## ON THE CONTINUITY OF PICKANDS CONSTANTS

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#### Abstract

For a non-negative separable random field  $Z(t), t \in \mathbb{R}^d$  satisfying some mild assumptions we show that

$$H_Z^{\delta} = \lim_{T \to \infty} \frac{1}{T^d} \mathbb{E} \left\{ \sup_{t \in [0,T]^d \cap \delta \mathbb{Z}^d} Z(t) \right\} < \infty$$

for  $\delta \geq 0$  where  $0\mathbb{Z}^d := \mathbb{R}^d$  and prove that  $H_Z^0$  can be approximated by  $H_Z^\delta$ if  $\delta$  tends to 0. These results extend the classical findings for the Pickands constants  $H_Z^\delta$ , defined for  $Z(t) = \exp\left(\sqrt{2}B_\alpha(t) - |t|^{2\alpha}\right), t \in \mathbb{R}$  with  $B_\alpha$  a standard fractional Brownian motion with Hurst parameter  $\alpha \in (0, 1]$ . The continuity of  $H_Z^\delta$  at  $\delta = 0$  is additionally shown for two particular extensions of Pickands constants.

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### 1. Introduction

The discrete Pickands constant  $H_Z^{\delta}$  is defined for a given positive  $\delta$  by

$$H_Z^{\delta} = \lim_{n \to \infty} \frac{1}{n\delta} \mathbb{E} \bigg\{ \sup_{1 \le k \le n} Z(k\delta) \bigg\} = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \bigg\{ \sup_{t \in [0,T] \cap \delta \mathbb{Z}} Z(t) \bigg\} \in (0,\infty),$$

where  $Z(t) = \exp(\sqrt{2}B_{\alpha}(t) - |t|^{2\alpha})$  and  $B_{\alpha}$  a standard fractional Brownian motion (fBm) with Hurst parameter  $\alpha \in (0, 1]$ . When  $\delta = 0$ , interpreting  $0\mathbb{Z}$  as  $\mathbb{R}$ ,  $H_Z^0$  can be defined in an analogous way, i.e.,

$$H_Z^0 = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left\{ \sup_{t \in [0,T]} Z(t) \right\} \in (0,\infty),$$

which is the classical Pickands constant appearing in the tail asymptotics of the distribution of supremum for a wide class of Gaussian processes, see e.g., [13, 26, 27]. Pickands pioneering method (see [26]) for the approximation of the tail distribution of supremum for a stationary Gaussian process relies strongly on a discretisation approach. A crucial element in Pickands methodology is the fact that

$$\lim_{\delta \downarrow 0} H_Z^\delta = H_Z^0. \tag{1.1}$$

Notably, the first attempt of Pickands to prove (1.1) contains a gap; a correct proof is given in [3][Thm B3], see also the comments after [3][Lem A3].

A systematic study of Pickands constants started with [13][Thm 1, Prop 2] showing that

$$H_Z^0 = \mathbb{E}\left\{\frac{\sup_{t\in\mathbb{R}} Z(t)}{\eta \sum_{t\in\eta\mathbb{Z}} Z(t)}\right\} = \mathbb{E}\left\{\frac{\sup_{t\in\mathbb{R}} Z(t)}{\int_{t\in\mathbb{R}} Z(t)\lambda(dt)}\right\}$$
(1.2)

is valid for any  $\eta > 0$ , where  $\lambda(\cdot)$  is the Lebesgue measure on  $\mathbb{R}$ .

Both expressions in (1.2) paved the way for simulation of Pickands constants and inspired new formulas for extremal indices of stationary time series, see e.g., [8, 9, 12, 19, 20, 29, 36].

Pickands constants for general Gaussian processes were considered first in [6], see also [7]. Later [10, 19] discussed extensions to general random fields (rf's) as we briefly outline next. Let therefore  $Z(t), t \in \mathbb{R}^d$  be a separable, non-negative rf such that

$$\mathbb{E}\{Z(t)\} = 1, \quad \forall t \in \mathbb{R}^d \tag{1.3}$$

and define the corresponding Pickands constant by

$$H_Z^{\delta} = \lim_{T \to \infty} T^{-d} \mathbb{E} \left\{ \sup_{t \in [0,T]^d \cap \delta \mathbb{Z}^d} Z(t) \right\}, \quad \delta \ge 0,$$
(1.4)

where we set  $\delta \mathbb{Z}^d := \mathbb{R}^d$  when  $\delta = 0$ . If  $\delta = 0$ , in order to avoid degenerated cases, we shall assume further that

$$\mathbb{E}\left\{\sup_{t\in[0,T]^d} Z(t)\right\} \in (0,\infty), \quad \forall T > 0.$$
(1.5)

Definition (1.4) might not be valid for a general Z since the limit might not exist. However, if for all compact sets  $K \subset \mathbb{R}^d$ 

$$\mathbb{E}\left\{\sup_{t\in K} Z(t+c)\right\} = \mathbb{E}\left\{\sup_{t\in K} Z(t)\right\}, \quad \forall c\in \mathbb{R}^d,$$
(1.6)

then as we shall show in Section 2 the constant  $H_Z^{\delta}$  is well-defined and finite for any  $\delta \geq 0$ . Notably, we know from [10, 19, 20] that the limit in (1.4) exists if there is a stationary max-stable process  $Y(t), t \in \mathbb{R}^d$  with unit Fréchet marginals which has spectral process Z in its de Haan representation (see e.g., [15, 18])

$$Y(t) = \max_{i \ge 1} \Gamma_i^{-1} Z^{(i)}(t), \quad t \in \mathbb{R}^d.$$
(1.7)

Here  $\Gamma_i = \sum_{k=1}^{i} \mathcal{V}_k$  with  $\mathcal{V}_k, k \geq 1$  mutually independent unit exponential rv's being independent of  $\{Z^{(i)}\}_{i=1}^{\infty}$ , which are independent copies of Z. The finite dimensional distributions (fidi's) of Y are given by

$$\mathbb{P}\{Y(t_1) \le x_1, \dots, Y(t_n) \le x_n\} = e^{-\mathbb{E}\{\max_{i \le n} Z(t_i)/x_i\}}, \quad x_i > 0, t_i \in \mathbb{R}^d, i \le n \quad (1.8)$$

and hence

$$(\mathbb{P}\{Y(t_1) \le x_1, \dots, Y(t_n) \le x_n\})^m = \mathbb{P}\{mY(t_1) \le x_1, \dots, mY(t_n) \le x_n\}$$

for all m > 0, which shows that the fidi's of Y are max-stable. Clearly, if Y is stationary, then  $\sup_{t \in K} Y(t + c)$  has the same law as  $\sup_{t \in K} Y(t)$  for all  $c \in \mathbb{R}^d$  and hence if Y has locally bounded sample paths, then (1.8) implies (1.5) and further (1.6) is valid. Pickands constants are closely related to extremal indices of the max-stable stationary rf Y. Indeed, under the assumption (1.5) and the finiteness of Pickands constants, the separability of Y implies

$$-\ln \mathbb{P}\left\{\sup_{t\in[0,T]^d\cap\delta\mathbb{Z}^d}Y(t)\leq rT^d\right\} = \frac{1}{rT^d}\mathbb{E}\left\{\sup_{t\in[0,T]^d\cap\delta\mathbb{Z}^d}Z(t)\right\} \to \frac{1}{r}H_Z^\delta$$
(1.9)

as  $T \to \infty$  for all r > 0. Thus, by definition, the extremal index of the stationary rf  $Y(t), t \in \delta \mathbb{Z}^d$  is equal to  $\delta^d H_Z^\delta \in [0, 1]$  for any  $\delta > 0$ . Clearly, (1.9) is an approximation of the distribution of supremum of Y. Such approximations are known for general stationary rf's. A prominent instance is Y being a symmetric  $\alpha$ -stable rf, see [31, 34, 35, 40] and the references therein.

A natural question that arises here is the relevance of general Pickands constants, both in extreme value theory of rf's and in stochastic modelling. As shown in [11], Pickands constants determined by  $Z(t) = \exp(W(t) - \sigma_W^2(t)/2)$ , where W is a centered Gaussian rf with stationary increments and variance function  $\sigma_W^2$ , appear naturally in risk and queueing theory. Moreover, as advocated in [20], Pickands constants related to a general non-Gaussian rf Z have appeared in the literature in numerous papers. In that context, considering a general rf Z is important since it unifies the study of extremal indices and Pickands constants.

The second question is: for what general Z does the limit (1.1) hold? The answer to this question is presented in Section 2. Such a result has two immediate consequences, namely:

- A) the discretisation method of Pickands for the study of extremes of Gaussian rf's can be utilised also for cases where the limiting constants are determined by general Gaussian rf's with stationary increments;
- B) the calculation of  $H_Z^0$  can be carried out by simulating  $H_Z^{\delta}$  for small  $\delta > 0$ . Therefore, it is further interesting to derive tractable formulas for  $H_Z^{\delta}$  as given by (2.3).

Organisation of the rest of the paper is the following. In Section 2 we first discuss the finiteness and the continuity of discrete Pickands constants (in Theorem 1) and then derive a formula corresponding to (1.2) (in Proposition 1), which shows in particular how to approximate  $H_Z^0$  using the fidi's of Z. Section 3 is concerned with two extensions. The first one is motivated by results related to symmetric  $\alpha$ -stable rf's derived in [34, 35], whereas the second extension is motivated by the following constant

$$\mathcal{H}_{Z}^{\delta} = \lim_{T \to \infty} T^{-d} \int_{0}^{1} \mathbb{E} \Biggl\{ \sup_{t \in [0,T]^{d} \cap \delta \mathbb{Z}^{d}} Z_{z}(t) \Biggr\} dz$$
(1.10)

defined in [5] for  $\delta = 0$ . Here  $Z_z, z \in [0, 1]$  is equal in law to  $\exp(W_z(t) - \sigma_z^2(t)/2)$ , with  $W_z$  a centered Gaussian rf with stationary increments, variance function  $\sigma_z^2$  and almost surely continuous sample paths. In Theorem 2 we establish the continuity of  $\mathcal{H}_Z^{\delta}$  at  $\delta = 0$ . Such a result is crucial for the applications of Pickands discretisation method in the study of extremes of locally stationary Gaussian rf's. It is also of certain relevance for the simulations of those constants. We have relegated all the proofs to Section 4.

#### 2. Main Results

Our first result establishes the finiteness of Pickands constants and (1.1) for a general rf Z under some weak restrictions. In particular our result is satisfied for Z such that  $Y(t), t \in \mathbb{R}^d$  defined in (1.7) is a stochastically continuous stationary maxstable rf with locally bounded sample paths. Hereafter we shall suppose that all rf's are defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . In the following, for a given stochastically continuous rf  $U(t), t \in \mathcal{T}$  we shall suppose that it is also separable and jointly measurable. In view of [16] a separable and jointly measurable version of a stochastically continuous rf exists.

**Theorem 1.** If  $Z(t), t \in \mathbb{R}^d$  is a non-negative stochastically continuous rf such that (1.3), (1.5) and (1.6) hold, then the constants defined in (1.4) are finite for all  $\delta \geq 0$  and further  $\lim_{\delta \downarrow 0} H_Z^{\delta} = H_Z^0$ .

In the following

$$S_{\delta}(Z) = \int_{t \in \delta \mathbb{Z}^d} Z(t) \lambda_{\delta}(dt), \quad \delta \ge 0,$$

where  $\lambda_0(dt) = \lambda(dt)$  is the Lebesgue measure on  $\mathbb{R}^d$  and for  $\delta > 0$   $\lambda_\delta(dt) = \delta^d \lambda(dt)$ with  $\lambda(dt)$  the counting measure on  $\delta \mathbb{Z}^d$ .

Clearly,  $S_{\delta}(Z)$  is a random variable (rv) for any  $\delta > 0$ . If  $\delta = 0$  (recall  $0\mathbb{Z}^d$  denotes simply  $\mathbb{R}^d$ ), since we consider Z to be jointly measurable, then supposing that (1.5) holds and using that Z is non-negative (recall we assume that the probability space is complete) it follows that  $S_0(Z)$  is a rv; see [16][Thm 2.7, 2.8] for details in case d = 1.

**Corollary 1.** Suppose that Z satisfies the assumptions of Theorem 1 and further  $\mathbb{P}\{\sup_{t\in\mathbb{R}^d} Z(t) > 0\} = 1$ . If  $S_\eta(Z) = \infty$  almost surely for some  $\eta \ge 0$ , then  $H_Z^{\delta} = 0$ 

for all  $\delta \geq 0$ . Conversely, if  $H_Z^{\delta} = 0$  for some  $\delta \geq 0$ , then  $S_{\eta}(Z) = \infty$  almost surely for all  $\eta \geq 0$ .

- **Remark 1.** i) The assumption  $\mathbb{P}\{\sup_{t\in\mathbb{R}^d} Z(t) > 0\} = 1$  in Corollary 1 cannot be removed. Taking for instance  $Z(t) = V/p, t \in \mathbb{R}^d$  with V a Bernoulli rv such that  $\mathbb{P}\{V=1\} = p \in (0,1)$ , we get  $H_Z^{\delta} = 0$  for all  $\delta \ge 0$ . However  $S_1(Z) = 0$  with probability 1-p > 0.
  - ii) Let  $Z(t), t \in \mathbb{R}^d$  be a stationary rf satisfying the assumptions of Theorem 1. By the definition and the fact that Z and  $\tilde{Z} = 1 + Z$  are both stationary, the corresponding Pickands constants exist and we simply have  $H_Z^{\delta} = H_{\tilde{Z}}^{\delta}$  for all  $\delta \geq 0$ . Clearly  $S_1(\tilde{Z}) = \infty$  and  $\mathbb{P}\{\sup_{t \in \mathbb{R}^d} \tilde{Z}(t) > 0\} = 1$ , hence Corollary 1 implies  $H_Z^{\delta} = 0$  and moreover  $S_{\eta}(Z) = \infty, \eta \geq 0$  if further  $\mathbb{P}\{\sup_{t \in \mathbb{R}^d} Z(t) > 0\} = 1$ .

In the rest of this section we consider  $Y(t), t \in \mathbb{R}^d$  a stationary max-stable rf as in the Introduction with spectral rf Z and de Haan representation (1.7). Suppose next that Z has sample paths almost surely in the space  $D = D(\mathbb{R}^d, [0, \infty))$  of generalised càdlàg functions  $f : \mathbb{R}^d \to [0, \infty)$  equipped with the  $\sigma$ -field  $\mathcal{D} = \sigma(\pi_t, t \in \mathcal{T}_0)$  generated by the projection maps  $\pi_t : \pi_t f = f(t), f \in D$  and  $\mathcal{T}_0$  a dense subset of  $\mathbb{R}^d$ . See e.g., [4, 37] for the definition and properties of generalised càdlàg functions. In view of [19][Thm 6.9] (take  $\alpha = 1$  and  $L = B^{-1}$  therein) the stationarity of Y is equivalent with the validity of

$$\mathbb{E}\{Z(h)F(Z)\} = \mathbb{E}\{Z(0)F(B^hZ)\}$$
(2.1)

for all  $h \in \mathbb{R}^d$  and all 0-homogeneous measurable functionals  $F : D \mapsto [0, \infty)$  (0homogeneous means  $F(cf) = F(f), \forall c > 0, f \in D$ ) with  $B^h Z(\cdot) = Z(\cdot - h), h \in \mathbb{R}^d$ . For discrete max-stable processes, (2.1) is stated in [29][Eq. (5.2)], see also [36] for other equivalent formulations. We note in passing that (2.1) is initially derived for Z as in Example 1 below in [12][Lem 5.2].

Clearly, if Z is stationary, then Y is stationary too. However this instance is not interesting, since as shown in Remark 1, ii)  $H_Z^0 = 0$  in this case. We discuss below two other examples such that max-stable rf Y is stationary.

**Example 1.** It is known from [19, 22] that if  $X(t) = W(t) - Var(W(t))/2, t \in \mathbb{R}^d$ with W a centered Gaussian rf with stationary increments and almost surely continuous sample paths, then both (1.3) and (1.6) hold with  $Z(t) = e^{X(t)}$ . Using for instance [39][Thm 1] we have that (1.5) holds. Moreover, in view of [23] the corresponding max-stable rf Y is stationary.

**Example 2.** Let  $L(t), t \in \mathbb{R}^d$  be a non-negative deterministic measurable function such that  $\int_{\mathbb{R}^d} L(t)\lambda(dt) = 1$ , with  $\lambda(dt)$  the Lebesgue measure on  $\mathbb{R}^d$ . It follows easily that  $Z(t) = L(t - N)/p(N), t \in \mathbb{R}^d$  with N an  $\mathbb{R}^d$ -valued rv having a positive density function  $p(t) > 0, t \in \mathbb{R}^d$  satisfies (1.6). Note that for this case condition (1.5) reads

$$\int_{x \in \mathbb{R}^d} \sup_{t \in [0,T]^d} L(t-x)\lambda(dx) < \infty, \quad \forall T > 0.$$
(2.2)

As shown in [20][Thm 1, Eq. (3.5)] without any further assumption on the maxstable stationary rf Y, for  $\delta = \eta > 0$  we have

$$H_Z^{\delta} = \mathbb{E}\left\{Z(0)\frac{\sup_{t \in \delta \mathbb{Z}^d} Z(t)}{S_{\eta}(Z)}\right\} \in [0,\infty).$$
(2.3)

In the next result we show that the above holds under some weak assumptions also for  $\delta = 0$ . It turns out that under (2.4) below we can obtain also the first formula in (1.2).

**Proposition 1.** Let  $Z(t), t \in \mathbb{R}^d$  be a non-negative rf with almost surely sample paths in D. If (1.3), (1.5) and (2.1) hold and  $H_Z^0 > 0$ , then (2.3) holds for  $\delta = \eta = 0$ . Moreover, (2.3) holds also for  $\delta = 0, \eta > 0$  or  $\delta > 0, \eta = k\delta, k \in \mathbb{N}$  if further

$$\{S_0(Z) < \infty\} \subset \{S_\eta(B^r Z) \in (0, \infty)\}, \quad \forall r \in \delta \mathbb{Z}^d$$
(2.4)

almost surely.

**Remark 2.** It is shown in the proof of Proposition 1 that (2.4) is implied by the assumption  $\mathbb{P}\{Z(0) > 0\} = 1$ , which is satisfied for the choice of Z as in Example 1. In particular, Proposition 1 extends [8][Thm 2 and Thm 3] and [13][Prop 2]. Moreover, as discussed in Example 3 below, condition (2.4) cannot be removed.

**Example 3.** Let  $L(t), t \in \mathbb{R}^d$  be as in Example 2 and suppose further that  $L(t) > 0, t \in \mathbb{R}^d$ , (2.2) holds and  $L \in D(\mathbb{R}^d, [0, \infty))$ . Then the claim of Theorem 1 follows. Moreover, Proposition 1 implies for all  $\eta > 0$ 

$$H_Z^0 = \int_{\mathbb{R}^d} L(s) \frac{\sup_{t \in \mathbb{R}^d} L(t+s)}{\eta^d \sum_{t \in \eta \mathbb{Z}^d} L(t+s)} \lambda(ds) = \sup_{t \in \mathbb{R}^d} L(t),$$
(2.5)

where the last equality follows by (2.3) with  $\delta = 0$  (recall  $\int_{\mathbb{R}^d} L(t)\lambda(dt) = 1$ ). It is mentioned in [13] that for d = 1 and L the standard Gaussian density on  $\mathbb{R}$ , the above identity could be verified numerically. As shown by Dmitry Zaporozhets (personal communications) the last two equalities in (2.5) can be derived utilising the translation invariance of both the Lebesgue measure and the counting measure, respectively and applying the Fubini-Tonelli theorem. Specifically, setting for simplicity below  $\eta = 1$ 

$$\begin{split} \int_{\mathbb{R}^d} \frac{L(s)}{\sum_{t \in \mathbb{Z}^d} L(t+s)} \lambda(ds) &= \sum_{r \in \mathbb{Z}^d} \int_{r+[0,1)^d} \frac{L(s)}{\sum_{t \in \mathbb{Z}^d} L(t+s)} \lambda(ds) \\ &= \sum_{r \in \mathbb{Z}^d} \int_{[0,1)^d} \frac{L(r+s)}{\sum_{t \in \mathbb{Z}^d} L(r+t+s)} \lambda(ds) \\ &= \int_{[0,1)^d} \sum_{r \in \mathbb{Z}^d} \frac{L(r+s)}{\sum_{t \in \mathbb{Z}^d} L(t+s)} \lambda(ds) = 1. \end{split}$$

Our initial proof of Proposition 1 was asymptotic in nature. A modification of Dmitry's arguments and (2.1) have led to the current proof of Proposition 1.

Note that if we take for instance  $L(t) = \mathbb{I}\{t \in [0,1]\}, d = 1$  with  $\mathbb{I}\{\cdot\}$  the indicator function, then  $H_Z^0$  cannot be given by (2.3) with  $\eta > 2$ , since the first equality in (2.5) gives  $H_Z^0 = \infty$ , which is a contradiction to the fact that  $H_Z^0 < \infty$ . For this choice of  $\eta$ we have that condition (2.4) is not satisfied.

#### 3. Two Extensions

Motivated by [34, 35] we consider first an alternative definition of Pickands constants  $H^{\delta}_{|f|,m}$  defined with respect to a collection of functions  $f_s : E \to \mathbb{R}$  with  $(E, \mathcal{E})$  a measurable space equipped with some  $\sigma$ -finite measure m. In the second part of this section we discuss the constant  $\mathcal{H}^{\delta}_Z$  defined in (1.10) proving also its continuity at  $\delta = 0$ .

# 3.1. Behaviour of $H^{\delta}_{|f|,m}$ at $\delta = 0$

Let *m* be a  $\sigma$ -finite measure on some measurable space  $(E, \mathcal{E})$  and  $f = f_s(z), s \in \mathbb{R}^d, z \in E$ real functions with  $f_s \in \mathcal{L}^1(m)$  ( $\mathcal{L}^{\alpha}(m)$  is the set of all functions *g* such that  $\int_E |g(x)|^{\alpha} m(dx) < \infty$  and  $\alpha > 0$ ). We define next for any  $\delta = 1/n, n \in \mathbb{N}$  or  $\delta = 0$ 

$$H^{\delta}_{|f|,m} = \lim_{T \to \infty} \frac{1}{T^d} \int_E \sup_{t \in [0,T]^d \cap \delta \mathbb{Z}^d} |f_t(z)| \, m(dz),$$

where  $0\mathbb{Z}^d$  equals  $\mathbb{Q}^d_*$  with  $\mathbb{Q}_*$  the set of dyadic rational numbers  $\{\frac{k}{2^n}: k \in \mathbb{Z}, n \in \mathbb{N}\}$ . Clearly, the above limit is in general not defined. In order to include also the case  $\delta = 0$  we shall assume further that

$$\int_{E} \sup_{t \in K \cap \mathbb{Q}^d_*} |f_t(z)| \, m(dz) < \infty \tag{3.1}$$

and similarly to (1.6)

$$\int_{E} \sup_{t \in K \cap \mathbb{Q}^d_*} |f_{t+c}(z)| \, m(dz) = \int_{E} \sup_{t \in K \cap \mathbb{Q}^d_*} |f_t(z)| \, m(dz), \quad \forall c \in \mathbb{Q}^d_*$$
(3.2)

hold for all compact sets  $K \subset \mathbb{R}^d$ .

If L is as in Example 2, then taking  $f_t(z) = L(z-t), t, z \in \mathbb{R}^d$ ,  $E = \mathbb{R}^d$  equipped with Borel  $\sigma$ -field and m(dz) the Lebesgue measure on  $\mathbb{R}^d$  we have that (3.1) holds and further (3.2) is satisfied since m(dz) is shift-invariant.

**Example 4.** Let  $Y(t), t \in \mathbb{R}^d$  be a symmetric  $\alpha$ -stable stationary rf with locally bounded sample paths,  $\alpha \in (0, 2)$  and representation

$$Y(t) = \int_E f_t(z) M(dz), \quad t \in \mathbb{R}^d,$$

where M is a symmetric  $\alpha$ -stable random measure on E with control measure m and  $f_t \in \mathcal{L}^{\alpha}(m), t \in \mathbb{R}^d$ , see [35]. It follows that both (3.1) and (3.2) hold, see [35] for the case d = 1 and [32] for the case d > 1.

With the same arguments as in the proof of Theorem 1, we have that under (3.1) and (3.2), constant  $H^{\delta}_{|f|,m}$  is finite, non-negative and further

$$\lim_{n \to \infty} H^{2^{-n}}_{|f|,m} = H^0_{|f|,m}.$$
(3.3)

**Remark 3.** For Y as in Example 4 explicit formulas for  $H^{\delta}_{|f|,m}$  are derived in [33–35]. Utilising the relation between  $\alpha$ -stable and max-stable processes, see [21], the aforementioned formulas imply (2.3) when  $\delta = \eta \geq 0$ .

# 3.2. Continuity of $\mathcal{H}_Z^{\delta}$ at $\delta = 0$

Let  $W_z(t), t \in \mathbb{R}^d, z \in [0, 1]$  be a centred Gaussian rf with stationary increments, almost surely continuous sample paths and variance function  $\sigma_z^2, z \in [0, 1]$  such that  $\sigma_z^2(0) = 0$  for all  $z \in [0,1]$ . We formulate below the assumptions on the variance functions  $\sigma_z^2$  imposed in Theorem 2 below. Specifically, we shall assume that

$$\lim_{w \to z} \sigma_w(t) = \sigma_z(t), \quad \forall z \in [0, 1], \forall t \in \mathbb{R}^d$$
(3.4)

and further for some  $C_0, C_\infty$  positive and  $\nu_0, \nu_\infty \in (0, 2]$ 

$$\limsup_{\|t\|\to 0} \frac{\sigma_z^2(t)}{\|t\|^{\nu_0}} \le C_0 \text{ and } \limsup_{\|t\|\to\infty} \frac{\sigma_z^2(t)}{\|t\|^{\nu_\infty}} \le C_\infty$$
(3.5)

hold for all  $z \in [0, 1]$ , where  $\|\cdot\|$  denotes the Euclidean norm on  $\mathbb{R}^d$ .

**Theorem 2.** If (3.4) and (3.5) are satisfied, then for any  $\delta \ge 0$  and  $Z_z(t) = \exp(W_z(t) - \sigma_z^2(t)/2)$ 

$$\mathcal{H}_{Z}^{\delta} = \lim_{T \to \infty} T^{-d} \int_{0}^{1} \mathbb{E} \left\{ \sup_{t \in [0,T]^{d} \cap \delta \mathbb{Z}^{d}} Z_{z}(t) \right\} dz = \int_{0}^{1} H_{Z_{z}}^{\delta} dz \in [0,\infty)$$
(3.6)

and furthermore  $\lim_{\delta \downarrow 0} \mathcal{H}_Z^{\delta} = \mathcal{H}_Z^0$ .

**Remark 4.** i) A sufficient condition for  $H_{Z_z}^{\delta}, z \in [0, 1]$  to be positive is

$$\lim_{\|t\|\to\infty} \frac{\sigma_z^2(t)}{\ln\|t\|} > 8d,\tag{3.7}$$

see [10]. If the above holds for  $z \in [0, 1]$  on a set with non-zero Lebesgue measure, then under the assumptions of Theorem 2 we have that  $\mathcal{H}_Z^{\delta} > 0$  for any  $\delta \ge 0$ .

ii) In order to define rf's  $W_z, z \in [0, 1]$  with stationary increments, we need to determine variance function  $\sigma_z^2(t) = Var(W_z(t+s) - W_z(t)), s, t \in \mathbb{R}^d$ , which is a negatively (or conditionally negatively) definite function. One instance is to start with a negatively definite variance function  $\sigma^2(t), t \in \mathbb{R}^d$  and then define  $\sigma_z(t) = q(z)\sigma(t)$  for some continuous function q not identical to zero. Such  $\sigma_z$ clearly satisfies (3.4) and if  $\sigma^2(t) \leq C ||t||^{\nu}, t \in \mathbb{R}^d$ , then (3.5) also holds. Another instance is to consider  $\sigma_z$  as in [5], namely

$$\sigma_z^2(t) = \|t\|^{\lambda} r_z(t/\|t\|), \quad \lambda \in (0,2], t \in \mathbb{R}^d,$$
(3.8)

where  $r_z$  is a non-negative function defined on the unit sphere on  $\mathbb{R}^d$  determined by the norm  $\|\cdot\|$ . On the continuity of Pickands constants

### 4. Proofs

PROOF OF THEOREM 1 In view of (1.5) and (1.6)

$$\mathbb{E}\left\{\sup_{t\in K\cap\delta\mathbb{Z}^d} Z(t+c)\right\} = \mathbb{E}\left\{\sup_{t\in K\cap\delta\mathbb{Z}^d} Z(t)\right\} < \infty$$
(4.1)

holds for all  $c \in \mathbb{R}^d$ ,  $\delta \geq 0$  and all compact sets  $K \subset \mathbb{R}^d$ . Next, assume for notational simplicity that d = 1. In the light of (4.1)  $\mathbb{E}\{\sup_{t \in [0,T] \cap \delta \mathbb{Z}} Z(t)\}, T > 0$  is a subadditive function of T for any  $\delta \geq 0$  implying that  $H_Z^{\delta}$  is a finite non-negative constant (recall Z is non-negative) in view of Fekete's lemma. Further, to show the second part of the theorem it is enough to take  $T = a_{\delta}n$  where  $a_{\delta} = \lfloor 1/\delta \rfloor \delta$  with  $\lfloor x \rfloor$  the integer part of x > 0 and  $n \in \mathbb{N}, \delta \in (0, 1/2)$ . Utilising (4.1) to derive the last equality in the next calculations we have

$$\mathbb{E}\left\{\sup_{t\in[0,T]}Z(t)\right\} - \mathbb{E}\left\{\max_{t\in[0,T]\cap\delta\mathbb{Z}}Z(t)\right\}$$

$$= \mathbb{E}\left\{\max_{1\leq i\leq n}\sup_{t\in[a_{\delta}(i-1),a_{\delta}i]}Z(t) - \max_{1\leq i\leq n}\max_{t\in[a_{\delta}(i-1),a_{\delta}i]\cap\delta\mathbb{Z}}Z(t)\right\}$$

$$\leq \mathbb{E}\left\{\max_{1\leq i\leq n}\left(\sup_{t\in[a_{\delta}(i-1),a_{\delta}i]}Z(t) - \max_{t\in[a_{\delta}(i-1),a_{\delta}i]\cap\delta\mathbb{Z}}Z(t)\right)\right\}$$

$$\leq \sum_{i=1}^{n}\mathbb{E}\left\{\sup_{t\in[a_{\delta}(i-1),a_{\delta}i]}Z(t) - \max_{t\in[a_{\delta}(i-1),a_{\delta}i]\cap\delta\mathbb{Z}}Z(t)\right\}$$

$$= n\mathbb{E}\left\{\sup_{t\in[0,a_{\delta}]}Z(t) - \max_{t\in[0,a_{\delta}]\cap\delta\mathbb{Z}}Z(t)\right\}.$$
(4.2)

The assumption that Z is stochastically continuous implies that any dense subset of  $\mathbb{R}^d$  is a separant for the separable rf Z (see e.g., [30][Thm 2.9]) and therefore since  $\lim_{\delta \downarrow 0} a_{\delta} = 1$  the following convergence in probability

$$\max_{t \in [0,a_{\delta}] \cap \delta \mathbb{Z}} Z(t) \to \sup_{t \in [0,1]} Z(t)$$

holds as  $\delta \downarrow 0$ . Hence by (1.5) and the dominated convergence theorem

$$\lim_{\delta \downarrow 0} \mathbb{E} \left\{ \max_{t \in [0, a_{\delta}] \cap \delta \mathbb{Z}} Z(t) \right\} = \mathbb{E} \left\{ \sup_{t \in [0, 1]} Z(t) \right\} < \infty.$$
(4.3)

Consequently (4.2) implies

$$\frac{1}{T} \mathbb{E}\left\{\max_{t\in[0,T]\cap\delta\mathbb{Z}} Z(t)\right\} \leq \frac{1}{T} \mathbb{E}\left\{\sup_{t\in[0,T]} Z(t)\right\}$$

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$$\leq a_{\delta}^{-1} \mathbb{E} \left\{ \sup_{t \in [0, a_{\delta}]} Z(t) - \max_{t \in [0, a_{\delta}] \cap \delta \mathbb{Z}} Z(t) \right\} + \frac{1}{T} \mathbb{E} \left\{ \max_{t \in [0, T] \cap \delta \mathbb{Z}} Z(t) \right\}.$$

Since  $H_Z^{\delta}$  exists and is finite for all  $\delta \geq 0$  we have

$$\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left\{ \sup_{t \in [0,T]} Z(t) \right\} = \lim_{n \to \infty} \frac{1}{na_{\delta}} \mathbb{E} \left\{ \sup_{t \in [0,na_{\delta}]} Z(t) \right\} = H_Z^0$$

and

$$\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left\{ \sup_{t \in [0,T] \cap \delta \mathbb{Z}} Z(t) \right\} = \lim_{n \to \infty} \frac{1}{na_{\delta}} \mathbb{E} \left\{ \sup_{t \in [0,na_{\delta}] \cap \delta Z} Z(t) \right\} = H_Z^{\delta} \le H_Z^0 < \infty.$$

Thus letting n to infinity we obtain from the above inequalities

$$0 \leq H_Z^0 - H_Z^\delta \leq a_\delta^{-1} \mathbb{E} \left\{ \sup_{t \in [0, a_\delta]} Z(t) - \max_{t \in [0, a_\delta] \cap \delta \mathbb{Z}} Z(t) \right\},$$
(4.4)

which together with (4.3) yields  $\lim_{\delta \downarrow 0} H_Z^{\delta} = H_Z^0$ .

PROOF OF COROLLARY 1 First, let us notice that  $H_Z^{\delta} = 0$  for some  $\delta \ge 0$  if and only if  $S_{\delta}(Z) = \infty$  almost surely, which is a direct implication of [15, 21] and [33–35]; this has been already discussed in [10, 17, 36] for the case d = 1 and [20] for the *d*-dimensional discrete setup. Note in passing that in [15] Z is such that  $\mathbb{P}\{\sup_{t\in\mathbb{R}^d} Z(t) > 0\} = 1$ . So if for some  $\eta > 0$  we have  $H_Z^{\eta} = 0$ , then  $S_{\eta/k}(Z) = \infty$  almost surely for any  $k \in N$  and  $H_Z^{\eta/k} = 0$ . Consequently, Theorem 1 implies that  $\lim_{k\to\infty} H_Z^{\eta/k} = H_Z^0 = 0$  and hence  $0 \le H_Z^{\eta} \le H_Z^0 = 0$  for all  $\eta \ge 0$ . Conversely, if for some  $\eta \ge 0$  we have  $S_{\eta}(Z) = \infty$ , then from the above equivalence  $H_Z^{\eta} = 0$  for all  $\eta \ge 0$ , hence the proof is complete.  $\Box$ 

PROOF OF PROPOSITION 1 The claim for  $\delta = \eta = 0$  follows as in [10] where d = 1 is considered and its proof is therefore omitted.

Let next  $D = D(\mathbb{R}^d, [0, \infty))$  be the space of generalised càdlàg functions  $f : \mathbb{R}^d \mapsto [0, \infty)$ , which can be equipped with a metric that turns it into a Polish space. The corresponding Borel  $\sigma$ -field in D denoted by  $\mathcal{D}$  agrees with the  $\sigma$ -field  $\sigma(\pi_t, t \in \mathcal{T}_0)$  for any  $\mathcal{T}_0$  a dense subset of  $\mathbb{R}^d$ , see e.g., [4][Thm 7.1]. Denote by  $\Theta(t), t \in \mathbb{R}^d$  a rf with almost surely sample paths in D defined by

$$\mathbb{P}\{\Theta \in A\} = \mathbb{E}\{Z(0)\mathbb{I}(Z/Z(0) \in A)\}/\mathbb{E}\{Z(0)\}, \quad \forall A \in \mathcal{D},$$

where we interpret 0 : 0 as 0 and set  $\mathbb{I}(x \in A)$  equals 1 or 0 if  $x \in A$  or  $x \notin A$ , respectively. Note that  $\mathbb{E}\{Z(0)\}$  equals 1, but we leave it since the same formula is applied to  $Z_*$  below. Note further that since D is Polish, by [38][Lemma on p. 1276] we can realise both Z and  $\Theta$  in the same complete non-atomic probability space which we assume for notational simplicity below. By (2.1) for any measurable functional  $F: D \mapsto [0, \infty]$  which is 0-homogeneous

$$\mathbb{E}\{\Theta(h)F(\Theta)\} = \mathbb{E}\{\mathbb{I}(Z(0) \neq 0)Z(h)F(Z)\} = \mathbb{E}\{\mathbb{I}(\Theta(-h) \neq 0)F(B^{h}\Theta)\}.$$
 (4.5)

In view of Corollary 1 the assumption  $H_Z^0 > 0$  implies  $\mathbb{P}\{S_\eta(Z) < \infty\} = p_\eta > 0$  for all  $\eta \ge 0$ . If  $p_0 = 1$ , then also  $\mathbb{P}\{S_0(\Theta) < \infty\} = 1$  follows and thus

$$\{S_{\eta}(\Theta) < \infty\} \subset \{S_0(\Theta) < \infty\}$$

$$(4.6)$$

almost surely.

Assume next that  $p_0 \in (0,1)$ . The rf  $Z_*(t) = Z(t)|S_0(Z) = \infty, t \in \mathbb{R}^d$  has almost surely sample paths in D and satisfies (2.1). Denote by  $\Theta_*$  the corresponding rf of  $Z_*$  defined by the change of measure as above. We have that  $\Theta_*$  has the same law as  $\Theta|S_0(\Theta) = \infty$ . Since  $S_0(Z_*) = \infty$  with probability 1, applying Corollary 1, we obtain  $S_\eta(Z_*) = \infty$  almost surely and thus  $S_\eta(\Theta_*) = \infty$  almost surely i.e.,

$$\mathbb{P}\{S_{\eta}(\Theta) = \infty, S_{0}(\Theta) = \infty\} / \mathbb{P}\{S_{0}(\Theta) = \infty\} = 1$$

(implying  $\{S_0(\Theta) = \infty\} \subset \{S_\eta(\Theta) = \infty\}$  almost surely), hence (4.6) holds. If  $\mathbb{P}\{S_0(Z) < \infty\} = 0$  then by Corollary 1  $\mathbb{P}\{S_\eta(Z) < \infty\} = 0$  for all  $\eta > 0$ , which in turn implies  $\mathbb{P}\{S_\eta(\Theta) < \infty\} = 0$ . The assumption  $p_0 = 0$  implies also  $\mathbb{P}\{S_0(\Theta) < \infty\} = 0$ , hence again (4.6) holds.

Next, suppose that  $\mathbb{P}{S_0(\Theta) < \infty} > 0$ . Since then  $\mathbb{P}{S_0(Z) < \infty} > 0$ , we can define as above  $Z_*(t) = Z(t)|S_0(Z) < \infty, t \in \mathbb{R}^d$ . The corresponding  $\Theta_*$  has the same law as  $\Theta|S_0(\Theta) < \infty$ . Since  $\mathbb{P}{S_0(Z_*) < \infty} = \mathbb{P}{S_0(\Theta_*) < \infty} = 1$ , by [36][Thm 2.8]

$$\mathbb{P}\left\{\lim_{\|t\|\to\infty,t\in\eta\mathbb{Z}^d}\Theta_*(t)=0\right\}=1$$

with  $\|\cdot\|$  some norm on  $\mathbb{R}^d$ . The latter is equivalent with  $\mathbb{P}\{S_\eta(\Theta_*) < \infty\} = 1$  see [20][Condition A2, A4]. As in the proof of [20][Lem A.2]  $\mathbb{P}\{S_0(\Theta) < \infty\} = 0 \iff p_0 = 0$  and thus the reverse inclusion to (4.6) holds implying

$$\{S_{\eta}(\Theta) < \infty\} = \{S_0(\Theta) < \infty\}, \ \forall \eta > 0$$

$$(4.7)$$

almost surely. Since further  $\mathbb{P}\{S_{\eta}(\Theta) > 0\} = 1$  for all  $\eta \ge 0$ , which follows from  $\mathbb{P}\{\Theta(0) = 1\} = 1$  and the fact that  $\Theta$  has paths in D almost surely, we have almost surely for all  $\delta, \eta \in [0, \infty)$ 

$$\frac{1}{S_{\eta}(\Theta)} = \frac{1}{S_{\eta}(\Theta)} \frac{S_{\delta}(\Theta)}{S_{\delta}(\Theta)} \Theta(0).$$
(4.8)

Set next  $S_{\eta}(f) = \int_{\eta \mathbb{Z}^d} f(t) \lambda_{\eta}(dt), \eta \ge 0$  and

$$M_{\eta}(f) = \sup_{t \in \eta \mathbb{Z}^d} f(t), \quad M_0(f) = \sup_{t \in \mathcal{T}_0} f(t), f \in D, \eta > 0.$$

Both maps  $S_{\eta}(\cdot)$  and  $M_{\eta}(\cdot), \eta \geq 0$  are measurable and by the separability of Z we have that  $M_0(Z)$  has the same law as  $\sup_{t \in \mathbb{R}^d} Z(t)$ . Hence using further the definition of  $\Theta$ and (4.8), by the Fubini-Tonelli theorem for all  $\delta \neq \eta, \delta \geq 0, \eta > 0$  we have

$$\mathbb{E}\left\{Z(0)\frac{\sup_{t\in\delta\mathbb{Z}^d}Z(t)}{S_{\eta}(Z)}\right\}$$

$$= \mathbb{E}\left\{\frac{\sup_{t\in\delta\mathbb{Z}^d}\Theta(t)}{S_{\eta}(\Theta)}\right\}$$

$$= \mathbb{E}\left\{\frac{\sup_{t\in\delta\mathbb{Z}^d}\Theta(t)}{S_{\eta}(\Theta)}\frac{S_{\delta}(\Theta)}{S_{\delta}(\Theta)}\Theta(0)\right\}$$

$$= \int_{\delta\mathbb{Z}^d}\mathbb{E}\left\{\frac{M_{\delta}(\Theta)}{S_{\delta}(\Theta)}\frac{\Theta(0)}{S_{\eta}(\Theta)}\Theta(s)\right\}\lambda_{\delta}(ds)$$

$$= \sum_{i\in\eta\mathbb{Z}^d}\int_{r\in[0,\eta)^d\cap\delta\mathbb{Z}^d}\mathbb{E}\left\{\frac{M_{\delta}(\Theta)}{S_{\delta}(\Theta)}\frac{\Theta(0)}{S_{\eta}(\Theta)}\Theta(i+r)\right\}\lambda_{\delta}(dr), \quad (4.9)$$

where in the last equality we used also the translation invariance of  $\lambda_{\delta}$ . Now by (4.5) for all  $i, r \in \mathbb{R}^d$ 

$$\mathbb{E}\bigg\{\Theta(i+r)\frac{M_{\delta}(\Theta)}{S_{\delta}(\Theta)}\frac{\Theta(0)}{S_{\eta}(\Theta)}\bigg\} = \mathbb{E}\bigg\{\frac{M_{\delta}(B^{r+i}\Theta)}{S_{\delta}(B^{r+i}\Theta)}\frac{\Theta(-i-r)}{S_{\eta}(B^{r+i}\Theta)}\bigg\}.$$
(4.10)

Note in passing that  $\Theta(-i-r)\mathbb{I}(\Theta(-i-r)\neq 0) = \Theta(-i-r)$  almost surely. If  $\delta = 0, \eta > 0$  or  $\delta > 0$  and  $\eta = k\delta, k \in \mathbb{N}$ , then almost surely for all  $i \in \eta \mathbb{Z}^d, r \in \delta \mathbb{Z}^d$ 

$$M_{\delta}(B^{r+i}\Theta) = M_{\delta}(\Theta), \ S_{\delta}(B^{r+i}\Theta) = S_{\delta}(\Theta), \ S_{\eta}(B^{r+i}\Theta) = S_{\eta}(B^{r}\Theta)$$
(4.11)

and from (2.4)

$$\mathbb{P}\{S_{\delta}(\Theta) < \infty\} = \mathbb{P}\{S_{0}(\Theta) < \infty\}$$
$$= \mathbb{E}\{Z(0)\mathbb{I}\{S_{0}(Z) < \infty\}\}$$

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$$= \mathbb{E}\{Z(0)\mathbb{I}\{S_0(Z) < \infty, S_\eta(B^r Z) \in (0,\infty)\}\}$$
$$= \mathbb{E}\{Z(0)\mathbb{I}\{S_\delta(Z) < \infty, S_\eta(B^r Z) \in (0,\infty)\}\}$$
$$= \mathbb{E}\{Z(0)\mathbb{I}\{S_\delta(Z/Z(0)) < \infty, S_\eta(B^r Z/Z(0)) \in (0,\infty)\}\}$$
$$= \mathbb{P}\{S_\delta(\Theta) < \infty, S_\eta(B^r \Theta) \in (0,\infty)\}, \ \forall r \in \delta \mathbb{Z}^d,$$

where the third last line above follows from (4.7). Consequently, almost surely

$$\frac{S_{\eta}(B^{r}\Theta)}{S_{\eta}(B^{r}\Theta)}\mathbb{I}\{S_{\delta}(\Theta) < \infty\} = \mathbb{I}\{S_{\delta}(\Theta) < \infty\}.$$
(4.12)

Hence, for these choices of  $\delta$  and  $\eta$ , (4.9)-(4.12) and the Fubini-Tonelli theorem yield

$$\mathbb{E}\left\{Z(0)\frac{\sup_{t\in\delta\mathbb{Z}^d}Z(t)}{S_{\eta}(Z)}\right\} = \int_{r\in[0,\eta)^d\cap\delta\mathbb{Z}^d} \mathbb{E}\left\{\frac{M_{\delta}(\Theta)}{S_{\delta}(\Theta)}\sum_{i\in\eta\mathbb{Z}^d}\frac{\Theta(-r-i)}{S_{\eta}(B^r\Theta)}\right\}\lambda_{\delta}(dr) \\
= \eta^{-d}\int_{r\in[0,\eta)^d\cap\delta\mathbb{Z}^d} \mathbb{E}\left\{\frac{M_{\delta}(\Theta)}{S_{\delta}(\Theta)}\frac{S_{\eta}(B^r\Theta)}{S_{\eta}(B^r\Theta)}\mathbb{I}\{S_{\delta}(\Theta)<\infty\}\right\}\lambda_{\delta}(dr) \\
= \eta^{-d}\int_{r\in[0,\eta)^d\cap\delta\mathbb{Z}^d} \mathbb{E}\left\{\frac{M_{\delta}(\Theta)}{S_{\delta}(\Theta)}\mathbb{I}\{S_{\delta}(\Theta)<\infty\}\right\}\lambda_{\delta}(dr) \\
= \mathbb{E}\left\{\frac{M_{\delta}(\Theta)}{S_{\delta}(\Theta)}\right\} \\
= \mathbb{E}\left\{Z(0)\frac{\sup_{t\in\delta\mathbb{Z}^d}Z(t)}{S_{\delta}(Z)}\right\},$$

hence the claim follows from (2.3).

Note that if  $\mathbb{P}{Z(0) > 0} = 1$ , by (2.1) for all  $t \in \mathbb{R}^d$ 

$$\mathbb{E}\{Z(t)\} = \mathbb{E}\{Z(t)\mathbb{I}\{Z(0) > 0\}\} = \mathbb{E}\{Z(0)\mathbb{I}\{Z(-t) > 0\}\} = \mathbb{E}\{Z(0)\}$$

and hence  $\mathbb{P}{Z(t) > 0} = \mathbb{P}{\Theta(t) > 0} = 1$  since  $\mathbb{E}{Z(t)} = \mathbb{E}{Z(0)} = 1$ . Consequently, applying (4.5) and utilising (4.7) for all  $r \in \mathbb{R}^d$ ,  $\eta > 0$ 

$$\mathbb{E}\{Z(-r)\mathbb{I}\{S_0(Z) < \infty, 0 < S_\eta(B^r Z) < \infty\}\}$$

$$= \mathbb{E}\{\Theta(-r)\mathbb{I}\{S_0(\Theta) < \infty, 0 < S_\eta(B^r \Theta) < \infty\}\}$$

$$= \mathbb{E}\{\mathbb{I}\{S_0(\Theta) < \infty, 0 < S_\eta(\Theta) < \infty\}\}$$

$$= \mathbb{E}\{\mathbb{I}\{S_0(\Theta) < \infty\}\}$$

$$= \mathbb{E}\{Z(0)\mathbb{I}\{S_0(Z) < \infty\}\}$$

$$= \mathbb{E}\{Z(-r)\mathbb{I}\{S_0(Z) < \infty\}\} \le 1$$

and thus (2.4) follows. The reason that the indicator function did not appear when we applied (4.5) in the above calculations is that  $\Theta(t) > 0$  almost surely for all  $t \in \mathbb{R}^d$ .  $\Box$ 

PROOF OF THEOREM 2 Let  $z \in [0,1]$  be fixed. We consider for simplicity the case d = 1 and set  $X_z(t) = W_z(t) - \sigma_z^2(t)/2$ . We show next that for any positive T the function

$$A_T(z) = \mathbb{E}\left\{\sup_{t\in[0,T]\cap\delta\mathbb{Z}}e^{X_z(t)}\right\}$$

is continuous in z and thus integrable for any  $\delta \geq 0$ . First note that by (1.5)

$$\mathbb{E}\left\{\sup_{t\in[0,T]\cap\delta\mathbb{Z}}e^{X_{z}(t)}\right\} = 1 + \int_{0}^{\infty}e^{s}\mathbb{P}\left\{\sup_{t\in[0,T]\cap\delta\mathbb{Z}}X_{z}(t) > s\right\}ds < \infty.$$

Since  $\sigma_z$  determines the covariance function of  $X_z$ , then (3.4) implies that fidi's of  $X_{z+h}$  converge weakly to those of  $X_z$  as  $h \to 0$ . Moreover, for some  $\varepsilon > 0$  and all  $|h| < \varepsilon$ , by (3.5)

$$Var\left(X_{z+h}(s) - X_{z+h}(t)\right) = \sigma_{z+h}^{2}(t-s) \le C ||t-s||_{0}^{\nu}$$

for some C > 0 and all  $t, s \in [0, T]$ . Consequently,  $X_{z+h}(t), t \in [0, T]$  is tight (with respect to h) and converges weakly in the space of real-valued continuous functions on [0, T] equipped with the uniform topology, see [28][Prop 9.7]. Hence by the continuous mapping theorem for almost all  $s \in \mathbb{R}$ 

$$\lim_{h \to 0} \mathbb{P}\left\{\sup_{t \in [0,T] \cap \delta \mathbb{Z}} X_{z+h}(t) > s\right\} = \mathbb{P}\left\{\sup_{t \in [0,T] \cap \delta \mathbb{Z}} X_z(t) > s\right\}.$$

Consequently, for all T > 0

$$\lim_{h \to 0} \mathbb{E} \left\{ \sup_{t \in [0,T] \cap \delta \mathbb{Z}} e^{X_{z+h}(t)} \right\} = \mathbb{E} \left\{ \sup_{t \in [0,T] \cap \delta \mathbb{Z}} e^{X_z(t)} \right\} = A_T(z) < \infty$$

implying the claim that  $A_T(z), z \in [0, 1]$  is continuous in z. The derivation above rests on the application of the dominated convergence theorem that can be justified by Borell-TIS inequality. Indeed for all  $z \in [0, 1]$ 

$$\sup_{t\in[0,T]\cap\delta\mathbb{Z}}e^{X_z(t)}\leq \sup_{t\in[0,T],v\in[0,1]}e^{W_v(t)}$$

and

$$\mathbb{E}\left\{\sup_{t\in[0,T],v\in[0,1]}e^{W_v(t)}\right\} = 1 + \int_0^\infty e^s \mathbb{P}\left\{\sup_{t\in[0,T],v\in[0,1]}W_v(t) > s\right\} ds.$$

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By Borell-TIS inequality and (3.5)

$$\mathbb{P}\left\{\sup_{t\in[0,T],v\in[0,1]}W_v(t)>s\right\} \le e^{-\frac{(s-Q_1)^2}{2Q_2}}$$

for sufficiently large s and  $Q_1$ ,  $Q_2$  positive constants, which justifies the limit above. Since by assumption (3.5) for some C > 0 and all  $z \in [0, 1]$ 

$$\sigma_z^2(t) \le C(\|t\|^{\nu_0} + \|t\|^{\nu_\infty}) =: \sigma^2(t),$$

then for the Gaussian process

$$X(t) = W_0(t) + W_{\infty}(t) - \sigma^2(t)/2, \ t \in [0, T],$$

with  $W_0$ ,  $W_\infty$  mutually independent centred Gaussian processes with continuous sample paths, stationary increments and variance function  $C||t||^{\nu_0}, C||t||^{\nu_\infty}$ , respectively, applying [10][Thm 3.1] we obtain for any  $z \in [0, 1]$ 

$$H_{Z_z}^{\delta} \le H_{\widetilde{Z}}^{\delta} \in (0,\infty),$$

where  $\widetilde{Z}(t) = \exp(X(t)), t \in [0, T]$  and

$$\mathbb{E}\left\{\sup_{t\in[0,T]\cap\delta\mathbb{Z}}e^{X_{z}(t)}\right\}\leq\mathbb{E}\left\{\sup_{t\in[0,T]\cap\delta\mathbb{Z}}e^{X(t)}\right\}<\infty.$$

Consequently, by the measurability of  $A_T(z), z \in [0, 1]$  and the dominated convergence theorem

$$\mathcal{H}_{Z}^{\delta} = \lim_{T \to \infty} T^{-d} \int_{0}^{1} \mathbb{E} \left\{ \sup_{t \in [0,T]^{d} \cap \delta \mathbb{Z}^{d}} e^{X_{z}(t)} \right\} dz = \int_{0}^{1} H_{Z_{z}}^{\delta} dz \le H_{\widetilde{Z}}^{\delta} < \infty.$$

Thus, in view of Theorem 1 we obtain

$$\lim_{\delta \downarrow 0} \mathcal{H}_Z^{\delta} = \int_0^1 \lim_{\delta \downarrow 0} H_{Z_z}^{\delta} dz = \int_0^1 H_{Z_z}^0 dz \ge 0,$$

which completes the proof.

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