# ON RANDOM QUADRATIC FORMS: SUPPORTS OF POTENTIAL LOCAL MAXIMA 

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#### Abstract

The selection model in population genetics is a dynamic system on the set of of the probability distributions $\mathbf{p}=\left(p_{1}, \ldots, p_{n}\right)$ of the alleles $A_{1}, \ldots, A_{n}$, with $p_{i}(t+1)$ proportional to $p_{i}(t)$ times $\sum_{j} f_{i, j} p_{j}(t)$, and $f_{i, j}=f_{j, i}$ interpreted as a fitness of the gene pair $\left(A_{i}, A_{j}\right)$. It is known that $\hat{\mathbf{p}}$ is a locally stable equilibrium iff $\hat{\mathbf{p}}$ is a strict local maximum of the quadratic form $\mathbf{p}^{T} \mathbf{f} \mathbf{p}$. Usually there are multiple local maxima and $\lim \mathbf{p}(t)$ depends on $\mathbf{p}(0)$. To address the question of a typical behavior of $\{\mathbf{p}(t)\}$, John Kingman considered the case when the $f_{i, j}$ are independent, $[0,1]$-uniform. He proved that with high probability (w.h.p.) no local maximum may have more than $2.49 n^{1 / 2}$ positive components, and reduced 2.49 to 2.14 for a non-biological case of exponentials on $[0, \infty)$. We show that the constant 2.14 serves a broad class of the smooth densities on $[0,1]$ with the increasing hazard rate. As for a lower bound, we prove that w.h.p. for all $k \leq 2 n^{1 / 3}$ there are many $k$-element subsets of $[n]$ that pass a partial test to be a support of a local maximum. Still it may well be that w.h.p. the actual supports are much smaller. In that direction we prove that w.h.p. (i) a support of a local maximum, that does not contain a support of a local equilibrium, is very unlikely to have size exceeding $(2 / 3) \log _{2} n$, and (ii) for the uniform fitnesses, there are super-polynomially many potential supports free of local equilibriums, of size close to $(1 / 2) \log _{2} n$.


## 1. Introduction and main results

The classic selection model in population genetics is a dynamic system on the set of the probability distributions $\mathbf{p}=\left(p_{1}, \ldots, p_{n}\right) \in \Delta_{n}:=\{\mathbf{x} \geq$ $\left.\mathbf{0}, \sum_{i \in[n]} x_{i}=1\right\}$ of the alleles $A_{1}, \ldots, A_{n}$ at the single locus:

$$
\begin{equation*}
p_{i}(t+1)=p_{i}(t) \cdot \frac{\sum_{j} f_{i, j} p_{j}(t)}{\sum_{r, s} f_{r, s} p_{r}(t) p_{s}(t)}, \quad i \in[n] . \tag{1.1}
\end{equation*}
$$

Here each $f_{r, s}=f_{s, r} \in[0,1]$ is interpreted as the fitness, i.e. the probability that the unordered gene pair $\left(A_{r}, A_{s}\right)$ survives to an adult age. While the dynamic behavior of $\mathbf{p}(t)$ in this model certainly depends on the fitness

[^0]matrix $\mathbf{f}=\left\{f_{r, s}\right\}$, it has long been known that the average fitness $V(\mathbf{p}(t)):=$ $\sum_{r, s} f_{r, s} p_{r}(t) p_{s}(t)$ strictly increases with $t$ unless $\mathbf{p}(t+1)=\mathbf{p}(t)$. Hofbauer and Sigmund 9 (i) characterized this property as a consequence of Fisher's Fundamental Theorem of Natural Selection [6], (ii) provided a full proof following Kingman [10], and (iii) sketched the different proofs given by Scheuer and Mandel [20, and Baum and Eagon [2].

Using the increase of the average fitness, it was later proven by various researchers that (i) $\mathbf{p}(\infty)=\lim _{t \rightarrow \infty} \mathbf{p}(t)$ exists for every initial gene distribution $\mathbf{p}(0)$, and (ii) $\mathbf{p}:=\mathbf{p}(\infty)$ is a fixed point of the mapping $\Phi(\cdot): \Delta_{n} \rightarrow \Delta_{n}$ defined by (1.1), with a property: for a nonempty $I \subseteq[n]$,

$$
\begin{equation*}
p_{i}=0,(i \notin I), \quad \sum_{j \in I} f_{i, j} p_{j} \equiv V(\mathbf{p}),(i \in I) . \tag{1.2}
\end{equation*}
$$

Remarkably, a fixed point $\mathbf{p}$ is a locally stable equilibrium iff $\mathbf{p}$ is a strict local maximum of $V(\cdot)$. There is no reason to expect that a local maximum is unique; so typically the limit $\mathbf{p}(\infty)$ depends on $\mathbf{p}(0)$.

For $\mathbf{p} \in \Delta_{n}$ to be a local maximum of $V(\cdot)$, three sets of conditions must be satisfied, Kingman [12]. If $I=I(\mathbf{p}):=\left\{i: p_{i}>0\right\}$, then

$$
\begin{align*}
\sum_{j \in I} f_{i, j} p_{j} & \equiv V(\mathbf{p}), \quad(i \in I), \\
\sum_{i, j \in I} f_{i, j} x_{i} x_{j} & \leq 0, \quad \forall\left\{x_{i}\right\}_{i \in I} \quad \text { with } \sum_{i \in I} x_{i}=0,  \tag{1.3}\\
\sum_{j \in I} f_{i, j} p_{j} & \leq V(\mathbf{p}), \quad(i \notin I) .
\end{align*}
$$

The second necessary condition applied to $\mathbf{x}$ such that $x_{i}=1, x_{j}=-1$, with the remaining $x_{k}=0$, easily yields

$$
\begin{equation*}
f_{i, j} \geq \frac{f_{i, i}+f_{j, j}}{2}, \quad i, j \in I, i \neq j \tag{1.4}
\end{equation*}
$$

To quote from [12]: "This condition uses internal stability alone, and takes no account of vulnerability to mutation".

The inequality (1.4) was earlier obtained by Lewontin, Ginzburg and Tuljapurkar as a corollary of a determinantal criterion applied to the system of ( $|I|-1$ ) linear equations for $p_{i}, i \in I \backslash\left\{i_{0}\right\}$, implicit in

$$
\sum_{j \in I} f_{i, j} p_{j} \equiv V(\mathbf{P}), \quad(i \in I), \quad \sum_{i \in I} p_{i}=1 .
$$

It was also asserted in 17] that $f_{i, j}<\max _{k}\left(f_{i, k}+f_{k, j}\right)$; the proof is valid under an additional condition $f_{i, j}>\max \left\{f_{i, i}, f_{j, j}\right\}$.

A subset $I$ meeting the condition (1.4) is a candidate to be a support set of a local maximum of $\mathbf{p}^{T} \mathbf{f} \mathbf{p}$. (We will use a term $K$-set for such sets $I$.) Kingman [12] posed a problem of analyzing these potential supports in a
typical case, i.e. when $f_{i, j}$ are i.i.d. random variables with range $[0,1]$. For the case when $f_{i, j}$ are $[0,1]$-uniform, he proved that with high probability (w.h.p.), i.e. with probability approaching one, $\max |I| \leq 2.49 n^{1 / 2}$ : so "the largest stable polymorphism will contain at most of the order of $n^{1 / 2}$ alleles". The key tool was the bound $\mathrm{P}\left(D_{I}\right) \leq \frac{1}{r!}, r:=|I|$, where $D_{I}$ is the event in (1.4). He found that, for a (non-biological) exponential distribution on $[0, \infty), \mathrm{P}\left(D_{I}\right)=\left(\frac{2}{r+1}\right)^{r} \ll \frac{1}{r!}$ and the constant 2.49 got reduced to 2.14 .

Haigh [7, 8] established the counterparts of some of Kingman's results for the case of a non-symmetric payoff matrix. For instance, he proved that for the density $e^{-x} / \sqrt{\pi x},(x>0)$, of $\chi_{1}^{2}$ distribution, with high probability, no evolutionarily stable strategy has support of size exceeding $1.64 n^{2 / 3}$. Kontogiannis and Spirakis [16] used the technique from Haig [8] to resolve the cases of uniform distribution and standard normal distribution left open there.

Recently, and independently of the work cited above, Chen and Peng [4] studied, in an operations research context of the random quadratic optimization problems, the probability of the events quite similar to, but different from $D_{I}$. The probability bounds include $\frac{2^{r}}{(r+1)!}$ (general continuous distribution), and $\frac{2^{r}}{(r+1)^{r}}$ (uniform distribution), $\frac{2^{r}}{(r-1)^{r}}$ (exponential distribution).

In [12] Kingman suggested that it should be interesting "to carry out a comparative analysis for other distributions of the $f_{i, j} "$, and conjectured, in [14], that "for every continuous distribution $F$ of $f$, there is a finite $\beta(F)=$ $\lim _{r \rightarrow \infty}\left\{r!P\left(D_{I}\right)\right\}^{1 / r}{ }^{\prime \prime}$. Whenever this limit exists, $\max |I| \leq 2.49 \beta(F) n^{1 / 2}$ w.h.p.; in general, $\max |I| \leq 4.98 n^{1 / 2}$ w.h.p.

In this paper we consider a relatively broad class of the distributions $F$, meeting the conditions: (I) $F(x)$ has a differentiable positive density $g(x)$, $x \in[0,1]$, such that $g^{\prime}(x) \leq 0$, and (II) the hazard ratio $\lambda(x):=\frac{g(x)}{1-F(x)}$ is increasing with $x$. The non-increasing linear density $g_{c}(x)=\frac{1-c x}{1-c / 2}, c \in[0,1]$ $\left(g_{0}(x) \equiv 1\right)$ meets these constraints, and so does $g(x)=\frac{c e^{-c x}}{1-e^{-c}}$, the density of the negative exponential distribution conditioned on $[0,1]$.

For $F$ meeting the conditions (I) and (II), we prove that

$$
\begin{equation*}
\left(\frac{2}{r+1}\right)^{r} \leq \mathrm{P}\left(D_{I}\right) \leq \frac{r^{r}}{\binom{r}{2}^{(r)}} \leq \frac{e}{2}\left(\frac{2}{r}\right)^{r} . \tag{1.5}
\end{equation*}
$$

In combination with Kingman's analysis of the exponential distribution on $[0, \infty)$, it follows from (1.5) that for every $F$ meeting the constraints above, we have $\max |I| \leq 2.14 n^{1 / 2}$ with high probability. We see also that, for every $F$ in question,

$$
\lim _{r \rightarrow \infty}\left\{r!\mathrm{P}\left(D_{I}\right)\right\}^{1 / r}=: \beta(F)=\frac{2}{e},
$$

proving not only that $\beta(F)$ exists, but also that $\beta(F)$ does not depend on $F$ in this class. This lends a certain support to Kingman's conjecture, [14, that $\lim _{r \rightarrow \infty}\left\{r!\mathrm{P}\left(D_{I}\right)\right\}^{1 / r}$ exists for every continuous $F$.

Suppose we restrict our attention to the minimal K-sets $I$, i.e. such that there is no $J \subset I,(|J| \geq 2)$, which supports a local equilibrium $\mathbf{p}=\left\{p_{i}\right\}_{i \in J}$, meeting the top two conditions in (1.3). Let $\mathcal{D}_{I}$ be the corresponding event. For the distributions $F$ from the class described above, we prove that

$$
\begin{equation*}
\mathrm{P}\left(\mathcal{D}_{I}\right) \leq 2^{-r^{2} / 2}\left(\frac{4 e}{r}\right)^{r / 2} \exp \left(\Theta\left(r^{1 / 3}\right)\right), \quad r:=|I| . \tag{1.6}
\end{equation*}
$$

Continuing with $\mathrm{P}\left(D_{I}\right)$, suppose that, in addition, $g^{(3)}(0)$ exists. Then

$$
\begin{equation*}
\mathrm{P}\left(D_{I}\right)=\left(1+O\left(r^{-\sigma}\right)\right)\left(\frac{2}{r}\right)^{r} \exp \left(\frac{g^{\prime}(0)}{g^{2}(0)}\right), \quad \forall \sigma<1 / 3, \tag{1.7}
\end{equation*}
$$

and if $\left|I_{1}\right|=\left|I_{2}\right|=r,\left|I_{1} \cap I_{2}\right|=k$, then

$$
\begin{equation*}
\mathrm{P}\left(D_{I_{1}} \cap D_{I_{2}}\right)=O(\mathrm{P}(r, k)), \quad \mathrm{P}(r, k):=r^{-6}\left(\frac{2}{r}\right)^{2(r-k-1)}\left(\frac{2}{2 r-k}\right)^{k-1}, \tag{1.8}
\end{equation*}
$$

uniformly for $r \geq 2$ and $k \in[1, r-1]$.
Let $X_{n, r}$ be the total number of K-sets of $[n]$ of cardinality $r$. We already know that w.h.p. $X_{n, r}=0$ for $r>2.14 n^{1 / 2}$, and that $\mathrm{E}\left[X_{n, r}\right] \rightarrow \infty$ for every $r<2.14 n^{1 / 2}$. We use the estimates (1.5), (1.7) and (1.8) to show that

$$
\frac{\operatorname{Var}\left(X_{n, r}\right)}{\mathrm{E}^{2}\left[X_{n, r}\right]}=O\left(n^{-2 / 3}\right), \quad 2 \leq r \leq r(n):=\left\lceil 2 n^{1 / 3}\right\rceil .
$$

It follows that

$$
\mathrm{P}\left(\bigcap_{\rho=2}^{r(n)}\left\{\left|\frac{X_{n, \rho}}{\mathrm{E}\left[X_{n, \rho}\right]}-1\right| \leq n^{-1 / 6+\varepsilon}\right\}\right)=1-O\left(n^{-2 \varepsilon}\right), \quad \varepsilon<1 / 6,
$$

i.e. w.h.p. the counts of the K -sets of size $r$ ranging from 2 to $r(n)$ are uniformly asymptotic to their expected values. In particular, setting $L_{n}=$ $\max \left\{\rho: X_{n, \rho}>0\right\}$, we have $\mathrm{P}\left(L_{n}>2 n^{1 / 3}\right) \rightarrow 1$, i.e. w.h.p. the size of the largest potential support of a local maximum is sandwiched between $2 n^{1 / 3}$ and $2.14 n^{1 / 2}$.

We cannot rule out a possibility that, with high probability, the actual supports of local maxima are considerably smaller. In fact, we use the bound (1.6) to show that, with probability $>1-n^{-a},(\forall a>0)$, there is no $K$-set of cardinality $>(2 / 3) \log _{2} n$ that contains, properly, a non-trivial support of a local equilibrium. Complementing this claim, we show that, with high probability, the number of $K$-sets of size $<0.5 \log _{2} n$ that do not contain the size 2 supports of local equilibriums is super-polynomially large.

The already cited paper [4] was preceded by Chen, Peng and Zhang [3]; both papers studied the likely behavior of an absolute minimum of a random quadratic form $\mathbf{x}^{T} Q \mathbf{x}$ for $\mathbf{x} \in \Delta_{n}$. Under the condition that the elements of $Q$ are i.i.d. random variables with a c.d.f. $F$ concave on its support, the support size of the absolute minimum point was shown to be bounded in probability, with its distribution tail decaying exponentially fast. In particular, it followed that, for $f_{i, j}$ uniform or positive-exponential on $[0,1]$, the absolute maximum of $\mathbf{p}^{T} \mathbf{f} \mathbf{p}$ is attained at a point of $\Delta_{n}$ with $N$, the number of positive components, satisfying $\mathrm{P}(N \geq k)=O\left(\rho^{k}\right), k>0$, as $n \rightarrow \infty$.

In view of all this information, it is tempting to conjecture that-for $f_{i, j}$ meeting the conditions (I) and (II) - the size of the largest support of a local maximum of $\mathbf{p}^{T} \mathbf{f} \mathbf{p}$ is, with high probability, of (poly)logarithmic order.

## 2. Proofs

### 2.1. Estimate of $\mathbf{P}\left(D_{I}\right)$.

Theorem 2.1. Suppose that $F$ (i) has a positive, non-increasing, differentiable density $g$, and (2) has a non-decreasing hazard ratio $\lambda(x)=\frac{g(x)}{1-F(x)}$. Then, with $a^{(b)}:=a(a+1) \cdots(a+b-1)$, we have

$$
\begin{equation*}
\left(\frac{2}{r+1}\right)^{r} \leq \mathrm{P}\left(D_{I}\right) \leq \frac{r^{r}}{\binom{r}{2}^{(r)}} \leq \frac{e}{2}\left(\frac{2}{r}\right)^{r} . \tag{2.1}
\end{equation*}
$$

In the special case of the uniform density $g(x) \equiv 1$, this bound improves Kingman's bound $\mathrm{P}\left(D_{I}\right) \leq \frac{1}{r!}$. It also shows that, for all $F$ meeting the conditions (i) and (ii),

$$
\lim _{r \rightarrow \infty}\left\{r!\mathrm{P}\left(D_{I}\right)\right\}^{1 / r}=\frac{2}{e} .
$$

Proof. As in [12], the probability of $D_{I}$, conditioned on $\left\{f_{i, i}=x_{i}, i \in I\right\}$, is

$$
\prod_{(i, j)} \mathrm{P}\left(f \geq \frac{x_{i}+x_{j}}{2}\right)=\prod_{(i, j)}\left(1-F\left(\frac{x_{i}+x_{j}}{2}\right)\right),
$$

where $i \neq j$.The function $1-F(x)$ is log-concave, since

$$
\frac{d}{d x} \log (1-F(x))=-\frac{g(x)}{1-F(x)}=-\lambda(x)
$$

is decreasing with $x$. (a) Lower bound. By Jensen inequality,

$$
\begin{aligned}
\prod_{(i, j)}\left(1-F\left(\frac{x_{i}+x_{j}}{2}\right)\right) & \geq \prod_{(i, j)}\left(1-F\left(x_{i}\right)\right)^{1 / 2}\left(1-F\left(x_{j}\right)\right)^{1 / 2} \\
& =\prod_{i=1}^{r}\left(1-F\left(x_{i}\right)\right)^{(r-1) / 2}
\end{aligned}
$$

Consequently

$$
\mathrm{P}\left(D_{I}\right) \geq \int_{\mathbf{x} \in[0,1]^{r}} \ldots \int_{i=1}^{r}\left(1-F\left(x_{i}\right)\right)^{(r-1) / 2} \prod_{i=1}^{r} g\left(x_{i}\right) d x_{i}
$$

and, switching to the variables $y_{i}=F\left(x_{i}\right)$,

$$
\begin{aligned}
\mathrm{P}\left(D_{I}\right) & \geq \int_{\mathbf{y} \in[0.1]^{n}} \ldots \int_{i=1} \prod_{i=1}^{r}\left(1-y_{i}\right)^{(r-1) / 2} d \mathbf{y} \\
& =\left(\int_{0}^{1}(1-y)^{(r-1) / 2} d y\right)^{r}=\left(\frac{2}{r+1}\right)^{r}
\end{aligned}
$$

(b) Upper bound. Again by Jensen inequality, denoting $s=\sum_{i} x_{i}$ we have

$$
\begin{align*}
\prod_{(i, j)}(1- & \left.F\left(\frac{x_{i}+x_{j}}{2}\right)\right)=\exp \left[\binom{r}{2} \sum_{(i, j)} \frac{1}{\binom{r}{2}} \log \left(1-F\left(\frac{x_{i}+x_{j}}{2}\right)\right)\right] \\
& \leq \exp \left[\binom{r}{2} \log \left(1-F\left(\frac{1}{r(r-1)} \sum_{(i, j)}\left(x_{i}+x_{j}\right)\right)\right)\right]  \tag{2.2}\\
& =\exp \left[\binom{r}{2} \log \left(1-F\left(\frac{s}{r}\right)\right)\right]=\left(1-F\left(\frac{s}{r}\right)\right)^{\binom{r}{2}}
\end{align*}
$$

Consequently

$$
\begin{equation*}
\mathrm{P}\left(D_{I}\right) \leq \int_{\mathbf{x} \in[0,1]^{r}} \ldots \int_{i}\left(1-F\left(\frac{s}{r}\right)\right)^{\binom{r}{2}} \prod_{i \in I} g\left(x_{i}\right) d x_{i} \tag{2.3}
\end{equation*}
$$

Again change the variables of integration, setting $y_{i}=F\left(x_{i}\right)$, so that $x_{i}=$ $F^{-1}\left(y_{i}\right)$, and $s=\sum_{i \in I} F^{-1}\left(y_{i}\right)$. Now

$$
\frac{d^{2}}{d y^{2}} F^{-1}(y)=-\frac{g^{\prime}(x)}{g(x)^{3}} \geq 0
$$

implying that $F^{-1}(y)$ is convex. Therefore, for each $t \leq r$, we have

$$
r^{-1} \min \left\{\sum_{i \in I} F^{-1}\left(y_{i}\right): \sum_{i \in I} y_{i}=t\right\}=F^{-1}\left(\frac{t}{r}\right)
$$

Hence

$$
\begin{aligned}
\max \left\{1-F\left(r^{-1}\right.\right. & \left.\left.\sum_{i \in I} x_{i}\right): \sum_{i \in I} y_{i}=t\right\} \\
& =1-\min \left\{F\left(r^{-1} \sum_{i \in I} F^{-1}\left(y_{i}\right)\right): \sum_{i \in I} y_{i}=t\right\} \\
= & 1-F\left(r^{-1} \min \left\{\sum_{i \in I} F^{-1}\left(y_{i}\right): \sum_{i \in I} y_{i}=t\right\}\right) \\
& =1-F\left(F^{-1}\left(\frac{t}{r}\right)\right)=1-\frac{t}{r}
\end{aligned}
$$

So (2.3) yields

$$
\mathrm{P}\left(D_{I}\right) \leq \int_{\mathbf{y} \in[0,1]^{r}} \ldots \int_{i}\left(1-\frac{t}{r}\right)^{\binom{r}{2}} \prod_{i \in I} d y_{i}
$$

Since

$$
\int_{\sum_{i} y_{i} \leq t} \cdots \int_{i \in I} \prod_{i} d y_{i}=\frac{t^{r}}{r!}
$$

we conclude that

$$
\begin{aligned}
\mathrm{P}\left(D_{I}\right) & \leq \int_{0}^{r}\left(1-\frac{t}{r}\right)^{\binom{r}{2}} \frac{t^{r-1}}{(r-1)!} d t \\
& =\frac{r^{r}}{(r-1)!} \int_{0}^{1}(1-\tau)^{\binom{r}{2}} \tau^{r-1} d \tau \\
& =\frac{r^{r}}{(r-1)!} \cdot \frac{\binom{r}{2}!(r-1)!}{\left(\binom{r}{2}+r\right)!}=\frac{r^{r}}{\binom{r}{2}^{(r)}} .
\end{aligned}
$$

Theorem 2.2. Suppose that, in addition to conditions (i), (ii), we have (iii): $g^{(3)}(0)$ exists. Then

$$
\mathrm{P}\left(D_{I}\right)=\left(1+O\left(r^{-\sigma}\right)\right)\left(\frac{2}{r}\right)^{r} \exp \left(\frac{g^{\prime}(0)}{g^{2}(0)}\right)
$$

for every $\sigma<1 / 3$.
To prove this claim, we shrink, in steps, the cube $[0,1]^{n}$ to a subset $C^{*}$ in such a way that (a) the integral of the product of $1-F\left(\frac{x_{i}+x_{j}}{2}\right)$ over $C^{*}$ sharply approximates that over $[0,1]^{n}$, and (b) the product itself admits a manageable approximation on $C^{*}$.

Given $C \subset[0,1]^{n}$, denote

$$
\mathrm{P}_{C}\left(D_{I}\right)=\int \ldots \int \prod_{\mathbf{x} \in C}\left(1-F\left(\frac{x_{i}+x_{j}}{2}\right)\right) d \mathbf{x}
$$

Lemma 2.3. Let

$$
C_{1}:=\left\{\mathbf{x} \in[0,1]^{n}:\left|\sum_{i=1}^{r} F\left(x_{i}\right)-2\right| \leq r^{-1 / 3}\right\}
$$

Then

$$
\mathrm{P}\left(D_{I}\right)-\mathrm{P}_{C_{1}}\left(D_{I}\right) \leq\left(\frac{2}{r}\right)^{r} \cdot \exp \left(-\frac{r^{1 / 3}}{10}\right)
$$

Proof. Let $\tau_{1,2}=\frac{2}{r} \mp r^{-4 / 3}$. From the proof of Theorem 2.1 it follows that

$$
\mathrm{P}\left(D_{I}\right)-\mathrm{P}_{C_{1}}\left(D_{I}\right) \leq \frac{r^{r}}{(r-1)!} \int_{\tau \in\left[\tau_{1}, \tau_{2}\right]^{c}}(1-\tau)^{\binom{r}{2}} \tau^{r-1} d \tau
$$

The (logconcave) integrand attains its maximum at $\tau_{\max }=\frac{2}{2+r} \in\left[\tau_{1}, \tau_{2}\right]$, and

$$
\max \left\{\frac{d^{2}}{d \tau^{2}}\left(\log (1-\tau)^{\binom{r}{2}} \tau^{r-1}\right): \tau \in\left[\tau_{1}, \tau_{2}\right]\right\} \leq-\frac{r^{3}}{4.1}
$$

Therefore the integral is at most

$$
\left(1-\frac{2}{2+r}\right)^{\binom{r}{2}}\left(\frac{2}{2+r}\right)^{r-1} \cdot \exp \left(-\frac{r^{1 / 3}}{9}\right)
$$

so that

$$
\begin{aligned}
\mathrm{P}\left(D_{I}\right)-\mathrm{P}_{C_{1}}\left(D_{I}\right) & \leq \frac{r^{r}}{(r-1)!}\left(1-\frac{2}{2+r}\right)^{\binom{r}{2}}\left(\frac{2}{2+r}\right)^{r-1} \cdot \exp \left(-\frac{r^{1 / 3}}{9}\right) \\
& \leq\left(\frac{2}{r}\right)^{r} \cdot \exp \left(-\frac{r^{1 / 3}}{10}\right)
\end{aligned}
$$

Next
Lemma 2.4. Let

$$
C_{2}:=\left\{\mathbf{x} \in C_{1}: \max _{i} \frac{F\left(x_{i}\right)}{\sum_{j} F\left(x_{j}\right)} \leq k \frac{\log r}{r}\right\}, \quad(k>1)
$$

Then

$$
\mathrm{P}_{C_{1}}\left(D_{I}\right)-\mathrm{P}_{C_{2}}\left(D_{I}\right) \leq\left(\frac{2}{r}\right)^{r} \cdot r^{-\alpha}, \quad \forall \alpha<k-1
$$

Proof. Similarly to the proof of Lemma 2.3,

$$
\mathrm{P}_{C_{1}}\left(D_{I}\right)-\mathrm{P}_{C_{2}}\left(D_{I}\right) \leq \int_{\max \frac{y_{i}}{t}>k \frac{\log r}{r}} \cdots \int_{i \in I}\left(1-\frac{t}{r}\right)^{\binom{r}{2}} \prod_{i \in I} d y_{i}
$$

Introduce $L_{1}, \ldots, L_{r}$ the lengths of the consecutive subintervals of $[0,1]$ obtained by sampling uniformly at random $r-1$ points in $[0,1]$. By Lemma 1 in [18, the integral above is at most

$$
\begin{aligned}
\mathrm{P}\left(\max L_{i} \geq k \frac{\log r}{r}\right) \int_{0}^{r} & \left(1-\frac{t}{r}\right)^{\binom{r}{2}} \frac{t^{r-1}}{(r-1)!} d t \\
& =\mathrm{P}\left(\max L_{i} \geq k \frac{\log r}{r}\right) \frac{r^{r}}{\binom{r}{2}^{(r)}} .
\end{aligned}
$$

And, introducing $U_{1}, \ldots, U_{r-1}$ the independent [0,1]-Uniforms, the probability factor is at most

$$
\begin{aligned}
r \mathrm{P}\left(L_{1} \geq k \frac{\log r}{r}\right) & =r \mathrm{P}\left(\min _{i} U_{i} \geq k \frac{\log r}{r}\right) \\
& =r\left(1-k \frac{\log r}{r}\right)^{r-1} \leq r \exp \left(-(r-1) k \frac{\log r}{r}\right) .
\end{aligned}
$$

One more reduction step defines the final

$$
\begin{equation*}
C^{*}=\left\{\mathrm{x} \in C_{2}:\left|\frac{r}{2} \frac{\sum_{i} F^{2}\left(x_{i}\right)}{\left(\sum_{j} F\left(x_{j}\right)\right)^{2}}-1\right| \leq r^{-\sigma}\right\}, \quad \sigma<1 / 3 \tag{2.4}
\end{equation*}
$$

Lemma 2.5.

$$
\mathrm{P}_{C_{2}}\left(D_{I}\right)-\mathrm{P}_{C^{*}}\left(D_{I}\right) \leq\left(\frac{2}{r}\right)^{r} \cdot \exp \left(-0.5 r^{1 / 3-\sigma}\right)
$$

Proof. Once again like the proofs of Lemmas 2.3, 2.4,

$$
\begin{aligned}
& \mathrm{P}_{C_{2}}\left(D_{I}\right)-\mathrm{P}_{C^{*}}\left(D_{I}\right) \leq \iiint\left(1-\frac{t}{r}\right)^{\binom{r}{2}} \prod_{i \in I} d y_{i} \\
& \left.\leq \mathrm{P}\left(\left|\frac{r}{2} \sum_{i} L_{i}^{2}-1\right|>r^{-\sigma}\right) \frac{\sum_{i} y_{i}^{2}}{\left(\sum_{j} y_{j}\right)^{2}}-1 \right\rvert\,>r^{-\sigma} \\
& \binom{r}{2}^{(r)}
\end{aligned}\left(\frac{2}{r}\right)^{r} \cdot \exp \left(-\Theta\left(r^{1 / 3-\sigma}\right)\right), ~ \$
$$

as the probability is at $\operatorname{most} \exp \left(-\Theta\left(r^{1 / 3-\sigma}\right)\right)$, (see Lemma 3.2 in [19]).
Note. A key to the proof of that Lemma 3.2 was a classic fact that $\left(L_{1}, \ldots, L_{r}\right)$ and $\left(\sum_{i} W_{i}\right)^{-1}\left(W_{1}, \ldots, W_{r}\right)$, $W_{j}$ being i.i.d. Exponentials), are equidistributed, Feller [5]. While both of the distribution tails of $\sum_{i} W_{j}$ decay exponentially, for the right tail of $\sum_{j} W_{j}^{2}$ we could prove only the bound $e^{-\Theta\left(r^{\delta}\right)}, \delta<1 / 3$. The obstacle here is that $\mathrm{E}\left[e^{z W^{2}}\right]=\infty$ for every $z>0$.

Combining Lemmas 2.3, 2.4 and 2.5, we obtain

## Corollary 2.6.

$$
\mathrm{P}\left(D_{I}\right)-\mathrm{P}_{C^{*}}\left(D_{I}\right) \leq\left(\frac{2}{r}\right)^{r} \cdot r^{-\alpha}, \quad \forall \alpha<k-1 .
$$

For $\mathrm{x} \in C^{*}$, we have $\max _{i} F\left(x_{i}\right) \leq \frac{3 k \log r}{r} \rightarrow 0$, which implies that $\max _{i} x_{i}=O\left(r^{-1} \log r\right) \rightarrow 0$. For $x=O\left(r^{-1} \log r\right)$, we have

$$
\begin{aligned}
F(x) & =g(0) x+\frac{1}{2} g^{(1)}(0) x^{2}+O\left(x^{3}\right) \\
& =g(0) x+\frac{1}{2} g^{(1)}(0) x^{2}+O\left(r^{-3} \log ^{3} r\right) .
\end{aligned}
$$

So

$$
\log (1-F(x))=-g(0) x-\frac{g^{\prime}(0)+g^{2}(0)}{2} x^{2}+O\left(r^{-3} \log ^{3} r\right)
$$

and with a bit of algebra

$$
\begin{align*}
& \log \left(1-F\left(\frac{x_{i}+x_{j}}{2}\right)\right)-\frac{\log \left(1-F\left(x_{i}\right)\right)+\log \left(1-F\left(x_{j}\right)\right)}{2} \\
& =\frac{g^{\prime}(0)+g^{2}(0)}{8}\left(x_{i}-x_{j}\right)^{2}+O\left(r^{-3} \log ^{3} r\right)  \tag{2.5}\\
& =\gamma\left(F\left(x_{i}\right)-F\left(x_{j}\right)\right)^{2}+O\left(r^{-3} \log ^{3} r\right), \quad \gamma:=\frac{g^{\prime}(0)+g^{2}(0)}{8 g^{2}(0)} .
\end{align*}
$$

Therefore

$$
\begin{aligned}
\prod_{(i, j)} \log \left(1-F\left(\frac{x_{i}+x_{j}}{2}\right)\right) & =\exp \left(\frac{r-1}{2} \sum_{i} \log \left(1-F\left(x_{i}\right)\right)\right) \\
& \times \exp \left(\gamma \sum_{(i, j)}\left(F\left(x_{i}\right)-F\left(x_{j}\right)\right)^{2}+O\left(r^{-1} \log r\right)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{r-1}{2} \sum_{i} \log \left(1-F\left(x_{i}\right)\right. & =-\frac{r-1}{2} \sum_{i}\left(F\left(x_{i}\right)+\frac{F^{2}\left(x_{i}\right)}{2}\right)+O\left(r^{-1} \log r\right), \\
\sum_{(i, j)}\left(F\left(x_{i}\right)-F\left(x_{j}\right)\right)^{2} & =r \sum_{i} F^{2}\left(x_{i}\right)-\left(\sum_{i} F\left(x_{i}\right)\right)^{2} .
\end{aligned}
$$

Hence on $C^{*}$ (see (2.4))

$$
\begin{aligned}
& \prod_{(i, j)} \log \left(1-F\left(\frac{x_{i}+x_{j}}{2}\right)\right)= \exp \left(-\frac{r-1}{2} \sum_{i} F\left(x_{i}\right)-\gamma\left(\sum_{i} F\left(x_{i}\right)\right)^{2}\right. \\
&\left.+\left(-\frac{r-1}{4}+\gamma r\right) \sum_{i} F^{2}\left(x_{i}\right)+O\left(r^{-1} \log r\right)\right) \\
&=\exp \left(-\frac{r-1}{2} \sum_{i} F\left(x_{i}\right)+\left(2 \gamma-\frac{1}{2}\right)\left(\sum_{i} F\left(x_{i}\right)\right)^{2}+O\left(r^{-\sigma}\right)\right) \\
&=\exp \left(-\frac{r}{2} \sum_{i} F\left(x_{i}\right)+\frac{g^{\prime}(0)}{g^{2}(0)}+O\left(r^{-\sigma}\right)\right) ;
\end{aligned}
$$

for the last equality we used the definition of $\gamma$ in (2.5).
Switching to the variables $y_{i}=F\left(x_{i}\right)$ and denoting $t=\sum_{i} y_{i}$, we obtain then

$$
\begin{gathered}
\mathrm{P}_{C^{*}}\left(D_{I}\right)=\int \underset{\mathbf{y} \in \mathcal{C}^{*}}{ } \ldots \int \exp \left(-\frac{r}{2} t+\frac{g^{\prime}(0)}{g^{2}(0)}+O\left(r^{-\sigma}\right)\right) d \mathbf{y} \\
\mathcal{C}^{*}:=\left\{\mathbf{y} \geq \mathbf{0}:|t-2| \leq r^{-1 / 3}, \max _{i} \frac{y_{i}}{t} \leq k \frac{\log r}{r},\left|\frac{r}{2 t^{2}} \sum_{i} y_{i}^{2}-1\right| \leq r^{-\sigma}\right\} .
\end{gathered}
$$

Notice that on $\mathcal{C}^{*}$ we have $\max _{i} y_{1} \rightarrow 0$, so that the omitted condition $\max _{i} y_{i} \leq 1$ would have been superfluous. By Lemma 3.1 in [19],

$$
\begin{gathered}
\int \cdots \int e^{-\frac{r t}{2}} d \mathbf{y} \\
=\int_{y \in \mathcal{C}^{*}} \frac{e^{-\frac{r t}{2}} t^{r-1}}{(r-1)!} \mathrm{P}\left(\max L_{i} \leq \min \left(t^{-1}, \frac{k \log r}{r}\right),\left|\frac{r}{2} \sum_{i} L_{i}^{2}-1\right| \leq r^{-\sigma}\right) d t \\
=\mathrm{P}\left(\max L_{i} \leq \frac{k \log r}{r},\left|\frac{r}{2} \sum_{i} L_{i}^{2}-1\right| \leq r^{-\sigma}\right)_{|t-2| \leq \frac{1}{r^{1 / 3}}} \frac{e^{-\frac{r t}{2}} t^{r-1}}{(r-1)!} d t
\end{gathered}
$$

From Lemma 2.4 and Lemma 2.5, and their proofs, we know that the probability factor is at least $1-r^{-\alpha}, \forall \alpha<k-1$. Furthermore, the integral
equals

$$
\begin{aligned}
& \int_{0}^{\infty} \frac{e^{-\frac{r t}{2}} t^{r-1}}{(r-1)!} d t-\int_{|t-2|>\frac{1}{r^{1 / 3}}} \frac{e^{-\frac{r t}{2}} t^{r-1}}{(r-1)!} d t \\
& =\left(\frac{2}{r}\right)^{r}-\frac{\left(\frac{2}{r}\right)^{r}}{(r-1)!} \int_{|\tau-r|>\frac{r^{2} / 3}{2}} e^{-\tau} \tau^{r-1} d \tau,
\end{aligned}
$$

and, by Chebyshev's inequality,

$$
\begin{aligned}
\frac{1}{(r-1)!} \int_{|\tau-r|>\frac{r^{2 / 3}}{2}} e^{-\tau} \tau^{r-1} d \tau & \leq \mathrm{P}\left(\mid \text { Poisson }(r-1)-(r-1) \left\lvert\,>\frac{r^{2 / 3}}{3}\right.\right) \\
& \leq \frac{9(r-1)}{r^{4 / 3}} \leq 9 r^{-1 / 3}
\end{aligned}
$$

So

$$
\int_{|t-2| \leq \frac{1}{r^{1 / 3}}} \frac{e^{-\frac{r t}{2}} t^{r-1}}{(r-1)!} d t=\left(1+O\left(r^{-1 / 3}\right)\right)\left(\frac{2}{r}\right)^{r} .
$$

Consequently

$$
\mathrm{P}_{\mathcal{C}^{*}}\left(D_{I}\right)=\left(1+O\left(r^{-\sigma}\right)\right)\left(\frac{2}{r}\right)^{r} \exp \left(\frac{g^{\prime}(0)}{g^{2}(0)}\right),
$$

for every $\sigma<1 / 3$. Combining this estimate with Corollary 2.6, we complete the proof of Theorem 2.2,
2.2. Estimate of $\mathbf{P}\left(D_{I_{1}} \cap D_{I_{2}}\right)$. Let $I_{1}, I_{2} \subset[n],\left|I_{j}\right|=r$. If $I_{1} \cap I_{2}=\emptyset$, then the events $D_{I_{1}}$ and $D_{I_{2}}$ are independent and so (by Theorem (2.2)

$$
\begin{equation*}
\mathrm{P}\left(D_{I_{1}} \cap D_{I_{2}}\right)=\mathrm{P}\left(D_{I_{1}}\right) \cdot \mathrm{P}\left(D_{I_{2}}\right)=\left(1+O\left(r^{-\sigma}\right)\right)\left(\frac{2}{r}\right)^{2 r} \exp \left(2 \frac{g^{\prime}(0)}{g^{2}(0)}\right) . \tag{2.6}
\end{equation*}
$$

Consider the case $\left|I_{1} \cap I_{2}\right|=k \in[1, r-1]$. By symmetry, we can assume that $I_{1}=\{1, \ldots, r\}$, and $I_{2}=\{r-k+1, \ldots, 2 r-k\}$. The probability of $D_{I_{1}} \cap D_{I_{2}}$, conditioned on the event $\left\{F_{i, i}=x_{i}: 1 \leq i \leq 2 r-k\right\}$, is

$$
\begin{align*}
\Psi(\mathbf{x}) & =\prod_{\substack{(i \neq j) \\
i, j \leq r}}\left(1-F\left(\frac{x_{i}+x_{j}}{2}\right)\right) \prod_{\substack{r<i \leq 2 r-k \\
r-k+1 \leq j<r}}\left(1-F\left(\frac{x_{i}+x_{j}}{2}\right)\right) \\
& \times \prod_{\substack{(i \neq j) \\
r \leq i, j \leq 2 r-k}}\left(1-F\left(\frac{x_{i}+x_{j}}{2}\right)\right) . \tag{2.7}
\end{align*}
$$

The three products contain, respectively, $\binom{r}{2},(r-k)(k-1)$ and $\binom{r-k+1}{2}$ factors. The total number of the factors is

$$
N(r, k)=\binom{r}{2}+(r-k)(k-1)+\binom{r-k+1}{2} .
$$

Now

$$
\begin{aligned}
& \sum_{\substack{(i \neq j) \\
1 \leq i, j \leq r}} \frac{x_{i}+x_{j}}{2}=\frac{r-1}{2} \sum_{i=1}^{r} x_{i}, \sum_{\substack{(i \neq j) \\
r \leq i, j \leq 2 r-k}} \frac{x_{i}+x_{j}}{2}=\frac{r-k}{2} \sum_{i=r}^{2 r-k} x_{i}, \\
& \sum_{\substack{r<i \leq 2 r-k \\
r-k+1 \leq j \leq r}} \frac{x_{i}+x_{j}}{2}=\frac{k-1}{2} \sum_{i=r+1}^{2 r-k} x_{i}+\frac{r-k}{2} \sum_{j=r-k+1}^{r-1} x_{j} .
\end{aligned}
$$

The total sum of the fractions $\frac{x_{i}+x_{j}}{2}$ is $\frac{r-1}{2} s_{1}+\frac{2 r-k-1}{2} s_{2}+\frac{r-1}{2} s_{3}$, where

$$
s_{1}=\sum_{i=1}^{r-k} x_{i}, \quad s_{2}=\sum_{i=r-k+1}^{r} x_{i}, \quad s_{3}=\sum_{i=r+1}^{2 r-k} x_{i},
$$

and the sum of the coefficients $\alpha_{i}$ by $x_{i}$ in the sum of those fractions is $N(r, k)$. By log-concavity of $1-F(x)$,

$$
\Psi(\mathbf{x}) \leq\left(1-F\left(\frac{\frac{r-1}{2} s_{1}+\frac{2 r-k-1}{2} s_{2}+\frac{r-1}{2} s_{3}}{N(r, k)}\right)\right)^{N(r, k)}
$$

As in the proof of Theorem [2.1, introduce $y_{i}=F\left(x_{i}\right), 1 \leq i \leq 2 r-k$, so that

$$
s_{1}=\sum_{i=1}^{r-k} F^{-1}\left(y_{i}\right), \quad s_{2}=\sum_{i=r-k+1}^{r} F^{-1}\left(y_{i}\right), \quad s_{3}=\sum_{i=r+1}^{2 r-k} F^{-1}\left(y_{i}\right) .
$$

Given $t_{1}, t_{2}, t_{3}$, by convexity of $F^{-1}$, we have

$$
\begin{aligned}
& \min \left\{\sum_{i=1}^{2 r-k} \frac{\alpha_{i}}{N(r, k)} F^{-1}\left(y_{i}\right): \sum_{i=1}^{r-k} y_{i}=t_{1}, \sum_{i=r-k+1}^{r} y_{i}=t_{2}, \sum_{i=r+1}^{2 r-k} y_{i}=t_{3}\right\} \\
& \quad \geq F^{-1}\left(\frac{\frac{r-1}{2} t_{1}}{N(r, k)}+\frac{\frac{2 r-k-1}{2} t_{2}}{N(r, k)}+\frac{\frac{r-k}{2} t_{3}}{N(r, k)}\right) .
\end{aligned}
$$

Consequently

$$
\begin{gathered}
\Psi(\mathbf{x}) \leq \Psi^{*}(\mathbf{t}):=\left(1-\frac{(r-1) t_{1}+(2 r-k-1) t_{2}+(r-1) t_{3}}{2 N(r, k)}\right)^{N(r, k)}, \\
t_{1}:=\sum_{i=1}^{r-k} F\left(x_{i}\right), t_{2}:=\sum_{i=r-k+1}^{r} F\left(x_{i}\right), t_{3}:=\sum_{i=r+1}^{2 r-k} F\left(x_{i}\right) .
\end{gathered}
$$

Therefore

$$
\begin{align*}
& \mathrm{P}\left(D_{I_{1}} \cap D_{I_{2}}\right)=\int_{\mathbf{x} \in[0,1]^{2 r-k}} \cdots \int_{t_{1} \leq r-k, t_{2} \leq k, t_{3} \leq r-k} \Psi(\mathbf{x}) d \mathbf{x} \leq \int_{t_{1}} \cdots \int^{*}(\mathbf{t}) d \mathbf{y} \\
& =\iiint_{t_{1}, t_{3} \leq r-k, t_{2} \leq k} \Psi^{*}(\mathbf{t}) \frac{t_{1}^{r-k-1}}{(r-k-1)!} \frac{t_{2}^{k-1}}{(k-1)!} \frac{t_{3}^{r-k-1}}{(r-k-1)!} d \mathbf{t} . \tag{2.8}
\end{align*}
$$

Introduce

$$
\tau_{1}=\frac{r-1}{2 N(r, k)} t_{1}, \tau_{2}=\frac{2 r-k-1}{2 N(r, k)} t_{2}, \tau_{3}=\frac{r-1}{2 N(r, k)} t_{3} .
$$

Since

$$
\begin{aligned}
t_{1} \leq r-k, t_{2} \leq k, t_{3} \leq r-k, \\
\frac{(r-1)(r-k)}{2 N(r, k)}+\frac{(2 r-k-1) k}{2 N(r, k)}+\frac{(r-1)(r-k)}{2 N(r, k)}=1,
\end{aligned}
$$

we see that $\tau_{1}+\tau_{2}+\tau_{3} \leq 1$. Switching to $\tau_{j}$, and denoting $N=N(r, k)$, we transform (2.8), into

$$
\begin{aligned}
& \mathrm{P}\left(D_{I_{1}} \cap D_{I_{2}}\right) \leq \frac{\left(\frac{2 N}{r-1}\right)^{r-k-1}}{(r-k-1)!} \cdot \frac{\left(\frac{2 N}{2 r-k-1}\right)^{k-1}}{(k-1)!} \cdot \frac{\left(\frac{2 N}{r-1}\right)^{r-k-1}}{(r-k-1)!} \\
& \times \iiint_{\tau_{1}+\tau_{2}+\tau_{3} \leq 1} \tau_{1}^{r-k-1} \tau_{2}^{k-1} \tau_{3}^{r-k-1}\left(1-\tau_{1}-\tau_{2}-\tau_{3}\right)^{N} d \tau_{1} d \tau_{2} d \tau_{3} \\
& \quad=\frac{N!\left(\frac{2 N}{r-1}\right)^{2(r-k-1)}\left(\frac{2 N}{2 r-k-1}\right)^{k-1}}{(N+2 r-k)!} \\
& \quad \leq N^{-3}\left(\frac{2}{r-1}\right)^{2(r-k-1)}\left(\frac{2}{2 r-k-1}\right)^{k-1} .
\end{aligned}
$$

(At the penultimate line we used the multidimensional extension of the beta integral, Andrews, Askey and Roy [1], Theorem 1.8.6.) Since $N=\Theta\left(r^{2}\right)$, we have then

$$
\begin{equation*}
\mathrm{P}\left(D_{I_{1}} \cap D_{I_{2}}\right)=O(\mathrm{P}(r, k)), \quad \mathrm{P}(r, k):=r^{-6}\left(\frac{2}{r}\right)^{2(r-k-1)}\left(\frac{2}{2 r-k}\right)^{k-1} \tag{2.9}
\end{equation*}
$$

2.3. Likely range of the maximum size of the K-set. Introduce $L_{n}=$ $\{\max |I|$ : (1.4) holds\}. Kingman [12], [13, [14 proved that, for $F=$ Uniform $[0,1]$, w.h.p. $L_{n} \leq n^{1 / 2}(\epsilon+o(1))$, where $\epsilon=\xi^{-1 / 2}(1-\xi)^{-1 / 2}$ and $\xi=2.49 \ldots$ is a positive root of $1-\xi=e^{-2 \xi}$. The proof consisted of showing that $\mathrm{P}\left(D_{I}\right) \leq \frac{1}{|T|!}$, and that

$$
\begin{equation*}
\mathrm{P}\left(L_{n} \geq r\right) \leq \frac{(n)_{s}}{(r)_{s}} \mathrm{P}\left(D_{I}\right), \quad|I|=s \leq r \tag{2.10}
\end{equation*}
$$

This inequality sharpens the (first-order moment) bound $\mathrm{P}\left(L_{n} \geq r\right) \leq$ $\binom{n}{r} \mathrm{P}\left(D_{I}\right),|I|=r$, by using the fact that every subset of a K-set is a K-set as well. Kingman also demonstrated that his exact formula $\mathrm{P}\left(D_{I}\right)=\left(\frac{2}{r+1}\right)^{r}$ for the negative exponential distribution on $[0, \infty)$ implied a better bound

$$
\begin{equation*}
L_{n} \leq n^{1 / 2}\left[\left(2 e^{-1}\right)^{1 / 2} \epsilon+o(1)\right], \quad\left(2 e^{-1}\right)^{1 / 2} \epsilon=2.14 \ldots \tag{2.11}
\end{equation*}
$$

Now, by Theorem [2.1, we have $\mathrm{P}\left(D_{I}\right) \leq \frac{e}{2}\left(\frac{2}{r}\right)^{r}$ for a wide class of the densities on $[0,1]$, that includes the uniform density and the exponential density restricted to $[0,1]$. Combining this Theorem and Kingman's proof for the exponential distribution, we obtain

Theorem 2.7. Under the conditions (i), (ii) of Theorem 2.1, w.h.p.

$$
L_{n} \leq n^{1 / 2}(2.14 \ldots+o(1))
$$

Armed with the bound (2.9) and the bounds in Theorem[2.1, we can prove a qualitatively matching lower bound.

Theorem 2.8. Let $X_{n, r}$ stand for the total number of $K$-sets of cardinality $r$. Introduce $r(n)=\left\lceil 2 n^{1 / 3}\right\rceil$. Then, under the conditions (i), (ii) and (iii) of Theorem 2.2.

$$
\mathrm{P}\left(\bigcap_{\rho=2}^{r(n)}\left\{\left|\frac{X_{n, \rho}}{\mathrm{E}\left[X_{n, \rho}\right]}-1\right| \leq n^{-1 / 6+\varepsilon}\right\}\right)=1-O\left(n^{-2 \varepsilon}\right), \quad \forall \varepsilon \in(0,1 / 6) .
$$

Consequently, $\min _{r \in[2, r(n)]} X_{n, r} \rightarrow \infty$ in probability, and so

$$
\lim _{n \rightarrow \infty} \mathrm{P}\left(L_{n} \geq 2 n^{1 / 3}\right)=1
$$

Proof. This time we use the second-order moment approach. By Theorem 2.1. for a generic set $I$ of cardinality $r \in[2, r(n)]$ we have

$$
\mathrm{E}\left[X_{n, r}\right]=\binom{n}{r} \mathrm{P}\left(D_{I}\right) \geq \frac{n^{r}}{2 r!}\left(\frac{2}{r+2}\right)^{r} \geq \text { const } n^{2} .
$$

The total number of ordered pairs $\left\{I_{1}, I_{2}\right\}$, with $\left|I_{1}\right|=\left|I_{2}\right|=r,\left|I_{1} \cap I_{2}\right|=k$, is

$$
\mathcal{N}(r, k)=\binom{n}{r}\binom{r}{k}\binom{n-r}{r-k} .
$$

Therefore, for a pair of generic sets $I_{1}, I_{2}$ meeting the conditions above,

$$
\begin{equation*}
\mathrm{E}\left[\left(X_{n, r}\right)_{2}\right]=\sum_{k=0}^{r-1} \mathcal{N}(r, k) \mathrm{P}\left(D_{I_{1}} \cap D_{I_{2}}\right) . \tag{2.12}
\end{equation*}
$$

Here $\mathrm{P}\left(D_{I_{1}} \cap D_{I_{2}}\right)=O(\mathrm{P}(r, k))$, with $\mathrm{P}(r, k)$ given in (2.9). After some elementary computations we obtain that

$$
\begin{gathered}
\max _{k \in[1, r-1]} \frac{\mathcal{N}(r, k+1) \mathrm{P}(r, k+1)}{\mathcal{N}(r, k) \mathrm{P}(r, k)} \\
\leq \frac{r^{2}}{2} \max _{k \in[1, r-1]} \frac{(r-k)^{2}}{(k+1)(2 r-k-1)(n-2 r+k+1)} \cdot \exp \left(\frac{k-1}{2 r-k-1}\right) \\
=\frac{r^{2}(r-1)}{8(n-2 r+2)} \leq e^{8 n^{-2 / 3}} ;
\end{gathered}
$$

(the second line maximum is attained at $k=1$ ). Consequently

$$
\begin{aligned}
\sum_{k=1}^{r-1} \mathcal{N}(r, k) \mathrm{P}(r, k) & \leq r e^{8 r n^{-2 / 3}} \mathcal{N}(r, 1) \mathrm{P}(r, 1) \leq 2 r \mathcal{N}(r, 1) \mathrm{P}(r, 1) \\
& \leq 2\binom{n}{r}\binom{n-r}{r-1}\left(\frac{2}{r}\right)^{2 r} \\
& =O\left(n^{-1} \mathcal{N}(r, 0) \mathrm{P}^{2}\left(D_{I}\right)\right)=O\left(\frac{r}{n} \mathrm{E}^{2}\left[X_{n, r}\right]\right)
\end{aligned}
$$

(For the last equality we used the lower bound for $\mathrm{P}\left(D_{I}\right)$ in Theorem 2.1.) Therefore, uniformly for $r \in[2, r(n)]$,

$$
\begin{equation*}
\frac{\sum_{k=1}^{r-1} \mathcal{N}(r, k) \mathrm{P}(r, k)}{\mathrm{E}^{2}\left[X_{n, r}\right]}=O\left(n^{-2 / 3}\right) \tag{2.13}
\end{equation*}
$$

From the equations (2.12) and (2.13), and $\mathrm{E}\left[X_{n, r}\right] \geq$ const $n^{2} \gg n^{2 / 3}$, we have

$$
\frac{\mathrm{E}\left[\left(X_{n, r}\right)_{2}\right]}{\mathrm{E}^{2}\left[X_{n, r}\right]}=1+O\left(n^{-2 / 3}\right) \Longrightarrow \frac{\operatorname{Var}\left(X_{n, r}\right)}{\mathrm{E}^{2}\left[X_{n, r}\right]}=O\left(n^{-2 / 3}\right) .
$$

By Chebyshev's inequality,

$$
\mathrm{P}\left(\left|\frac{X_{n, r}}{\mathrm{E}\left[X_{n, r}\right]}-1\right| \leq \delta\right) \geq 1-O\left(\delta^{-2} n^{-2 / 3}\right) \rightarrow 1
$$

uniformly for all $\delta \gg n^{-1 / 3}$ and $r \in[2, r(n)]$. Therefore

$$
\sum_{r=2}^{r(n)} \mathrm{P}\left(\left|\frac{X_{n, r}}{\mathrm{E}\left[X_{n, r}\right]}-1\right| \geq \delta\right)=O\left(\delta^{-2} n^{-1 / 3}\right) \rightarrow 0
$$

which implies: for $\varepsilon \in(0,1 / 6)$,

$$
\mathrm{P}\left(\bigcap_{r=2}^{r(n)}\left\{\left|\frac{X_{n, r}}{\mathrm{E}\left[X_{n, r}\right]}-1\right| \leq n^{-1 / 6+\varepsilon}\right\}\right)=1-O\left(n^{-2 \varepsilon}\right) .
$$

2.4. Estimate of $\mathbf{P}\left(\mathcal{D}_{I}\right)$. Recall that the event $\mathcal{D}_{I}$ happens iff $I$ is a $K$-set and no $J \subset I$, with $|J| \geq 2$, supports a local equilibrium $\mathbf{p}=\left\{p_{i}\right\}_{i \in J}>\mathbf{0}$, $\left(\sum_{i \in J} p_{i}=1\right)$.

Let the event $D_{I}$ holds, so that $f_{u, v} \geq\left(f_{u, u}+f_{v, v}\right) / 2$ for all $u, v \in I$. So $D_{J}$ holds for every $J \subseteq I$. Suppose that for some $i \neq j$ in $I$ we have $f_{i, j}>\max \left\{f_{i, i}, f_{j, j}\right\}$. Set $J=\{i, j\}$ and

$$
p_{i}:=\frac{f_{i, j}-f_{j, j}}{2 f_{i, j}-f_{i, i}-f_{j, j}}>0, \quad p_{j}=\frac{f_{i, j}-f_{i, i}}{2 f_{i, j}-f_{i, i}-f_{j, j}}>0
$$

Then $\mathbf{p}=\left(p_{i}, p_{j}\right)$ is a non-trivial local equilibrium, and this cannot happen on the event $\mathcal{D}_{I}$. Thus

$$
\mathcal{D}_{I} \subseteq \bigcap_{(i \neq j): i, j \in I}\left\{\frac{f_{i, i}+f_{j, j}}{2} \leq f_{i, j} \leq \max \left\{f_{i, i}, f_{j, j}\right\}\right\}
$$

Consequently we obtain

$$
\begin{equation*}
\mathrm{P}\left(\mathcal{D}_{I} \mid f_{i, i}=x_{i}, i \in I\right) \leq \prod_{(i \neq j): i, j \in I}\left[F\left(\max \left\{x_{i}, x_{j}\right\}\right)-F\left(\frac{x_{i}+x_{j}}{2}\right)\right] \tag{2.14}
\end{equation*}
$$

Introduce $y_{i}=F\left(x_{i}\right)$, i.e. $x_{i}=F^{-1}\left(y_{i}\right),(i \in I)$. Then $F\left(\max \left\{x_{i}, x_{j}\right\}\right)=$ $\max \left\{y_{i}, y_{j}\right\}$, and (since $F^{-1}(y)$ is convex),

$$
\begin{aligned}
F\left(\frac{x_{i}+x_{j}}{2}\right) & =F\left(\frac{F^{-1}\left(y_{i}\right)+F^{-1}\left(y_{j}\right)}{2}\right) \\
& \geq F\left(F^{-1}\left(\frac{y_{i}+y_{j}}{2}\right)\right)=\frac{y_{i}+y_{j}}{2}
\end{aligned}
$$

Therefore

$$
\mathrm{P}\left(\mathcal{D}_{I} \mid f_{i, i}=x_{i}, i \in I\right) \leq \prod_{(i \neq j): i, j \in I} \frac{\left|y_{i}-y_{j}\right|}{2}
$$

implying

$$
\begin{equation*}
\mathrm{P}\left(\mathcal{D}_{I}\right) \leq 2^{-r(r-1) / 2} \int_{\mathbf{y} \in[0,1]^{r}} \cdots \int_{(i \neq j): i, j \in I} \prod_{i}\left|y_{i}-y_{j}\right| d \mathbf{y}, \quad r:=|I| \tag{2.15}
\end{equation*}
$$

Since the integral is below 1, we see that

$$
\begin{equation*}
\mathrm{P}\left(\mathcal{D}_{I}\right) \leq 2^{-r(r-1) / 2} \tag{2.16}
\end{equation*}
$$

Hence
Corollary 2.9. With probability $\geq 1-n^{-a}$, $(\forall a>0)$, there is no $K$-set of cardinality $\geq r_{n}:=\left\lceil 2 \log _{2} n\right\rceil$, that contains, properly, the support of a non-trivial local equilibrium.

Proof. By (2.16) the expected number of $K$-sets in question is, at most, of order

$$
\binom{n}{r_{n}} 2^{-r_{n}^{2} / 2} \leq \frac{1}{r_{n}!} \leq n^{-a}, \quad \forall a>0 .
$$

We can do better though. The integral in (2.15) is a special case of Selberg's remarkable integral, [1], Section 8.1: in particular, for $\alpha>0, \beta>$ $0, \gamma \geq 0$,

$$
\begin{align*}
& \int_{\mathbf{y} \in[0,1]^{r}} \cdots \prod_{i \in I}\left\{y_{i}^{\alpha-1}\left(1-y_{i}\right)^{\beta-1}\right\} \prod_{(i \neq j): i, j \in I}\left|y_{i}-y_{j}\right|^{2 \gamma} d \mathbf{y} \\
= & \prod_{j=1}^{r} \frac{\Gamma(\alpha+(j-1) \gamma) \Gamma((\beta+(j-1) \gamma) \Gamma(1+j \gamma)}{\Gamma(\alpha+\beta+(r+j-2) \gamma) \Gamma(1+\gamma)} . \tag{2.17}
\end{align*}
$$

So we have

$$
\mathrm{P}\left(\mathcal{D}_{I}\right) \leq 2^{-r(r-1) / 2} \mathcal{S}(r), \quad \mathcal{S}(r):=\prod_{j=1}^{r} \frac{\Gamma^{2}(1+(j-1) / 2) \Gamma(1+j / 2)}{\Gamma(1+(r+j) / 2) \Gamma(3 / 2)} .
$$

Using the Stirling formula

$$
\Gamma(1+z)=\sqrt{2 \pi z}\left(\frac{z}{e}\right)^{z}\left(1+O\left(z^{-1}\right)\right), \quad z \rightarrow \infty
$$

and applying the Euler summation formula to the logarithm of the resulting product, one can show that, for some constants $\eta_{1}, \eta_{2}$,

$$
\begin{equation*}
\mathcal{S}(r)=2^{-r^{2}} \exp \left(\eta_{1} r \log r+\eta_{2} r+O(\log r)\right) \tag{2.18}
\end{equation*}
$$

We have proved
Lemma 2.10. There exist constants $\eta_{1}^{*}, \eta_{2}^{*}$ such that

$$
\mathrm{P}\left(\mathcal{D}_{I}\right) \leq 2^{-\frac{3}{2} r^{2}} \exp \left(\eta_{1}^{*} r \log r+\eta_{2}^{*} r+O(\log r)\right), \quad r:=|I| .
$$

So $\mathrm{P}\left(\mathcal{D}_{I}\right)$ is of order $2^{-\frac{3(1+o(1))}{2} r^{2}}$, at most. This leads immediately to a better upper bound fot the maximum size of a $K$-set free of supports of local equilibriums.

Theorem 2.11. With probability $\geq 1-\exp \left(-\Theta\left(\varepsilon \log ^{2} n\right)\right)$, there is no $K$-set of cardinality $\geq r_{n}^{*}:=\left\lceil(2 / 3+\varepsilon) \log _{2} n\right\rceil$, that properly contains the support of a non-trivial local equilibrium.

The sharp formula (2.18) allows us to show that with high probability there exist many $K$ sets of the logarithmic size that do not contain the size 2 supports of local equilibriums in the case when $f_{i, j}$ are uniform.

Given a set $I,|I| \geq 3$, let $\mathcal{D}_{I}^{*}$ be the event that $I$ is a $K$-set meeting the above, less stringent, requirement. For brevity, we call such $I$ a $K^{*}$-set. Instead of the inequality (2.14), here we have the equality

$$
\begin{equation*}
\mathrm{P}\left(\mathcal{D}_{I}^{*} \mid f_{i, i}=x_{i}, i \in I\right)=\prod_{(i \neq j): i, j \in I}\left[F\left(\max \left\{x_{i}, x_{j}\right\}\right)-F\left(\frac{x_{i}+x_{j}}{2}\right)\right] . \tag{2.19}
\end{equation*}
$$

For the uniform fitnesses the RHS in (2.19) is the product of the $\left|x_{i}-x_{j}\right| / 2$. So, by (2.17) and (2.18),

$$
\begin{equation*}
\mathrm{P}\left(\mathcal{D}_{I}^{*}\right)=2^{-\frac{3}{2} \rho^{2}} \exp \left(\eta_{1}^{*} \rho \log \rho+\eta_{2}^{*} \rho+O(\log \rho)\right), \quad \rho:=|I| . \tag{2.20}
\end{equation*}
$$

Let $X_{n, r}^{*}$ denote the total number of the $K^{*}$-sets of cardinality $r$. Then the expected number of the $K^{*}$-sets of cardinality $r$ is $\mathrm{E}\left[X_{n, r}^{*}\right]=\binom{n}{r} \mathrm{P}\left(\mathcal{D}_{I}^{*}\right)$, $(|I|=r)$. This expectation is easily shown to be of order $\geq \exp \left(\Theta\left(\varepsilon \log ^{2} n\right)\right)$, thus super-polynomially large, if $r=\left[(2 / 3)(1-\varepsilon) \log _{2} n\right], \varepsilon \in(0,1)$. In fact, we are about to prove that $X_{n, r}^{*}$ is likely to be this large if $r<0.5 \log _{2} n$.

Theorem 2.12. For $r=\left[(0.5-\varepsilon) \log _{2} n\right],(\varepsilon<1 / 4)$, we have

$$
\mathrm{P}\left(X_{n, r}^{*} \geq \exp \left(\Theta\left(\varepsilon \log ^{2} n\right)\right)\right) \geq 1-O\left(n^{-2 \varepsilon+O(\log \log n /(\log n))}\right) .
$$

Proof. We use the proof of Theorem 2.8 as a rough template. Given $0 \leq$ $k \leq r-1$, let

$$
I_{1}=I_{1}(k) \equiv\{1, \ldots, r\}, \quad I_{2}=I_{2}(k)=\{r-k+1, \ldots, 2 r-k\} ;
$$

so $\left|I_{1}\right|=\left|I_{2}\right|=r$ and $\left|I_{1} \cap I_{2}\right|=k$. Then, by symmetry,
$\mathrm{E}\left[\left(X_{n, r}^{*}\right)_{2}\right]=\sum_{k=0}^{r-1} \mathcal{N}(r, k) \mathrm{P}\left(\mathcal{D}_{I_{1}(k)}^{*} \cap \mathcal{D}_{I_{2}(k)}^{*}\right), \quad \mathcal{N}(r, k)=\binom{n}{r}\binom{r}{k}\binom{n-r}{r-k}$.
To bound $\mathrm{P}\left(\mathcal{D}_{I_{1}(k)}^{*} \cap \mathcal{D}_{I_{2}(k)}^{*}\right)$, observe that, denoting by $(i, j)$ a generic, unordered pair $(i \neq j)$, we have

$$
\begin{gathered}
\mathrm{P}\left(\mathcal{D}_{I_{1}(k)}^{*} \cap \mathcal{D}_{I_{2}(k)}^{*} \mid f_{i, i}=x_{i}, i \in I_{1} \cup I_{2}\right)=\quad \prod_{\substack{(i, j) \\
i, j \in[1, r]\lfloor[r-k+1,2 r-k]}} \frac{\left|x_{i}-x_{j}\right|}{2} \\
\leq 2^{-(r)_{2}+\binom{k}{2}} \prod_{\substack{(i, j) \\
i, j \in[1, r-k]}}\left|x_{i}-x_{j}\right| \\
\times \prod_{\substack{(i, j) \\
i, j \in[r-k+1, r]}}\left|x_{i}-x_{j}\right| \prod_{\substack{(i, j) \\
i, j \in[r+1,2 r-k]}}\left|x_{i}-x_{j}\right| .
\end{gathered}
$$

Unconditioning and using (2.19), we obtain:

$$
\begin{align*}
& \mathrm{P}\left(\mathcal{D}_{I_{1}(k)}^{*} \cap \mathcal{D}_{I_{2}(k)}^{*}\right)=\mathcal{P}^{*}(r, k) e^{O(\log r)},  \tag{2.21}\\
\mathcal{P}^{*}(r, k):= & 2^{-(r)_{2}+\binom{k}{2}} \cdot 2^{-2(r-k)^{2}-k^{2}} \\
& \times \exp \left[2 \eta_{1}^{*}(r-k) \log (r-k)+\eta_{1}^{*} k \log k+2 \eta_{2}^{*}(r-k)+\eta_{2}^{*} k\right] .
\end{align*}
$$

It follows that

$$
\begin{aligned}
\frac{\mathcal{N}(r, k+1) \mathrm{P}\left(\mathcal{D}_{I_{1}(k+1)}^{*} \cap \mathcal{D}_{I_{2}(k+1)}^{*}\right)}{\mathcal{N}(r, k) \mathrm{P}\left(\mathcal{D}_{I_{1}(k)}^{*} \cap \mathcal{D}_{I_{2}(k)}^{*}\right)} & \leq \frac{2^{2 r}}{n} \exp (O(\log r)) \\
& \leq n^{-2 \varepsilon+o(1)} \rightarrow 0
\end{aligned}
$$

since $r \leq(0.5-\varepsilon) \log _{2} n$. Consequently

$$
\frac{\mathrm{E}\left[\left(X_{n, r}^{*}\right)_{2}\right]}{\mathcal{N}(r, 0) \mathrm{P}^{2}\left(\mathcal{D}_{I_{1}(0)}^{*}\right)} \leq 1+n^{-2 \varepsilon+o(1)}
$$

Since

$$
\begin{aligned}
& \mathcal{N}(r, 0)=\left(1+O\left(r^{2} / n\right)\right)\binom{n}{r}^{2} \\
& \mathrm{E}\left[X_{n, r}^{*}\right]=\mathrm{P}\left(\mathcal{D}_{I_{1}^{*}}(0)\right)\binom{n}{r} \geq \exp \left(\Theta\left(\log ^{2} n\right)\right),
\end{aligned}
$$

the Chebyshev inequality completes the proof.
Acknowledgment. About thirty years ago John Kingman gave a lecture on stable polymorphisms at Stanford University. The talk made a deep, lasting impression on me. At that time Don Knuth introduced me to his striking formula for the expected number of stable matchings via a highlydimensional integral, [15]. Despite the world of difference between the stable polymorphisms and the stable matchings, the multidimensional integrals expressing the probability of respective stability conditions are of a similar kind.

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[^0]:    Date: October 7, 2018.
    2010 Mathematics Subject Classification. 34E05, 60C05.
    Key words and phrases. stable polymorphisms, random fitnesses, asymptotics .

