COMPARISON OF HIT-AND-RUN, SLICE SAMPLER AND RANDOM WALK METROPOLIS

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Abstract

Different Markov chains can be used for approximate sampling of a distribution given by an unnormalized density function with respect to the Lebesgue measure. The hit-and-run, (hybrid) slice sampler and random walk Metropolis algorithm are popular tools to simulate such Markov chains. We develop a general approach to compare the efficiency of these sampling procedures by the use of a partial ordering of their Markov operators, the covariance ordering. In particular, we show that the hit-and-run and the simple slice sampler are more efficient than a hybrid slice sampler based on hit-and-run which, itself, is more efficient than a (lazy) random walk Metropolis algorithm.

Keywords: hit-and-run; slice sampler; random walk Metropolis; covariance ordering

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1. Introduction

In many scenarios of Bayesian statistics, statistical physics and other branches of applied sciences, see [33, 17, 4], it is of interest to sample on \mathbb{R}^d with respect to a distribution π . In particular, we assume that π is given by an unnormalized density. More precisely, let $K \subseteq \mathbb{R}^d$ be an open, measurable set and $\rho: K \to (0, \infty)$ be a positive, bounded and with respect to the Lebesgue measure an almost everywhere continuous function with $\int_K \rho(x) dx \in (0, \infty)$. Then, define the probability measure π

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through ρ by

$$\pi(A) = \frac{\int_A \rho(x) \, \mathrm{d}x}{\int_K \rho(x) \, \mathrm{d}x}$$

for all measurable $A \subseteq K$.

Maybe the most successful approach to approximate π is the construction of a suitable Markov chain. The hit-and-run algorithm, the random walk Metropolis, the simple slice sampler and hybrid slice sampler provide such construction methods. A crucial question is: Which one of these algorithm should be used?

This question is of course related to the speed of convergence of the Markov chain sampling and any answer depends very much on the imposed assumptions. In general it is difficult to derive explicit estimates of this speed of convergence. But, it might be possible to prove that one algorithm is better than another. This motivates the idea of the comparison of Markov chains.

The first comparison result of Markov chains is due to Peskun [29]. There, a partial ordering on finite state spaces is invented, where one transition kernel has higher order than another one if the former dominates the latter off the diagonal. This order was later extended by Tierney [39] to general state spaces. However, for the Markov chains we have in mind it seems not possible to use this off-diagonal ordering. We consider a partial ordering on the set of linear operators, see [12, p. 470]. In the context of Markov chains this ordering is called covariance ordering, see [20, 21]. Let $L_2(\pi)$ be the Hilbert space of functions with finite stationary variance and assume that $P_1, P_2: L_2(\pi) \to L_2(\pi)$ are two self-adjoint linear operators. Then we say, $P_1 \leq P_2$ if and only if

$$\langle P_1 f, f \rangle_{\pi} \ge \langle P_2 f, f \rangle_{\pi}, \quad f \in L_2(\pi).$$

Here the inner-product of $L_2(\pi)$ is given by

$$\langle f_1, f_2 \rangle_{\pi} = \int_K f_1(x) f_2(x) \, \mathrm{d}\pi(x), \quad f_1, f_2 \in L_2(\pi)$$

The operator P_1 is called positive if $P_1 \ge 0$, that is, $\langle P_1 f, f \rangle_{\pi} \ge 0$ for any $f \in L_2(\pi)$. Reversible transition kernels of given Markov chains induce self-adjoint Markov operators and we can compare these operators.

Let P_1, P_2 be two of such Markov operators and assume that $0 \le P_1 \le P_2$. There are a number of consequences for the corresponding Markov chains: For example, the

spectral gap of P_2 is smaller than that of P_1 . The spectral gap of a Markov chain is a quantity which is closely related to the speed of convergence to π , see [3, 36]. Another example is concerning the stationary asymptotic variance of sample averages. For i = 1, 2 let $S_n^{(i)}(f) = \frac{1}{n} \sum_{k=1}^n f(X_k^{(i)})$ with Markov chain $(X_k^{(i)})_{k \in Nature}$ starting in stationarity corresponding to P_i and an arbitrary function $f \in L_2(\pi)$. These sample averages give approximations of the mean $Ef = \int_K f \, d\pi$ and, in [39, 20] it is observed that, if $P_1 \leq P_2$, then

$$\mathbb{V}(f, P_2) \le \mathbb{V}(f, P_1).$$

Here $\mathbb{V}(f, P_i) := \lim_{n \to \infty} n \cdot \mathbb{E} |S_n^{(i)}(f) - Ef|^2$ is the stationary asymptotic variance of $S_n^{(i)}(f)$. In other words, the Markov chain of P_2 is (asymptotically) more efficient than that of P_1 . For more details to implications of $P_1 \leq P_2$ we refer to Section 2.1.1.

The goal of this article is to compare the hit-and-run algorithm, the (lazy) random walk Metropolis, the simple slice sampler and a hybrid slice sampler based on hit-andrun according to this partial ordering. To do so we develop a systematic approach for the comparison of Markov chains which can be written by a suitable two step procedure.

The intuition behind our approach is the following. It is a simple and well-known observation that, if the self-adjoint and positive Markov operators P_1 and P_2 satisfy $P_1P_2 = P_2$, then $P_1 \leq P_2$. However, due to the quite restrictive assumptions, this technique can only be used in very few cases. A useful generalization of this technique, which was invented in a special case in [41], can be used whenever we have the representation $P_i = R\tilde{P}_i R^*$ for certain operators R and \tilde{P}_i , i = 1, 2. In this case, and if $\tilde{P}_1\tilde{P}_2 = \tilde{P}_2$, one can also conclude that $P_1 \leq P_2$, see Lemma 1. Our comparison inequalities therefore follow once we have established such representations for the Markov chains under consideration.

The algorithms

Let us briefly explain the algorithms. Roughly, a transition of the hit-and-run algorithm works as follows. Choose randomly a line through the current state and sample according to π restricted to this line. Thus, instead of sampling with respect to π on $K \subseteq \mathbb{R}^d$ hit-and-run only uses sampling with respect to π on 1-dimensional lines through the state space, which is feasible in many cases .

In contrast, the simple slice sampler chooses a suitable d-dimensional set, a superlevel set of ρ , depending on the current state and then, samples the next state of the Markov chain uniformly distributed on this super-level set. Sampling of the uniform distribution on a d-dimensional set is often not efficiently implementable. This is why this Markov chain is mostly of theoretical interest.

The hybrid slice sampler we are interested in overcomes this problem by replacing the uniform sampling by one step of a hit-and-run algorithm on the super-level set: First, choose a line through the current state uniformly at random and then generate the next state uniformly distributed on the intersection of that line with the super-level set. We call this procedure hybrid slice sampler based on hit-and-run. Intuitively, it is clear that the simple slice sampler is better than that hybrid one. The intuition for the comparison of the hit-and-run algorithm and the hybrid slice sampler based on hit-and-run is not so obvious. Observe that this particular hybrid sampler can also be interpreted as choosing first a line and, then, performing a simple slice sampling step according the distribution of π restricted to that line. This observation leads us to the fact that the hit-and-run algorithm is better.

Finally, let us consider the random walk Metropolis. Assume that we have a proposal density q on \mathbb{R}^d and let $x \in K$ be the current state. A transition works as follows: Generate $z \in \mathbb{R}^d$ according the distribution determined by q and accept x + z as the next state with probability min $\{1, \rho(x + z)/\rho(x)\}$ if $x + z \in K$. Otherwise stay at x. It is well known, see [10, 35], that the random walk Metropolis can be interpreted as a certain (hybrid) slice sampling procedure, which runs a random walk according to q on the super-level set with uniform limit distribution. We want to compare the random walk Metropolis and the hybrid slice sampler based on a hit-and-run. Thus, the question is whether the uniform hit-and-run step is better than the random walk step on the super-level set. It turns out that this is indeed the case.

For example, if ρ is a log-concave density, then the restriction of π to a line has also a log-concave density and one can use different acceptance/rejection methods to sample such a distribution on the line efficiently. For details we refer to [7, Sect. 2.4.2] see also [19, 37].

Main results

Now let us formulate the main results. To compare the above Markov chains we develop in Section 3 a general approach which might be of interest on its own. There, in Lemma 1 conditions for two suitably defined Markov operators P_1, P_2 are stated which imply that $P_1 \leq P_2$. This lemma is the main ingredient for the comparison argument. Its application leads us to Theorem 5. For the Markov operators M, U, H, S of the (lazy) random walk Metropolis with rotational invariant proposal q, the hybrid slice sampler based on hit-and-run, the hit-and-run algorithm and the simple slice sampler, respectively, we have

$$\begin{split} M &\leq U \leq H, \\ M &\leq U \leq S. \end{split}$$

Thus, the random walk Metropolis is less efficient than the hit-and-run algorithm and simple slice sampler. The hybrid slice sampler based on hit-and-run we propose lies concerning efficiency in between.

Outline

The paper is organized as follows. In the next section we introduce the notation we use, comment on the partial ordering, present consequences for the Markov chains and state the algorithms we study in detail. In Section 3 we invent a new approach how to compare Markov chains with a specific structure. Section 4 contains the application of the former developed comparison arguments. Finally, in Section 5, we give some concluding remarks and discuss open problems.

2. Preliminaries

Let $L_2(\pi)$ be the Hilbert space of functions $f: K \to \mathbb{R}$ with finite norm $||f||_{\pi} = \langle f, f \rangle_{\pi}^{1/2}$, where the inner-product of $f_1, f_2 \in L_2(\pi)$ is denoted by

$$\langle f_1, f_2 \rangle_{\pi} = \int_K f_1(x) f_2(x) \, \mathrm{d}\pi(x).$$

Let P be a transition kernel on K which is reversible with respect to π and let $(X_n)_{n \in Nature}$ be a Markov chain with transition kernel P, i.e.

$$P(x,A) = \mathbb{P}(X_{k+1} \in A \mid X_k = x)$$

almost surely for all $k \in Nature$ and $A \subseteq K$. The corresponding Markov operator, also denoted by P, is given by

$$Pf(x) = \int_{K} f(y)P(x, \mathrm{d}y) \tag{1}$$

for $f \in L_2(\pi)$. Obviously, $P(x, A) = P\mathbf{1}_A(x)$ for all $A \subseteq K$, where $\mathbf{1}_A$ denotes the indicator function of A. Note that, by the reversibility, $P : L_2(\pi) \to L_2(\pi)$ is self-adjoint. We say that a (Markov) operator P on $L_2(\pi)$ is *positive* if $\langle Pf, f \rangle_{\pi} \geq 0$ for all $f \in L_2(\pi)$.

2.1. On the ordering and consequences

With this notation we define on the set of Markov operators the following partial ordering. For Markov operators P_1 and P_2 we write

 $P_1 \leq P_2$

if and only if $\langle P_1 f, f \rangle_{\pi} \geq \langle P_2 f, f \rangle_{\pi}$ for all $f \in L_2(\pi)$. In the following let us motivate why the consideration of this ordering is particularly meaningful for Markov chains.

2.1.1. Consequences for the speed of convergence There are many ways to measure the speed of convergence of the distribution of a Markov chain towards its stationary distribution. Probably the most desirable quantity is the *total variation distance* of νP^n and π , i.e.

$$\|\nu P^n - \pi\|_{tv} = \sup_{A \subseteq K} |\nu P^n(A) - \pi(A)|,$$

where $\nu P^n(A) = \int_K P^n(x, A) d\nu(x)$ is the distribution of the Markov chain with transition kernel P and initial distribution ν after n steps. However, estimating the total variation distance is quite delicate and, in practice, it is usually much easier to derive bounds on it by more analytic quantities, like isoperimetric constants or certain norms of P, see e.g. [15, 18] and [23]. Many of these quantities are defined by

$$c_{\mathcal{M}}(P) = \inf_{f \in \mathcal{M}} \langle (I - P)f, f \rangle_{\pi}$$

for certain sets of functions $\mathcal{M} \subset L_2(\pi)$, where If := f. Hence, the proof of $P_1 \leq P_2$ is enough to obtain $c_{\mathcal{M}}(P_1) \leq c_{\mathcal{M}}(P_2)$ for every choice of \mathcal{M} . Prominent examples are the Comparison of Markov chains

- spectral gap: $\mathcal{M} = \{f : \|f\|_{\pi} = 1, \int_{K} f \, \mathrm{d}\pi = 0\}$
- conductance: $\mathcal{M} = \{ f \colon f = \frac{\mathbf{1}_A}{\sqrt{\pi(A)}}, \ \pi(A) \in (0, 1/2], \ A \subset K \}$
- log-Sobolev constant: $\mathcal{M} = \{f: \int_K f^2 \log(f^2/||f||_{\pi}^2) d\pi = 1\}.$

There are some more quantities of this form, like the best constant in a Nash inequality, average and blocking conductance. For details see e.g. [6] and [26].

In what follows, we will prove $P_1 \leq P_2$ for a couple of Markov operators and, hence, that P_2 is "faster" than P_1 in all the above senses.

2.1.2. Consequences for sample averages The property $P_1 \leq P_2$ has also consequences for the worst case mean square error and the asymptotic stationary variance of Markov chain Monte Carlo methods.

Let $(X_k^{(i)})_{k \in Nature}$ be a Markov chain with transition kernel P_i , i = 1, 2, and initial distribution π , and define the Markov chain Monte Carlo method

$$S_n^{(i)}(f) = \frac{1}{n} \sum_{k=1}^n f(X_k^{(i)}), \qquad i = 1, 2,$$

which gives an approximation to $Ef := \int_K f \, d\pi$ for $f \in L_2(\pi)$. By virtue of spectral theoretic arguments one can show that

$$\sup_{\|f\|_{\pi} \le 1} Econometrica \left| S_n^{(2)}(f) - Ef \right|^2 \le \sup_{\|f\|_{\pi} \le 1} Econometrica \left| S_n^{(1)}(f) - Ef \right|^2$$

Another interesting consequence is stated in [21, Theorem 6]. There it is proven that $P_1 \leq P_2$ if and only if

$$\mathbb{V}(f, P_2) \le \mathbb{V}(f, P_1),$$

for each individual $f \in L_2(\pi)$, where $\mathbb{V}(f, P_i) := \lim_{n \to \infty} n \cdot \mathbb{E}|S_n^{(i)}(f) - Ef|^2$ is the asymptotic stationary variance (which can also be considered as asymptotic mean square error).

2.2. The algorithms

We present the different algorithms we consider in detail and provide relevant literature.

One argues with Corollary 3.27 and Lemma 2.12 from [36], as well as the fact that the spectral gap of P_1 is smaller than that of P_2 .

2.2.1. *Hit-and-run algorithm* The hit-and-run algorithm proposed by Smith [38] is well studied in different settings, see [5, 8, 13, 15, 18, 14, 37].

Informally it samples at each step on a randomly chosen 1-dimensional line with respect to the corresponding conditional distribution. Let \mathbb{S}_{d-1} be the Euclidean unit sphere in \mathbb{R}^d . For $x \in K$ and $\theta \in \mathbb{S}_{d-1}$ we define

$$L(x,\theta) = \{x + s\theta \in K \mid s \in \mathbb{R}\}\$$

as the chord through x in direction θ . A transition from x to y of the hit-and-run algorithm works as follows: Generate a set $L(x, \theta)$ by choosing θ with the uniform distribution on the sphere and, then, choose $y \in L(x, \theta)$ according to the distribution determined by ρ conditioned on the chord $L(x, \theta)$. A transition of hit-and-run is given by Algorithm 1.

Algorithm 1. (Hit-and-run)

Transition from current state x to next state y:

- 1. Sample $\theta \sim \text{Uniform}(\mathbb{S}_{d-1})$.
- 2. Sample $y \sim H_{\theta}(x, \cdot)$, where

$$H_{\theta}(x,A) = \frac{\int_{L(x,\theta)} \mathbf{1}_{A}(z)\rho(z) \,\mathrm{d}z}{\int_{L(x,\theta)} \rho(z) \,\mathrm{d}z}$$

Note that the integral in H_{θ} is over a 1-dimensional subset of \mathbb{R}^d and the integration is with respect to the 1-dimensional Lebesgue measure. For $x \in K$ and $A \subseteq K$ the transition kernel, say H, of the hit-and-run algorithm is determined by

$$H(x, A) = \int_{\mathbb{S}_{d-1}} H_{\theta}(x, A) \,\mathrm{d}\sigma(\theta),$$

where $\sigma = \text{Uniform}(\mathbb{S}_{d-1})$ denotes the uniform distribution on the sphere. It is well known that this transition kernel is reversible with respect to π , see for example [5].

An important special case of the hit-and-run algorithm above is given if the density is an indicator function, for example $\rho = \mathbf{1}_K$. In this case, the hit-and-run algorithm is reversible with respect to the uniform distribution on K. Thus, under weak regularity assumptions, the uniform distribution is the (unique) stationary distribution, see [5]. We call this special case uniform hit-and-run. Let us mention that we use the uniform hit-and-run in the next section. 2.2.2. Simple and hybrid slice sampling Slice sampling belongs to the class of auxiliary variable algorithms that are defined by a Markov chain on an extended state space, see [10, 31, 22, 25, 33, 27] and the references therein.

We consider the simple slice sampler and a hybrid slice sampler based on hit-andrun. A single transition of the simple slice sampler is presented in Algorithm 2.

Algorithm 2. (Simple slice sampler)

Transition from current state x to next state y.

- 1. Sample $t \sim \text{Uniform}(0, \rho(x))$.
- 2. Sample $y \sim \text{Uniform}(K(t))$, where

$$K(t) = \{x \in K \mid \rho(x) > t\}$$

is the super-level set of ρ determined by t.

The transition kernel, say S, corresponding to Algorithm 2 is

$$S(x,A) = \frac{1}{\rho(x)} \int_0^{\rho(x)} \frac{\operatorname{vol}_d(A \cap K(t))}{\operatorname{vol}_d(K(t))} \,\mathrm{d}t,$$

where vol_d denotes the *d*-dimensional Lebesgue measure. The simple slice sampler exhibits quite robust convergence properties, see [31, 25]. However, a crucial drawback is that the second step is difficult to implement. Because of this we consider the following hybrid slice sampler based on hit-and-run. The idea is to replace the second step of the simple slice sampler by a Markov chain transition according to the uniform hit-and-run algorithm in K(t), see Algorithm 3.

Algorithm 3. (Hybrid slice sampler based on hit-and-run)

Transition from current state x to next state y.

- 1. Sample $t \sim \text{Uniform}(0, \rho(x))$ and $\theta \sim \text{Uniform}(\mathbb{S}_{d-1})$ independently.
- 2. Sample $y \sim \text{Uniform}(L_t(x,\theta))$, where

$$L_t(x,\theta) = \{x + r\theta \in K(t) \mid r \in \mathbb{R}\}\$$

is the chord through x in direction θ restricted to K(t).

The transition kernel, say U, of this hybrid slice sampler is given by

$$U(x,A) = \frac{1}{\rho(x)} \int_0^{\rho(x)} \int_{\mathbb{S}_{d-1}} \frac{|L_t(x,\theta) \cap A|}{|L_t(x,\theta)|} \,\mathrm{d}\sigma(\theta) \,\mathrm{d}t$$

where $|\cdot|$ here denotes the 1-dimensional Lebesgue measure. This modification is tempting since the uniform hit-and-run algorithm is, at least in some scenarios, implementable. For example, when the super-level sets are convex (ρ is quasi-conave) by a bisection method one can roughly approximate the intersection points of the line $x + r\theta$ with K(t) and use a 1-dimensional acceptance/rejection approach to sample the uniform distribution on $L_t(x, \theta)$. Further, the simple slice sampler and the hybrid slice sampler are reversible with respect to π , see for example in [16].

2.2.3. Random walk Metropolis The random walk Metropolis in \mathbb{R}^d provides an easy to implement method for Markov chain sampling. Further, it is well studied and different convergence results are known, see for example [34, 24, 11].

To guarantee that a certain operator is positive we consider a lazy version of a random walk Metropolis. Further, we assume in the following that $q: \mathbb{R}^d \to [0, \infty)$ is a rotational invariant probability density on \mathbb{R}^d , i.e. $q(r\theta_1) = q(r\theta_2)$ for $\theta_1, \theta_2 \in \mathbb{S}_{d-1}$. The rotational invariance guarantees that q is symmetric. Let us provide two examples which satisfy the rotational invariance.

Example 2.1. Let $x \in \mathbb{R}^d$, $\delta > 0$ and \mathbb{B}_d be the Euclidean unit ball with $\kappa_d := \operatorname{vol}_d(\mathbb{B}_d)$. Then we can set $q(x) := \mathbf{1}_{\mathbb{B}_d}(\delta^{-1}x)/(\delta^d\kappa_d)$ and the corresponding random walk Metropolis is known as δ -ball walk.

Example 2.2. Again, let $x \in \mathbb{R}^d$ and set $q(x) := \exp(-|x|^2/2)/(2\pi)^{d/2}$. The corresponding random walk Metropolis is known as Gaussian random walk.

In Example 2.1 the proposal q depends on a parameter δ . In connection to this we want to mention that in recent years optimal scaling results concerning a parameter of the proposal of the random walk Metropolis attracted a lot of attention, see [30, 32, 28].

Now a single transition of the (lazy) random walk Metropolis with proposal q is described in Algorithm 4.

Algorithm 4. (Random walk Metropolis)

Transition from current state x to next state y with proposal q.

- 1. Sample $z \sim q$ and $u_1, u_2 \sim \text{Uniform}(0, 1)$ independently.
- 2. If $x + z \in K$, $u_1 \le 1/2$ and $u_2 < \min\{1, \rho(x+z)/\rho(x)\}$ then accept the proposal, and set y := x + z, else reject the proposal and set y := x.

To simplify the notation we define the acceptance probability of a proposed state y = x + z as

$$\alpha(x,y) = \min\left\{1, \frac{\rho(y)}{\rho(x)}\right\}$$

for $x, y \in K$ and $\alpha(x, y) = 0$ otherwise. Then, the transition kernel is given by

$$M(x,A) = \frac{1}{2} \int_K \mathbf{1}_A(y) \,\alpha(x,y) \,q(y-x) \,\mathrm{d}y,$$

for $A \subseteq K$ with $x \notin A$ and $M(x, \{x\}) = 1 - M(x, K \setminus \{x\})$.

3. Auxiliary variable Markov chains

We develop a systematic approach how to compare Markov chains which can be described by a suitable two-step procedure. That many Markov chains are of this form was already observed in [2] and the idea of a comparison of this type was developed in [40, 41] in a specific setting.

For this let \mathcal{A} be an arbitrary (index) set equipped with a σ -finite measure λ . By assumption we have a function $s \colon K \times \mathcal{A} \to [0, \infty)$ which satisfies:

- for all $x \in K$ we have that $s(x, \cdot)$ is a probability density function according to λ ;
- for all $a \in \mathcal{A}$ we have that $s(\cdot, a)$ is integrable according to π .

Define for almost all $a \in \mathcal{A}$ (concerning λ) a probability measure π_a on K induced by s(x, a), i.e.

$$\pi_a(A) = \frac{\int_A s(x, a) \,\mathrm{d}\pi(x)}{\int_K s(x, a) \,\mathrm{d}\pi(x)}, \qquad A \subseteq K.$$

In addition, we assume that

- for every $a \in \mathcal{A}$ we have an equivalence relation \sim_a and by $[x]_a = \{y \in K : x \sim_a y\}$ we denote the equivalence class of K with respect to \sim_a to which x belongs;
- for $(\lambda$ -almost) every $a \in \mathcal{A}$ we have a transition kernel P_a on $(K, \mathcal{B}(K))$, such that $P_a(x, A) = 0$ for each $x \in K$ and $A \subseteq K \setminus [x]_a$.

With this we can define a Markov chain in K for which a single transition, starting from $x \in K$, can be written as the following procedure:

- 1) Sample $a \in \mathcal{A}$ according to the distribution with density $s(x, \cdot)$.
- 2) Generate the next state with respect to $P_a(x, \cdot)$.

That is, the transition kernel P is given by

$$P(x,A) = \int_{\mathcal{A}} P_a(x,A) s(x,a) \,\mathrm{d}\lambda(a) \tag{2}$$

for $A \subseteq K$.

Remark 1. Since the support of the measure $P_a(x, \cdot)$ is contained in $[x]_a$ and $[x]_a = [y]_a$ if $y \in [x]_a$, we can interpret a Markov chain with transition kernel P_a on K and initial state x as a Markov chain on $[x]_a$.

Clearly, for every Markov chain a transition can be written in this form. (Simply, set $P_a(x, \cdot) := P(x, \cdot)$ and take $[x]_a \equiv K$ for arbitrary \mathcal{A} and λ .) However, there might also be more interesting equivalence relations and scenarios as we illustrate for the hit-and-run transition kernel.

Example 3.1. Here let $\mathcal{A} := \mathbb{S}_{d-1}$ and $\lambda := \sigma$. Then, for every $a \in \mathcal{A}$ let $x \sim_a y$ if and only if $x \in L(y, a)$, such that $[x]_a := L(x, a)$. Further, for $(x, a) \in K \times \mathcal{A}$ set s(x, a) := 1, which implies $\pi_a = \pi$ for all $a \in \mathcal{A}$. For (λ -almost every) $a \in \mathcal{A}$ we have that

$$H_a(x,A) = \frac{\int_{[x]_a} \rho(y) \mathbf{1}_A(y) \, \mathrm{d}y}{\int_{[x]_a} \rho(y) \, \mathrm{d}y}, \qquad A \subseteq K,$$

is a transition kernel on K and we can write the Markov operator $H: L_2(\pi) \to L_2(\pi)$ of the hit-and-run algorithm in the form (2) as

$$Hf(x) = \int_{\mathcal{A}} \int_{K} f(y) H_{a}(x, \mathrm{d}y) \,\mathrm{d}\lambda(a).$$

In some cases it is even possible to represent the transitions of two different Markov chains in the form (2) with the same equivalence relations \sim_a and s(x, a), so that the corresponding "inner" transition kernels are much easier to analyze. We show that a suitable relation of these "inner" kernels is enough to compare the original Markov chains. **Lemma 1.** Assume that for $(\lambda$ -almost) all $a \in \mathcal{A}$ there are transition kernels $P_a^{(1)}, P_a^{(2)}$ on K such that

- 1. $P_a^{(1)}$ and $P_a^{(2)}$ are self-adjoint operators on $L_2(\pi_a)$;
- 2. $P_a^{(1)}$ is positive on $L_2(\pi_a)$;
- 3. $P_a^{(1)}P_a^{(2)} = P_a^{(2)}$.

Then, for the operators $P_1, P_2: L_2(\pi) \to L_2(\pi)$ defined by

$$P_i f(x) = \int_{\mathcal{A}} P_a^{(i)} f(x) \, s(x, a) \, \mathrm{d}\lambda(a), \quad i \in \{1, 2\}$$

holds $P_1 \leq P_2$.

Proof. First, we show that P_i can be written as a product of suitable operators. For this let μ be a probability measure on $K \times A$ that is given by

$$\mu(B) := \int_{K} \int_{\mathcal{A}} \mathbf{1}_{B}(x, a) \, s(x, a) \, \mathrm{d}\lambda(a) \, \mathrm{d}\pi(x) \tag{3}$$

for $B \subseteq K \times \mathcal{A}$, and let $L_2(\mu)$ be the Hilbert space of functions $g \colon K \times \mathcal{A} \to \mathbb{R}$ with finite norm $\|g\|_{\mu} = \langle g, g \rangle_{\mu}^{1/2}$, where the inner-product of $g, h \in L_2(\mu)$ is defined by

$$\langle g, h \rangle_{\mu} = \int_{K \times \mathcal{A}} g(x, a) h(x, a) d\mu(x, a) = \int_{K} \int_{\mathcal{A}} g(x, a) h(x, a) s(x, a) d\lambda(a) d\pi(x)$$

Further, let $R: L_2(\mu) \to L_2(\pi)$ be the operator that is given by

$$Rg(x) = \int_{\mathcal{A}} g(x,a) \, s(x,a) \, \mathrm{d}\lambda(a), \qquad g \in L_2(\mu).$$
(4)

Since

$$\langle f, Rg \rangle_{\pi} = \int_{K} \int_{\mathcal{A}} f(x) g(x, a) s(x, a) d\lambda(a) d\pi(x)$$

we obtain that the adjoint operator $R^* \colon L_2(\pi) \to L_2(\mu)$ satisfies

$$R^*f(x,a) = f(x), \qquad f \in L_2(\pi),$$
 (5)

for $a \in \mathcal{A}$ and $x \in K$. It is useful to write $g_a(x) = g(x, a)$ for $g \in L_2(\mu)$ and fixed $a \in \mathcal{A}$. Clearly, $g_a \in L_2(\pi_a)$ for almost every $a \in \mathcal{A}$ (with respect to λ), such that $P_a^{(i)}g_a$ is well-defined. Let

$$\widetilde{P}_i g(x,a) = \int_K g(y,a) P_a^{(i)}(x,\mathrm{d}y), \qquad g \in L_2(\mu), \tag{6}$$

and note that the operator satisfies $\widetilde{P}_i: L_2(\mu) \to L_2(\mu)$. An immediate consequence of the definitions is

$$P_i = R\widetilde{P}_i R^*, \qquad i \in \{1, 2\}.$$

$$\tag{7}$$

We prove different properties of \tilde{P}_1 and \tilde{P}_2 . For $i \in \{1, 2\}$ we show that \tilde{P}_i is selfadjoint on $L_2(\mu)$. Note that by the assumptions we know that $P_a^{(i)}$ is self-adjoint on $L_2(\pi_a)$. With

$$C_a = \int_K s(x, a) \,\mathrm{d}\pi(x)$$

we have for $g, h \in L_2(\mu)$ that

$$\begin{split} \langle \widetilde{P}_i g, h \rangle_{\mu} &= \int_K \int_{\mathcal{A}} \int_K g(y, a) \, P_a^{(i)}(x, \mathrm{d}y) \, h(x, a) \, s(x, a) \, \mathrm{d}\lambda(a) \, \mathrm{d}\pi(x) \\ &= \int_{\mathcal{A}} C_a \int_K \int_K g(y, a) \, P_a^{(i)}(x, \mathrm{d}y) \, h(x, a) \, \mathrm{d}\pi_a(x) \, \mathrm{d}\lambda(a) \\ &= \int_{\mathcal{A}} C_a \langle P_a^{(i)} g_a, h_a \rangle_{\pi_a} \, \mathrm{d}\lambda(a) = \int_{\mathcal{A}} C_a \langle g_a, P_a^{(i)} h_a \rangle_{\pi_a} \, \mathrm{d}\lambda(a) \\ &= \langle g, \widetilde{P}_i h \rangle_{\mu}. \end{split}$$

By the same line of arguments, for $g \in L_2(\mu)$ we have

$$\langle \widetilde{P}_1 g, g \rangle_{\mu} = \int_{\mathcal{A}} C_a \langle P_a^{(1)} g_a, g_a \rangle_{\pi_a} \, \mathrm{d}\lambda(a) \ge 0,$$

such that \widetilde{P}_1 preserves the positivity of $P_a^{(1)}$, i.e. \widetilde{P}_1 is positive on $L_2(\mu)$.

Further, for $g \in L_2(\mu)$ we obtain

$$\widetilde{P}_1 \widetilde{P}_2 g(x, a) = P_a^{(1)} P_a^{(2)} g_a(x) = P_a^{(2)} g_a(x) = \widetilde{P}_2 g(x, a).$$

By the self-adjointness of \widetilde{P}_1 and \widetilde{P}_2 we also have $\widetilde{P}_2\widetilde{P}_1 = \widetilde{P}_2$.

Now we gathered all tools together to prove the assertion. By the positivity, we know that \tilde{P}_1 has a unique positive square root N, i.e. $N^2 = \tilde{P}_1$. It is well known that N commutes with every operator that commutes with \tilde{P}_1 , see e.g. [12, Theorem 9.4-2], in particular $N\tilde{P}_2 = \tilde{P}_2N$. We obtain

$$\begin{split} \langle P_2 f, f \rangle_{\pi} &= \langle R P_2 R^* f, f \rangle_{\pi} = \langle P_2 R^* f, R^* f \rangle_{\mu} \\ &= \langle \widetilde{P}_1 \widetilde{P}_2 R^* f, R^* f \rangle_{\mu} = \langle \widetilde{P}_2 N R^* f, N R^* f \rangle_{\mu} \\ &\leq \langle N R^* f, N R^* f \rangle_{\mu} = \langle \widetilde{P}_1 R^* f, R^* f \rangle_{\mu} = \langle P_1 f, f \rangle_{\tau} \end{split}$$

where the inequality comes from $\|\widetilde{P}_2\| \leq 1$, which is true since \widetilde{P}_2 is a Markov operator.

Thus, for each of the intended comparisons, say between P_1 and P_2 , we have to

• find representations

$$P_i f(x) = \int_{\mathcal{A}} P_a^{(i)} f(x) s(x, a) \,\mathrm{d}\lambda(a), \quad i \in \{1, 2\};$$

• check the assumptions 1.-3. of Lemma 1, i.e. reversibility of $P_a^{(i)}$ with respect to π_a , positivity of $P_a^{(1)}$ and $P_a^{(1)}P_a^{(2)} = P_a^{(2)}$.

Remark 2. Let us comment on the necessity and possible generalizations of the assumptions in Lemma 1. The second assumption, i.e., the positivity of $P_a^{(1)}$, is most likely an artifact of the proof technique and we conjecture that the result holds without this assumption. However, in the proof it is essential and we were not able to remove it. The same problem already appeared in [40, 41]. The third assumption of the lemma says, roughly speaking, that a step of the Markov chain with corresponding operator $P_a^{(1)}$ "cannot be seen" if followed by $P_a^{(2)}$. As the last steps of the proof show, this could be replaced, e.g., by the weaker assumption that $\tilde{P}_1 \leq \tilde{P}_2$, i.e., $\langle \tilde{P}_2 g, g \rangle_{\mu} \leq \langle \tilde{P}_1 g, g \rangle_{\mu}$ for all $g \in L_2(\mu)$. However, at least in the examples we consider, the relatively easy-to-check assumption 3. is already fulfilled.

4. Main result

In this section we apply Lemma 1 to prove the following theorem.

Theorem 5. Let M, U, H, S be the Markov operators of the (lazy) random walk Metropolis with rotational invariant proposal q, the hybrid slice sampler based on hit-and-run, the hit-and-run algorithm and the simple slice sampler. Then

$$M \le U \le H,$$
$$M \le U \le S.$$

Before proving the different inequalities in the statement we start with recalling some notion and state a useful lemma. Recall that \mathbb{S}_{d-1} denotes the Euclidean unit sphere in \mathbb{R}^d and σ is the uniform distribution on \mathbb{S}_{d-1} . For $\theta \in \mathbb{S}_{d-1}$, $x \in \mathbb{R}^d$ and $t \in [0, \infty)$ let

$$K(t) = \{x \in K \mid \rho(x) > t\}$$

be the super-level set of ρ determined by t and

$$L_t(x,\theta) = \{x + \theta r \in K(t) \mid r \in \mathbb{R}\}\$$

be the chord in K(t) through x in direction θ . Moreover, define $L(x, \theta) := L_0(x, \theta)$ and, for a set $A \subseteq K = K(0)$, let

$$\Pi_{\theta}(A) = \{ y \in \mathbb{R}^d \mid y \perp \theta, \ L(y, \theta) \cap A \neq \emptyset \}$$

be the orthogonal projection of A to the hyperplane that is orthogonal to θ . We obtain the following useful results by an application of the Theorem of Fubini and the integral transformation to polar coordinates.

Lemma 2. Let $t \ge 0$ and $\theta \in \mathbb{S}_{d-1}$. For Lebesgue integrable $f: K(t) \to \mathbb{R}$ we have

$$\int_{K(t)} f(x) \,\mathrm{d}x = \int_{\Pi_{\theta}(K(t))} \int_{L_t(x,\theta)} f(y) \,\mathrm{d}y \,\mathrm{d}x,\tag{8}$$

and for fixed $x \in \mathbb{R}^d$ holds

$$\int_{K(t)} f(y) \,\mathrm{d}y = \frac{d\kappa_d}{2} \int_{\mathbb{S}_{d-1}} \int_{L_t(x,\theta)} f(y) \left| x - y \right|^{d-1} \,\mathrm{d}y \,\mathrm{d}\sigma(\theta) \tag{9}$$

where $\kappa_d = \operatorname{vol}_d(\mathbb{B}_d)$ denotes the volume of the d-dimensional Euclidean unit ball.

Note that in both identities the inner integral on the ride-hand-side is over a 1dimensional subset of \mathbb{R}^d and the integration is with respect to the 1-dimensional Lebesgue measure.

The different inequalities in Theorem 5 will be proven in the following sections. There we use the notation of Section 3.

4.1. Hit-and-run vs. hybrid slice sampler:

For the hit-and-run we use the scenario described in Example 3.1. Hence, let $\mathcal{A} := \mathbb{S}_{d-1}$ and $\lambda := \sigma$. Further, for $(x, a) \in K \times \mathcal{A}$ set s(x, a) := 1 such as $[x]_a := L_0(x, a)$. This implies $\pi_a = \pi$ for all $a \in \mathcal{A}$. By

$$H_a(x,A) = \frac{\int_{[x]_a} \rho(y) \mathbf{1}_A(y) \,\mathrm{d}y}{\int_{[x]_a} \rho(y) \,\mathrm{d}y}, \qquad A \subseteq K,$$

we can write the Markov operator $H: L_2(\pi) \to L_2(\pi)$ of the hit-and-run algorithm as

$$Hf(x) = \int_{\mathcal{A}} \int_{K} f(y) H_{a}(x, \mathrm{d}y) \,\mathrm{d}\lambda(a).$$

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Let

$$U_a(x,A) = \frac{1}{\rho(x)} \int_0^{\rho(x)} \int_{L_t(x,a)} \frac{\mathbf{1}_A(y)}{|L_t(x,a)|} \, \mathrm{d}y \, \mathrm{d}t$$

and observe that $L_t(x, a) = [x]_a \cap K(t)$. With this we can write the Markov operator $U: L_2(\pi) \to L_2(\pi)$ of the hybrid slice sampler based on hit-and-run as

$$Uf(x) = \int_A \int_K f(y) U_a(x, \mathrm{d}y) \,\mathrm{d}\lambda(a).$$

Thus we have a common representation. Now we check the assumptions (1).-(3). of Lemma 1:

1. To the reversibility of H_a and U_a with respect to π_a :

We only show reversibility of U_a , since reversibility of H_a follows by the same line of arguments. For $A, B \subseteq K$ it is enough to prove

$$\int_{A} U_a(x,B)\rho(x) \,\mathrm{d}x = \int_{B} U_a(x,A)\rho(x) \,\mathrm{d}x. \tag{10}$$

By the application of (8) with t = 0, the equality $\mathbf{1}_{[0,\rho(y))}(t) = \mathbf{1}_{K(t)}(y)$ and the fact that $y \in [x]_a$ implies $L_t(y, a) = L_t(x, a)$ we obtain

$$\int_{A} U_{a}(x,B)\rho(x) \, \mathrm{d}x = \int_{\Pi_{a}(K)} \int_{[x]_{a}} \mathbf{1}_{A}(y) U_{a}(y,B)\rho(y) \, \mathrm{d}y \, \mathrm{d}x$$
$$= \int_{\Pi_{a}(K)} \int_{[x]_{a}} \mathbf{1}_{A}(y) \int_{0}^{\rho(y)} \int_{L_{t}(y,a)} \frac{\mathbf{1}_{B}(z)}{|L_{t}(y,a)|} \, \mathrm{d}z \, \mathrm{d}t \, \mathrm{d}y \, \mathrm{d}x$$
$$= \int_{\Pi_{a}(K)} \int_{0}^{\infty} \frac{1}{|L_{t}(x,a)|} \, |L_{t}(x,a) \cap A| \, |L_{t}(x,a) \cap B| \, \mathrm{d}t \, \mathrm{d}x.$$

Observe that in the right-hand-side A and B are interchangeable which proves (10).

2. To the positivity of U_a :

By similar arguments as in the proof of (10) we obtain with $c = \int_K \rho(x) \, \mathrm{d}x$ that

$$\langle U_a f, f \rangle_{\pi} = \frac{1}{c} \int_{\Pi_a(K)} \int_0^\infty \frac{1}{|L_t(x, a)|} \left(\int_{L_t(x, a)} f(y) \, \mathrm{d}y \right)^2 \, \mathrm{d}t \, \mathrm{d}x \ge 0.$$

3. To $U_aH_a = H_a$:

From the definition of the Markov kernels H_a it is obvious that $y \in [x]_a$ implies $H_a(y, A) = H_a(x, A)$ for all $A \subseteq K$. This implies

$$U_a H_a(x, A) = \frac{1}{\rho(x)} \int_0^{\rho(x)} \int_{L_t(x, a)} H_a(y, A) \frac{\mathrm{d}y}{|L_t(x, a)|} \,\mathrm{d}t = H_a(x, A)$$

and hence $U_a H_a = H_a$.

Thus, as a direct consequence of Lemma 1 we obtain $U \leq H$.

4.2. Simple vs. hybrid slice sampler:

Here, we derive another representation of the hybrid slice sampler adapted to simple slice sampling.

For this let $\mathcal{A} := [0, \infty)$ and λ be the 1-dimensional Lebesgue measure. Further, for $(x, a) \in K \times \mathcal{A}$ set $s(x, a) := \mathbf{1}_{[0, \rho(x))}(a) / \rho(x)$ such as $[x]_a := K(a)$. Observe that

$$\pi_a(A) = \frac{\operatorname{vol}_d(A \cap K(a))}{\operatorname{vol}_d(K(a))}.$$

For any $x \in K$ let

$$S_a(x,A) = \pi_a(A), \quad A \subseteq K$$

Clearly, the Markov operator $S: L_2(\pi) \to L_2(\pi)$ of the simple slice sampler can be written as

$$Sf(x) = \int_{\mathcal{A}} \int_{K} f(y) S_a(x, \mathrm{d}y) s(x, a) \, \mathrm{d}\lambda(a).$$

Note that, with these notations, we have $L_a(x,\theta) = [x]_a \cap L(x,\theta)$ and let

$$U_a(x,A) = \int_{\mathbb{S}_{d-1}} \int_{L_a(x,\theta)} \frac{\mathbf{1}_A(y)}{|L_a(x,\theta)|} \, \mathrm{d}y \, \mathrm{d}\sigma(\theta)$$

With this we can write the Markov operator $U: L_2(\pi) \to L_2(\pi)$ of the hybrid slice sampler based on hit-and-run as

$$Uf(x) = \int_{\mathcal{A}} \int_{K} f(y) U_a(x, \mathrm{d}y) s(x, a) \,\mathrm{d}\lambda(a).$$

Again, we have a common representation and it remains to check the assumptions 1.-3. of Lemma 1:

1. To the reversibility of S_a and U_a with respect to π_a :

Since $S_a(x, A) = \pi_a(A)$ reversibility of S_a is obvious. Observe that π_a is the uniform distribution on $[x]_a = K(a)$ and $U_a(x, \cdot)$ performs a uniform hit-and-run step on K(a) which is known to be reversible, see for example [36, Lemma 4.10].

2. To the positivity of U_a :

With $c = \operatorname{vol}_d(K(a))$, by the application of (8) and by the fact that for $y \in$

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 $L_a(x,\theta)$ follows $L_a(x,\theta) = L_a(y,\theta)$ holds

$$\langle U_a f, f \rangle_{\pi_a} = \frac{1}{c} \int_{K(a)} \int_{\mathbb{S}_{d-1}} \int_{L_a(x,\theta)} \frac{f(z)f(x)}{|L_a(x,\theta)|} \, \mathrm{d}z \, \mathrm{d}\sigma(\theta) \, \mathrm{d}x$$

$$= \frac{1}{c} \int_{\mathbb{S}_{d-1}} \int_{\Pi_{\theta}(K(a))} \int_{L_a(x,\theta)} \int_{L_a(y,\theta)} \frac{f(y)f(z)}{|L_a(y,\theta)|} \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}\sigma(\theta)$$

$$= \frac{1}{c} \int_{\mathbb{S}_{d-1}} \int_{\Pi_{\theta}(K(a))} \frac{1}{|L_a(x,\theta)|} \left(\int_{L_a(x,\theta)} f(y) \, \mathrm{d}y \right)^2 \, \mathrm{d}x \, \mathrm{d}\sigma(\theta) \ge 0.$$

3. To $U_a S_a = S_a$:

This follows immediately from $S_a = \pi_a$ and the reversibility of U_a with respect to π_a .

Thus, as a direct consequence of Lemma 1 we obtain $U \leq S$.

4.3. Hybrid slice sampler vs. Metropolis:

Again we need a suitable representation for the hybrid slice sampler based on hitand-run.

Here, let $\mathcal{A} := \mathbb{S}_{d-1} \times [0, \infty)$ and λ be the product measure of σ and the 1dimensional Lebesgue measure. Further, for $x \in K$ and $(a_1, a_2) \in \mathcal{A}$, set $s(x, a_1, a_2) = \mathbf{1}_{[0,\rho(x)]}(a_2)/\rho(x)$ such as $[x]_{(a_1,a_2)} := L_{a_2}(x, a_1)$. Observe that

$$\pi_{(a_1,a_2)}(A) = \frac{\operatorname{vol}_d(A \cap K(a_2))}{\operatorname{vol}_d(K(a_2))}, \quad A \subseteq K,$$

is the uniform distribution in $K(a_2)$. By

$$U_{(a_1,a_2)}(x,A) = \int_{[x]_{(a_1,a_2)}} \mathbf{1}_A(y) \frac{\mathrm{d}y}{\left| [x]_{(a_1,a_2)} \right|}$$

we have a representation of the Markov operator $U: L_2(\pi) \to L_2(\pi)$ of the hybrid slice sampler by

$$Uf(x) = \int_{\mathcal{A}} \int_{K} f(y) U_a(x, \mathrm{d}y) s(x, a) \,\mathrm{d}\lambda(a)$$

with $a \in \mathcal{A}$. Now we have to represent the random walk Metropolis in the same fashion. For this let us define

$$\eta(x,y) = \frac{d\kappa_d}{2} q(y-x) |y-x|^{d-1}, \quad x,y \in \mathbb{R}^d$$

and for $A \subseteq K$ let

$$M_a(x,A) = \frac{1}{2} \int_{[x]_a} \mathbf{1}_A(y) \,\eta(x,y) \,\mathrm{d}y + \mathbf{1}_A(x) \left(1 - \frac{1}{2} \int_{[x]_a} \eta(x,y) \,\mathrm{d}y \right).$$

Here it is essential that q is rotational invariant, namely this property assures that $\int_{[x]_a} \eta(x, y) \, dy \leq 1$. Note that M_a is again a lazy transition kernel, since $M_a(x, \{x\}) \geq 1/2$. For $x \notin A$ by (9) we have

$$\begin{split} M(x,A) &= \frac{1}{2} \int_{K} \mathbf{1}_{A}(y) \, \alpha(x,y) \, q(y-x) \, \mathrm{d}y \\ &= \frac{1}{2\rho(x)} \int_{0}^{\rho(x)} \int_{K} \mathbf{1}_{[0,\rho(y)]}(t) \mathbf{1}_{A}(y) \, q(y-x) \, \mathrm{d}y \, \mathrm{d}t \\ &= \frac{1}{2\rho(x)} \int_{0}^{\rho(x)} \int_{K(t)} \mathbf{1}_{A}(y) \, q(y-x) \, \mathrm{d}y \, \mathrm{d}t \\ &= \frac{d\kappa_{d}}{4\rho(x)} \int_{\mathbb{S}_{d-1}} \int_{0}^{\rho(x)} \int_{L_{t}(x,\theta)} \mathbf{1}_{A}(y) q(x-y) \, |x-y|^{d-1} \, \mathrm{d}y \, \mathrm{d}t \, \mathrm{d}\sigma(\theta) \\ &= \int_{\mathcal{A}} M_{a}(x,A) s(x,a) \, \mathrm{d}\lambda(a). \end{split}$$

Thus, the transition kernel of the random walk Metropolis has the desired representation. It remains to check the conditions 1.-3. of Lemma 1:

1. To the reversibility of M_a and U_a with respect to π_a :

This follows again by a suitable application of (8). Let $a = (a_1, a_2)$. Due to its simple form, reversibility of U_a is obvious. It is enough to show for disjoint $A, B \subseteq K$ that

$$\int_{A\cap K(a_2)} M_a(x,B) \,\mathrm{d}x = \int_{B\cap K(a_2)} M_a(x,A) \,\mathrm{d}x.$$

Since $L_{a_2}(x, a_1) = L_{a_2}(y, a_1)$ if $y \in L_{a_2}(x, a_1)$ we have

$$\begin{split} &\frac{1}{2} \int_{A \cap K(a_2)} \int_{[x]_a} \mathbf{1}_B(y) \eta(x, y) \, \mathrm{d}y \, \mathrm{d}x \\ &= \frac{1}{2} \int_{K(a_2)} \int_{L_{a_2}(x, a_1)} \mathbf{1}_A(x) \mathbf{1}_B(y) \eta(x, y) \, \mathrm{d}y \, \mathrm{d}x \\ &= \frac{1}{2} \int_{\Pi_{a_1}(K(a_2))} \int_{L_{a_2}(x, a_1)} \int_{L_{a_2}(y, a_1)} \mathbf{1}_A(y) \mathbf{1}_B(z) \eta(y, z) \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \\ &= \frac{1}{2} \int_{\Pi_{a_1}(K(a_2))} \int_{L_{a_2}(x, a_1)} \int_{L_{a_2}(x, a_1)} \mathbf{1}_A(y) \mathbf{1}_B(z) \eta(y, z) \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \end{split}$$

By the symmetry of q, i.e. q(y-z) = q(z-y) follows $\eta(y,z) = \eta(z,y)$. This leads to the reversibility.

2. To the positivity of M_a :

By the fact that M_a is a lazy transition kernel we have positivity.

3. To $M_a U_a = U_a$:

By the fact that $y \in [x]_a$ implies $[x]_a = [y]_a$ and hence $U_a(x, A) = U_a(y, A)$ for all $A \subseteq K$ we obtain

$$\begin{aligned} M_a U_a(x, A) \\ &= \frac{1}{2} \int_{[x]_a} U_a(y, A) \, \eta(x, y) \, \mathrm{d}y + U_a(x, A) \left(1 - \frac{1}{2} \int_{[x]_a} \eta(x, y) \, \mathrm{d}y \right) \\ &= U_a(x, A). \end{aligned}$$

Thus, as a direct consequence of Lemma 1 we obtain $M \leq U.$

5. Concluding remarks

In this article we have presented a technique to compare the efficiency of Markov chains of a specific type. Using this technique we provide two comparison hierarchies according to a partial ordering of Markov operators of four prominent Markov chains for sampling general distributions in \mathbb{R}^d . The comparison with respect to the partial ordering leads to comparison results according to different criterions, for example the spectral gap, the conductance or the log-Sobolev constant, cf. Section 2.1.1.

Let us mention here that the computational cost for the simulation of each individual Markov chain is not taken into account. There seems to be a trade-off between efficiency and computional cost which should be further investigated. We leave this open for future work.

Finally, there are three open problems related to the considered Markov chains.

First, what is about the relation of the hit-and-run and the simple slice sampler? It is easy to see that there cannot be a general result as in the other cases. For this consider two examples:

- If π is the uniform distribution on K, then one step of the simple slice sampler is enough to sample π, while for the hit-and-run algorithm it is not (as long as d > 1). Hence, in this situation, hit-and-run is worse.
- 2. Let d = 1. Then, the hit-and-run algorithm samples π in one step, regardless of π . Hence, in this situation hit-and-run is better.

It seems to be interesting to find cases where hit-and-run is better. This is because, we guess that in general the computational cost of the simple slice sampler is (if it is at all implementable) much higher.

The second open problem concerns reverse inequalities. That is, if a Markov chain is better than another, how much better is it? This seems to be a delicate question and we can answer it only for toy examples. The techniques that were used in [41] in a discrete setting do not seem to work here.

The last problem we want to mention: Is there a similar hierarchy for the *mixing time*? That is, the number of steps that are needed to make the total variation distance "small", cf. Section 2.1.1. Certainly, the answer to this question additionally depends on the choice of the initial distribution. But the (quite analytical) techniques of this paper, see also [29, 16, 41, 1], do not seem to be suitable for this purpose. One interesting approach in this direction for Markov chains on discrete state spaces is given in [9].

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