Conservations of first-order reflections

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Abstract

The set theory $\mathsf{KP}\Pi_{N+1}$ for Π_{N+1}-reflecting universes is shown to be Π_{N+1}-conservative over iterations of Π_N-recursively Mahlo operations for each $N \geq 2$.

1 Introduction

It is well known that the set of weakly Mahlo cardinals below a weakly compact cardinal is stationary. Furthermore any weakly compact cardinal κ is in the diagonal intersection $\kappa \in M^{\triangle} = \bigcap\{M(M^{\alpha}) : \alpha < \kappa\}$ for the α-th iterate M^{α} of the Mahlo operation M, where $\kappa \in M(X)$ iff $X \cap \kappa$ is stationary in κ.

The same holds for the recursive analogues of the indescribable cardinals, *reflecting ordinals* introduced by Richter-Aczel [12]. First let us recall the ordinals briefly. For a full account of the admissible set theory, see [8].

Δ_0 denotes the set of bounded formulas in the language $\{\in\}$ of set theories. Then the classes Σ_{i+1}, Π_{i+1} are defined recursively as usual. Each class Σ_{i+1}, Π_{i+1} is defined to be closed under bounded quantifications $\exists x \in a, \forall x \in a$.

The axioms of the Kripke-Platek set theory with the axiom of infinity, denoted $\mathsf{KP}\omega$, are Extensionality, Foundation schema, Pair, Union, Δ_0-Separation, Δ_0-Collection, and the axiom of infinity. Note that except Foundation schema, each axiom in $\mathsf{KP}\omega$ is a Π_2-formula.

For set-theoretic formulas φ, let $P \models \varphi :\Leftrightarrow (P, \in) \models \varphi$.

In what follows, let V denote a transitive and wellfounded model of $\mathsf{KP}\omega$, which is a universe in discourse. $P, Q, \ldots$ denote non-empty transitive sets in $V \cup \{V\}$.

A Π_i-recursively Mahlo operation for $2 \leq i < \omega$, is defined through a universal Π_i-formula $\Pi_i(a)$:

$$P \in RM_i(\mathcal{X}) \quad :\Leftrightarrow \quad \forall b \in P[P \models \Pi_i(b) \to \exists Q \in \mathcal{X} \cap P(b \in Q \models \Pi_i(b))]$$
$$\text{(read:} P \text{ is } \Pi_i\text{-reflecting on } \mathcal{X}.)$$

For the universe V, $V \in RM_i(\mathcal{X})$ denotes $\forall b[\Pi_i(b) \to \exists Q \in \mathcal{X}(b \in Q \models \Pi_i(b))]$. Suppose that there exists a first-order sentence φ such that $P \in \mathcal{X} \Leftrightarrow P \models \varphi$

for any transitive $P \in V \cup \{V\}$. Then $RM_i(\mathcal{X})$ is Π_{i+1}, i.e., there exists a Π_{i+1}-sentence $rm_i(\mathcal{X})$ such that $P \in RM_i(\mathcal{X})$ iff $P \models rm_i(\mathcal{X})$ for any transitive set P.

The iteration of RM_i along a definable relation $\prec$ is defined as follows.

$$P \in RM_i(a; \prec) :\Leftrightarrow a \in P \in \bigcap \{RM_i(RM_i(b; \prec)) : b \in P \models b \prec a\}.$$

Again $P \in RM_i(a; \prec)$ is a Π_{i+1}-relation.

Let Ord denote the class of ordinals in V. Let us write RM_i^α for $RM_i(\alpha; <)$ and ordinals $\alpha \in Ord$. A transitive set P is said to be Π_i*-reflecting* if $P \in RM_i = RM_i^1$.

$P \in RM_{i+1}$ is much stronger than $P \in RM_i$: Assume $P \in RM_{i+1}$ and $P \models \Pi_i(b)$ for $b \in P$. Then $P \in RM_i$ and $P \models rm_i \wedge \Pi_i(b)$ for the Π_{i+1}-sentence rm_i such that $P \in RM_i$ iff $P \models rm_i$. Hence there exists a $Q \in P$ such that $Q \models rm_i \wedge \Pi_i(b)$, i.e., $Q \in RM_i \,\&\, Q \models \Pi_i(b)$. This means $P \in RM_i^2 = RM_i(RM_i)$. Moreover P is in the diagonal intersection of RM_i, $P \in RM_i^{\triangle}$, i.e., $P \in \bigcap\{RM_i^\beta : \beta \in P \cap Ord\}$, and so on.

In particular a set theory $\mathrm{KP}\Pi_{i+1}$ for universes in RM_{i+1} proves the consistency of a set theory for universes in $RM_i^{\triangle}$.

In this paper we address a problem: How far can we iterate lower recursively Mahlo operations in higher reflecting universes? In [1] we gave a sketchy proof of the following Theorem 1.1, which is implicit in ordinal analyses in [2, 4].

Theorem 1.1 *For each $N \geq 2$ there exists a Σ_1-relation $\lhd_N$ on ω such that the set theory $\mathsf{KP}\ell$ for limits of admissibles proves the transfinite induction schema for $\lhd_N$ up to each $a \in \omega$, and $\mathsf{KP}\Pi_{N+1}$ is Π^1_1 (on ω)-conservative over the theory*

$$\mathsf{KP}\ell + \{V \in RM_N(a; \lhd_N) : a \in \omega\}.$$

Theorem 1.1 suffices to approximate $\mathsf{KP}\Pi_{N+1}$ proof-theoretically in terms of iterations of Π_N-recursively Mahlo operations. However $V \in RM_N(a; \prec)$ is a Π_{N+1}-formula for Σ_{N+1}-relation $\prec$, and the class Π^1_1 on ω is smaller than Π_{N+1}.

In this paper the set theory $\mathsf{KP}\Pi_{N+1}$ for Π_{N+1}-reflecting universes is shown to be Π_{N+1}-conservative over iterations of Π_N-recursively Mahlo operations RM_N for each $N \geq 2$. This result will be extended in [6, 7] to the indescribable cardinals over $\mathsf{ZF} + (V = L)$.

2 Conservation

Let $Ord \subset V$ denote the class of ordinals, $Ord^\varepsilon \subset V$ and $<^\varepsilon$ be Δ-predicates such that for any transitive and wellfounded model V of $\mathsf{KP}\omega$, $<^\varepsilon$ is a well ordering of type $\varepsilon_{\Omega+1}$ on Ord^ε for the order type Ω of the class Ord in V. Specifically let us encode 'ordinals' $\alpha < \varepsilon_{\Omega+1}$ by codes $\lceil \alpha \rceil \in Ord^\varepsilon$ as follows. $\lceil \alpha \rceil = \langle 0, \alpha \rangle$ for $\alpha \in Ord$, $\lceil \Omega \rceil = \langle 1, 0 \rangle$, $\lceil \omega^\alpha \rceil = \langle 2, \lceil \alpha \rceil \rangle$ for $\alpha > \Omega$, and $\lceil \alpha \rceil = \langle 3, \lceil \alpha_1 \rceil, \ldots, \lceil \alpha_n \rceil \rangle$ if $\alpha = \alpha_1 + \cdots + \alpha_n > \Omega$ with $\alpha_1 \geq \cdots \geq \alpha_n$, $n > 1$

and $\exists\beta_i(\alpha_i = \omega^{\beta_i})$ for each α_i. Then $\lceil\omega_n(\Omega+1)\rceil \in Ord^\varepsilon$ denotes the code of the 'ordinal' $\omega_n(\Omega+1)$.

$<^\varepsilon$ is assumed to be a canonical ordering such that $\mathsf{KP}\omega$ proves the fact that $<^\varepsilon$ is a linear ordering, and for any formula φ and each $n<\omega$,

$$\mathsf{KP}\omega \vdash \forall x(\forall y <^\varepsilon x\, \varphi(y) \to \varphi(x)) \to \forall x <^\varepsilon \lceil\omega_n(\Omega+1)\rceil\varphi(x) \tag{1}$$

For a definition of Δ-predicates Ord^ε and $<^\varepsilon$, and a proof of (1), cf. [5].

Proposition 2.1 $\mathsf{KP}\omega$ *proves that if* $P \in RM_N(\beta; <^\varepsilon)$*, then* $\forall\alpha <^\varepsilon \beta(\alpha \in P \to P \in RM_N(\alpha; <^\varepsilon))$.

Proof. This is seen from the fact that $<^\varepsilon$ is transitive in $\mathsf{KP}\omega$. □

Theorem 2.2 *For each* $N \geq 2$, $\mathsf{KP}\Pi_{N+1}$ *is* Π_{N+1}*-conservative over the theory*

$$\mathsf{KP}\omega + \{V \in RM_N(\lceil\omega_n(\Omega+1)\rceil; <^\varepsilon) : n \in \omega\}.$$

From (1) we see that $\mathsf{KP}\Pi_{N+1}$ proves $V \in RM_N(\lceil\omega_n(\Omega+1)\rceil; <^\varepsilon)$ for each $n \in \omega$. The converse is proved in section 3.

Proposition 2.3 *For any class* Γ *of* Π_{N+1}*-sentences, there exists a* Σ_{N+1}*-sentence* A *such that* $\mathsf{KP}\Pi_{N+1} \vdash A$*, and* $\mathsf{KP}\omega + \Gamma \not\vdash A$ *unless* $\mathsf{KP}\omega + \Gamma$ *is inconsistent.*

Proof. This follows from the essential unboundedness theorem due to G. Kreisel and A. Lévy [10]. In this proof let $\vdash A :\Leftrightarrow \mathsf{KP}\omega \vdash A$ and Pr denote a standard provability predicate for $\mathsf{KP}\omega$. Also $\mathrm{Tr}_{\Pi_{N+1}}$ denotes a partial truth definition of Π_{N+1}-sentences.

Then let A be a Σ_{N+1}-sentence saying that 'I am not provable from any true Π_{N+1}-sentence', $\vdash A \leftrightarrow \forall x \in \omega[\mathrm{Tr}_{\Pi_{N+1}}(x) \to \neg\mathrm{Pr}(x \dot{\to} \lceil A\rceil)]$, where $\dot{\to}$ denotes a recursive function such that $\lceil A\rceil \dot{\to} \lceil B\rceil = \lceil A \to B\rceil$ for codes $\lceil A\rceil$ of formulas A.

Suppose $\mathsf{KP}\omega + \Gamma \vdash A$. Pick a $C \in \Gamma$ so that $\vdash C \to A$. Then $\mathsf{KP}\omega + \Gamma \vdash \mathrm{Tr}_{\Pi_{N+1}}(\lceil C\rceil) \wedge \mathrm{Pr}(\lceil C \to A\rceil)$. Hence $\mathsf{KP}\omega + \Gamma \vdash \neg A$.

In what follows argue in $\mathsf{KP}\Pi_{N+1}$. Suppose A is false, and let C be any true Π_{N+1}-sentence. Since the universe V is Π_{N+1}-reflecting, there exists a transitive model $P \in V$ of $\mathsf{KP}\omega + \{C, \neg A\}$, which shows that $\mathsf{KP}\omega + \{C, \neg A\}$ is consistent. In other words, $\neg\mathrm{Pr}(\lceil C \to A\rceil)$. Therefore $\mathsf{KP}\Pi_{N+1} \vdash \neg A \to A$. □

Thus Theorem 2.2 is optimal with respect to the class Π_{N+1} of formulas provided that $\mathsf{KP}\Pi_{N+1}$ is consistent.

Corollary 2.4 *For each* $N \geq 3$, $\mathsf{KP}\Pi_{N+1}$ + (Power) + (Σ_{N-2}-Separation) + (Π_{N-2}-Collection) *is* Π_{N+1}*-conservative over the theory* $\mathsf{KP}\omega + \{V \in RM_N(\lceil\omega_n(\Omega+1)\rceil; <^\varepsilon) : n \in \omega\}$ + (Power) + (Σ_{N-2}-Separation) + (Π_{N-2}-Collection).

Proof. This follows from Theorem 2.2 and the facts that the axiom Power is a Π_3-sentence $\forall a \exists b \forall x \subset a(x \in b)$, and Σ_i-Separation or Π_i-Collection are Π_{i+2}-formulas. $\Box$

Let us announce an extension of Theorem 2.2 in [6, 7] to the indescribable cardinals over $\mathsf{ZF} + (V = L)$.

Let $<^\varepsilon$ be an ε-ordering as above. Let M_N denote the Π^1_N-Mahlo operation defined for sets S of ordinals and uncountable regular cardinals κ: $\kappa \in M_N(S)$ iff $S \cap \kappa$ is Π^1_N-indescribable in κ. The Π^1_{N+1}-indescribability is proof-theoretically reducible to iterations of an operation along initial segments of $<^\varepsilon$ over $\mathsf{ZF}+(V = L)$. The operation is a mixture of the Π^1_N-Mahlo operation M_N and Mostowski collapsings.

For $\alpha <^\varepsilon \varepsilon_{\mathcal{K}+1}$ and finite sets $\Theta \subset_{fin} (\mathcal{K} + 1)$, Π_{n+1}-classes $Mh^\alpha_n[\Theta]$ are defined so that the following holds.

In Theorem2.5 $\mathcal{K}$ is intended to denote the least Π^1_{N+1}-indescribable cardinal, and Ω the least weakly inaccessible cardinal above $\mathcal{K}$.

Theorem 2.5 (The case $N = 0$ in [6], and the general case in [7].)

1. For each $n < \omega$,

$$\mathsf{ZF} + (V = L) + (\mathcal{K} \text{ is } \Pi^1_{N+1}\text{-indescribable}) \vdash \mathcal{K} \in Mh_n^{\omega_n(\Omega+1)}[\emptyset].$$

2. For any Σ^1_{N+2}-sentences φ, if

$$\mathsf{ZF} + (V = L) + (\mathcal{K} \text{ is } \Pi^1_{N+1}\text{-indescribable}) \vdash \varphi^{L_\mathcal{K}},$$

then we can find an $n < \omega$ such that

$$\mathsf{ZF} + (V = L) + (\mathcal{K} \in Mh_n^{\omega_n(\Omega+1)}[\emptyset]) \vdash \varphi^{L_\mathcal{K}}.$$

The classes $Mh^\alpha_n[\Theta]$ are defined from iterated Skolem hulls $\mathcal{H}_{\alpha,n}(X)$, through which we described the limit of $\mathsf{ZF} + (V = L)$-provable countable ordinals in [5] as follows.

Theorem 2.6 ([5].)

$$\begin{aligned} |\mathsf{ZF} + (V = L)|_{\omega_1} &:= \inf\{\alpha \leq \omega_1 : \forall\varphi[\mathsf{ZF} + (V = L) \vdash \exists x \in L_{\omega_1}\, \varphi \Rightarrow \exists x \in L_\alpha\, \varphi]\} \\ &= \Psi_{\omega_1}\varepsilon_{\Omega+1} := \sup\{\Psi_{\omega_1,n}\omega_n(\Omega + 1) : n < \omega\}. \end{aligned}$$

In Theorem 2.6, Ω is intended to denote the least weakly inaccessible cardinal.

3 Proof of Theorem 2.2

In this section we prove Theorem 2.2. Our proof is extracted from M. Rathjen's ordinal analyses of Π_3-reflection in [11].

Let $N \geq 2$ denote a fixed integer. The axioms of the set theory $\mathsf{KP}\Pi_{N+1}$ for Π_{N+1}-reflecting universes are those of $\mathsf{KP}\omega$ and the axiom for Π_{N+1}-reflection: for Π_{N+1}-formulas φ, $\varphi(a) \to \exists c[ad^c \land a \in c \land \varphi^c(a)]$, where ad denotes a Π_3-sentence such that $P \models ad$ iff P is a transitive model of $\mathsf{KP}\omega$, and φ^c denotes the result of restricting any unbounded quantifiers $\exists x, \forall x$ in φ to $\exists x \in c, \forall x \in c$, resp.

$\mathsf{KP}i$ denotes the set theory for recursively inaccessible sets, which is obtained from $\mathsf{KP}\omega$ by adding the axiom $\forall x \exists y[x \in y \land ad^y]$.

Throughout this section we work in an intuitionistic fixed point theory $\mathrm{FiX}^i(\mathsf{KP}i)$ over $\mathsf{KP}i$. The intuitionistic theory $\mathrm{FiX}^i(\mathsf{KP}i)$ is introduced in [5], and shown to be a conservative extension of $\mathsf{KP}i$. Let us reproduce definitions and results on $\mathrm{FiX}^i(\mathsf{KP}i)$ here.

Fix an X-strictly positive formula $\mathcal{Q}(X,x)$ in the language $\{\in, =, X\}$ with an extra unary predicate symbol X. In $\mathcal{Q}(X,x)$ the predicate symbol X occurs only strictly positive. This means that the predicate symbol X does not occur in the antecedent φ of implications $\varphi \to \psi$ nor in the scope of negations $\neg$ in $\mathcal{Q}(X,x)$. The language of $\mathrm{FiX}^i(\mathsf{KP}i)$ is $\{\in, =, Q\}$ with a fresh unary predicate symbol Q. The axioms in $\mathrm{FiX}^i(\mathsf{KP}i)$ consist of the following:

1. All provable sentences in $\mathsf{KP}i$ (in the language $\{\in, =\}$).

2. Induction schema for any formula φ in $\{\in, =, Q\}$:

$$\forall x(\forall y \in x\, \varphi(y) \to \varphi(x)) \to \forall x\, \varphi(x) \tag{2}$$

3. Fixed point axiom:

$$\forall x[Q(x) \leftrightarrow \mathcal{Q}(Q,x)].$$

The underlying logic in $\mathrm{FiX}^i(\mathsf{KP}i)$ is defined to be the intuitionistic (first-order predicate) logic (with equality).

(2) yields the following Lemma 3.1.

Lemma 3.1 *Let $<^{\varepsilon}$ denote a Δ_1-predicate mentioned in the beginning of section 2. For each $n < \omega$ and each formula φ in $\{\in, =, Q\}$,*

$$\mathrm{FiX}^i(\mathsf{KP}i) \vdash \forall x(\forall y <^{\varepsilon} x\, \varphi(y) \to \varphi(x)) \to \forall x <^{\varepsilon} \omega_n(I+1)\varphi(x).$$

The following Theorem 3.2 is seen as in [3, 5].

Theorem 3.2 $\mathrm{FiX}^i(\mathsf{KP}i)$ *is a conservative extension of* $\mathsf{KP}i$.

In what follows we work in $\mathrm{FiX}^i(\mathsf{KP}i)$.

Let V denote a transitive and wellfounded model of $\mathsf{KP}\omega$. Consider the language $\mathcal{L}_V = \{\in\} \cup \{c_a : a \in V\}$ where c_a denotes the name of the set $a \in V$. We identify the set a with its name c_a.

Our proof proceeds as follows. Assume that $\mathsf{KP}\Pi_{N+1} \vdash A$ for a Π_{N+1}-sentence A. $\mathsf{KP}\Pi_{N+1}$ is embedded to an infinitary system R_N formulated in one-sided sequent calculus, and cut inferences are eliminated, which results in an infinitary derivation of height $\alpha < \varepsilon_{\Omega+1}$ with an inference rule (Ref_{N+1}) for Π_{N+1}-reflection. Then A is seen to be true in $P \in RM_N(\alpha; <^{\varepsilon})$.

In one-sided sequent calculi, formulas are generated from atomic formulas and their negations $a \in b, a \not\in b$ by propositional connectives $\lor, \land$ and quantifiers $\exists, \forall$. It is convenient here to have bounded quantifications $\exists x \in a, \forall x \in a$ besides unbounded ones $\exists x, \forall x$. The negation $\neg A$ of formulas A is defined recursively by de Morgan's law and elimination of double negations. Also $(A \to B) :\equiv (\neg A \lor B)$.

$\Gamma, \Delta, \ldots$ denote finite sets of sentences, called *sequents* in the language $\mathcal{L}_V$. Γ, Δ denotes the union $\Gamma \cup \Delta$, and Γ, A the union $\Gamma \cup \{A\}$. A finite set Γ of sentences is intended to denote the disjunction $\bigvee \Gamma := \bigvee \{A : A \in \Gamma\}$. Γ is *true* in $P \in V \cup \{V\}$ iff $\bigvee \Gamma$ is true in P iff $\bigvee \Gamma^P$ is true.

Classes $\Delta_0, \Sigma_{i+1}, \Pi_{i+1}$ of sentences in $\mathcal{L}_V$ are defined as usual.

We assign disjunctions or conjunctions to sentences as follows. When a disjunction $\bigvee (A_\iota)_{\iota \in J}$ [a conjunction $\bigwedge (A_\iota)_{\iota \in J}$] is assigned to A, we denote $A \simeq \bigvee (A_\iota)_{\iota \in J}$ [$A \simeq \bigwedge (A_\iota)_{\iota \in J}$], resp.

Definition 3.3 *1.* For a Δ_0-sentence M

$$M :\simeq \left\{ \begin{array}{ll} \bigvee (A_\iota)_{\iota \in J} & \text{if } M \text{ is false in } V \\ \bigwedge (A_\iota)_{\iota \in J} & \text{if } M \text{ is true in } V \end{array} \right. \text{ with } J := \emptyset.$$

In what follows consider the unbounded sentences.

2. $(A_0 \lor A_1) :\simeq \bigvee (A_\iota)_{\iota \in J}$ and $(A_0 \land A_1) :\simeq \bigwedge (A_\iota)_{\iota \in J}$ with $J := 2$.

3. $\exists x \in a\, A(x) :\simeq \bigvee (A(b))_{b \in J}$ and $\forall x \in a\, A(x) :\simeq \bigwedge (A(b))_{b \in J}$ with $J := a$.

4. $\exists x\, A(x) :\simeq \bigvee (A(b))_{b \in J}$ and $\forall x\, A(x) :\simeq \bigwedge (A(b))_{b \in J}$ with $J := V$.

Definition 3.4 The *depth* $\mathrm{dp}(A) < \omega$ of $\mathcal{L}_V$-sentences A is defined recursively as follows.

1. $\mathrm{dp}(A) = 0$ if $A \in \Delta_0$.

In what follows consider unbounded sentences A.

2. $\mathrm{dp}(A) = \max\{\mathrm{dp}(A_i) : i < 2\} + 1$ if $A \equiv (A_0 \circ A_1)$ for $\circ \in \{\lor, \land\}$.

3. $\mathrm{dp}(A) = \mathrm{dp}(B(\emptyset)) + 1$ if $A \in \{(Qx\, B(x)), (Qx \in a\, B(x)) : a \in V\}$ *for* $Q \in \{\exists, \forall\}$.

Definition 3.5 *1.* For $\mathcal{L}_V$-sentences A, $\mathsf{k}(A) := \{a \in V : c_a \text{ occurs in } A\}$.

2. For sets Γ of sentences, $\mathsf{k}(\Gamma) := \bigcup \{\mathsf{k}(A) : A \in \Gamma\}$.

3. For $\iota \in V$ and a transitive model $P \in V \cup \{V\}$ of $\mathsf{KP}\omega$, $P(\iota) \in V \cup \{V\}$ denotes the smallest transitive model of $\mathsf{KP}\omega$ such that $P \cup \{\iota\} \subset P(\iota)$, cf. [8].

For finite lists $\vec{a} = (a_1, \ldots, a_n)$, $P(\vec{a}) := (\cdots P(a_1) \cdots)(a_n)$.

Inspired by operator controlled derivations due to W. Buchholz [9], let us define a relation $P \vdash^{\alpha}_{m} \Gamma$ for transitive models $P \in V \cup \{V\}$ of $\mathsf{KP}\omega$. The relation $P \vdash^{\alpha}_{m} \Gamma$ is defined as a fixed point of a strictly positive formula H

$$H(P, \alpha, m, \Gamma) \Leftrightarrow P \vdash^{\alpha}_{m} \Gamma$$

in $\mathrm{FiX}^{i}(\mathsf{KP}i)$.

Note that P contains the code $\langle 1, 0\rangle = \lceil \Omega \rceil$, and is closed under ordinal addition $(\alpha, \beta) \mapsto \alpha + \beta$, exponentiation $\alpha \mapsto \omega^{\alpha}$ for $\alpha, \beta \in Ord^{\varepsilon}$ and $a \mapsto rank(a)$ for $rank(a) = \sup\{rank(b) + 1 : b \in a\}$.

Definition 3.6 *Let $P \in V \cup \{V\}$ be a transitive model of $\mathsf{KP}\omega$, $\alpha < \varepsilon_{\Omega+1}$ and $m < \omega$.*

$P \vdash^{\alpha}_{m} \Gamma$ holds if

$$\mathsf{k}(\Gamma) \cup \{\alpha\} \subset P \tag{3}$$

and one of the following cases holds:

($\bigvee$) *there is an $A \in \Gamma$ such that $A \simeq \bigvee(A_{\iota})_{\iota \in J}$, and for an $\iota \in J$ and an $\alpha(\iota) < \alpha$, $P \vdash^{\alpha(\iota)}_{m} \Gamma, A_{\iota}$.*

$$\frac{P \vdash^{\alpha(\iota)}_{m} \Gamma, A_{\iota}}{P \vdash^{\alpha}_{m} \Gamma} \ (\bigvee)$$

($\bigwedge$) *there is an $A \in \Gamma$ such that $A \simeq \bigwedge(A_{\iota})_{\iota \in J}$, and for any $\iota \in J$, there is an $\alpha(\iota)$ such that $\alpha(\iota) < \alpha$ and $P(\iota) \vdash^{\alpha(\iota)}_{m} \Gamma, A_{\iota}$.*

$$\frac{\{P(\iota) \vdash^{\alpha(\iota)}_{m} \Gamma, A_{\iota} : \iota \in J\}}{P \vdash^{\alpha}_{m} \Gamma} \ (\bigwedge)$$

(cut) *there are C and α_0, α_1 such that $\mathrm{dp}(C) < m$, $\alpha_0, \alpha_1 < \alpha$, and $P \vdash^{\alpha_0}_{m} \Gamma, \neg C$ and $P \vdash^{\alpha_1}_{m} C, \Gamma$.*

$$\frac{P \vdash^{\alpha_0}_{m} \Gamma, \neg C \quad P \vdash^{\alpha_1}_{m} C, \Gamma}{P \vdash^{\alpha}_{m} \Gamma} \ (cut)$$

(Ref_{N+1}) *there are $A(c) \in \Pi_{N+1}$ and $\alpha_0, \alpha_1 < \alpha$ such that $P \vdash^{\alpha_0}_{m} \Gamma, A(c)$ and $P \vdash^{\alpha_1}_{m} \forall z[ad^{z} \to c \in z \to \neg A^{z}], \Gamma$.*

$$\frac{P \vdash^{\alpha_0}_{m} \Gamma, A(c) \quad P \vdash^{\alpha_1}_{m} \forall z[ad^{z} \to c \in z \to \neg A^{z}], \Gamma}{P \vdash^{\alpha}_{m} \Gamma} \ (Ref_{N+1})$$

In what follows, let us fix an integer n_0 and restrict (codes of) ordinals to $\alpha <^{\varepsilon} \lceil \omega_{n_0}(\Omega + 1)\rceil$. n_0 is chosen from the given finite proof of a Π_{N+1}-sentence A in $\mathsf{KP}\Pi_{N+1}$, cf. Corollary 3.9 (Embedding). Since n_0 is a constant, we see from Lemma 3.1 that $\mathrm{FiX}^{i}(\mathsf{KP}i)$ proves transfinite induction schema up to $\lceil \omega_{n_0}(\Omega + 1)\rceil$ for any formula in which the derivability relation $P \vdash^{\alpha}_{m} \Gamma$ may occur.

Proposition 3.7 *Let $P' \supset P$ be a transitive model of* $\mathsf{KP}\omega$*, $m \leq m' < \omega$ and $\mathsf{k}(\Delta) \cup \{\alpha'\} \subset P'$. If $P \vdash^{\alpha}_{m} \Gamma$, then $P' \vdash^{\alpha'}_{m'} \Gamma, \Delta$.*

In embedding $\mathsf{KP}\Pi_{N+1}$ in the infinitary calculus, it is convenient to formulate $\mathsf{KP}\Pi_{N+1}$ in (finitary) one-sided sequent calculus of the language $\{\in, 0\}$ with the individual constant 0 for the empty set. Axioms are logical ones $\Gamma, \neg A, A$ for any formulas A, and axioms in the theory $\mathsf{KP}\Pi_{N+1}$. Inference rules are $(\vee)$, $(\wedge)$ for propositional connectives, $(b\exists)$, $(b\forall)$ for bounded quantifications, $(\exists)$, $(\forall)$ for unbounded quantifications, and (cut). For details, see the proof of the following Lemma 3.8.

Though the following Lemmas 3.8, 3.10 and 3.11 are seen as in [9], we give proofs of them for readers' convenience.

Let $(m, \vec{a}) := \Omega \cdot m + 3rank(a_1)\# \cdots \# 3rank(a_n)$ for $\vec{a} = (a_1, \ldots, a_n)$ and the natural (commutative) sum $\alpha\#\beta$ of ordinals α, β.

Lemma 3.8 *Suppose $\mathsf{KP}\Pi_{N+1} \vdash \Gamma(\vec{x})$, where free variables occurring in the sequent are among the list $\vec{x}$. Then there is an $m < \omega$ such that for any $\vec{a} \subset V$ and any transitive model $P \in V \cup \{V\}$ of* $\mathsf{KP}\omega$*, $P(\vec{a}) \vdash^{(m,\vec{a})}_{m} \Gamma(\vec{a})$.*

Proof. First consider the logical axiom $\Gamma(\vec{x}), \neg A(\vec{x}), A(\vec{x})$. We see that for any $\vec{a}$

$$P(\vec{a}) \vdash^{2d}_{0} \Gamma(\vec{a}), \neg A(\vec{a}), A(\vec{a}) \tag{4}$$

by induction on $d = \mathrm{dp}(A)$.

Then by Proposition 3.7 we have $P(\vec{a}) \vdash^{(2d,\vec{a})}_{2d} \Gamma(\vec{a}), \neg A(\vec{a}), A(\vec{a})$.

If $d = 0$, then $A \in \Delta_0$ and one of $\neg A(\vec{a})$ and $A(\vec{a})$ is true. Hence by $(\bigwedge)$ we have $P(\vec{a}) \vdash^{0}_{0} \Gamma(\vec{a}), \neg A(\vec{a}), A(\vec{a})$.

Next consider the case when $A \equiv (\exists y\, B(\vec{x}, y)) \notin \Delta_0$ with $\mathrm{dp}(B(\vec{x}, y)) = d - 1$. By IH(=Induction Hypothesis) we have for any $\vec{a} \subset V$ and any $b \in V$, $P(\vec{a} * (b)) \vdash^{2d-2}_{0} \Gamma(\vec{a}), \neg B(\vec{a}, b), B(\vec{a}, b)$, where $(a_1, \ldots, a_n) * (b) = (a_1, \ldots, a_n, b)$. $(\bigvee)$ yields $P(\vec{a} * (b)) \vdash^{2d-1}_{0} \Gamma(\vec{a}), \neg B(\vec{a}, b), \exists y\, B(\vec{a}, y)$. Hence $(\bigwedge)$ with $P(\vec{a} * (b)) = P(\vec{a})(b)$ yields $P(\vec{a}) \vdash^{2d}_{0} \Gamma(\vec{a}), \neg \exists y\, B(\vec{a}, y), \exists y\, B(\vec{a}, y)$.

The cases $A \equiv (\exists y \in a\, B(\vec{x}, y)) \notin \Delta_0$ and $A \equiv (B_0 \vee B_1) \notin \Delta_0$ are similar. Thus (4) was shown.

Second consider the inference rule $(\exists)$ with $\exists y\, A(\vec{x}, y) \in \Gamma(\vec{x})$

$$\frac{\Gamma(\vec{x}), A(\vec{x}, t)}{\Gamma(\vec{x})}\ (\exists)$$

When t is a variable y, we can assume that y is an x_i in the list $\vec{x}$, for otherwise substitute 0 for y. By IH there is an m such that $P(\vec{a}) \vdash^{(m,\vec{a})}_{m} \Gamma(\vec{a}), A(\vec{a}, t')$ where $t' \equiv a_i$ if $t \equiv x_i$, and $t' \equiv 0$ otherwise. Thus $P(\vec{a}) \vdash^{(m+1,\vec{a})}_{m+1} \Gamma(\vec{a})$.

Third consider the inference rule $(\forall)$ with $\forall y\, A(\vec{x}, y) \in \Gamma(\vec{x})$

$$\frac{\Gamma(\vec{x}), A(\vec{x}, y)}{\Gamma(\vec{x})}\ (\forall)$$

where the variable y does not occur in $\Gamma(\vec{x})$. IH yields for an m, $P(\vec{a}*(b)) \vdash_m^{(m,\vec{a}*(b))} \Gamma(\vec{a}), A(\vec{a},b)$. $(\bigwedge)$ with $(m+1,\vec{a}) > (m,\vec{a}*(b))$ yields $P(\vec{a}) \vdash_{m+1}^{(m+1,\vec{a})} \Gamma(\vec{a})$.

The following cases are similarly seen.

$$\frac{\Gamma, t \in s \quad \Gamma, B(\vec{x},t)}{\Gamma, \exists y \in s\, B(\vec{x},y)}\ (b\exists)\ \frac{\Gamma, y \notin s, B(\vec{x},y)}{,\Gamma, \forall y \in s\, B(\vec{x},y)}\ (b\forall)\ \frac{\Gamma, A_0, A_1}{,\Gamma, A_0 \vee A_1}\ (\vee)\ \frac{\Gamma, A_0 \quad \Gamma, A_1}{,\ \Gamma, A_0 \wedge A_1}\ (\wedge)$$

In a cut inference

$$\frac{\Gamma(\vec{x}), \neg A(\vec{x}) \quad A(\vec{x}), \Gamma(\vec{x})}{\Gamma(\vec{x})}\ (cut)$$

if the cut formula $A(\vec{x})$ has free variables $\vec{y}$ other than $\vec{x}$, then substitute 0 for $\vec{y}$.

In what follows let us suppress parameters.

Fourth consider the axioms other than Foundation. For example, consider the Δ_0-Collection $\forall x \in a \exists y\, A(x,y) \to \exists z \forall x \in a \exists y \in z\, A(x,y)$ for $A \in \Delta_0$ and $a \in V$. Since $P(a)$ is a transitive model of $\mathsf{KP}\omega$ and $a \in P(a)$, pick a $b \in P(a)$ such that $\forall x \in a \exists y\, A(x,y) \to \forall x \in a \exists y \in b\, A(x,y)$ holds in $P(a)$. Then $\neg\forall x \in a \exists y\, A(x,y) \vee \forall x \in a \exists y \in b\, A(x,y)$ is a true Δ_0-sentence. Hence $P(a) \vdash_0^0 \neg\forall x \in a \exists y\, A(x,y), \forall x \in a \exists y \in b\, A(x,y)$. Three $(\bigvee)$'s yield $P(a) \vdash_0^3 \forall x \in a \exists y\, A(x,y) \to \exists z \forall x \in a \exists y \in z\, A(x,y)$.

Next consider the axiom $A(c) \to \exists z[ad^z \wedge c \in z \wedge A^z]$ for $A \in \Pi_{N+1}$. We have by (4) for $d = \mathrm{dp}(A)$

$$\frac{P(c) \vdash_0^{2d} \neg A(c), A(c) \quad P(c) \vdash_0^2 \forall z[ad^z \to c \in z \to \neg A^z], \exists z[ad^z \wedge c \in z \wedge A^z]}{P(c) \vdash_0^{2d+1} \neg A(c), \exists z[ad^z \wedge c \in z \wedge A^z]}\ (Ref_{N+1})$$

In this way we see that there are cut-free infinitary derivations of finite heights deducing axioms in $\mathsf{KP}\Pi_{N+1}$ other than Foundation.

Finally consider the Foundation. Let $d = \mathrm{dp}(A)$ and $B \equiv (\neg\forall x(\forall y \in x\, A(y) \to A(x)))$. We show by induction on $rank(a)$ that

$$P(a) \vdash_0^{2d+3rank(a)} B, \forall x \in a\, A(x) \tag{5}$$

By IH we have for any $b \in a$, $P(b) \vdash_0^{2d+3rank(b)} B, \forall x \in b\, A(x)$. Thus we have by (4)

$$\dfrac{\dfrac{P(b) \vdash_0^{2d+3rank(b)} B, \forall x \in b\, A(x) \quad P(b) \vdash_0^{2d} \neg A(b), A(b)}{P(b) \vdash_0^{2d+3rank(b)+1} B, \forall x \in b\, A(x) \wedge \neg A(b), A(b)}\ (\bigwedge)}{P(b) \vdash_0^{2d+3rank(b)+2} B, A(b)}\ (\bigvee)$$

Therefore (5) was shown.

$$\frac{\{P(a,b) \vdash_0^{2d+3rank(b)+2} B, A(b) : b \in a\}}{P(a) \vdash_0^{2d+3rank(a)} B, \forall x \in a\, A(x)}\ (\bigwedge)$$

$\square$

Corollary 3.9 (Embedding) *If* $\mathsf{KP}\Pi_{N+1} \vdash A$ *for a sentence* A*, then there is an* $m < \omega$ *such that for any transitive model* $P \in V \cup \{V\}$ *of* $\mathsf{KP}\omega$*,* $P \vdash^{\Omega \cdot m}_{m} A$*.*

Lemma 3.10 (Reduction) *Let* $C \simeq \bigvee(C_\iota)_{\iota \in J}$*. Then*

$$(P \vdash^{\alpha}_{m} \Delta, \neg C) \,\&\, (P \vdash^{\beta}_{m} C, \Gamma) \,\&\, (\mathrm{dp}(C) \leq m) \Rightarrow P \vdash^{\alpha+\beta}_{m} \Delta, \Gamma$$

Proof. This is seen by induction on β.

Consider first the case when C is a Δ_0-sentence. Then C is false and $J = \emptyset$. From $P \vdash^{\beta}_{m} C, \Gamma$ we see that $P \vdash^{\beta}_{m} \Gamma$. $\beta \leq \alpha + \beta$ yields $P \vdash^{\alpha+\beta}_{m} \Delta, \Gamma$.

Next assume that the last inference rule in $P \vdash^{\beta}_{m} C, \Gamma$ is a $(\bigvee)$ with the main formula $C \not\in \Delta_0$:

$$\frac{P \vdash^{\beta(\iota)}_{m} C, C_\iota, \Gamma}{P \vdash^{\beta}_{m} C, \Gamma} \ (\bigvee)$$

where $\iota \in J$ and $\beta(\iota) < \beta$. We can assume that ι occurs in C_ι. Otherwise set $\iota = 0$. Thus $\iota \in P$ by (3). On the other hand we have $P(\iota) \vdash^{\alpha}_{m} \Delta, \neg C_\iota$ by inversion, and hence $P \vdash^{\alpha}_{m} \Delta, \neg C_\iota$ by $\iota \in P$.

IH yields $P \vdash^{\alpha+\beta(\iota)}_{m} C_\iota, \Delta, \Gamma$. A cut inference with $P \vdash^{\alpha}_{m} \Delta, \neg C_\iota$ and $\mathrm{dp}(C_\iota) < \mathrm{dp}(C) \leq m$ yields $P \vdash^{\alpha+\beta}_{m} \Delta, \Gamma$. □

Lemma 3.11 (Predicative Cut-elimination) $P \vdash^{\alpha}_{m+1} \Gamma \Rightarrow P \vdash^{\omega^{\alpha}}_{m} \Gamma$*.*

Proof. This is seen by induction on α using Reduction 3.10 and the fact: $\beta < \alpha \Rightarrow \omega^{\beta} + \omega^{\beta} \leq \omega^{\alpha}$. □

For $\alpha <^{\varepsilon} \lceil \omega_n(\Omega+1) \rceil$, set $RM^{\alpha}_{N} := RM_N(\alpha; <^{\varepsilon})$.

Proposition 3.12 *Let* $\Gamma \subset \Pi_{N+1}$ *and* $P \in RM^{\alpha}_{N}$ *be a transitive model of* $\mathsf{KP}i$*. Assume*

$$\exists \xi, x \in P(0 <^{\varepsilon} \xi <^{\varepsilon} \alpha \wedge \forall Q \in RM^{\xi}_{N} \cap P(x \in Q \models \mathsf{KP}i \to \Gamma \text{ is true in } Q)).$$

Then Γ *is true in* P*.*

Proof. By $P \in RM^{\alpha}_{N}$ we have $P \in RM_N(RM^{\xi}_{N})$ for any $\xi \in P$ such that $\xi <^{\varepsilon} \alpha$.

Suppose the Σ_{N+1}-sentence $\varphi := \bigwedge \neg\Gamma := \bigwedge\{\neg\theta : \theta \in \Gamma\}$ is true in P. Then for any $\xi \in P$ with $\xi <^{\varepsilon} \alpha$ and $x \in P$ there exists a transitive model $Q \in RM^{\xi}_{N} \cap P$ of $\mathsf{KP}i$ minus Δ_0-Collection such that $x \in Q$ and φ is true in Q. Let $0 <^{\varepsilon} \xi <^{\varepsilon} \alpha$. Then any $Q \in RM^{\xi}_{N}$ is Π_N-reflecting for $N \geq 2$, and hence is a model of Δ_0-Collection. □

Lemma 3.13 (Elimination of (Ref_{N+1})) *Let* $\Gamma \subset \Pi_{N+1}$*. Suppose* $P_0 \vdash^{\alpha}_{0} \Gamma$*,* $P_0 \in P$ *and* $P \in RM^{\alpha}_{N}$ *for a transitive model* P *of* $\mathsf{KP}i$*. Then* Γ *is true in* P*.*

Proof. This is seen by induction on α. Let $P_0 \vdash^{\alpha}_0 \Gamma$, $P_0 \in P$ and $P \in RM^{\alpha}_N$ be a transitive model P of $\mathsf{KP}i$. Note that any sentence occurring in the witnessed derivation of $P_0 \vdash^{\alpha}_0 \Gamma$ is Π_{N+1}.

Case 1. First consider the case when the last inference is a (Ref_{N+1}): By (3) we have $\{\alpha_\ell, \alpha_r\} \subset P_0 \subset P$, $\max\{\alpha_\ell, \alpha_r\} <^{\varepsilon} \alpha$, $A \in \Pi_{N+1}$.

$$\frac{P_0 \vdash^{\alpha_\ell}_0 \Gamma, A(c) \quad P_0 \vdash^{\alpha_r}_0 \forall z[ad^z \to c \in z \to \neg A^z(c)], \Gamma}{P_0 \vdash^{\alpha}_0 \Gamma}\ (Ref_{N+1})$$

If $\alpha_\ell = 0$, then there is a $B \in \Gamma \cup A(c)$ such that $B \simeq \bigwedge(B_\iota)_{\iota\in\emptyset}$, i.e., B is either a true Δ_0-sentence or a sentence $\forall x \in \emptyset\, C(x)$. In each case we can assume $B \in \Gamma$, and $B^P \in \Gamma^P$ is true.

In what follows assume $0 <^{\varepsilon} \alpha_\ell$. We can assume that c occurs in $A(c)$, and hence $c \in P_0$.

By Proposition 2.1 we have $P \in RM^{\alpha_r}_N$. From IH we see that

$$\text{either } \forall z \in P[ad^z \to c \in z \to \neg A^z(c)] \text{ or } \bigvee \Gamma^P \text{ is true.} \tag{6}$$

On the other hand by IH we have for any $Q \in RM^{\alpha_\ell}_N \cap P$ with $c \in P_0 \in Q \models \mathsf{KP}i$ that either $\bigvee \Gamma^Q$ is true or $A(c)^Q$ is true. By (6) for any $Q \in RM^{\alpha_\ell}_N \cap P$ with $P_0 \in Q \models \mathsf{KP}i$, $\bigvee \Gamma^Q \vee \bigvee \Gamma^P$ is true. From Proposition 3.12 with $0 <^{\varepsilon} \alpha_\ell$ we see that $\bigvee \Gamma^P$ is true.

Case 2. Second consider the case when the last inference is a $(\bigwedge)$: we have $A \simeq \bigwedge(A_\iota)_{\iota\in J}$, $A \in \Gamma$, and $\alpha(\iota) < \alpha$ for any $\iota \in J$

$$\frac{\{P_0(\iota) \vdash^{\alpha(\iota)}_0 \Gamma, A_\iota : \iota \in J\}}{P_0 \vdash^{\alpha}_0 \Gamma}\ (\bigwedge)$$

For any $\iota \in P$ we have $P_0(\iota) \in P$ since P is assumed to be a limit of admissibles.

IH yields for any $\iota \in P$ that either $\bigvee \Gamma^P$ is true or A^P_ι is true. If $J = V$, then we are done. If $J = a \in V$, then $a \in P_0 \subset P$ by (3) , and hence $a \subset P$.

Case 3. Third consider the case when the last inference is a $(\bigvee)$: we have $A \simeq \bigvee(A_\iota)_{\iota\in J}$, $A \in \Gamma$, and $\alpha(\iota) < \alpha$ for an $\iota \in J$

$$\frac{P_0 \vdash^{\alpha(\iota)}_0 \Gamma, A_\iota}{P_0 \vdash^{\alpha}_0 \Gamma}\ (\bigvee)$$

IH yields that either $\bigvee \Gamma^P$ is true or A^P_ι is true. Consider the case when $J = V$. We can assume that ι occurs in A_ι. Then $\iota \in P_0 \subset P$. Hence $\bigvee \Gamma^P$ is true. $\Box$

Let us prove Theorem 2.2. Let $N \geq 2$, and A be a Π_{N+1}-sentence provable in $\mathsf{KP}\Pi_{N+1}$. By Embedding 3.9 and Predicative Cut-elimination 3.11 we have for an $m < \omega$, $L_{\omega_1^{CK}} \vdash^{\omega_m(\Omega\cdot m)}_0 A$ with $\emptyset = \mathsf{k}(A)$, and $L_{\omega_1^{CK}} \vdash^{\omega_n(\Omega+1)}_0 A$ for an

$n > m$ with $n < n_0$. If $V \in RM_N^{\omega_n(\Omega+1)}$, then $L_{\omega_1^{CK}} \in V \models \mathsf{KP}i$, and A is true (in V) by Elimination of (Ref_{N+1}) 3.13.

By formalizing the above proof in $\mathrm{FiX}^i(\mathsf{KP}i)$ with Lemma 3.1 yields

$$\mathrm{FiX}^i(\mathsf{KP}i) \vdash V \in RM_N(\lceil \omega_n(\Omega+1) \rceil; <^\varepsilon) \to A$$

and then by Theorem 3.2 and $\mathsf{KP}\omega \vdash V \in RM_N(\lceil \omega_n(\Omega+1) \rceil; <^\varepsilon) \to \forall x \exists y [x \in y \wedge ad^y]$

$$\mathsf{KP}\omega \vdash V \in RM_N(\lceil \omega_n(\Omega+1) \rceil; <^\varepsilon) \to A.$$

In the formalization note that we have in $\mathrm{FiX}^i(\mathsf{KP}i)$, a partial truth definition of Π_{N+1}-sentences.

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