# Perron-Frobenius theory for kernels and Crump-Mode-Jagers processes with macro-individuals 

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#### Abstract

Perron-Frobenius theory developed for irreducible non-negative kernels deals with so-called $R$-positive recurrent kernels. If kernel $M$ is $R$-positive recurrent, then the main result determines the limit of the scaled kernel iterations $R^{n} M^{n}$ as $n \rightarrow \infty$. In the Nummelin's monograph [10] this important result is proven using a regeneration method whose major focus is on $M$ having an atom. In the special case when $M=P$ is a stochastic kernel with an atom, the regeneration method has an elegant explanation in terms of an associated split chain.

In this paper we give a new probabilistic interpretation of the general regeneration method in terms of multi-type Galton-Watson processes producing clusters of particles. Treating clusters as macro-individuals, we arrive at a single-type Crump-Mode-Jagers process with a naturally embedded renewal structure.


Keywords: irreducible non-negative kernels, multi-type Galton-Watson process, $R$-positive recurrent kernel

## 1 Introduction

A Galton-Watson (GW) process describes random fluctuations of the numbers of independently reproducing particles counted generation-wise, see [1]. Given a measurable type space $(E, \mathcal{E})$, the multi-type GW process is defined as a measure-valued Markov chain $\left\{\Xi_{n}\right\}_{n=0}^{\infty}$, where $\Xi_{n}(A)$ gives the number of $n$-th generation particles whose types lie in the set $A \in \mathcal{E}$, see [6, Ch 3]. Given a current state

$$
\Xi_{n}=\sum_{i=1}^{Z_{n}} \delta_{x_{i}}, \quad \delta_{x}(A):=1_{\{x \in A\}},
$$

where $Z_{n}=\Xi_{n}(E)$ is the number of particles in the $n$-th generation and $x_{1}, x_{2}, \ldots$ are the types of these particles, the next state of the Markov chain is determined in terms of the
offspring to $Z_{n}$ particles

$$
\Xi_{n+1}=\sum_{i=1}^{Z_{n}} \Xi_{i, n}^{\left(x_{i}\right)}, \quad \Xi_{i, n}^{\left(x_{i}\right)} \stackrel{d}{=} \Xi^{\left(x_{i}\right)}
$$

The random measure $\Xi_{i, n}^{\left(x_{i}\right)}$, describing the allocation of a group of siblings over the type space $E$, is assumed to be independent of everything else except for the maternal type $x_{i}$.

A key characteristic of the multi-type GW process is its reproduction kernel $M$ defining the expected number of offspring found in a given subset of the type space

$$
\begin{equation*}
M(x, A)=\mathrm{E}\left[\Xi^{(x)}(A)\right], \quad x \in E, \quad A \in \mathcal{E} \tag{1}
\end{equation*}
$$

as a function of the maternal type $x$. Denote by $M^{n}$ the iterations of the reproduction kernel:

$$
\begin{equation*}
M^{0}(x, A)=\delta_{x}(A), \quad M^{n}(x, A)=\int M^{n-1}(y, A) M(x, \mathrm{~d} y), \quad n \geq 1 \tag{2}
\end{equation*}
$$

here and elsewhere in this paper, the integrals are taken over the whole type space $E$, unless specified otherwise. Then, for the multi-type GW process with the initial state $\Xi_{0}$, we get

$$
\mathrm{E}\left[\Xi_{n}(A)\right]=\int M^{n}(x, A) \mu_{0}(\mathrm{~d} x), \quad \mu_{0}=\mathrm{E}\left[\Xi_{0}\right]
$$

The asymptotic properties of the multi-type GW processes are studied on the basis of the Perron-Frobenius theorem dealing with the limiting behaviour of the expectation kernels and producing an asymptotic formula of the form

$$
M^{n}(x, A) \sim \rho^{n} \frac{h(x) \pi(A)}{\int h(y) \pi(d y)}, \quad n \rightarrow \infty
$$

see [8, Ch 6]. In the classical case of finitely many types, $M$ is a matrix and $\rho$ is its largest, the so-called Perron-Frobenius eigenvalue. Depending on whether $\rho<1$, $\rho=1$, or $\rho>1$, we distinguish among subcritical, critical, or supercritical GW processes.

The Perron-Frobenius theory for the irreducible non-negative kernels is build around the so-called regeneration method, see [2] and especially [10]. A key step of the regeneration method deals with $M$ having an atom, see Section 2 for key definitions. In the special case, when $M=P$ is a stochastic kernel with an atom, one can write

$$
\begin{equation*}
P(x, A)=p(x, A)+g(x) \gamma(A) \tag{3}
\end{equation*}
$$

where $\gamma(E)=1,0 \leq g(x) \leq 1$ and $p(x, E)=1-g(x)$ for all $x \in E$. The transition probabilities defined by such a kernel $P(x, d y)$ describe a split chain, whose transition from a given state $x$ is governed either by $\gamma(\mathrm{d} y)$ or $p(x, \mathrm{~d} y) /(1-g(x))$ depending on a random outcome of a $g(x)$-coin tossing [10, Ch. 4.4]. After each $\gamma$-transition step, the future evolution of the split chain becomes independent from the past and present states, so that the sequence of such regeneration events forms a renewal process with a delay. Then, it remains to apply the basic renewal theory to establish the Perron-Frobenius theorem for stochastic kernels.

In this paper we suggest a probabilistic interpretation of the general regeneration method (when kernel $M$ is not necessarily stochastic) in terms of a certain class of multi-type GW processes which we call GW processes with clusters, see Section 3. In Section 4 we show that a GW process with clusters has an intrinsic structure of the single-type Crump-Mode-Jagers (CMJ) process with discrete time [5]. In Sections 5 and 6we give a proof of a suitable version of the Perron-Frobenius theorem for the kernels with an atom, see Theorem 13, using the regeneration property of the renewal process embedded into the CMJ process. Section 7 contains an illuminating example of a GW process with clusters.

## 2 Irreducible kernels

In this section we give a summary of basic definitions and results presented in [10], including Theorems 2.1, 5.1, 5.2, and Propositions 2.4, 2.8, 3.4.

Consider a measurable type space $(E, \mathcal{E})$ assuming that $\sigma$-algebra $\mathcal{E}$ is countably generated. We denote by $\mathcal{M}_{+}$the set of $\sigma$-finite measures $\phi$ on $(E, \mathcal{E})$, and write $\phi \in \mathcal{M}^{+}$if $\phi \in \mathcal{M}_{+}$and $\phi(E) \in(0, \infty]$.
Definition $1 A$ (non-negative) kernel on $(E, \mathcal{E})$ is a map $M: E \times \mathcal{E} \rightarrow[0, \infty)$ such that for any fixed $A \in \mathcal{E}$, the function $M(\cdot, A)$ is measurable, and on the other hand, $M(x, \cdot) \in \mathcal{M}_{+}$ for any fixed $x \in E$. For a pair $(x, A) \in(E, \mathcal{E})$, we write $x \rightarrow A$ if

$$
M^{n}(x, A)>0 \text { for some } n \geq 1
$$

Kernel $M$ is called irreducible, if there is such a measure $\phi \in \mathcal{M}^{+}$, that for any $x \in E$, we have $x \rightarrow A$ whenever $\phi(A)>0$. Measure $\phi$ is then called an irreducibility measure for $M$.

If measure $\phi^{\prime} \in \mathcal{M}^{+}$is absolutely continuous with respect to an irreducibility measure $\phi$, then $\phi^{\prime}$ is itself an irreducibility measure. For an irreducible kernel $M$, there always exists a maximal irreducible measure $\psi$ such that any other irreducibility measure $\phi$ is absolutely continuous with respect to $\psi$.

For an irreducible kernel $M$ with a maximal irreducible measure $\psi$, there is a decomposition of the form

$$
\begin{equation*}
M^{n_{0}}(x, A)=m(x, A)+g(x) \gamma(A), \quad \text { for all } x \in E, A \in \mathcal{E}, \tag{4}
\end{equation*}
$$

where
$\gamma$ is an irreducibility measure for $M$,
$g$ is a measurable non-negative function such that $\int g(x) \psi(d x)>0$,
$m$ is a another kernel on $(E, \mathcal{E})$,
$n_{0}$ is a positive integer number.
Definition 2 If (4) holds with $n_{0}=1$, so that

$$
\begin{equation*}
M(x, A)=m(x, A)+g(x) \gamma(A), \quad x \in E, \quad A \in \mathcal{E} \tag{5}
\end{equation*}
$$

then the kernel $M$ is said to have an atom $(g, \gamma)$.

Given (4), put

$$
\begin{equation*}
F(s)=\sum_{n=1}^{\infty} F_{n} s^{n}, \quad F_{n}=\iint g(y) M^{n-1}(x, \mathrm{~d} y) \gamma(d x) \tag{6}
\end{equation*}
$$

Definition 3 Define the convergence parameter $R \in[0, \infty)$ of the irreducible kernel $M$ by

$$
F(s)<\infty \text { for } s<R, \text { and } F(s)=\infty \text { for } s>R .
$$

If $F(R)<\infty$, then kernel $M$ is called $R$-transient, if $F(R)=\infty$, then kernel $M$ is called $R$-recurrent.

Definition 4 A non-negative measurable function $h$ which is not identically infinite is called $R$-subinvariant for $M$ if

$$
h(x) \geq R \int h(y) M(x, \mathrm{~d} y), \quad \text { for all } x \in E
$$

An $R$-subinvariant function is called $R$-invariant if

$$
h(x)=R \int h(y) M(x, \mathrm{~d} y), \quad \text { for all } x \in E
$$

A measure $\pi \in \mathcal{M}^{+}$such that $\int g(y) \pi(d y) \in(0, \infty)$ is called $R$-subinvariant for $M$ if

$$
\pi(A) \geq R \int M(x, A) \pi(d x), \quad \text { for all } A \in \mathcal{E}
$$

An $R$-subinvariant meaure is called $R$-invariant if

$$
\pi(A)=R \int M(x, A) \pi(d x), \quad \text { for all } A \in \mathcal{E}
$$

Suppose $M$ is $R$-recurrent. The function $h$ and the measure $\pi$ defined by

$$
\begin{equation*}
h(x)=\sum_{n=1}^{\infty} R^{n n_{0}} \int g(y) m^{n-1}(x, \mathrm{~d} y), \quad \pi(A)=\sum_{n=1}^{\infty} R^{n n_{0}} \int m^{n-1}(x, A) \gamma(\mathrm{d} x) \tag{7}
\end{equation*}
$$

are $R$-invariant for $M$, scaled in such a way that

$$
\begin{equation*}
\int h(x) \gamma(\mathrm{d} x)=\int g(y) \pi(\mathrm{d} y)=1 \tag{8}
\end{equation*}
$$

For any $R$-subinvariant function $\tilde{h}$ satisfying $\int \tilde{h}(x) \gamma(\mathrm{d} x)=1$, we have

$$
\tilde{h}=h \quad \psi \text {-everywhere } \quad \text { and } \quad \tilde{h} \geq h \quad \text { everywhere } .
$$

The measure $\pi$ is the unique $R$-subinvariant measure satisfying (8).

Definition 5 An $R$-recurrent kernel $M$ is called $R$-positive recurrent if the $R$-invariant function and measure $(h, \pi)$ satisfy $\int h(y) \pi(d y)<\infty$. If $\int h(y) \pi(d y)=\infty$, then $M$ is called $R$-null recurrent.

Definition 6 Kernel $M$ has period $d$ if $d$ is the smallest positive integer such that there is a sequence of non-empty disjoint sets $\left(D_{0}, D_{1}, \ldots D_{d-1}\right)$ having the following property

$$
\text { if } x \in D_{i}, \text { then } M\left(x, E \backslash D_{j}\right)=0 \quad \text { for } j=i+1(\bmod d), \quad i=0, \ldots, d-1
$$

We call kernel $M$ aperiodic if its period $d=1$.
In the periodic case with $d \geq 2$, provided $M$ is irreducible and satisfies (4), there is an index $i, 0 \leq i \leq d-1$, such that $g=0$ over all $D_{j}$ except $D_{i}$. Furthermore,

$$
\gamma\left(E \backslash D_{j}\right)=0 \text { for } j=i+n_{0}(\bmod d)
$$

## 3 GW processes with clusters

As will be explained later in this section, the following definition yields the above mentioned split chain construction in the particular case when $\mathrm{P}\left(\Xi^{(x)}(E)=1\right)=1$.

Definition 7 Consider a multi-type GW process $\left\{Z_{n}\right\}_{n=0}^{\infty}$ whose reproduction measure can be decomposed into a sum of a random number of integer-valued random measures

$$
\begin{equation*}
\Xi^{(x)}=\xi^{(x)}+\sum_{i=1}^{N^{(x)}} \tau_{i} . \tag{9}
\end{equation*}
$$

Let each $\tau_{i}$ be independent of everything else and have a common distribution $\tau_{i} \stackrel{d}{=} \tau$.
(i) Such a multi-type $G W$ process will be called a $G W$ process with clusters.
(ii) Each group of particles behind a measure $\tau_{i}$ in (9) will be called a cluster, so that $N^{(x)}$ gives the number of clusters produced by a single particle of type $x$. Simple clusters correspond to the case $\mathrm{P}(\tau(E)=1)=1$.
(iii) A multi-type $G W$ process with the reproduction measure $\xi^{(x)}$ will be called a stem process.

Given (9) and

$$
\begin{equation*}
\mathrm{E} \xi^{(x)}(A)=m(x, A), \quad \mathrm{E} N^{(x)}=g(x), \quad \mathrm{E} \tau(A)=\gamma(A) \tag{10}
\end{equation*}
$$

by the total expectation formula, we see that the kernel (1) satisfies (5). Note that we allow for dependence between $\xi^{(x)}$ and $N^{(x)}$. Definition 7 puts no restrictions on the reproduction kernel $m$ of the stem process. The example from Section 7 presents a case with $E=[0, \infty)$, where the kernel $m$ is reducible, in that for any ordered pair of types $(x, y)$, where $x<y$, type $x$ particles (within the stem process) may produce type $y$ particles but not otherwise.

Consider a GW process with simple clusters such that

$$
\begin{aligned}
& \mathrm{P}\left(\xi^{(x)}(E)=0\right)=g(x), \quad \mathrm{P}\left(\xi^{(x)}(E)=1\right)=1-g(x), \quad g(x) \in[0,1] \\
& N^{(x)}=1_{\left\{\xi^{(x)}(E)=0\right\}}, \quad x \in E \\
& \tau=\delta_{Y}, \quad \mathrm{P}(Y \in A)=\gamma(A)
\end{aligned}
$$

In this case each particle produces exactly one offspring, and the GW process tracks the type of the regenerating particle. Using (5), we find that $M=P$ is a stochastic kernel satisfying (3) with

$$
p(x, A)=(1-g(x)) \mathrm{P}\left(\xi^{(x)}(A)=1 \mid \xi^{(x)}(E)=1\right)
$$

As a result we get a split chain corresponding to a stochastic kernel. Notice that the associated stem process is a pure death multi-type GW process.

An important family of GW processes with simple clusters is formed by linear-fractional multi-type GW processes, see [7, 11]. This family is framed by the following additional conditions

$$
\begin{aligned}
& \mathrm{P}\left(\xi^{(x)}(E)=0\right)+\mathrm{P}\left(\xi^{(x)}(E)=1\right)=1 \\
& N^{(x)}=N \cdot 1_{\left\{\xi^{(x)}(E)=1\right\}}, \quad \text { where } N \text { has a geometric distribution, } \\
& \tau=\delta_{Y} .
\end{aligned}
$$

In this case (5) holds with

$$
m(x, A)=\mathrm{P}\left(\xi^{(x)}(A)=1\right), \quad g(x)=\mathrm{E} N \cdot \mathrm{P}\left(\xi^{(x)}(A)=1\right), \quad \gamma(A)=\mathrm{P}(Y \in A)
$$

Here again, the stem process is a pure death multi-type GW process.

## 4 Embedded CMJ process

The key assumption of Definition 7 guarantees that the procreation of particles constituting a cluster is independent of the other parts of the GW process with clusters. The main idea of this paper is to treat each cluster as a newborn CMJ individual, which reminds the construction of macro-individuals in the sibling dependence setting of [9].

Consider the stem process starting from a single cluster at time 0 and denote by $L \in[1, \infty]$ its extinction time. Put $X_{0}=1$ and let $X_{n}$ stand for the number of new clusters generated at time $n$ by the particles in the stem process born at time $n-1, n \geq 1$. Observe that

$$
f_{n}:=\mathrm{E}\left(X_{n}\right)=\iint g(y) m^{n-1}(x, d y) \gamma(d x) .
$$

We treat the random vector $\left(X_{1}, \ldots, X_{L}\right)$ as the life record of the initial individual in an embedded CMJ process, see Figure1. A CMJ individual during its life of length $L$ at different ages produces random numbers of offspring, cf [5]. Such independently reproducing CMJ



Figure 1: Embedding a CMJ individual into a multi-type GW process stemming from a single cluster of size $Z_{0}=3$. Left panel. Solid lines represent the lineages of the stem process which dies out by time $L=6$. Dashed lines delineate the daughter clusters directly generated by the stem process. We see that $X_{1}=3$ with $\tau_{1}(E)=0, \tau_{2}(E)=1, \tau_{3}(E)=3$. Right panel. The summary of the individual life: $\left(X_{1}, \ldots, X_{L}\right)=(3,2,2,0,2,1)$.
individuals build a population with overlapping generations (in contrast to GW particles living one unit of time, so that there is no time overlap between generations).

Throughout this paper we assume

$$
\begin{equation*}
f\left(s_{0}\right) \in(0, \infty) \text { for some } s_{0}>0, \text { where } f(s)=\sum_{n=1}^{\infty} f_{n} s^{n} \tag{11}
\end{equation*}
$$

so that on one hand, that $f_{n}>0$ for some $n \geq 1$, and on the other hand, the radius of convergence

$$
r=\inf \{s \geq 0: f(s)=\infty\}
$$

is positive. The assumption $r>0$ prohibits very fast growing sequences of the type $f_{n}=e^{n^{2}}$.
Definition 8 Given (11), define a parameter $R \in(0, \infty)$ as $R=r$ if $f(r)<1$, and as the unique positive solution of the equation $f(R)=1$ if $f(r) \geq 1$.

Since $f(R) \leq 1$, the sequence $\left(f_{n} R^{n}\right)$ can be viewed as a (possibly defective) distribution on the lattice $\{1,2, \ldots\}$. This is the distribution of the inter-arrival time for the renewal process naturally embedded into the CMJ process defined above. The renewal process is interpreted as the consecutive ages at childbearing as one tracks a single ancestral lineage backwards in time. Given $f(R)=1$, the mean inter-arrival time for the embedded renewal process equals

$$
\sum_{n=1}^{\infty} n f_{n} R^{n}=R f^{\prime}(R)
$$

and is interpreted as the average age at childbearing or the mean generation length for the CMJ process, see [4].

Focussing on the current waiting time of such a discrete renewal process, we get an irreducible Markov chain with the state space $\{0,1, \ldots\}$. The following observation concerning this Markov chain is straightforward.

Proposition 9 The embedded renewal process is transient if $f(r)<1$, and recurrent if $f(r) \geq 1$. Let $R$ be defined by Definition 8. If $f(r)>1$, then $R \in(0, r), f^{\prime}(R)<\infty$, and the embedded renewal process is positive recurrent. If $f(r)=1$, then the embedded renewal process is either positive recurrent or null recurrent depending on whether $f^{\prime}(r)<\infty$ or $f^{\prime}(r)=\infty$.

Let $W_{n}$ be the number of newborn individuals at time $n$ in the embedded CMJ process started from a single newborn individual, or in other words, the total number of clusters emerging at time $n$ in the original GW process starting from a single cluster. Clearly,

$$
F_{n}:=\mathrm{E}\left(W_{n}\right)=\iint g(y) M^{n-1}(x, \mathrm{~d} y) \gamma(d x)
$$

Theorem 10 Consider a kernel $M$ with atom ( $g, \gamma$ ). Parameter $R$ from Definition 8 coincides with the convergence parameter of the kernel M. Moreover,
(i) if $f(r)<1$, then $R=r, f(R)<1$, and $F(R)<\infty$, so that $M$ is $R$-transient,
(ii) if $f(r) \geq 1$, then $f(R)=1$ and $F(R)=\infty$, so that $M$ is $R$-recurrent,
(iii) if $f(R)=1$, then either $f^{\prime}(R)=\infty$ so that $M$ is $R$-null recurrent, or $f^{\prime}(R) \in(0, \infty)$, so that $M$ is $R$-positive recurrent.

Proof. Using the law of total expectation it is easy to justify the following recursion

$$
F_{n}=f_{n}+f_{n-1} F_{1}+\ldots+f_{1} F_{n-1}
$$

This leads to the equality for generating functions

$$
F(s)=f(s)+f(s) F(s),
$$

which yields

$$
\begin{equation*}
F(s)=\frac{f(s)}{1-f(s)} \quad \text { for } s \text { such that } f(s)<1 \tag{12}
\end{equation*}
$$

From here and in view of Definition 3, it is obvious that the first statement is valid. Parts (i) and (ii) follow immediately. Part (iii) is proven in Section 5 .

Remark. For a general starting configuration of particles $Z_{0}$, putting $\mu_{0}=\mathrm{E} Z_{0}$, we get

$$
\begin{aligned}
& \tilde{f}_{n}:=\mathrm{E} X_{n}=\iint g(y) m^{n-1}(x, d y) \mu_{0}(d x), \\
& \tilde{F}_{n}:=\mathrm{E} Y_{n}=\iint g(y) M^{n-1}(x, \mathrm{~d} y) \mu_{0}(d x)
\end{aligned}
$$

The corresponding generating functions

$$
\tilde{f}(s)=\sum_{n=1}^{\infty} \tilde{f}_{n} s^{n}, \quad \tilde{F}(s)=\sum_{n=1}^{\infty} \tilde{F}_{n} s^{n}
$$

are connected by

$$
\begin{equation*}
\tilde{F}(s)=\frac{\tilde{f}(s)}{1-f(s)} \quad \text { for } s \text { such that } f(s)<1 \tag{13}
\end{equation*}
$$

(To obtain this relation, observe that

$$
\tilde{F}_{n}=\tilde{f}_{n}+\tilde{f}_{n-1} F_{1}+\ldots+\tilde{f}_{1} F_{n-1}
$$

which gives $\tilde{F}(s)=\tilde{f}(s)(1+F(s))$, and it remains to apply (12).)
As mentioned above, under the special initial condition $Z_{0} \stackrel{d}{=} \tau$, the embedded CMJ process starts from a single newborn individual. For a general $Z_{0}$, the embedded CMJ process has an immigration component characterised by the generating function $\tilde{f}(s)$. By immigration we mean the inflow of new clusters generated by the stem process starting from $Z_{0}$ particles.

## 5 Null and positive recurrence of a kernel with atom

Consider a non-negative kernel $M$ with atom $(g, \gamma)$, and put

$$
M_{s}(x, A)=\sum_{n=1}^{\infty} s^{n} M^{n-1}(x, A), \quad m_{s}(x, A)=\sum_{n=1}^{\infty} s^{n} m^{n-1}(x, A), \quad s \geq 0
$$

so that the earlier introduced generating functions $F$ and $f$ can be presented as

$$
F(s)=\iint g(y) M_{s}(x, \mathrm{~d} y) \gamma(\mathrm{d} x), \quad f(s)=\iint g(y) m_{s}(x, \mathrm{~d} y) \gamma(\mathrm{d} x) .
$$

Denote

$$
\begin{equation*}
h_{s}(x)=\int g(y) m_{s}(x, \mathrm{~d} y), \quad \pi_{s}(A)=\int m_{s}(x, A) \gamma(\mathrm{d} x) \tag{14}
\end{equation*}
$$

and observe that

$$
\int h_{s}(x) \gamma(\mathrm{d} x)=\int g(y) \pi_{s}(\mathrm{~d} y)=f(s), \quad \int h_{s}(y) \pi_{s}(\mathrm{~d} y)=s^{2} f^{\prime}(s)
$$

The latter equality requires the following argument

$$
\begin{aligned}
\int h_{s}(x) \pi_{s}(\mathrm{~d} x) & =\iiint g(y) m_{s}(x, \mathrm{~d} y) m_{s}(z, \mathrm{~d} x) \gamma(\mathrm{d} z) \\
& =\iint g(y) m_{s}^{2}(z, \mathrm{~d} y) \gamma(\mathrm{d} z)=\sum_{n=1}^{\infty} n s^{n+1} f_{n}=s^{2} f^{\prime}(s)
\end{aligned}
$$

where we used the relation

$$
\begin{aligned}
s^{-2} m_{s}^{2}(y, A) & =\int s^{-1} m_{s}(x, A) s^{-1} m_{s}(y, \mathrm{~d} x)=\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \int s^{n} m^{n}(x, A) s^{k} m^{k}(y, \mathrm{~d} x) \\
& =\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} s^{n+k} m^{n+k}(y, A)=\sum_{j=0}^{\infty}(j+1) s^{j} m^{j}(y, A) .
\end{aligned}
$$

Lemma 11 Consider a kernel with atom (5). If a positive $s$ is such that $f(s) \leq 1$, then the function $h_{s}$ and the measure $\pi_{s}$, defined by (14), satisfy

$$
\begin{align*}
& \int h_{s}(y) M(x, \mathrm{~d} y)=s^{-1} h_{s}(x)-(1-f(s)) g(x)  \tag{15}\\
& \int M(y, A) \pi_{s}(\mathrm{~d} y)=s^{-1} \pi_{s}(A)-(1-f(s)) \gamma(A) \tag{16}
\end{align*}
$$

so that they are s-subinvariant function and measure for the kernel $M$.
Proof. By (5), we have

$$
\begin{aligned}
\int m_{s}(y, A) M(x, \mathrm{~d} y) & =\sum_{n=1}^{\infty} s^{n} m^{n}(x, A)+g(x) \int m_{s}(y, A) \gamma(\mathrm{d} y) \\
& =s^{-1} m_{s}(x, A)-\delta_{x}(A)+g(x) \pi_{s}(A)
\end{aligned}
$$

which implies relation (15):

$$
\int h_{s}(y) M(x, \mathrm{~d} y)=\iint g(w) m_{s}(y, \mathrm{~d} w) M(x, \mathrm{~d} y)=s^{-1} h_{s}(x)-g(x)+g(x) f(s) .
$$

Similarly, from

$$
\begin{aligned}
\int M(y, A) m_{s}(x, \mathrm{~d} y) & =\sum_{n=1}^{\infty} s^{n} m^{n}(x, A)+\gamma(A) \int g(y) m_{s}(x, \mathrm{~d} y) \\
& =s^{-1} m_{s}(x, A)-\delta_{x}(A)+\gamma(A) h_{s}(x)
\end{aligned}
$$

we arrive at relation (16).
Lemma 11 yields the following statement which in turn provides the proof of part (iii) of Theorem 10 (recall Definition 5).

Corollary 12 Consider an $R$-recurrent kernel $M$ with atom $(g, \gamma)$. If $f(R)=1$, then $h=h_{R}$ and $\pi=\pi_{R}$ are $R$-invariant function and measure satisfying relation (7) with $n_{0}=1$, relation (8), as well as

$$
\int h(y) \pi(\mathrm{d} y)=R^{2} f^{\prime}(R)
$$

Observe that

$$
\begin{equation*}
h(x)=\sum_{n=1}^{\infty} R^{n} \int g(y) m^{n-1}(x, d y) \tag{17}
\end{equation*}
$$

is the expected $R$-discounted number of clusters ever produced by the stem process starting from a single particle of type $x$. From this angle, $h(x)$ can be interpreted as the reproductive value of type $x$. On the other hand,

$$
\begin{equation*}
\pi(A)=\sum_{n=1}^{\infty} R^{n} \int m^{n-1}(x, A) \gamma(\mathrm{d} x) \tag{18}
\end{equation*}
$$

is the expected $R$-discounted number of particles whose type belongs to $A$ and which appear in the stem process starting from a single cluster of particles. As shown next, see Theorem 13, the measure $\pi$ can be viewed as an asymptotically stable distribution for the types of particles in the GW process with clusters.

## 6 Perron-Frobenius theorem for kernels with atom

Theorem 13 Consider an aperiodic $R$-positive recurrent kernel $M$ with atom ( $g, \gamma$ ). Let $h$ and $\pi$ be given by (17) and (18). If $(x, A)$ are such that

$$
\begin{equation*}
R^{n} m^{n}(x, A) \rightarrow 0, \quad n \rightarrow \infty \tag{19}
\end{equation*}
$$

then

$$
\begin{equation*}
R^{n} M^{n}(x, A) \rightarrow \frac{h(x) \pi(A)}{R^{2} f^{\prime}(R)}, \quad n \rightarrow \infty \tag{20}
\end{equation*}
$$

If $h(x)<\infty$, then condition (19) holds for any $A$ such that

$$
\begin{equation*}
A \subset\{y: g(y) \geq \epsilon\} \text { for some } \epsilon>0 \tag{21}
\end{equation*}
$$

To prove this result we need two lemmas. In the end of this section we give a remark addressing condition (19).

Lemma 14 Consider a kernel $M$ with atom $(g, \gamma)$. If $s>0$ is such that $f(s)<1$, then

$$
\begin{equation*}
M_{s}(x, A)=m_{s}(x, A)+\frac{h_{s}(x) \pi_{s}(A)}{1-f(s)} \quad \text { for all } x \in E, A \in \mathcal{E} \tag{22}
\end{equation*}
$$

Proof. By (5), we have the recursion

$$
\begin{aligned}
M^{n}(x, A)= & g(x) \int M^{n-1}(y, A) \gamma(\mathrm{d} y)+\int M^{n-1}(y, A) m(x, \mathrm{~d} y) \\
= & g(x) \int M^{n-1}(y, A) \gamma(\mathrm{d} y)+\int g(y) m(x, \mathrm{~d} y) \int M^{n-2}(z, A) \gamma(\mathrm{d} z) \\
& +\int M^{n-2}(z, A) m^{2}(x, d z) \\
= & \sum_{i=1}^{n} \int g(y) m^{i-1}(x, \mathrm{~d} y) \int M^{n-i}(y, A) \gamma(\mathrm{d} y)+m^{n}(x, A)
\end{aligned}
$$

which in terms of generating functions gives

$$
M_{s}(x, A)=m_{s}(x, A)+h_{s}(x) \int M_{s}(y, A) \gamma(\mathrm{d} y)
$$

and after integration,

$$
\int M_{s}(x, A) \gamma(\mathrm{d} x)=\frac{\pi_{s}(A)}{1-f(s)}
$$

Combining the last two relations we get (22). Observe also that the last formula yields (12).

Lemma 15 Let

$$
a(s)=\sum_{n=0}^{\infty} a_{n} s^{n}, \quad b(s)=\sum_{n=0}^{\infty} b_{n} s^{n}, \quad c(s)=\sum_{n=0}^{\infty} c_{n} s^{n},
$$

be three generating functions for non-negative sequences connected by

$$
c(s)=\frac{b(s)}{1-a(s)}
$$

If sequence $\left\{a_{n}\right\}$ is aperiodic with $a(1)=1, a^{\prime}(1) \in(0, \infty)$, then

$$
c_{n} \rightarrow \frac{b(1)}{a^{\prime}(1)}, \quad n \rightarrow \infty
$$

Proof. This is a well-known result from Chapter XIII. 4 in [3].

Proof of Theorem 13. $R$-positive recurrence implies $f(R)=1$ and $f^{\prime}(R) \in(0, \infty)$. Due to $f(R)=1$, we can rewrite (22) as

$$
M_{\hat{s}}(x, A)-m_{\hat{s}}(x, A)=\frac{b(s)}{1-a(s)},
$$

where $\hat{s}=s R$ and

$$
a(s)=f(s R), \quad b(s)=h_{\hat{s}}(x) \pi_{\hat{s}}(A)
$$

so that $a^{\prime}(1)=R f^{\prime}(R), b(1)=h(x) \pi(A)$. Applying Lemma 15, we find that as $n \rightarrow \infty$,

$$
R^{n}\left(M^{n}(x, A)-m^{n}(x, A)\right) \rightarrow \frac{h(x) \pi(A)}{R^{2} f^{\prime}(R)}
$$

which combined with condition (19) yields the main assertion. The stated sufficient condition for (19) is verified using

$$
\sum_{n=1}^{\infty} R^{n} m^{n-1}(x, A) \leq \sum_{n=1}^{\infty} R^{n} \int 1_{\{y: g(y)>\epsilon\}} m^{n-1}(x, d y) \leq \epsilon^{-1} h(x)<\infty
$$

Remark. To illustrate the role of the condition (19), consider the kernel (5) with

$$
m(x, A)=g_{1}(x) \gamma_{1}(A)
$$

assuming

$$
\int g_{1}(x) \gamma_{1}(d x)=a_{1}, \quad \int g(x) \gamma(d x)=a, \quad \int g_{1}(x) \gamma(d x)=\int g(x) \gamma_{1}(d x)=0
$$

where $a_{1}>a>0$. In this particular case, we have

$$
M^{n}(x, A)=m^{n}(x, A)+a^{n} g(x) \gamma(A), \quad m^{n}(x, A)=a_{1}^{n} g_{1}(x) \gamma_{1}(A)
$$

and clearly,

$$
M^{n}(x, A) \sim \begin{cases}a_{1}^{n} g_{1}(x) \gamma_{1}(A), & \text { if } g_{1}(x) \gamma_{1}(A)>0 \\ a^{n} g(x) \gamma(A), & \text { if } g_{1}(x) \gamma_{1}(A)=0 \text { and } g(x) \gamma(A)>0\end{cases}
$$

Turning to the generating function defined by (6) we find

$$
F_{n}=\iint g(y) M^{n-1}(x, \mathrm{~d} y) \gamma(d x)=a^{n+1}, \quad F(s)=\frac{a^{2} s}{1-a s} .
$$

This yields $R=a^{-1}$ and we see that condition (19) is not valid for $(x, A)$ such that $g_{1}(x) \gamma_{1}(A)>0$. On the other hand, if $g(x)<\infty$ and $A$ satisfies 21), then

$$
0=\int g(x) \gamma_{1}(d x) \geq \int_{A} g(x) \gamma_{1}(d x) \geq \epsilon \gamma_{1}(A)
$$

so that $\gamma_{1}(A)=0$ and therefore $R^{n} M^{n}(x, A) \rightarrow g(x) \gamma(A)$.

## 7 3-parameter GW process with clusters

Here we construct a transparent example of a GW process with clusters having the type space $E=[0, \infty)$. Its positive recurrent reproduction kernel is fully specified by just three parameters $a, c \in(0, \infty)$, and $b \in(-1, \infty)$ :

$$
M(x, \mathrm{~d} y)=a e^{x-y} 1_{\{y \geq x\}} \mathrm{d} y+c e^{-b x} \delta_{0}(\mathrm{~d} y)
$$

This kernel satisfies (5) with

$$
\begin{equation*}
m(x, \mathrm{~d} y)=a e^{x-y} 1_{\{y \geq x\}} \mathrm{d} y, \quad g(x)=c e^{-b x}, \quad \gamma(A)=\delta_{0}(A), \tag{23}
\end{equation*}
$$

implying that each cluster consists of a single particle of type 0 .
The full specification of our example refers to a continuous time Markov branching process modeling the size of a population of Markov particles having the unit life-length mean and offspring mean $a$. The main idea is to count the Markov particles generation-wise, and to define the type of a Galton-Watson particle as the birth-time of the corresponding Markov particle. The corresponding stem process $\left\{\xi_{n}\right\}_{n \geq 0}$ is defined by

$$
\xi_{n}(A)=\text { the number of } n \text {-generation Markov particles born in the time period } A,
$$

so that its conditionally on the parent's birth time $x$,

$$
m^{n}(x,[0, t])=a^{n} \mathbb{P}\left(x+T_{1}+\ldots+T_{n} \leq t\right)=a^{n} \mathbb{P}\left(N_{t-x} \geq n\right), \quad \text { for } t>x
$$

where $T_{i}$ are independent exponentials with unit mean and $\left\{N_{t}\right\}_{t \geq 0}$ is the standard Poisson process.

Proposition 16 Consider the above described multi-type $G W$ process with clusters characterised by (23). Then we have

$$
\begin{equation*}
f(s)=\frac{r c s}{r-s}, \quad r=\frac{1+b}{a}, \quad R=\frac{r}{1+c r} . \tag{24}
\end{equation*}
$$

The process is supercritical if $c>\frac{r-1}{r}$, critical if $c=\frac{r-1}{r}$, and subcritical if $c<\frac{r-1}{r}$.
Convergence (20) holds for $A=[0, t], t \in[0, \infty)$, with the right hand side equal to

$$
e^{-b x}\left(R \delta_{0}(\mathrm{~d} y)+a R^{2} e^{(a R-1) y} \mathrm{~d} y\right)
$$

If $R a<1$, then (20) holds even for $A=E$ with the right hand side equal to $\frac{R e^{-b x}}{1-a R}$.
Proof. Referring to the underlying Poisson process, we find that for $s \neq 1 / a$,

$$
\begin{aligned}
m_{s}(0,[0, t]) & =s \sum_{n=0}^{\infty} s^{n} a^{n} \sum_{k=n}^{\infty} \mathbb{P}\left(N_{t}=k\right)=s \sum_{k=0}^{\infty} \mathbb{P}\left(N_{t}=k\right) \frac{1-(a s)^{k+1}}{1-a s} \\
& =\frac{s}{1-a s}\left(1-a s \mathbb{E}(a s)^{N_{t}}\right)=\frac{s}{1-a s}\left(1-a s e^{t(a s-1)}\right)
\end{aligned}
$$

More generally, we have

$$
m_{s}(x,[0, t])=m_{s}(0,[0, t-x])=\frac{s}{1-a s}\left(1-a s e^{(t-x)(a s-1)}\right) 1_{\{t \geq x\}}
$$

so that

$$
m_{s}(x, \mathrm{~d} y)=s \delta_{x}(\mathrm{~d} y)+a s^{2} e^{(a s-1)(y-x)} 1_{\{y \geq x\}} \mathrm{d} y
$$

By (14)

$$
h_{s}(x)=\int g(y) m_{s}(x, \mathrm{~d} y)=s c e^{-b x}+\operatorname{cas}^{2} \int_{0}^{\infty} e^{h(a s-1)} e^{-b(x+u)} d u=f(s) e^{-b x}
$$

where $f(s)$ satisfies (24). Since $f(r)=\infty$, the stated value $R=\frac{r}{1+c r}$ is found from the equation $f(R)=1$.

Applying once again (14), we find

$$
\pi_{s}(\mathrm{~d} y)=\int m_{s}(x, \mathrm{~d} y) \gamma(\mathrm{d} x)=m_{s}(0, \mathrm{~d} y)=s \delta_{0}(\mathrm{~d} y)+a s^{2} e^{(a s-1) y} \mathrm{~d} y
$$

To check this and previously obtained expressions, we verify the general formula for the integral

$$
\int h_{s}(x) \pi_{s}(\mathrm{~d} x)=s f(s)+f(s) a s^{2} \int e^{-(1+b) x} e^{a s x} \mathrm{~d} x=\frac{r s f(s)}{r-s}=\frac{r^{2} s^{2}}{(r-s)^{2}}=s^{2} f^{\prime}(s)
$$

With

$$
h(x)=e^{-b x}, \quad \pi(\mathrm{~d} x)=R \delta_{0}(\mathrm{~d} x)+a R^{2} e^{(a R-1) x} \mathrm{~d} x
$$

Theorem 13 specialised to the current example says that for $t \in[0, \infty)$,

$$
\begin{aligned}
R^{n} M^{n}(x,[0, t]) & \rightarrow e^{-b x}\left(R+a R^{2} \int_{0}^{t} e^{(a R-1) y} \mathrm{~d} y\right) \\
& =e^{-b x}\left(R+\frac{a R^{2}}{a R-1}\left(e^{(a R-1) t}-1\right)\right)=e^{-b x} \frac{a R^{2} e^{(a R-1) t}-R}{a R-1}, \quad n \rightarrow \infty .
\end{aligned}
$$

If $a R<1$ and $A=E$, then condition (19) holds since

$$
\pi(E)=R+\frac{a R^{2}}{1-a R}=\frac{R}{1-a R}<\infty
$$

and $R^{n} m^{n}(x, E)=(R a)^{n} \rightarrow 0$.
Remark. If we further specialize this example by letting the stem process to be the Yule process, then we have $a=2$. If furthermore, $b=2$ and $c<\frac{r-1}{r}=\frac{1}{3}$, then the corresponding GW process with clusters is subcritical, despite the total number of particles in the Yule process is infinite.

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