{r} \mp r^{-4/3}$. From the proof of Theorem 2.1 it follows that

$$P(D_I) - P_{C_1}(D_I) \le \frac{r^r}{(r-1)!} \int_{\tau \in [\tau_1, \tau_2]^c} (1-\tau)^{\binom{r}{2}} \tau^{r-1} d\tau.$$

The (logconcave) integrand attains its maximum at $\tau_{\max} = \frac{2}{2+r} \in [\tau_1, \tau_2]$, and

$$\max\left\{\frac{d^2}{d\tau^2} \left(\log(1-\tau)^{\binom{r}{2}}\tau^{r-1}\right) : \tau \in [\tau_1, \tau_2]\right\} \le -\frac{r^3}{4.1}$$

Therefore the integral is at most

$$\left(1-\frac{2}{2+r}\right)^{\binom{r}{2}}\left(\frac{2}{2+r}\right)^{r-1}\cdot\exp\left(-\frac{r^{1/3}}{9}\right),$$

so that

$$P(D_I) - P_{C_1}(D_I) \le \frac{r^r}{(r-1)!} \left(1 - \frac{2}{2+r}\right)^{\binom{r}{2}} \left(\frac{2}{2+r}\right)^{r-1} \cdot \exp\left(-\frac{r^{1/3}}{9}\right)$$
$$\le \left(\frac{2}{r}\right)^r \cdot \exp\left(-\frac{r^{1/3}}{10}\right).$$

Next

Lemma 2.4. Let

$$C_2 := \left\{ \mathbf{x} \in C_1 : \max_i \frac{F(x_i)}{\sum_j F(x_j)} \le k \frac{\log r}{r} \right\}, \quad (k > 1).$$

Then

$$P_{C_1}(D_I) - P_{C_2}(D_I) \le \left(\frac{2}{r}\right)^r \cdot r^{-\alpha}, \quad \forall \alpha < k-1.$$

Proof. Similarly to the proof of Lemma 2.3,

$$P_{C_1}(D_I) - P_{C_2}(D_I) \le \int_{\max \frac{y_i}{t} > k \frac{\log r}{r}} \left(1 - \frac{t}{r}\right)^{\binom{r}{2}} \prod_{i \in I} dy_i.$$

Introduce L_1, \ldots, L_r the lengths of the consecutive subintervals of [0, 1] obtained by sampling uniformly at random r-1 points in [0, 1]. By Lemma 1 in [18], the integral above is at most

$$P\left(\max L_i \ge k \frac{\log r}{r}\right) \int_0^r \left(1 - \frac{t}{r}\right)^{\binom{r}{2}} \frac{t^{r-1}}{(r-1)!} dt$$
$$= P\left(\max L_i \ge k \frac{\log r}{r}\right) \frac{r^r}{\binom{r}{2}}^{(r)}$$

And, introducing U_1, \ldots, U_{r-1} the independent [0, 1]-Uniforms, the probability factor is at most

$$r \operatorname{P}\left(L_{1} \ge k \frac{\log r}{r}\right) = r \operatorname{P}\left(\min_{i} U_{i} \ge k \frac{\log r}{r}\right)$$
$$= r \left(1 - k \frac{\log r}{r}\right)^{r-1} \le r \exp\left(-(r-1)k \frac{\log r}{r}\right).$$

One more reduction step defines the final

(2.4)
$$C^* = \left\{ \mathbf{x} \in C_2 : \left| \frac{r}{2} \frac{\sum_i F^2(x_i)}{\left(\sum_j F(x_j)\right)^2} - 1 \right| \le r^{-\sigma} \right\}, \quad \sigma < 1/3.$$

Lemma 2.5.

$$\mathbf{P}_{C_2}(D_I) - \mathbf{P}_{C^*}(D_I) \le \left(\frac{2}{r}\right)^r \cdot \exp\left(-0.5r^{1/3-\sigma}\right).$$

Proof. Once again like the proofs of Lemmas 2.3, 2.4,

$$P_{C_2}(D_I) - P_{C^*}(D_I) \leq \int \cdots \int \left(1 - \frac{t}{r}\right)^{\binom{r}{2}} \prod_{i \in I} dy_i$$
$$\left| \frac{r}{2} \frac{\sum_i y_i^2}{\left(\sum_j y_j\right)^2} - 1 \right| > r^{-\sigma}$$
$$\leq P\left(\left| \frac{r}{2} \sum_i L_i^2 - 1 \right| > r^{-\sigma} \right) \frac{r^r}{\binom{r}{2}} \leq \left(\frac{2}{r}\right)^r \cdot \exp\left(-\Theta(r^{1/3-\sigma})\right),$$

as the probability is at most $\exp(-\Theta(r^{1/3-\sigma}))$, (see Lemma 3.2 in [19]). \Box

Note. A key to the proof of that Lemma 3.2 was a classic fact that (L_1, \ldots, L_r) and $(\sum_i W_i)^{-1}(W_1, \ldots, W_r)$, $(W_j$ being i.i.d. Exponentials), are equidistributed, Feller [5]. While both of the distribution tails of $\sum_i W_j$ decay exponentially, for the right tail of $\sum_j W_j^2$ we could prove only the bound $e^{-\Theta(r^{\delta})}$, $\delta < 1/3$. The obstacle here is that $\mathbb{E}[e^{zW^2}] = \infty$ for every z > 0.

Combining Lemmas 2.3, 2.4 and 2.5, we obtain

Corollary 2.6.

$$P(D_I) - P_{C^*}(D_I) \le \left(\frac{2}{r}\right)^r \cdot r^{-\alpha}, \quad \forall \alpha < k-1.$$

For $\mathbf{x} \in C^*$, we have $\max_i F(x_i) \leq \frac{3k \log r}{r} \to 0$, which implies that $\max_i x_i = O(r^{-1} \log r) \to 0$. For $x = O(r^{-1} \log r)$, we have

$$F(x) = g(0)x + \frac{1}{2}g^{(1)}(0)x^2 + O(x^3)$$

= $g(0)x + \frac{1}{2}g^{(1)}(0)x^2 + O(r^{-3}\log^3 r).$

 So

$$\log(1 - F(x)) = -g(0)x - \frac{g'(0) + g^2(0)}{2}x^2 + O(r^{-3}\log^3 r),$$

and with a bit of algebra

(2.5)
$$\log\left(1 - F\left(\frac{x_i + x_j}{2}\right)\right) - \frac{\log(1 - F(x_i)) + \log(1 - F(x_j))}{2}$$
$$= \frac{g'(0) + g^2(0)}{8} (x_i - x_j)^2 + O(r^{-3}\log^3 r)$$
$$= \gamma(F(x_i) - F(x_j))^2 + O(r^{-3}\log^3 r), \quad \gamma := \frac{g'(0) + g^2(0)}{8g^2(0)}.$$

Therefore

$$\prod_{(i,j)} \log\left(1 - F\left(\frac{x_i + x_j}{2}\right)\right) = \exp\left(\frac{r-1}{2}\sum_i \log(1 - F(x_i))\right)$$
$$\times \exp\left(\gamma \sum_{(i,j)} (F(x_i) - F(x_j))^2 + O(r^{-1}\log r)\right),$$

where

$$\frac{r-1}{2} \sum_{i} \log(1 - F(x_i)) = -\frac{r-1}{2} \sum_{i} \left(F(x_i) + \frac{F^2(x_i)}{2} \right) + O(r^{-1}\log r),$$
$$\sum_{(i,j)} (F(x_i) - F(x_j))^2 = r \sum_{i} F^2(x_i) - \left(\sum_{i} F(x_i)\right)^2.$$

Hence on C^* (see (2.4))

$$\prod_{(i,j)} \log\left(1 - F\left(\frac{x_i + x_j}{2}\right)\right) = \exp\left(-\frac{r-1}{2}\sum_i F(x_i) - \gamma\left(\sum_i F(x_i)\right)^2 + \left(-\frac{r-1}{4} + \gamma r\right)\sum_i F^2(x_i) + O(r^{-1}\log r)\right)\right)$$
$$= \exp\left(-\frac{r-1}{2}\sum_i F(x_i) + \left(2\gamma - \frac{1}{2}\right)\left(\sum_i F(x_i)\right)^2 + O(r^{-\sigma})\right)$$
$$= \exp\left(-\frac{r}{2}\sum_i F(x_i) + \frac{g'(0)}{g^2(0)} + O(r^{-\sigma})\right);$$

for the last equality we used the definition of γ in (2.5).

Switching to the variables $y_i = F(x_i)$ and denoting $t = \sum_i y_i$, we obtain then

$$P_{C^*}(D_I) = \int_{\mathbf{y} \in \mathcal{C}^*} \int \exp\left(-\frac{r}{2}t + \frac{g'(0)}{g^2(0)} + O(r^{-\sigma})\right) d\mathbf{y},$$
$$\mathcal{C}^* := \left\{\mathbf{y} \ge \mathbf{0} : |t-2| \le r^{-1/3}, \max_i \frac{y_i}{t} \le k \frac{\log r}{r}, \left|\frac{r}{2t^2} \sum_i y_i^2 - 1\right| \le r^{-\sigma}\right\}.$$

Notice that on \mathcal{C}^* we have $\max_i y_1 \to 0$, so that the omitted condition $\max_i y_i \leq 1$ would have been superfluous. By Lemma 3.1 in [19],

$$\int \dots \int e^{-\frac{rt}{2}} d\mathbf{y}$$

= $\int_{|t-2| \le \frac{1}{r^{1/3}}} \frac{e^{-\frac{rt}{2}} t^{r-1}}{(r-1)!} \operatorname{P}\left(\max L_i \le \min\left(t^{-1}, \frac{k \log r}{r}\right), \left|\frac{r}{2} \sum_i L_i^2 - 1\right| \le r^{-\sigma}\right) dt$
= $\operatorname{P}\left(\max L_i \le \frac{k \log r}{r}, \left|\frac{r}{2} \sum_i L_i^2 - 1\right| \le r^{-\sigma}\right) \int_{|t-2| \le \frac{1}{r^{1/3}}} \frac{e^{-\frac{rt}{2}} t^{r-1}}{(r-1)!} dt.$

From Lemma 2.4 and Lemma 2.5, and their proofs, we know that the probability factor is at least $1 - r^{-\alpha}$, $\forall \alpha < k - 1$. Furthermore, the integral equals

$$\int_{0}^{\infty} \frac{e^{-\frac{rt}{2}}t^{r-1}}{(r-1)!} dt - \int_{|t-2| > \frac{1}{r^{1/3}}} \frac{e^{-\frac{rt}{2}}t^{r-1}}{(r-1)!} dt$$
$$= \left(\frac{2}{r}\right)^{r} - \frac{\left(\frac{2}{r}\right)^{r}}{(r-1)!} \int_{|\tau-r| > \frac{r^{2/3}}{2}} e^{-\tau}\tau^{r-1} d\tau,$$

and, by Chebyshev's inequality,

$$\frac{1}{(r-1)!} \int_{|\tau-r| > \frac{r^{2/3}}{2}} e^{-\tau} \tau^{r-1} d\tau \le \operatorname{P}\left(\left|\operatorname{Poisson} (r-1) - (r-1)\right| > \frac{r^{2/3}}{3}\right)$$
$$\le \frac{9(r-1)}{r^{4/3}} \le 9r^{-1/3}.$$

 So

$$\int_{|t-2| \le \frac{1}{r^{1/3}}} \frac{e^{-\frac{rt}{2}}t^{r-1}}{(r-1)!} dt = \left(1 + O(r^{-1/3})\right) \left(\frac{2}{r}\right)^r.$$

Consequently

$$P_{\mathcal{C}^*}(D_I) = \left(1 + O(r^{-\sigma})\right) \left(\frac{2}{r}\right)^r \exp\left(\frac{g'(0)}{g^2(0)}\right),$$

for every $\sigma < 1/3$. Combining this estimate with Corollary 2.6, we complete the proof of Theorem 2.2.

2.2. Estimate of $\mathbf{P}(D_{I_1} \cap D_{I_2})$. Let $I_1, I_2 \subset [n], |I_j| = r$. If $I_1 \cap I_2 = \emptyset$, then the events D_{I_1} and D_{I_2} are independent and so (by Theorem 2.2)

(2.6)
$$P(D_{I_1} \cap D_{I_2}) = P(D_{I_1}) \cdot P(D_{I_2}) = (1 + O(r^{-\sigma})) \left(\frac{2}{r}\right)^{2r} \exp\left(2\frac{g'(0)}{g^2(0)}\right).$$

Consider the case $|I_1 \cap I_2| = k \in [1, r-1]$. By symmetry, we can assume that $I_1 = \{1, \ldots, r\}$, and $I_2 = \{r - k + 1, \ldots, 2r - k\}$. The probability of $D_{I_1} \cap D_{I_2}$, conditioned on the event $\{F_{i,i} = x_i : 1 \le i \le 2r - k\}$, is

(2.7)
$$\Psi(\mathbf{x}) = \prod_{\substack{(i\neq j)\\i,j\leq r}} \left(1 - F\left(\frac{x_i + x_j}{2}\right)\right) \prod_{\substack{r < i\leq 2r-k\\r-k+1\leq j< r}} \left(1 - F\left(\frac{x_i + x_j}{2}\right)\right) \times \prod_{\substack{(i\neq j)\\r\leq i,j\leq 2r-k}} \left(1 - F\left(\frac{x_i + x_j}{2}\right)\right).$$

The three products contain, respectively, $\binom{r}{2}$, (r-k)(k-1) and $\binom{r-k+1}{2}$ factors. The total number of the factors is

$$N(r,k) = \binom{r}{2} + (r-k)(k-1) + \binom{r-k+1}{2}.$$

Now

$$\sum_{\substack{(i\neq j)\\1\leq i,\,j\leq r}} \frac{x_i + x_j}{2} = \frac{r-1}{2} \sum_{i=1}^r x_i, \quad \sum_{\substack{(i\neq j)\\r\leq i,\,j\leq 2r-k}} \frac{x_i + x_j}{2} = \frac{r-k}{2} \sum_{i=r}^{2r-k} x_i$$
$$\sum_{\substack{r$$

The total sum of the fractions $\frac{x_i+x_j}{2}$ is $\frac{r-1}{2}s_1 + \frac{2r-k-1}{2}s_2 + \frac{r-1}{2}s_3$, where

$$s_1 = \sum_{i=1}^{r-k} x_i, \ s_2 = \sum_{i=r-k+1}^r x_i, \ s_3 = \sum_{i=r+1}^{2r-k} x_i,$$

and the sum of the coefficients α_i by x_i in the sum of those fractions is N(r, k). By log-concavity of 1 - F(x),

$$\Psi(\mathbf{x}) \le \left(1 - F\left(\frac{\frac{r-1}{2}s_1 + \frac{2r-k-1}{2}s_2 + \frac{r-1}{2}s_3}{N(r,k)}\right)\right)^{N(r,k)}$$

As in the proof of Theorem 2.1, introduce $y_i = F(x_i)$, $1 \le i \le 2r - k$, so that

$$s_1 = \sum_{i=1}^{r-k} F^{-1}(y_i), \ s_2 = \sum_{i=r-k+1}^r F^{-1}(y_i), \ s_3 = \sum_{i=r+1}^{2r-k} F^{-1}(y_i).$$

Given t_1, t_2, t_3 , by convexity of F^{-1} , we have

$$\min\left\{\sum_{i=1}^{2r-k} \frac{\alpha_i}{N(r,k)} F^{-1}(y_i) : \sum_{i=1}^{r-k} y_i = t_1, \sum_{i=r-k+1}^r y_i = t_2, \sum_{i=r+1}^{2r-k} y_i = t_3\right\}$$
$$\geq F^{-1}\left(\frac{\frac{r-1}{2}t_1}{N(r,k)} + \frac{\frac{2r-k-1}{2}t_2}{N(r,k)} + \frac{\frac{r-k}{2}t_3}{N(r,k)}\right).$$

Consequently

$$\Psi(\mathbf{x}) \le \Psi^*(\mathbf{t}) := \left(1 - \frac{(r-1)t_1 + (2r-k-1)t_2 + (r-1)t_3}{2N(r,k)}\right)^{N(r,k)},$$
$$t_1 := \sum_{i=1}^{r-k} F(x_i), \ t_2 := \sum_{i=r-k+1}^r F(x_i), \ t_3 := \sum_{i=r+1}^{2r-k} F(x_i).$$

Therefore

(2.8)
$$P(D_{I_1} \cap D_{I_2}) = \int \cdots \int \Psi(\mathbf{x}) \, d\mathbf{x} \leq \int \cdots \int \Psi^*(\mathbf{t}) \, d\mathbf{y}$$
$$= \iiint_{t_1, t_3 \leq r-k, t_2 \leq k} \Psi^*(\mathbf{t}) \frac{t_1^{r-k-1}}{(r-k-1)!} \frac{t_2^{k-1}}{(k-1)!} \frac{t_3^{r-k-1}}{(r-k-1)!} \, d\mathbf{t}.$$

Introduce

$$\tau_1 = \frac{r-1}{2N(r,k)}t_1, \ \tau_2 = \frac{2r-k-1}{2N(r,k)}t_2, \ \tau_3 = \frac{r-1}{2N(r,k)}t_3$$

Since

$$t_1 \le r - k, \ t_2 \le k, \ t_3 \le r - k,$$
$$\frac{(r-1)(r-k)}{2N(r,k)} + \frac{(2r-k-1)k}{2N(r,k)} + \frac{(r-1)(r-k)}{2N(r,k)} = 1.$$

we see that $\tau_1 + \tau_2 + \tau_3 \leq 1$. Switching to τ_j , and denoting N = N(r, k), we transform (2.8), into

$$P(D_{I_1} \cap D_{I_2}) \leq \frac{\left(\frac{2N}{r-1}\right)^{r-k-1}}{(r-k-1)!} \cdot \frac{\left(\frac{2N}{2r-k-1}\right)^{k-1}}{(k-1)!} \cdot \frac{\left(\frac{2N}{r-1}\right)^{r-k-1}}{(r-k-1)!} \times \iint_{\tau_1+\tau_2+\tau_3 \leq 1} \tau_1^{r-k-1} \tau_2^{r-k-1} (1-\tau_1-\tau_2-\tau_3)^N d\tau_1 d\tau_2 d\tau_3$$
$$= \frac{N! \left(\frac{2N}{r-1}\right)^{2(r-k-1)} \left(\frac{2N}{2r-k-1}\right)^{k-1}}{(N+2r-k)!} \leq N^{-3} \left(\frac{2}{r-1}\right)^{2(r-k-1)} \left(\frac{2}{2r-k-1}\right)^{k-1}.$$

(At the penultimate line we used the multidimensional extension of the beta integral, Andrews, Askey and Roy [1], Theorem 1.8.6.) Since $N = \Theta(r^2)$, we have then

(2.9)
$$P(D_{I_1} \cap D_{I_2}) = O(P(r,k)), \quad P(r,k) := r^{-6} \left(\frac{2}{r}\right)^{2(r-k-1)} \left(\frac{2}{2r-k}\right)^{k-1}.$$

2.3. Likely range of the maximum size of the K-set. Introduce $L_n = \{\max | I| : (1.4) \text{ holds} \}$. Kingman [12], [13], [14] proved that, for F = Uniform[0,1], w.h.p. $L_n \leq n^{1/2}(\epsilon + o(1))$, where $\epsilon = \xi^{-1/2}(1-\xi)^{-1/2}$ and $\xi = 2.49...$ is a positive root of $1-\xi = e^{-2\xi}$. The proof consisted of showing that $P(D_I) \leq \frac{1}{|I|!}$, and that

(2.10)
$$P(L_n \ge r) \le \frac{(n)_s}{(r)_s} P(D_I), \quad |I| = s \le r.$$

This inequality sharpens the (first-order moment) bound $P(L_n \ge r) \le \binom{n}{r} P(D_I), |I| = r$, by using the fact that every subset of a K-set is a K-set as well. Kingman also demonstrated that his *exact* formula $P(D_I) = \left(\frac{2}{r+1}\right)^r$ for the negative exponential distribution on $[0, \infty)$ implied a better bound

(2.11)
$$L_n \le n^{1/2} [(2e^{-1})^{1/2} \epsilon + o(1)], \quad (2e^{-1})^{1/2} \epsilon = 2.14...$$

Now, by Theorem 2.1, we have $P(D_I) \leq \frac{e}{2} \left(\frac{2}{r}\right)^r$ for a wide class of the densities on [0, 1], that includes the uniform density and the exponential density restricted to [0, 1]. Combining this Theorem and Kingman's proof for the exponential distribution, we obtain

Theorem 2.7. Under the conditions (i), (ii) of Theorem 2.1, w.h.p.

$$L_n \le n^{1/2} (2.14... + o(1)).$$

Armed with the bound (2.9) and the bounds in Theorem 2.1, we can prove a qualitatively matching lower bound.

Theorem 2.8. Let $X_{n,r}$ stand for the total number of K-sets of cardinality r. Introduce $r(n) = \lceil 2n^{1/3} \rceil$. Then, under the conditions (i), (ii) and (iii) of Theorem 2.2,

$$P\left(\bigcap_{\rho=2}^{r(n)} \left\{ \left| \frac{X_{n,\rho}}{\mathbb{E}[X_{n,\rho}]} - 1 \right| \le n^{-1/6+\varepsilon} \right\} \right) = 1 - O(n^{-2\varepsilon}), \quad \forall \varepsilon \in (0, 1/6).$$

Consequently, $\min_{r \in [2,r(n)]} X_{n,r} \to \infty$ in probability, and so

$$\lim_{n \to \infty} \mathbf{P}(L_n \ge 2n^{1/3}) = 1.$$

Proof. This time we use the second-order moment approach. By Theorem 2.1, for a generic set I of cardinality $r \in [2, r(n)]$ we have

$$\mathbf{E}[X_{n,r}] = \binom{n}{r} \mathbf{P}(D_I) \ge \frac{n^r}{2r!} \left(\frac{2}{r+2}\right)^r \ge \text{const } n^2$$

The total number of ordered pairs $\{I_1, I_2\}$, with $|I_1| = |I_2| = r$, $|I_1 \cap I_2| = k$, is

$$\mathcal{N}(r,k) = \binom{n}{r} \binom{r}{k} \binom{n-r}{r-k}$$

Therefore, for a pair of generic sets I_1 , I_2 meeting the conditions above,

(2.12)
$$E[(X_{n,r})_2] = \sum_{k=0}^{r-1} \mathcal{N}(r,k) P(D_{I_1} \cap D_{I_2}).$$

Here $P(D_{I_1} \cap D_{I_2}) = O(P(r,k))$, with P(r,k) given in (2.9). After some elementary computations we obtain that

$$\leq \frac{\max_{k \in [1, r-1]} \frac{\mathcal{N}(r, k+1) \operatorname{P}(r, k+1)}{\mathcal{N}(r, k) \operatorname{P}(r, k)}}{(k+1)(2r-k-1)(n-2r+k+1)} \cdot \exp\left(\frac{k-1}{2r-k-1}\right)$$
$$= \frac{r^2(r-1)}{8(n-2r+2)} \leq e^{8n^{-2/3}};$$

(the second line maximum is attained at k = 1). Consequently

$$\sum_{k=1}^{r-1} \mathcal{N}(r,k) \operatorname{P}(r,k) \leq r e^{8rn^{-2/3}} \mathcal{N}(r,1) \operatorname{P}(r,1) \leq 2r \mathcal{N}(r,1) \operatorname{P}(r,1)$$
$$\leq 2\binom{n}{r} \binom{n-r}{r-1} \left(\frac{2}{r}\right)^{2r}$$
$$= O\left(n^{-1} \mathcal{N}(r,0) \operatorname{P}^{2}(D_{I})\right) = O\left(\frac{r}{n} \operatorname{E}^{2}\left[X_{n,r}\right]\right).$$

(For the last equality we used the lower bound for $P(D_I)$ in Theorem 2.1.) Therefore, uniformly for $r \in [2, r(n)]$,

(2.13)
$$\frac{\sum_{k=1}^{r-1} \mathcal{N}(r,k) \operatorname{P}(r,k)}{\operatorname{E}^2[X_{n,r}]} = O(n^{-2/3}).$$

From the equations (2.12) and (2.13), and $\mathbb{E}[X_{n,r}] \ge \text{const } n^2 \gg n^{2/3}$, we have

$$\frac{\mathrm{E}[(X_{n,r})_2]}{\mathrm{E}^2[X_{n,r}]} = 1 + O(n^{-2/3}) \Longrightarrow \frac{\mathrm{Var}(X_{n,r})}{\mathrm{E}^2[X_{n,r}]} = O(n^{-2/3}).$$

By Chebyshev's inequality,

$$P\left(\left|\frac{X_{n,r}}{\mathbb{E}[X_{n,r}]} - 1\right| \le \delta\right) \ge 1 - O\left(\delta^{-2}n^{-2/3}\right) \to 1,$$

uniformly for all $\delta \gg n^{-1/3}$ and $r \in [2, r(n)]$. Therefore

$$\sum_{r=2}^{r(n)} \mathbb{P}\left(\left|\frac{X_{n,r}}{\mathbb{E}[X_{n,r}]} - 1\right| \ge \delta\right) = O\left(\delta^{-2}n^{-1/3}\right) \to 0,$$

which implies: for $\varepsilon \in (0, 1/6)$,

$$P\left(\bigcap_{r=2}^{r(n)} \left\{ \left| \frac{X_{n,r}}{\mathbb{E}[X_{n,r}]} - 1 \right| \le n^{-1/6+\varepsilon} \right\} \right) = 1 - O(n^{-2\varepsilon}).$$

2.4. Estimate of $\mathbf{P}(\mathcal{D}_I)$. Recall that the event \mathcal{D}_I happens iff I is a K-set and no $J \subset I$, with $|J| \ge 2$, supports a local equilibrium $\mathbf{p} = \{p_i\}_{i \in J} > \mathbf{0}$, $(\sum_{i \in J} p_i = 1)$.

Let the event D_I holds, so that $f_{u,v} \ge (f_{u,u} + f_{v,v})/2$ for all $u, v \in I$. So D_J holds for every $J \subseteq I$. Suppose that for some $i \ne j$ in I we have $f_{i,j} > \max\{f_{i,i}, f_{j,j}\}$. Set $J = \{i, j\}$ and

$$p_i := \frac{f_{i,j} - f_{j,j}}{2f_{i,j} - f_{i,i} - f_{j,j}} > 0, \quad p_j = \frac{f_{i,j} - f_{i,i}}{2f_{i,j} - f_{i,i} - f_{j,j}} > 0.$$

Then $\mathbf{p} = (p_i, p_j)$ is a non-trivial local equilibrium, and this cannot happen on the event \mathcal{D}_I . Thus

$$\mathcal{D}_I \subseteq \bigcap_{(i \neq j): i, j \in I} \left\{ \frac{f_{i,i} + f_{j,j}}{2} \le f_{i,j} \le \max\{f_{i,i}, f_{j,j}\} \right\}.$$

Consequently we obtain

(2.14)
$$P(\mathcal{D}_I \mid f_{i,i} = x_i, i \in I) \leq \prod_{(i \neq j): i, j \in I} \left[F(\max\{x_i, x_j\}) - F\left(\frac{x_i + x_j}{2}\right) \right].$$

Introduce $y_i = F(x_i)$, i.e. $x_i = F^{-1}(y_i)$, $(i \in I)$. Then $F(\max\{x_i, x_j\}) = \max\{y_i, y_j\}$, and (since $F^{-1}(y)$ is convex),

$$F\left(\frac{x_i + x_j}{2}\right) = F\left(\frac{F^{-1}(y_i) + F^{-1}(y_j)}{2}\right)$$
$$\geq F\left(F^{-1}\left(\frac{y_i + y_j}{2}\right)\right) = \frac{y_i + y_j}{2}$$

Therefore

$$\mathbb{P}(\mathcal{D}_I \mid f_{i,i} = x_i, i \in I) \le \prod_{(i \neq j): i,j \in I} \frac{|y_i - y_j|}{2}$$

implying

(2.15)
$$P(\mathcal{D}_I) \le 2^{-r(r-1)/2} \int_{\mathbf{y} \in [0,1]^r} \int_{(i \ne j): i, j \in I} \prod_{|y_i - y_j|} d\mathbf{y}, \quad r := |I|.$$

Since the integral is below 1, we see that

(2.16) $P(\mathcal{D}_I) \le 2^{-r(r-1)/2}.$

Hence

Corollary 2.9. With probability $\geq 1 - n^{-a}$, $(\forall a > 0)$, there is no K-set of cardinality $\geq r_n := \lceil 2 \log_2 n \rceil$, that contains, properly, the support of a non-trivial local equilibrium.

Proof. By (2.16) the expected number of K-sets in question is, at most, of order

$$\binom{n}{r_n} 2^{-r_n^2/2} \le \frac{1}{r_n!} \le n^{-a}, \quad \forall a > 0.$$

We can do better though. The integral in (2.15) is a special case of Selberg's remarkable integral, [1], Section 8.1: in particular, for $\alpha > 0$, $\beta > 0$, $\gamma \ge 0$,

(2.17)
$$\int_{\mathbf{y}\in[0,1]^r} \prod_{i\in I} \left\{ y_i^{\alpha-1} (1-y_i)^{\beta-1} \right\} \prod_{(i\neq j): i,j\in I} |y_i - y_j|^{2\gamma} d\mathbf{y}$$
$$= \prod_{j=1}^r \frac{\Gamma(\alpha + (j-1)\gamma) \Gamma((\beta + (j-1)\gamma) \Gamma(1+j\gamma))}{\Gamma(\alpha + \beta + (r+j-2)\gamma) \Gamma(1+\gamma)}.$$

So we have

$$P(\mathcal{D}_I) \le 2^{-r(r-1)/2} \mathcal{S}(r), \quad \mathcal{S}(r) := \prod_{j=1}^r \frac{\Gamma^2(1+(j-1)/2) \Gamma(1+j/2)}{\Gamma(1+(r+j)/2) \Gamma(3/2)}.$$

Using the Stirling formula

$$\Gamma(1+z) = \sqrt{2\pi z} \left(\frac{z}{e}\right)^z (1+O(z^{-1})), \quad z \to \infty,$$

and applying the Euler summation formula to the logarithm of the resulting product, one can show that, for some constants η_1 , η_2 ,

(2.18)
$$S(r) = 2^{-r^2} \exp(\eta_1 r \log r + \eta_2 r + O(\log r)).$$

We have proved

Lemma 2.10. There exist constants η_1^* , η_2^* such that

$$P(\mathcal{D}_I) \le 2^{-\frac{3}{2}r^2} \exp(\eta_1^* r \log r + \eta_2^* r + O(\log r)), \quad r := |I|.$$

So $P(\mathcal{D}_I)$ is of order $2^{-\frac{3(1+o(1))}{2}r^2}$, at most. This leads immediately to a better upper bound for the maximum size of a K-set free of supports of local equilibriums.

Theorem 2.11. With probability $\geq 1 - \exp(-\Theta(\varepsilon \log^2 n))$, there is no K-set of cardinality $\geq r_n^* := \lceil (2/3 + \varepsilon) \log_2 n \rceil$, that properly contains the support of a non-trivial local equilibrium.

The sharp formula (2.18) allows us to show that with high probability there exist many K sets of the logarithmic size that do not contain the size 2 supports of local equilibriums in the case when $f_{i,j}$ are uniform.

Given a set I, $|I| \ge 3$, let \mathcal{D}_I^* be the event that I is a K-set meeting the above, less stringent, requirement. For brevity, we call such I a K^* -set. Instead of the inequality (2.14), here we have the equality

(2.19)
$$P(\mathcal{D}_{I}^{*} \mid f_{i,i} = x_{i}, i \in I) = \prod_{(i \neq j): i, j \in I} \left[F\left(\max\{x_{i}, x_{j}\}\right) - F\left(\frac{x_{i} + x_{j}}{2}\right) \right].$$

For the uniform fitnesses the RHS in (2.19) is the product of the $|x_i - x_j|/2$. So, by (2.17) and (2.18),

(2.20)
$$P(\mathcal{D}_{I}^{*}) = 2^{-\frac{3}{2}\rho^{2}} \exp\left(\eta_{1}^{*}\rho\log\rho + \eta_{2}^{*}\rho + O(\log\rho)\right), \quad \rho := |I|.$$

Let $X_{n,r}^*$ denote the total number of the K^* -sets of cardinality r. Then the expected number of the K^* -sets of cardinality r is $\mathbb{E}[X_{n,r}^*] = \binom{n}{r} \mathbb{P}(\mathcal{D}_I^*)$, (|I| = r). This expectation is easily shown to be of order $\geq \exp(\Theta(\varepsilon \log^2 n))$, thus super-polynomially large, if $r = [(2/3)(1-\varepsilon)\log_2 n]$, $\varepsilon \in (0,1)$. In fact, we are about to prove that $X_{n,r}^*$ is likely to be this large if $r < 0.5 \log_2 n$.

Theorem 2.12. For $r = [(0.5 - \varepsilon) \log_2 n]$, $(\varepsilon < 1/4)$, we have

$$\mathbb{P}\Big(X_{n,r}^* \ge \exp\big(\Theta(\varepsilon \log^2 n)\big)\Big) \ge 1 - O\big(n^{-2\varepsilon + O(\log \log n/(\log n))}\big).$$

Proof. We use the proof of Theorem 2.8 as a rough template. Given $0 \le k \le r - 1$, let

$$I_1 = I_1(k) \equiv \{1, \dots, r\}, \quad I_2 = I_2(k) = \{r - k + 1, \dots, 2r - k\};$$

so $|I_1| = |I_2| = r$ and $|I_1 \cap I_2| = k$. Then, by symmetry,

$$\mathbf{E}[(X_{n,r}^*)_2] = \sum_{k=0}^{r-1} \mathcal{N}(r,k) \ \mathbf{P}(\mathcal{D}_{I_1(k)}^* \cap \mathcal{D}_{I_2(k)}^*), \quad \mathcal{N}(r,k) = \binom{n}{r} \binom{r}{k} \binom{n-r}{r-k}.$$

To bound $P(\mathcal{D}^*_{I_1(k)} \cap \mathcal{D}^*_{I_2(k)})$, observe that, denoting by (i, j) a generic, unordered pair $(i \neq j)$, we have

$$P(\mathcal{D}_{I_{1}(k)}^{*} \cap \mathcal{D}_{I_{2}(k)}^{*} | f_{i,i} = x_{i}, i \in I_{1} \cup I_{2}) = \prod_{\substack{(i,j) \\ i,j \in [1,r] \cup [r-k+1,2r-k]}} \frac{|x_{i} - x_{j}|}{2}$$

$$\leq 2^{-(r)_{2} + \binom{k}{2}} \prod_{\substack{(i,j) \\ i,j \in [1,r-k]}} |x_{i} - x_{j}| | |x_{i} - x_{j}|$$

$$\times \prod_{\substack{(i,j) \\ i,j \in [r-k+1,r]}} |x_{i} - x_{j}| \prod_{\substack{(i,j) \\ i,j \in [r+1,2r-k]}} |x_{i} - x_{j}|$$

Unconditioning and using (2.19), we obtain: (2.21)

$$P(\mathcal{D}_{I_1(k)}^* \cap \mathcal{D}_{I_2(k)}^*) = \mathcal{P}^*(r,k)e^{O(\log r)},$$

$$\mathcal{P}^*(r,k) := 2^{-(r)_2 + \binom{k}{2}} \cdot 2^{-2(r-k)^2 - k^2}$$

$$\times \exp\left[2\eta_1^*(r-k)\log(r-k) + \eta_1^*k\log k + 2\eta_2^*(r-k) + \eta_2^*k\right].$$

It follows that

$$\frac{\mathcal{N}(r,k+1) \operatorname{P}\left(\mathcal{D}_{I_{1}(k+1)}^{*} \cap \mathcal{D}_{I_{2}(k+1)}^{*}\right)}{\mathcal{N}(r,k) \operatorname{P}\left(\mathcal{D}_{I_{1}(k)}^{*} \cap \mathcal{D}_{I_{2}(k)}^{*}\right)} \leq \frac{2^{2r}}{n} \exp\left(O(\log r)\right)$$
$$\leq n^{-2\varepsilon + o(1)} \to 0,$$

since $r \leq (0.5 - \varepsilon) \log_2 n$. Consequently

$$\frac{\mathrm{E}\left[(X_{n,r}^*)_2\right]}{\mathcal{N}(r,0) \mathrm{P}^2(\mathcal{D}_{I_1(0)}^*)} \le 1 + n^{-2\varepsilon + o(1)}.$$

Since

$$\mathcal{N}(r,0) = \left(1 + O(r^2/n)\right) \binom{n}{r}^2,$$

$$\mathbf{E}[X_{n,r}^*] = \mathbf{P}(\mathcal{D}_{I_1^*}(0)) \binom{n}{r} \ge \exp(\Theta(\log^2 n)),$$

0

the Chebyshev inequality completes the proof.

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