The Number of Atomic Models of Uncountable Theories

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Abstract

We show there exists a complete theory in a language of size continuum possessing a unique atomic model which is not constructible. We also show it is consistent with $ZFC + \aleph_1 < 2^{\aleph_0}$ that there is a complete theory in a language of size \aleph_1 possessing a unique atomic model which is not constructible. Finally we show it is consistent with $ZFC + \aleph_1 < 2^{\aleph_0}$ that for every complete theory T in a language of size \aleph_1 , if T has uncountable atomic models but no constructible models, then T has 2^{\aleph_1} atomic models of size \aleph_1 .

1 Introduction

There are several model-theoretic notions of "smallness," namely: a model M is atomic if every tuple $\overline{a} \in M$ has its type isolated by a single formula; a model M is prime if for every $N \equiv M$, there is an elementary embedding of M into N; and a model M is constructible if there is a sequence $M = (a_{\alpha} : \alpha < \alpha^*)$ such that each $tp(a_{\alpha}/\{a_{\beta} : \beta < \alpha\})$ is isolated by a single formula.

If we are just interested in the complete theories in countable languages, then then these notions all coincide, by an old theorem of Vaught [14] (essentially):

Theorem 1.1. For models of T a countable complete theory, the notions "countable atomic," "prime" and "constructible" coincide. Such a model exists if and only if the isolated types are dense in the Stone spaces $S^n(\emptyset)$ for all n; when they exist they are unique up to isomorphism.

When we ask about theories in uncountable languages, things get harder. We have the following examples:

• Laskowski and Shelah [5]: there is a complete theory T in a language of size \aleph_2 , such that the isolated types are dense in $S^n(\emptyset)$ for all n, but T has no atomic models.

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- Knight [3]: there is a complete theory in a language of size \aleph_1 , with atomic models but no prime models.
- Folklore: there is a complete theory in a language of size continuum, with prime models but no atomic models. Namely $Th(2^{\omega}, \mathbf{f}, U_n)_{f \in 2^{\omega}, n \in \omega}$, where $\mathbf{f}(g) = f \oplus g \mod 2$, and $U_n(g)$ holds iff g(n) = 1.
- Shelah [12]: there is a complete theory in a language of size \aleph_1 , with models that are atomic but not prime, and with models that are prime but not constructible, and with a constructible model. Namely $Th(\omega^{\omega_1}, E_{\alpha} : \alpha < \omega_1)$, where $\eta E_{\alpha} \tau$ iff $\eta \upharpoonright_{\alpha} = \tau \upharpoonright_{\alpha}$.

The following, one of the few positive results, was proved by Ressayre, see for example [6]:

Theorem 1.2. Let T be a complete theory in an arbitrary language. If T has a constructible model M, then M is unique up to isomorphism; it is furthermore prime and atomic. Also, the construction sequence for M can be chosen of order type |T|.

And the following was proved independently by Knight [3], Kueker [4] and Shelah [9]:

Theorem 1.3. Let T be a complete theory in a language of size \aleph_1 . Then T has an atomic model if and only if the isolated types are dense in $S^n(\emptyset)$ for all n.

In this paper we are specifically interested in looking at the atomic models of T; we wonder when, for example, there exists a constructible model. Knight's example above shows that the answer is "not always" but we would like to say more. In fact Knight's example has 2^{\aleph_1} models of size \aleph_1 . We wonder if this is a necessary feature: that is, suppose T is a complete theory in a language of size κ , with a unique atomic model M of size $\leq \kappa$. Must M be constructible?

By Vaught's Theorem 1.1, for $\kappa = \aleph_0$ we know this to be true. We introduce the following examples to show it is false for $\kappa = 2^{\aleph_0}$.

First Example: Theorem 3.1. There is a complete theory in a language of size continuum, with a unique atomic model, which is not prime. (Hence there are no prime models.)

Second Example: Remark 4.6. There is a complete theory in a language of size continuum, with a unique atomic model, which is furthermore prime, but which is not constructible.

Do we need continuum? This is only interesting if $\aleph_1 < 2^{\aleph_0}$; and with that assumption it turns out to be independent of ZFC. In particular we have the following theorems:

Third Example: Theorem 4.1. IT is consistent with $ZFC + \aleph_1 < 2^{\aleph_0}$ that there is a complete theory in a language of size \aleph_1 , with a unique atomic model, which is furthermore prime, but which is not constructible.

Theorem 1.4. It is consistent with $ZFC + \aleph_1 < 2^{\aleph_0}$ that whenever T is a complete theory in a language of size \aleph_1 , if T has atomic models but no constructible models, then T has 2^{\aleph_1} atomic models of size \aleph_1 .

The paper is organized as follows:

In Section 2 we explain the various set-theoretic tools we use in the paper, and give sharper statements of the Third Example and of Theorem 1.4. The First Example is given in Section 3, the Second and Third Examples are given in Section 4 and in Section 5 we prove Theorem 1.4.

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2 Background, and Statement of Results

We first review the set-theoretic notions required for the consistency proofs. [7] serves as a general reference.

2.1 Ladder Systems

Let $\Lambda \subseteq \omega_1$ be the limit ordinals. Suppose $S \subset \Lambda$ is stationary. A ladder system $(L_{\alpha} : \alpha \in S)$ is a sequence of subsets of ω_1 such that for each $\alpha \in S$, $L_{\alpha} \subset \alpha$ is cofinal and of order type ω . $(L_{\alpha} : \alpha \in S)$ has the uniformization property if for every sequence $(f_{\alpha} : \alpha \in S)$ of functions $f_{\alpha} : L_{\alpha} \to 2$, there is some $f : \omega_1 \to 2$ such that for all $\alpha \in S$, $\{b \in L_{\alpha} : f_{\alpha}(\beta) \neq f(\beta)\}$ is finite.

We have the following, proven by Devlin-Shelah [2]:

Theorem 2.1. Martin's Axiom, together with $\aleph_1 < 2^{\aleph_0}$, implies that every ladder system on Λ has the uniformization property (and hence that every ladder system on any stationary S has the uniformization property.)

In particular $ZFC + \aleph_1 < 2^{\aleph_0} +$ "every ladder system on Λ has the uniformiation property" is equiconsistent with ZFC.

The uniformization property was originally introduced to analyze Whitehead groups. Namely, Shelah showed in [13] that there is a non-free Whitehead group of size \aleph_1 if and only if for some stationary $S \subset \Lambda$, some ladder system on S has the uniformization property.

We sharpen the Third Example as follows:

Third Example, Sharp Version. Suppose there is some stationary $S \subseteq \omega_1$ that admits a ladder system. Then there is a theory T in a language of size \aleph_1 such that T has a unique atomic model, which is furthermore prime, but which is not constructible.

2.2 The Weak Diamond Principle

If $S \subseteq \omega_1$ is stationary, then $\Phi(S)$ ("weak diamond on S") is the combinatorial guessing-principle which states that for every $F: 2^{<\omega_1} \to 2$, there is some $g: S \to 2$ such that for every $f: \omega_1 \to 2$, the set $\{\alpha \in S: F(f \upharpoonright_{\alpha}) = g(\alpha)\}$ is stationary. So the smaller S is, the stronger $\Phi(S)$ is; $\Phi(\omega_1)$ is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$.

Definition 2.2. Let Φ^* abbreviate: for all stationary $S \subset \omega_1$, $\Phi(S)$ holds.

It is easy to show that, for example, $\Phi(S)$ holds if and only if for every $F: (2 \times 2 \times \omega_1)^{<\omega_1} \to 2$, there is some $g: S \to 2$ such that for every $f_0, f_1: \omega_1 \to 2$ and for every $h: \omega_1 \to \omega_1$, the set $\{\alpha \in S: F(f_0 \upharpoonright_{\alpha}, f_1 \upharpoonright_{\alpha}, h \upharpoonright_{\alpha}) = g(\alpha)\}$ is stationary.

These principles were introduced by Devlin and Shelah [2], where they proved the following theorems:

Theorem 2.3. 1. $\Phi(\omega_1)$ is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$.

- 2. Suppose $\Phi(S)$ holds. Then we can write S as the disjoint union of stationary sets $(S_{\alpha}: \alpha < \omega_1)$ such that $\Phi(S_{\alpha})$ holds for each α .
- 3. Suppose $S \subseteq \Lambda$ is stationary. If $\Phi(S)$ holds then no ladder system on S has the uniformization property.

In view of the first item, Φ^* is a strengthening of $2^{\aleph_0} < 2^{\aleph_1}$.

2.3 The Covering Number

Let $\mathsf{Cov}(\mathcal{K})$ be the covering number of the σ -ideal of meager sets: i.e. the least κ such that 2^{ω} is the union of κ -many closed nowhere dense sets. This is a well-understood cardinal invariant of the continuum. In particular $\omega < \mathsf{Cov}(\mathcal{K}) \leq 2^{\aleph_0}$, and if Martin's Axiom holds then $\mathsf{Cov}(\mathcal{K}) = 2^{\aleph_0}$.

So $\Phi^* \wedge \mathsf{Cov}(\mathcal{K}) \geq \aleph_2$ says that $\aleph_1 < 2^{\aleph_0} < 2^{\aleph_1}$ in a strong way. This assertion is consistent: let \mathbb{P} be the forcing notion $\mathbb{P}_0 \times \mathbb{P}_1$ where $\mathbb{P}_0 = \mathrm{Fn}(\omega_2, 2, \omega)$ and $\mathbb{P}_1 = \mathrm{Fn}(\omega_3, 2, \omega_1)$. (Here $\mathrm{Fn}(X, Y, \kappa)$ is the set of all partial functions f with domain $\subseteq X$ and range $\subseteq Y$, and with $|f| < \kappa$.) \mathbb{P} is the standard forcing notion for arranging $2^{\aleph_0} = \aleph_2$, $2^{\aleph_1} = \aleph_3$, starting from GCH. Then we have:

Theorem 2.4. Suppose $\mathbb{V} \models GCH$ and G is \mathbb{P} -generic over \mathbb{V} . Then $\mathbb{V}[G] \models \Phi^* \land \mathsf{Cov}(\mathcal{K}) \geq \aleph_2$.

Proof. It is shown in [11] (Theorem 2.11 from the appendix) that $\mathbb{V}[G] \models \Phi^*$.

Note that these forcing notions all preserve cardinals, so we can refer to ω_1 , etc., without ambiguity.

Let G_1 be \mathbb{P}_1 -generic over \mathbb{V} . Working in $\mathbb{V}[G_1]$, we show that if G_0 is \mathbb{P}_0 -generic over $\mathbb{V}[G_1]$ then $\mathbb{V}[G_1][G_0] \models \mathsf{Cov}(\mathcal{K}) \geq \aleph_2$.

Indeed, suppose $(C_{\alpha} : \alpha < \omega_1)$ is a sequence in $\mathbb{V}[G_1][G_0]$ of closed nowhere dense subsets of $(\omega^{\omega})^{\mathbb{V}[G_1][G_0]}$. Let $x_{\alpha} \in (\omega^{\omega})^{\mathbb{V}[G_1][G_0]}$ encode C_{α} .

In $V[G_1]$, write $\omega_2 = I \cup J$ where I, J are disjoint, $|I| \leq \aleph_1$, and such that setting $H_0 = G_0 \upharpoonright_I$, $H_1 = G_0 \upharpoonright_J$, we have that $(x_\alpha : \alpha < \omega_1) \subset V[G_1][H_0]$. Choose a real $x \in V[G_1][G_0] = V[G_1][H_0][H_1]$ such that x is Cohen over $V[G_1][H_0]$; then $x \notin C_\alpha$ for all $\alpha < \omega_1$, showing that $\bigcup_{\alpha \in V} C_\alpha \neq (\omega^\omega)^{V[G_0][G_1]}$.

2.4 The Main Theorem

For T a complete theory in a countable language, the question of the number of atomic models of T of size \aleph_1 has been closely investigated. First of all, such models exist if and only if T has a (unique) countable atomic model, which furthermore has a proper atomic extension. Assuming this, let \mathbf{K}_T be the class of atomic models of T.

Now say that \mathbf{K}_T is ω -stable if $S^n_{at}(M)$ is countable for all n, where M is some countable atomic model of T, and $S^n_{at}(M)$ is the set of all n-types $p(\overline{x}) \in S^n(M)$ such that $M\overline{a}$ is atomic whenever \overline{a} realizes $p(\overline{x})$.

Then we have the following theorems of Shelah [8] [10] (or see [1] for an exposition):

Theorem 2.5. Suppose $2^{\aleph_0} < 2^{\aleph_1}$, and \mathbf{K}_T is not ω -stable. Then T has 2^{\aleph_1} nonisomorphic models of size \aleph_1 .

It is not known if the assumption $2^{\aleph_0} < 2^{\aleph_1}$ is necessary here. On the other hand, if \mathbf{K}_T is ω -stable, then we have a strong enough structure theory to determine e.g. when \mathbf{K}_T is \aleph_1 -categorical.

Now, our main theorem (Theorem 2.8 below) will be essentially a generalization of Theorem 2.5, and will follow the same general proof outline, which we now describe.

Namely, the proof of Theorem 2.5 splits into cases depending on whether \mathbf{K}_T has the amalgation property at \aleph_0 . Here, an amalgamation problem at \aleph_0 (for \mathbf{K}_T) is a triple (M_0, M_1, M_2) where each M_i is a countable atomic model of T, and $M_0 \leq M_i$ for i = 1, 2. A solution to the amalgamation problem is a triple (M_3, f_1, f_2) where M_3 is a countable atomic model of T, and $f_i : M_i \leq M_3$, and $f_1 \upharpoonright_{M_0} = f_2 \upharpoonright_{M_0}$. \mathbf{K}_T has the amalgamation property at \aleph_0 if every amalgamation problem at \aleph_0 has a solution.

So to prove Theorem 2.5, we first consider the case where \mathbf{K}_T fails the amalgamation property at \aleph_0 , and then the case where \mathbf{K}_T has the amalgamation property at \aleph_0 but is not ω -stable.

But it is worth noting that we have the following Corollary 19.14 from [1]:

Theorem 2.6. If \mathbf{K}_T is ω -stable then \mathbf{K}_T has the amalgamation property at \aleph_0 .

We will also want the following strengthening (an easy consequence of Corollary 24.4 from [1]). To state it conveniently we work in a monster model \mathfrak{C} of T. Say that a set $A \subset \mathfrak{C}$ is atomic if every finite tuple from A realizes an isolated type.

Theorem 2.7. Suppose \mathbf{K}_T is ω -stable, and (A_0, A_1, A_2) is a triple of countable atomic sets with $A_0 \subseteq A_i$ for i = 1, 2. Suppose $S_{at}(A_0)$ is countable. Then (A_0, A_1, A_2) can be amalgamated by some countable atomic set A_3 .

Let T be a complete theory in a language of size \aleph_1 , and let \mathbf{K}_T be its class of atomic models. In Section 5 we define the notion " \mathbf{K}_T is club totally transcendental," generalizing the definition of ω -stability for \mathcal{L} countable. We then prove our main theorem, a sharpening of Theorem 1.4:

Theorem 2.8. Suppose Φ^* holds, and $Cov(\mathcal{K}) \geq \aleph_2$. Suppose T is a complete theory in a language of size \aleph_1 with atomic models, and \mathbf{K}_T is not club totally transcendental. Then T has 2^{\aleph_1} atomic models of size \aleph_1 .

The hypotheses can be understood as follows: the First and Second Examples require CH (to matter at \aleph_1), which $Cov(\mathcal{K}) \geq \aleph_2$ prevents; and the Third Example requires the existence of ladder systems with the uniformization property, which Φ^* prevents.

The proof of Theorem 2.8 follows the same outline as that of Theorem 2.5. Namely, we will say what it means for \mathbf{K}_T to have the club amalgamation property, then split into two cases, depending on whether \mathbf{K}_T fails the club amalgamation property, or else \mathbf{K}_T has the club amalgamation property but it not club totally transcendental.

As in the countable case we can actually show that club totally transcendental implies the club amalgamation property; this is discussed in Section 5.3. However this is not technically needed for the proof.

Finally, one obtains Theorem 1.4 quickly, since if T has no constructible models then \mathbf{K}_T is not club totally transcendental; see Section 5.2.

3 Unique Atomic Model that is not Prime

In this section I construct the First Example: namely an atomic model $\mathfrak{A} \models T$, in a language of size continuum, which has a unique atomic model that is not prime.

Given $\eta \in 2^{<\omega_1}$, let $\ell g(\eta)$ be its length, i.e. its domain.

Let $\mathcal{L} = (U_{\alpha}, \pi_{\alpha\beta}, \eta) : \beta \leq \alpha < \omega_1, \eta \in 2^{<\omega_1}$ where each U_{α} is a unary relation symbol, each $\eta : U_{\ell g(\eta)} \to U_{\ell g(\eta)}$ is a unary function symbol, and each $\pi_{\alpha\beta} : U_{\alpha} \to U_{\beta}$ is a unary function symbol. (Formally, since we are using single-sorted logic, each of these function symbols will be total, but we let their values be trivial outside their domain.)

We turn $2^{<\omega_1}$ into an \mathcal{L} -structure $\mathfrak{A} = (2^{<\omega_1}, U_{\alpha}, \pi_{\alpha\beta}, \boldsymbol{\eta} : \beta \leq \alpha < \omega_1, \eta \in 2^{<\omega_1})$ as follows:

- Interpret U_{α} as 2^{α} , i.e. all $\eta \in 2^{<\omega_1}$ with $\ell g(\eta) = \alpha$;
- Given $\tau \in 2^{\alpha}$ and $\beta \leq \alpha$, interpret $\pi_{\alpha\beta}(\tau)$ as $\tau \upharpoonright_{\beta}$;
- Given $\eta \in 2^{\alpha}$ and $\tau \in U_{\alpha}$ interpret $\eta(\tau)$ as $\eta \oplus \tau$, where the addition is pointwise mod 2.

Let T be the complete theory of \mathfrak{A} .

Theorem 3.1. \mathfrak{A} is the unique atomic model of T, and it is not prime.

The proof goes as follows. First we establish that $\mathfrak A$ is the unique atomic model of T. Then we give an axiomatization of T, and use it to exhibit a model $\mathfrak B$ of T into which $\mathfrak A$ does not embed; in fact $\mathfrak B$ will omit $tp_{\mathfrak A}(\overline{0}_{\alpha}:\alpha<\omega_1)$, where $\overline{0}_{\alpha}\in 2^{\alpha}$ is the zero sequence.

Lemma 3.2. We write down some straightforward observations:

- Given $\eta, \tau \in 2^{\alpha}, \eta \tau = \tau \eta$;
- Given $\eta \in 2^{\alpha}$ and $\beta \leq \alpha$, $\pi_{\alpha\beta} \eta = \eta \upharpoonright_{\beta} \pi_{\alpha\beta}$;
- Given $\gamma \leq \beta \leq \alpha$, $\pi_{\beta\gamma}\pi_{\alpha\beta} = \pi_{\alpha\gamma}$, and $\pi_{\alpha\alpha}$ is the identity on 2^{α} ;
- Given $\nu \in 2^{\omega_1}$, the map $f_{\nu} : \mathfrak{A} \to \mathfrak{A}$ defined by $f_{\nu}(\eta) = \nu \upharpoonright_{\ell g(\eta)} (\eta)$ is an automorphism of \mathfrak{A} ;
- For all $\eta \in 2^{\alpha}$, $\tau \in 2^{\beta}$, if $\alpha \geq \beta$ then τ is in the definable closure η , namely $\tau = \boldsymbol{\tau} \pi_{\alpha\beta} \boldsymbol{\eta}(\eta)$.
- In particular, for all η, τ , either η is in the definable closure of τ or vice versa.

Lemma 3.3. Suppose $\overline{\eta} = \eta_0 \dots \eta_{n-1}$ is a finite sequence from $2^{<\omega_1}$. Write $\alpha_i = \ell g(\eta_i)$; we can suppose $\alpha_0 \ge \alpha_i$ for all i < n. Then the formula

$$\phi_{\overline{\eta}}(x_0 \dots x_{n-1}) := U_{\alpha_0}(x_0) \wedge \bigwedge_{i < n} x_i = \eta_i \pi_{\alpha_0 \alpha_i} \eta_0(x_0)$$

isolates $tp_{\mathfrak{A}}(\overline{\eta})$. In particular \mathfrak{A} is an atomic model of T.

Proof. It is clear that $\mathfrak{A} \models \phi_{\overline{\eta}}(\overline{\eta})$. Conversely, suppose $\mathfrak{A} \models \phi_{\overline{\eta}}(\tau_0, \dots, \tau_{n-1})$. Let $\nu := (\tau_0 \oplus \eta_0) \cap \overline{0} \in 2^{\omega_1}$. Then $f_{\nu} : \mathfrak{A} \cong \mathfrak{A}$ (defined above) is an automorphism of \mathfrak{A} taking $\overline{\eta}$ to $\overline{\tau}$, so they have the same type.

Lemma 3.4. \mathfrak{A} is the unique atomic model of T.

Proof. Suppose $\mathfrak{B} \models T$ is atomic; say $\mathfrak{B} = (B, U_{\alpha}^*, \pi_{\alpha\beta}^*, \eta^* : \eta \in 2^{<\omega_1}, \beta \leq \alpha < \omega_1).$

Note that Lemma 3.3 characterizes all the complete isolated types of T. In particular $B = \bigcup U_{\alpha}^*$.

We define by induction on $\alpha < \omega_1$ an element $b_{\alpha} \in U_{\alpha}^*$ such that for all $\beta \leq \alpha < \omega_1$, $b_{\beta} = \pi_{\alpha\beta}^*(b_{\alpha})$.

There is a unique element of U_0^* , so we let that element be b_0 .

Suppose we have defined b_{α} . Then let $b_{\alpha+1}$ be either of the two elements in $U_{\alpha+1}^*$ that restrict to b_{α} .

Finally, suppose $\alpha < \omega_1$ is a limit, and we have defined b_{β} for all $\beta < \omega_1$. Let $b \in U_{\alpha}^*$ be arbitrary. For each $\beta < \alpha$, let $\eta_{\beta} \in 2^{\beta}$ be the unique function with $b_{\beta} = \eta_{\beta} \pi_{\alpha\beta}(b)$. Then $\eta_{\beta} \subseteq \eta_{\gamma}$ for $\beta \leq \gamma < \alpha$. Define $\eta = \bigcup_{\beta < \alpha} \eta_{\beta}$, and define $b_{\alpha} = \eta(b)$. This works, clearly.

So we have $(b_{\alpha}: \alpha < \omega_1)$ as desired. For each $\alpha < \omega_1$, let $\overline{0}_{\alpha} \in \mathfrak{A}$ be the zero sequence of length α . Then $f: \overline{0}_{\alpha} \mapsto b_{\alpha}$ is a partial elementary map from \mathfrak{A} into \mathfrak{B} . So f extends to a partial elementary map g from the definable closure of $\{\overline{0}_{\alpha}: \alpha < \omega_1\}$ in \mathfrak{A} to the definable closure of $\{b_{\alpha}: \alpha < \omega_1\}$.

But note that the definable closure of each $\overline{0}_{\alpha}$ contains all of U_{α} , and the definable closure of each b_{α} contains all of U_{α}^* . Hence $g:\mathfrak{A}\cong\mathfrak{B}$.

Now we provide an axiomatization of T.

Definition 3.5. Let T_0 consist of the consequences of the following axioms.

- (I) Suppose $\phi(x)$ is a quantifier-free formula of \mathcal{L} with only the variable x free. Suppose $\mathfrak{A} \models \forall x \phi(x)$. Then " $\forall x \phi(x)$ " is an axiom.
- (II) " $\exists ! x : U_0(x)$ ".
- (III) For all α , " $\forall x: U_{\alpha}(x) \to \exists^{-2}y: (U_{\alpha+1}(y) \land \pi_{\alpha+1}\alpha(y) = x)$ ".
- (IV) For all $\alpha < \beta$, " $\forall x : U_{\alpha}(x) \to \exists y : (U_{\beta}(y) \land \pi_{\beta\alpha}(y) = x)$ ".

Obviously $\mathfrak{A} \models T_0$.

Lemma 3.6. $T_0 = T$, i.e. T_0 is complete.

Proof. (Sketch.) It suffices to show that for sufficiently rich finite fragments $\mathcal{L}' \subseteq \mathcal{L}$, $T_0 \upharpoonright_{\mathcal{L}'}$ is \aleph_0 -categorical.

Temporarily define a template to be a sequence $\overline{G} = (G_{\alpha} : \alpha \in X)$ where:

- $X \subseteq \omega_1$ is finite and closed under immediate predecessors, and $0 \in X$;
- Each G_{α} is a finite subgroup of $(2^{\alpha}, \oplus)$, containing the set of all $\eta \in 2^{\alpha}$ which are zero outside of X;
- For $\beta \leq \alpha$ both in X, $G_{\alpha} \upharpoonright_{\beta} = G_{\beta}$.

Given a template $\overline{G} = (G_{\alpha} : \alpha \in X)$, let $\mathcal{L}_{\overline{G}} \subseteq \mathcal{L}$ be defined as follows: $U_{\alpha} \in \mathcal{L}_{\overline{G}}$ iff $\alpha \in X$; $\pi_{\alpha\beta} \in \mathcal{L}_{\overline{G}}$ iff $\alpha, \beta \in X$; and $\eta \in \mathcal{L}_{\overline{G}}$ iff $\ell g(\eta) \in X$ and $\eta \in G_{\ell g(\eta)}$. Let $T_{\overline{G}} = T_0 \upharpoonright_{\mathcal{L}_{\overline{G}}}$.

Then it is easy to see that each $T_{\overline{G}}$ is \aleph_0 -categorical; note for example that $T_{\overline{G}}$ proves there are infinitely many unsorted elements (i.e. elements that are not in any U_{α} for $\alpha \in X$) and that these elements are absolutely indiscernible over the rest of the model.

The following lemma concludes the proof of Theorem 3.1.

Lemma 3.7. \mathfrak{A} is not a prime model of T.

Proof. We define a model $\mathfrak{B} = (B, U_{\alpha}^*, \pi_{\alpha\beta}^*, \eta^* : \eta \in 2^{<\omega_1}, \beta \leq \alpha < \omega_1) \models T$ into which \mathfrak{A} does not embed.

- Let B be the set of all pairs (τ, s) where:
 - $-\tau \in 2^{<\omega_1};$
 - $-s \in \omega_1^{<\omega}$ is a finite, strictly increasing sequence of ordinals, with $|s| \ge 2$;
 - -s(0)=0, $s(1)=\omega$, and for all $n\geq 1$, s(n) is a limit ordinal;
 - $s(|s| 2) \le \ell g(\tau) < s(|s| 1).$
- Suppose $(\tau, s) \in B$. Then let $U_{\alpha}^*(\tau, s)$ hold iff $\tau \in 2^{\alpha}$.
- Suppose $(\tau, s) \in U_{\alpha}^*$ and $\eta \in 2^{\alpha}$. Then let $\eta^*(\tau, s) = (\eta \oplus \tau, s)$.
- Suppose $(\tau, s) \in U_{\alpha}^*$ and $\beta \leq \alpha$. Let n be such that $s(n-2) \leq \beta < s(n-1)$. Let $\pi_{\alpha\beta}^*(\tau, s) = (\tau \upharpoonright_{\beta}, s \upharpoonright_n)$.

It is routine to check that \mathfrak{B} is a model of Axiom Schemas II-IV. To check Axiom Schema I: suppose $\mathfrak{B} \models \exists x \phi(x)$, where $\phi(x)$ is a quantifier-free \mathcal{L} -formula. Say $\mathfrak{B} \models \phi(\eta, s)$. Let A_0 be the definable closure of η in \mathfrak{A} (i.e., all $\tau \in \mathfrak{A}$ with $\ell g(\tau) \leq \ell g(\eta)$) and let B_0 be the definable closure of (η, s) in \mathfrak{B} (i.e., all $(\tau, t) \in \mathfrak{B}$ with $\ell g(\tau) \leq \ell g(\eta)$ and $t \subseteq s$). Then the map $\Phi : B_0 \to A_0$ taking (τ, t) to τ is a partial isomorphism from B_0 onto A_0 . Hence $\mathfrak{A} \models \phi(\eta)$, so $\mathfrak{A} \models \exists x \phi(x)$.

So $\mathfrak{B} \models T$. Suppose towards a contradiction that $f: \mathfrak{A} \to \mathfrak{B}$ were an elementary embedding. Let $\overline{0}_{\alpha}$ be the zero sequence of length α in \mathfrak{A} , for each $\alpha < \omega_1$; and let $(\eta_{\alpha}, s_{\alpha}) = f(\overline{0}_{\alpha})$. Then we have for all $\alpha < \beta$, $\pi_{\beta\alpha}^*(\eta_{\beta}, s_{\beta}) = (\eta_{\alpha}, s_{\alpha})$. In particular, for all $\alpha < \beta$, $s_{\alpha} \subseteq s_{\beta}$.

Hence $(s_{\alpha}: \alpha < \omega_1)$ eventually stabilizes; say $s_{\alpha} = s_{\beta} = s$ for all $\alpha, \beta \geq \alpha_0$. Let $\alpha_1 = \max(s(|s|-1), \alpha_0)$. Then $\ell g(\eta_{\alpha_1}) \geq s_{\alpha}(|s_{\alpha}|-1)$, contradicting the definition of B

4 Unique Atomic Models that are Prime but not Constructible

In this section, I show the following:

Theorem 4.1. Third Example: Suppose for some stationary $S \subset \Lambda$, some ladder system $(L_{\alpha} : \alpha \in S)$ has the uniformization property. Then from this ladder system we can define a theory T in a language \mathcal{L} of size \aleph_1 , such that T has a unique atomic model, which is additionally prime, yet which is not constructible.

A small tweak (see Remark 4.6 below) gives the Second Example.

The idea is to make an example similar to the first example, except we replace the tree $(2^{<\omega_1},<)$ with a much smaller tree, in fact a tree of height $\omega+1$. (In neither example is < itself part of the language.)

Fix a stationary $S \subset \Lambda$ and a ladder system $(L_{\alpha} : \alpha \in S)$ with the uniformization property. Let $\nu_{\alpha} : \omega \to L_{\alpha}$ be the strictly increasing enumeration.

Let J_0 be the set of all strictly increasing functions $\eta_0 : \alpha \to \omega_1$, where $\alpha \le \omega$, and if $\alpha = \omega$ then $\eta_0 = \nu_\beta$ for some $\beta \in S$. So J_0 is a tree of height $\omega + 1$ under \subset .

Let $J_1 = \{\eta_1 \in 2^{\leq \omega} : \eta_1 \text{ has finite support}\}$. J_1 is also tree of height $\omega + 1$, under initial segment \subset .

Let $J = J_0 \otimes J_1$ be the product tree of pairs $\eta = (\eta_0, \eta_1)$, where $\eta_0 \in S_0$ and $\eta_1 \in S_1$ and $|\eta_0| = |\eta_1|$; we say $\eta \leq \tau$ if $\eta_i \leq \tau_i$ for each i < 2.

Given $\eta \in J$, we view η as a sequence with domain $\alpha \leq \omega$, and write $\ell g(\eta)$, $\eta \upharpoonright_n$, etc. accordingly. If $\eta, \tau, \ldots \in J$ then always $\eta = (\eta_0, \eta_1), \tau = (\tau_0, \tau_1), \ldots$

Let \mathcal{L} be the language $\{U_{\eta_0}, \pi_{\alpha}, \boldsymbol{\eta} : \eta_0 \in J_0, \eta \in J, \alpha \leq \omega\}$, where each U_{η_0} is a unary predicate, and each $\pi_{\alpha}, \boldsymbol{\eta}$ are unary function symbols. $\boldsymbol{\eta}$ will be a map $U_{\eta_0} \to U_{\eta_0}$.

We turn J into a \mathcal{L} -structure \mathfrak{A} as follows. Let $U_{\eta_0} = \{ \tau \in J : \tau_0 = \eta_0 \}$. Given $\eta \in J$ and $\alpha \leq \omega$, let $\pi_{\alpha}(\eta) = \eta \upharpoonright_{\alpha}$ (so π_{ω} is the identity map). Finally, given $\eta, \tau \in J$ with $\eta_0 = \tau_0$, define $\eta \tau = (\eta_0, \eta_1 \oplus \tau_1 \mod 2)$.

Let T be the complete theory of \mathfrak{A} . The claim is that this works.

Given $\eta_0, \eta_1 \in J_0$, then let $d(\eta_0, \eta_1)$ be the greatest $\alpha \leq \omega$ such that $\alpha \leq \ell g(\eta_0)$ and $\alpha \leq \ell g(\eta_1)$ and $\eta_0 \upharpoonright_{\alpha} = \eta_1 \upharpoonright_{\alpha}$.

Lemma 4.2. Let $\overline{\eta} = (\eta^0, \eta^1, \dots, \eta^{n-1})$ be a tuple from J. For each i, j let $\alpha_{ij} = d(\eta_0^i, \eta_0^j)$. Let $\phi_{\overline{\eta}}(x_0, \dots, x_{n-1})$ be the following formula:

$$\bigwedge_{i < n} U_{\eta_0^i}(x_i) \wedge \bigwedge_{i \neq j < n} \pi_{\alpha_{ij}}(\boldsymbol{\eta^i} x_i) = \pi_{\alpha_{ij}}(\boldsymbol{\eta^j} x_j).$$

Then $\mathfrak{A} \models \phi_{\overline{\eta}}(\overline{\eta})$ and moreover $\phi_{\overline{\eta}}(\overline{x})$ is complete.

Proof. It is clear that $\mathfrak{A} \models \phi_{\overline{\eta}}(\overline{\eta})$. We show that the formula is complete by defining, for each pair $\overline{\eta}, \overline{\tau}$ with $\mathfrak{A} \models \phi_{\overline{\eta}}(\overline{\tau})$, an automorphism $\sigma_{\overline{\eta}, \overline{\tau}} : \mathfrak{A} \cong \mathfrak{A}$ taking $\overline{\eta}$ to $\overline{\tau}$. We do this inductively on $n = |\overline{\eta}|$.

For n = 0 define $\sigma_{\emptyset,\emptyset} = id_{\mathfrak{A}}$.

Suppose we have defined $\sigma_{\overline{\eta},\overline{\tau}}$ for all $|\overline{\eta}|, |\overline{\tau}| \leq n$. Let $\overline{\eta} = (\eta^0, \dots, \eta^n)$ be given, and suppose $\mathfrak{A} \models \phi_{\overline{\eta}}(\overline{\tau})$. We can suppose, by applying $\sigma_{(\eta^0,\dots,\eta^{n-1}),(\tau^0,\dots,\tau^{n-1})}^{-1}$ to $\overline{\tau}$, that $\eta_i = \tau_i$ for each i < n. So we want to find some $\sigma : \mathfrak{A} \cong \mathfrak{A}$ such that $\sigma(\eta^i) = \eta^i$ for each i < n, and $\sigma(\eta^n) = \tau^n$.

Let α_{ij} be as in the definition of $\phi_{\overline{\eta}}(\overline{x})$: $\alpha_{ij} = d(\eta_0^i, \eta_0^j)$.

We know that $\eta_0^n = \tau_0^n$. If $\eta_1^n = \tau_1^n$ then we are done, so suppose $\eta_1^n \neq \tau_1^n$. Let $m < \omega$ be the least value at which they differ (so m is greatest such that $\eta_1^n \upharpoonright_m = \tau_1^n \upharpoonright_m$).

Then for each i < n, $\alpha_{in} \le m$, since by the (i,n) clause of $\phi_{\overline{\eta}}$ we have that $(\eta_1^n \oplus \eta_1^n) \upharpoonright_{\alpha_{in}} = (\eta_1^i \oplus \eta_1^i) \upharpoonright_{\alpha_{in}} = (\eta_1^n \oplus \tau_1^n) \upharpoonright_{\alpha_{in}}$.

Define σ as follows: suppose $\eta \in \mathfrak{A}$. Then $\sigma(\eta) = \tau$ where $\tau_0 = \eta_0$, where $\tau_1(k) = \eta_1(k) + \eta_1^n(k) + \tau_1^n(k) \mod 2$ for $k < d(\eta_0, \eta_0^n)$, and $\tau_1(k) = \eta_1(k)$ for $k \ge d(\eta_0, \eta_0^n)$. Then it is simple to check that σ has the desired properties.

From this it is clear that the algebraic closure of the emptyset $acl(\emptyset)^{\mathfrak{A}}$ is just $\{\eta \in J : \ell q(\eta) < \omega\}$. Denote this set as X.

We define an auxiliary \mathcal{L} -structure $\mathfrak{M} = (\mathbf{J}, U_{\eta_0}, \pi_{\alpha}, \boldsymbol{\eta} : \boldsymbol{\eta} \in J, \alpha \leq \omega)$ similarly to \mathfrak{A} : namely $\mathbf{J} = J_0 \otimes 2^{\leq \omega}$, with the natural operations. So \mathfrak{A} is a substructure of \mathfrak{M} .

In fact $\mathfrak{A} \leq \mathfrak{M}$ but we won't need this.

Given a sequence $\mathcal{F} = (f_{\alpha} : \alpha \in S)$, where each $f_{\alpha} \in 2^{\omega}$, define $J_{\mathcal{F}}$ to be be X, together with all pairs $(\eta_0, \eta_1) \in \mathbf{J}$ where $\eta_0 = \nu_{\alpha}$ is the canonical enumeration of L_{α} (defined at the beginning of the section) and where η_1 differs only finitely often from f_{α} . Define $\mathfrak{A}_{\mathcal{F}}$ to be the substructure of \mathfrak{M} with domain $J_{\mathcal{F}}$.

Note that $\mathfrak{A} = \mathfrak{A}_{(\overline{0}:\alpha \in S)}$.

Lemma 4.3. Each $\mathfrak{A}_{\mathcal{F}} \cong \mathfrak{A}$.

Proof. Fix $\mathcal{F} = (f_{\alpha} : \alpha \in S)$. Define $s_{\alpha} : L_{\alpha} \to 2$ by $s_{\alpha}(\nu_{\alpha}(n)) = f_{\alpha}(n)$. By the uniformization property we can choose some $s : \omega_1 \to 2$ such that s differs from each s_{α} only finitely often. Define $\sigma : \mathfrak{M} \cong \mathfrak{M}$ by $\sigma(\eta_0, \eta_1) = (\tau_0, \tau_1)$, where $\tau_0 = \eta_0$ and where $\tau_1(n) = \eta_1(n) + s(\eta_0(n)) \mod 2$, for each $n < \ell g(\eta)$.

Then σ is clearly an automorphism of \mathfrak{M} , and moreover restricts to an isomorphism from \mathfrak{A} to $\mathfrak{A}_{\mathcal{F}}$.

Lemma 4.4. \mathfrak{A} is the unique atomic model of T, and is furthermore prime.

Proof. Fix $\mathfrak{N} = (N, U_{\eta_0}^*, \pi_{\alpha}^*, \eta^* : \eta \in J, \alpha \leq \omega) \models T$. We can suppose $acl(\emptyset)^{\mathfrak{N}} = X$. I find some \mathcal{F} such that $\mathfrak{A}_{\mathcal{F}}$ embeds \mathfrak{N} , which suffices to show that \mathfrak{A} is prime.

Indeed, for each $\alpha \in S$, choose $a_{\alpha} \in U_{\nu_{\alpha}}^*$. Let $f_{\alpha} \in 2^{\omega}$ be defined by $f_{\alpha}(n) = \pi_m^*(a_{\alpha})(n)$ for some (any) m > n.

Let $\mathcal{F} = (f_{\alpha} : \alpha \in S)$. Then by Lemma 4.2, the map $\sigma_0 : (\nu_{\alpha}, f_{\alpha}) \mapsto a_{\alpha}$ is a partial elementary map from $\mathfrak{A}_{\mathcal{F}}$ to \mathfrak{N} . So it extends to a partial elementary map $\sigma : dcl((\nu_{\alpha}, f_{\alpha}) : \alpha \in S)^{\mathfrak{A}_{\mathcal{F}}} \to dcl(a_{\alpha} : \alpha \in S)^{\mathfrak{N}}$. But then clearly σ has domain all of $A_{\mathcal{F}}$. Hence $\sigma : \mathfrak{A}_{\mathcal{F}} \preceq \mathfrak{N}$.

To see that \mathfrak{A} is the unique atomic model of T, note that if \mathfrak{N} is atomic, then σ is also surjective, again by Lemma 4.2.

We conclude the proof of Theorem 4.1 with the following

Lemma 4.5. \mathfrak{A} is not constructible.

Proof. Suppose $(\eta^{\alpha}: \alpha < \omega_1)$ were a construction of \mathfrak{A} (it suffices to consider this order type by Theorem 1.2). Let $J_{\alpha} = \{\eta^{\beta}: \beta < \alpha\}$. Let $C \subset \omega_1$ be the club set of all $\alpha < \omega_1$ such that $J_{\alpha} = \{\eta \in J : \sup(\eta_0) < \alpha\}$. Choose $\alpha \in S \cap C$. Let $\beta \geq \alpha$ be least with $\eta_0^{\beta} = \nu_{\alpha}$, i.e. with $\eta^{\beta} \in U_{\nu_{\alpha}}$.

By Lemma 4.2, it is clear that for any set B such that $B \supset \{\eta \in J : \eta_0 \subset \nu_\alpha, \ell g(\eta) < \omega\}$ and $B \cap U_{\nu_\alpha} = \emptyset$, that $tp(\eta^\beta/B)$ is nonisolated. In particular $tp(\eta^\beta/J_\beta)$ is nonisolated.

Remark 4.6. Second Example: rewind back to the beginning of the section, and define instead J_1 to be the entire space $2^{\leq \omega}$. Then we have without any special combinatorics that \mathfrak{A} is the unique atomic model of T, and is prime, but is not constructible (although the language \mathcal{L} now has size continuum).

5 Producing Many Atomic Models of Size \aleph_1

5.1 Setup

Fix throughout this section a complete theory T in a language \mathcal{L} of cardinality \aleph_1 , such that T has atomic models. Write $\mathcal{L} = \bigcup_{\alpha < \omega_1} \mathcal{L}_{\alpha}$ as the union of a continuous increasing chain of countable languages, and let $T_{\alpha} = T \upharpoonright_{\mathcal{L}_{\alpha}}$.

Recall that an \mathcal{L} -formula $\phi(\overline{x})$ is T-complete if it is consistent with T and for every formula $\psi(\overline{x})$, $T + \phi(\overline{x})$ decides $\psi(\overline{x})$; equivalently, $\phi(\overline{x})$ isolates a single point in the Stone space $S^n(\emptyset)$. Since T has atomic models, for every \mathcal{L} -formula $\phi(\overline{x})$ consistent with T, there is a T-complete formula $\psi(\overline{x})$ that implies $\phi(\overline{x})$. So we can choose a club set $\mathbf{C}_0 \subseteq \omega_1$ such that for every $\alpha \in \mathbf{C}_0$ and for every \mathcal{L}_{α} -formula $\phi(\overline{x})$, if $\phi(\overline{x})$ is consistent with T then $\phi(\overline{x})$ has a T-complete extension $\psi(\overline{x})$, which is itself an \mathcal{L}_{α} formula.

It follows that for each $\alpha \in \mathbf{C}_0$, T_{α} has atomic models (though possibly not uncountable atomic models); and an \mathcal{L}_{α} -formula $\phi(\overline{x})$ is T-complete if and only if it is T_{α} complete.

Let $\mathfrak C$ be a monster model of T. We use standard model-theoretic notation: A, B, C, ... will range over parameter sets, and M, N, ... will range over elementary submodels of $\mathfrak C$. If $\overline a \in \mathfrak C$ and $\phi(\overline x)$ is an $\mathcal L$ -formula we write $\models \phi(\overline a)$ for $\mathfrak C \models \phi(\overline a)$. If $A \subset \mathfrak C$ is a set then $S^n(A)$ denotes the space of n-types over A, and S(A) denotes $\bigcup_n S^n(A)$. If we write $f: A \to B$ it is implied that f is partial $\mathcal L$ -elementary.

Define an atomic set to be a countable set $A \subset \mathfrak{C}$ such that every tuple $\overline{a} \in A$ has $tp(\overline{a})$ isolated by a single \mathcal{L} -formula. Say that A is an α -atomic set if moreover this formula can be chosen in \mathcal{L}_{α} .

If A is α -atomic, for some $\alpha \in \mathbf{C}_0$, then say that A is an α -base if $A \upharpoonright_{\mathcal{L}_{\alpha}} \preceq \mathfrak{C} \upharpoonright_{\mathcal{L}_{\alpha}}$. A is a base if it is an α -base for some $\alpha \in \mathbf{C}_0$. (Here we are using the term base as in "amalgamation base.")

Note that for each $\alpha \in \mathbf{C}_0$, α -bases exist and are unique up to isomorphism. Also, if $f: \omega \to \mathbf{C}_0$ is increasing, and A_n is an increasing chain of f(n)-bases, then $\bigcup_n A_n$ is a $\bigcup_n f(n)$ -base. Similarly, if $f: \omega_1 \to \mathbf{C}_0$ is increasing and cofinal, and A_α is an increasing chain of $f(\alpha)$ -bases, then $\bigcup_n A_n$ is an atomic model of T.

For each atomic set A and for each n, let $S_{at}^n(A)$ be the set of all atomic types over A (i.e. all types $p(\overline{x}) \in S^n(A)$ such that whenever \overline{a} realizes $p(\overline{x})$, $A\overline{a}$ is atomic). This is a dense subset of $S^n(A)$; give it the subspace topology. Let $S_{at}(A) = \bigcup_n S_{at}^n(A)$ with the disjoint union topology.

For each atomic set A and for each $\beta \in \mathbf{C}_0$, define $S_{at}^{n,\beta}(A)$ to be the set of all types $p(\overline{x}) \in S_{at}^n(A)$, such that whenever \overline{a} realizes $p(\overline{x})$, $A\overline{a}$ is β -atomic. Give $S_{at}^{n,\beta}(A)$ the subspace topology.

Lemma 5.1. • For each $\beta \in \mathbf{C}_0$, the topology on $S_{at}^{n,\beta}(A)$ is generated by the $\mathcal{L}_{\beta}(A)$ formulas.

Proof. Let \mathcal{O} be a basic open subset of $S_{at}^n(A)$; say $\mathcal{O} = \{p(\overline{x}) \in S_{at}^n(A) : p(\overline{x}) \models \phi(\overline{x}, \overline{a})\}$ where $\phi(\overline{x}, \overline{a})$ is an \mathcal{L} -formula. Suppose $p(\overline{x}) \in S_{at}^{n,\beta}(A) \cap \mathcal{O}$. We can choose a complete $\mathcal{L}_{\beta}(\overline{a})$ -formula $\psi(\overline{x}, \overline{a})$ such that $p(\overline{x}) \models \psi(\overline{x}, \overline{a})$. Let $\mathcal{U} = \{q(\overline{x}) \in S_{at}^n(A) : q(\overline{x}) \models \psi(\overline{x}, \overline{a})\}$. Then $p \in \mathcal{U} \subseteq \mathcal{O}$ as desired.

• For each β , $S_{at}^{n,\beta}(A)$ is a Polish space (or empty), and is closed in $S_{at}^n(A)$.

Proof. Closure is clear. To see that it is a Polish space, let $\mathfrak{C}' = \mathfrak{C} \upharpoonright \mathcal{L}_{\beta}$ and let $X = S^n(A)$ computed in \mathfrak{C}' . Then X is a Polish space and $S^{n,\beta}_{at}(A)$ is naturally embedded as a G_{δ} subset of X.

Let \mathbf{K}_T be the class of atomic models of T. We now define what it means for \mathbf{K}_T to be club totally transcendental:

For $\alpha \in \mathbf{C}_0$, say that \mathbf{K}_T is totally transcendental at α if, letting A be any α -base, we have that $S_{at}(A)$ is scattered, i.e. has no perfect subset. Equivalently \mathbf{K}_T is totally transcendental at α if for each n and for each $\beta \in \mathbf{C}_0$, $S_{at}^{n,\beta}(A)$ is countable. Let the transcendence spectrum of \mathbf{K}_T , $\operatorname{Spec}_{\mathbf{K}_T}(t.t.)$, denote the set of all $\alpha \in \mathbf{C}_0$ at which \mathbf{K}_T is totally transcendental.

Definition 5.2. \mathbf{K}_T is *club totally transcendental* if $\mathrm{Spec}_{\mathbf{K}_T}(t.t.)$ contains a club.

We aim to prove:

Theorem 2.8. Suppose Φ^* holds and $Cov(\mathcal{K}) \geq \aleph_2$. Suppose further that \mathbf{K}_T is not club totally transcendental. Then T has 2^{\aleph_1} atomic models of size \aleph_1 .

5.2 Club Totally Transcendental Property and the Existence of Constructible Models

Given $\alpha \in \mathbf{C}_0$, note that if \mathbf{K}_T is totally transcendental at α , then the isolated types are dense in $S_{at}(A)$, where A is any α -base. The converse of course can fail drastically: say T has \aleph_1 -many sorts, each a model of DLO.

We can relate all this to constructible models as follows. Let the constructible spectrum of \mathbf{K}_T , $\operatorname{Spec}_{\mathbf{K}_T}(CS)$, be the set of all $\alpha \in \mathbf{C}_0$ such that the isolated types are dense in $S_{at}(A)$, where A is any α -base. So by the preceding, $\operatorname{Spec}_{\mathbf{K}_T}(t.t.) \subseteq \operatorname{Spec}_{\mathbf{K}_T}(CS)$. Moreover:

Theorem 5.3. T has a constructible model if and only if $\operatorname{Spec}_{\mathbf{K}_T}(CS)$ contains a club. In particular, if \mathbf{K}_T is club totally transcendental, then T has a constructible model.

Proof. First suppose T has a constructible model M; say $M = (a_{\alpha} : \alpha < \omega_1)$ is a construction (with repetitions if M is countable). Let $A_{\alpha} := \{a_{\beta} : \beta < \alpha\}$. Then the set $C = \{\alpha \in \mathbf{C}_0 : A_{\alpha} \text{ is an } \alpha\text{-base}\}$ is club. Let $\alpha \in C$. Then M is atomic over A_{α} by Theorem 1.2 (applied to the theory $T(c_a : a \in A_{\alpha})$ in the language $\mathcal{L}(c_a : a \in A_{\alpha})$ where

we add constants for elements of A_{α}), which shows that the isolated types are dense in $S_{at}(A)$. Hence $\alpha \in \operatorname{Spec}_{\mathbf{K}_T}(CS)$, so $\operatorname{Spec}_{\mathbf{K}_T}(CS)$ contains a club.

Conversely, suppose $\operatorname{Spec}_{\mathbf{K}_T}(CS) \supseteq \hat{C}$, C a club; we can suppose $C \subseteq \mathbf{C}_0$. We define an increasing, continuous chain of atomic sets $(A_{\gamma} : \gamma \in C')$ where $C' = \{\gamma_{\alpha} : \alpha < \omega_1\} \subseteq C$ is club and each A_{α} is an α -base. We will further have that for all $\alpha < \omega_1$, $A_{\gamma_{\alpha+1}}$ is atomic over $A_{\gamma_{\alpha}}$. Finally, for each $\alpha < \omega_1$, we will have a construction $A_{\gamma_{\alpha}} = (a_{\beta} : \beta < \gamma_{\alpha})$. As implied by the notation, for $\alpha < \alpha'$, the construction of $A_{\gamma_{\alpha}}$ is an initial segment of the construction of $A_{\gamma_{\alpha'}}$.

Note that this will suffice, since setting $M := \bigcup_{\alpha} A_{\gamma_{\alpha}}$, we have M is a constructible model of T, as witnessed by $(a_{\beta} : \beta < \omega_1)$.

Let γ_0 be the least infinite element of C and let $A_{\gamma_0} = (a_\beta : \beta < \gamma_0)$ be any γ_0 -base. Take unions at limit stages.

Suppose we have defined γ_{α} and $A_{\gamma_{\alpha}} = (a_{\beta} : \beta < \gamma_{\alpha})$. Write $A = A_{\gamma_{\alpha}}$. Since the isolated types are dense in $S_{at}(A)$ we can choose an $M^{\alpha} \models T$, $M^{\alpha} \supseteq A$ and M^{α} atomic over A by Theorem 1.3 (applied to the theory $T(c_{a} : a \in A_{\alpha})$ in the language $\mathcal{L}(c_{a} : a \in A_{\alpha})$). It is possible that M^{α} is countable or even $M^{\alpha} = A$, but in any case we can enumerate $M^{\alpha} = (a_{\beta}^{\alpha} : \beta < \omega_{1})$ so that for all $\beta < \gamma_{\alpha}$, $a_{\beta}^{\alpha} = a_{\beta}$. For each $\delta < \omega_{1}$, let $B_{\delta} = \{a_{\beta}^{\alpha} : \beta < \delta\}$. Then the set of all δ such that B_{δ} is a δ -base is club, so we can choose some such δ with $\delta \in C$ and $\delta > \gamma_{\alpha}$. Let $\gamma_{\alpha+1} = \delta$ and let $A_{\gamma_{\alpha}+1} = B_{\delta}$ and define $a_{\beta} = a_{\beta}^{\alpha}$ for all $\gamma_{\alpha} \leq \beta < \gamma_{\alpha+1}$.

Hence, as a corollary of Theorem 2.8, we will get

Theorem 1.4. Suppose Φ^* holds and $Cov(\mathcal{K}) \geq \aleph_2$. Suppose further that T has no constructible models. Then T has 2^{\aleph_1} atomic models of size \aleph_1 .

5.3 Club Totally Transcendental Property and Amalgamation

The proof of the main theorem will split into two cases: first, where \mathbf{K}_T fails the club amalgamation property (to be defined below), and second, where \mathbf{K}_T has the club amalgamation property but is not club totally transcendental. As in the countable language case we will actually have that if \mathbf{K}_T is club totally transcendental then \mathbf{K}_T has the club amalgamation property.

Let $\alpha \in \mathbf{C}_0$. An amalgamation problem at α is a triple (A_0, A_1, A_2) of (countable) atomic sets, such that A_0 is an α -base, and $A_0 \subseteq A_i$ for i = 1, 2. A solution is a triple (A_3, f_1, f_2) such that A_3 is an atomic set, $f_i : A_i \to A_3$ are elementary, and the f_i 's agree on A_0 . We say that \mathbf{K}_T has the amalgamation property at α if every amalgation problem at α has a solution. We let the amalgamation spectrum of \mathbf{K}_T , $\operatorname{Spec}_{\mathbf{K}_T}(AP)$, denote the set of all $\alpha \in \mathbf{C}_0$ at which \mathbf{K}_T has the amalgamation property.

Definition 5.4. \mathbf{K}_T has the club amalgamation property if $\operatorname{Spec}_{\mathbf{K}_T}(AP)$ contains a club.

Theorem 5.5. Suppose \mathbf{K}_T is club totally transcendental. Then \mathbf{K}_T has the club amalgamation property.

Proof. We show that $\operatorname{Spec}_{\mathbf{K}_T}(t.t.) \subseteq \operatorname{Spec}_{\mathbf{K}_T}(AP)$, which suffices.

Indeed, let $\alpha \in \operatorname{Spec}_{\mathbf{K}_T}(t,t)$ and let (A_0, A_1, A_2) be an amalgamation property at α . Choose $\beta > \alpha$ so that $\beta \in \operatorname{Spec}_{\mathbf{K}_T}(t,t)$ and each A_i is β -atomic. Now $S_{at}^{\beta}(A_0)$ is countable, hence the isolated types are dense in $S_{at}^{\beta}(A_0)$. So by applying Theorem 2.7 to the theory $T \upharpoonright_{\mathcal{L}_{\beta}}$ we get a solution.

5.4 Promises

Our idea for constructing many models is the following: we will produce a tree $(A_{\mathbf{s}}: \mathbf{s} \in 2^{<\omega_1})$ of bases, such that if we set $M_{\eta} := \bigcup_{\alpha} A_{\eta \upharpoonright \alpha}$ for $\eta \in 2^{\omega_1}$, then each M_{η} is an atomic model of T.

We will also be producing, for each $\mathbf{s} \in 2^{<\omega_1}$, a set $\Phi_{\mathbf{s}} \subseteq S_{at}(A_{\mathbf{s}})$, such that every $\eta \supseteq \mathbf{s}$ has M_{η} omits $\Phi_{\mathbf{s}}$. I.e. we are "promising" to omit these types. Typically $\Phi_{\mathbf{s}}$ will be a union of \aleph_1 -many closed nowhere dense sets, so in order to omit it we will need $\mathsf{Cov}(\mathcal{K}) \geq \aleph_2$.

By an appropriate failure of amalgamation, we will have that for each $\mathbf{s} \in 2^{<\omega_1}$, there is no M, f_0, f_1 such that: $f_i : A_{\mathbf{s}^{\frown}(i)} \to M, f_0 \upharpoonright_{A_{\mathbf{s}}} = f_1 \upharpoonright_{A_{\mathbf{s}}}$, and M omits each $f_i(\Phi_{\mathbf{s}^{\frown}(i)})$.

Then we will apply a diagonalization argument using Φ^* to get that $\{M_{\eta} : \eta \in 2^{\omega_1}\}$ contains 2^{\aleph_1} distinct isomorphism types.

In this subsection we develop some general machinery for building the tree $(A_{\mathbf{s}}, \Phi_{\mathbf{s}} : \mathbf{s} \in 2^{<\omega_1})$ and extracting 2^{\aleph_1} models of size \aleph_1 . For the following, the reader should note that the special case $\mathbb{P} = \emptyset$ is actually an important example.

Definition 5.6. A system of promises is a set \mathbb{P} such that:

• Every $\Gamma \in \mathbb{P}$ is a nonempty subset of $S_{at}(A)$ for a (unique) base A. Write $A = \text{dom}(\Gamma)$.

If A is an atomic set and $\Gamma \in P$, then say that A omits Γ if $A \supseteq \text{dom}(\Gamma)$ and for all $\overline{a} \in A$, $tp(\overline{a}/\text{dom}(\Gamma)) \not\in \Gamma$. If $\Phi \subseteq \mathbb{P}$ is countable and A is an atomic set then say that A omits Φ if A omits Γ for all $\Gamma \in \Phi$.

- (Invariance) \mathbb{P} is closed under $Aut(\mathfrak{C})$.
- (Extendibility) Suppose A is an atomic set, and $\Phi \subseteq \mathbb{P}$ is countable such that A omits Φ . Then for arbitrarily large $\alpha \in \mathbb{C}_0$ there is an α -base $B \supseteq A$ such that B omits Φ .

Suppose \mathbb{P} is a system of promises. Then a \mathbb{P} -atomic set (\mathbb{P} -base, (α, \mathbb{P}) -atomic set, (α, \mathbb{P}) -base) is a pair (A, Φ) where A is an atomic set (base, α -atomic set, α -base) and $\Phi \subset \mathbb{P}$ is countable and A omits Φ .

If (A_0, Φ_0) and (A_1, Φ_1) are \mathbb{P} -atomic sets, say that (A_1, Φ_1) extends (A_0, Φ_0) , and write that $(A_0, \Phi_0) \subseteq (A_1, \Phi_1)$, if $A_0 \subseteq A_1$ and $\Phi_0 \subseteq \Phi_1$.

A \mathbb{P} -amalgamation problem is a triple of \mathbb{P} -atomic sets $(A_0, \Phi_0), (A_1, \Phi_1), (A_2, \Phi_2)$) where (A_0, Φ_0) is a \mathbb{P} -base and each (A_i, Φ_i) extends (A_0, Φ_0) . We call (A_0, Φ_0) is called the base of the problem.

A solution to the above problem is a sequence $((B, \Psi), f_1, f_2)$ where (B, Ψ) is a \mathbb{P} -atomic set, and $f_1: A_1 \to B$ and $f_2: A_2 \to B$ are both the identity on A, and $f_1(\Phi_1) \cup f_2(\Phi_2) \subseteq \Psi$.

Note that by the invariance property of promise systems, if two amalgamation problems are isomorphic then one has a solution if and only if the other does.

For $\alpha \in \mathbf{C}_0$, we say that \mathbb{P} has the amalgamation property at α if there is some (α, \mathbb{P}) -base (A, Φ) , such that every \mathbb{P} -amalgamation problem with base (A, Φ) has a solution.

Let the amalgamation spectrum of \mathbb{P} , $\operatorname{Spec}_{\mathbf{K}_T}(\mathbb{P})$, be the set of $\alpha \in \mathbf{C}_0$ such that \mathbb{P} has the amalgamation property at α . We say that \mathbb{P} has the club amalgamation property if $\operatorname{Spec}_{\mathbf{K}_T}(\mathbb{P})$ contains a club.

In particular, if $\mathbb{P} = \emptyset$ then $\operatorname{Spec}_{\mathbf{K}_T}(\mathbb{P}) = \operatorname{Spec}_{\mathbf{K}_T}(AP)$, and so \mathbb{P} has the club amalgamation property iff \mathbf{K}_T has the club amalgamation property.

The proof of the following (in a different context) is due originally to Shelah [8], see [1] Theorem 17.11 for a nice exposition.

Lemma 5.7. Suppose Φ^* holds, and T admits a system of promises \mathbb{P} which fails the club amalgamation property. Then T has 2^{\aleph_1} atomic models of size \aleph_1 . (In fact we just need $\Phi(\omega_1 \backslash \operatorname{Spec}_{\mathbf{K}_T}(\mathbb{P}))$ to hold.)

The rest of this subsection is a proof of the lemma. Note that T has at most one countable atomic model, so it suffices to show that T has 2^{\aleph_1} atomic models of size $\leq \aleph_1$.

Let $S = \mathbf{C}_0 \backslash \operatorname{Spec}_{\mathbf{K}_T}(\mathbb{P})$. We are assuming that S is stationary; thus $\Phi(S)$ holds (and in particular $2^{\aleph_0} < 2^{\aleph_1}$).

The proof splits into two cases.

Case A. There exist \mathbb{P} -bases $(A_0, \Phi_0) \subseteq (A, \Phi)$, such that for every \mathbb{P} -base $(B, \Psi) \supseteq (A, \Phi)$, there exist \mathbb{P} -bases (B_0, Ψ_0) and (B_1, Ψ_1) extending (B, Ψ) , such that the \mathbb{P} -amalgamation problem $((A_0, \Phi_0), (B_0, \Psi_0), (B_1, \Psi_1))$ has no solution.

In this case we build inductively a system $(A_{\mathbf{s}}, \Phi_{\mathbf{s}}, \alpha_{\mathbf{s}} : \mathbf{s} \in 2^{<\omega_1})$ such that:

- For each $\mathbf{s} \in 2^{<\omega_1}$, $\alpha_{\mathbf{s}} \in \mathbf{C}_0$ and $(A_{\mathbf{s}}, \Phi_{\mathbf{s}})$ is an $(\alpha_{\mathbf{s}}, \mathbb{P})$ -base.
- For $\mathbf{s} \subseteq \mathbf{t}$, $\alpha_{\mathbf{s}} < \alpha_{\mathbf{t}}$, and $(A_{\mathbf{s}}, \Phi_{\mathbf{s}}) \subseteq (A_{\mathbf{t}}, \Phi_{\mathbf{t}})$.
- For each $\mathbf{s} \in 2^{<\omega_1}$ of limit length, $\alpha_{\mathbf{s}} = \bigcup_{\mathbf{t} \subset \mathbf{s}} \alpha_{\mathbf{t}}$ and $A_{\mathbf{s}} = \bigcup_{\mathbf{t} \subset \mathbf{s}} A_{\mathbf{t}}$ and $\Phi_{\mathbf{s}} = \bigcup_{\mathbf{t} \subset \mathbf{s}} \Phi_{\mathbf{t}}$.
- For each $\mathbf{s} \in 2^{<\omega_1}$, the \mathbb{P} -amalgamation problem $((A_0, \Phi_0), (A_{\mathbf{s}^{\frown}(0)}, \Phi_{\mathbf{s}^{\frown}(0)}), (A_{\mathbf{s}^{\frown}(1)}, \Phi_{\mathbf{s}^{\frown}(1)}))$ has no solution.

For each $\eta \in 2^{\omega_1}$, let $M_{\eta} = \bigcup_{\alpha < \omega_1} A_{\eta \upharpoonright_{\alpha}}$, an atomic model of T of size $\leq \aleph_1$. Then for each $\eta \neq \tau$, $(M_{\eta}, a : a \in A_0) \ncong (M_{\tau}, a : a \in A_0)$. Hence $\{M_{\eta} : \eta \in 2^{\aleph_1}\}$ represents

 2^{\aleph_1} different isomorphism types if we add countably many constants. Since $2^{\aleph_0} < 2^{\aleph_1}$ it follows that $\{M_{\eta} : \eta \in 2^{\aleph_1}\}$ represents 2^{\aleph_1} different isomorphism types.

Case B. (The negation of Case A.) For all \mathbb{P} -bases $(A_0, \Phi_0) \subseteq (A, \Phi)$ there is $(B, \Psi) \supseteq (A, \Phi)$ such that for all (B_0, Ψ_0) and (B_1, Ψ_1) extending (B, Ψ) , the \mathbb{P} -amalgamation problem $((A_0, \Phi_0), (B_0, \Psi_0), (B_1, \Psi_1))$ has a solution.

In this case we inductively build a system $(A_s, \Phi_s, \alpha_s : s \in 2^{<\omega_1})$ such that:

- For each $\mathbf{s} \in 2^{<\omega_1}$, $\alpha_{\mathbf{s}} \in \mathbf{C}_0$ and $(A_{\mathbf{s}}, \Phi_{\mathbf{s}})$ is an $(\alpha_{\mathbf{s}}, \mathbb{P})$ -base.
- For $\mathbf{s} \subseteq \mathbf{t}$, $\alpha_{\mathbf{s}} < \alpha_{\mathbf{t}}$, and $(A_{\mathbf{s}}, \Phi_{\mathbf{s}}) \subset (A_{\mathbf{t}}, \Phi_{\mathbf{t}})$.
- For each $\mathbf{s} \in 2^{<\omega_1}$ of limit length, $\alpha_{\mathbf{s}} = \bigcup_{\mathbf{t} \subset \mathbf{s}} \alpha_{\mathbf{t}}$ and $A_{\mathbf{s}} = \bigcup_{\mathbf{t} \subset \mathbf{s}} A_{\mathbf{t}}$ and $\Phi_{\mathbf{s}} = \bigcup_{\mathbf{t} \subset \mathbf{s}} \Phi_{\mathbf{t}}$.
- For each $\mathbf{s} \in 2^{<\omega_1}$ with $\alpha_{\mathbf{s}} \in S$, the \mathbb{P} -amalgamation problem $((A_{\mathbf{s}}, \Phi_{\mathbf{s}}), (A_{\mathbf{s}^{\frown}(0)}, \Phi_{\mathbf{s}^{\frown}(0)}), (A_{\mathbf{s}^{\frown}(1)}, \Phi_{\mathbf{s}^{\frown}(1)}))$ has no solution.
- For each $\mathbf{s} \in 2^{<\omega_1}$, for each $i \in 2$ and for each pair of \mathbb{P} -bases (B_0, Ψ_0) and (B_1, Ψ_1) extending $(A_{\mathbf{s}^{\frown}(i)}, \Phi_{\mathbf{s}^{\frown}(i)})$, the \mathbb{P} -amalgamation problem $((A_{\mathbf{s}}, \Phi_{\mathbf{s}}), (B_0, \Psi_0), (B_1, \Psi_1))$ has a solution.

For each $\eta \in 2^{\omega_1}$ let $M_{\eta} := \bigcup_{\alpha < \omega_1} A_{\mathbf{s} \upharpoonright \alpha}$, an atomic model of T of size $\leq \aleph_1$. I claim that

in fact each M_{η} has size exactly \aleph_1 . Indeed, fix $\eta \in 2^{\omega_1}$. Then there are uncountably many $\alpha < \omega_1$ such that $\alpha = \alpha_{\eta \upharpoonright_{\alpha}} \in S$, so it suffices to show that for each such α , $A_{\eta \upharpoonright_{\alpha}}$ is strictly contained in $A_{\eta \upharpoonright_{\alpha+1}}$. Suppose not; set $A = A_{\eta \upharpoonright_{\alpha}} = A_{\eta \upharpoonright_{\alpha+1}}$ and set $\Phi = \Phi_{\eta \upharpoonright_{\alpha+1}} \supseteq \Phi_{\eta \upharpoonright_{\alpha}}$. Then (A, Φ) is an (α, \mathbb{P}) base, but every \mathbb{P} -amalgamation problem with base (A, Φ) must have a solution by the final requirement above, contradicting $\alpha \in S$.

Choose bijections $\sigma_{\eta}: M_{\eta} \to \omega_1$, such that for all $\eta, \tau \in 2^{\omega_1}$ with $\eta \upharpoonright_{\alpha} = \tau \upharpoonright_{\alpha} = \mathbf{s}$ say, we have that $\sigma_{\eta} \upharpoonright_{A_{\mathbf{s}}} = \sigma_{\tau} \upharpoonright_{A_{\mathbf{s}}} := \sigma_{\mathbf{s}}$.

We view $(2\times2\times\omega_1)^{<\omega_1}$ as a subset of $2^{<\omega_1}\times2^{<\omega_1}\times\omega_1^{<\omega_1}$. Define $F:(2\times2\times\omega_1)^{<\omega_1}\to 2$ by $F(\mathbf{s},\mathbf{t},h)=1$ if:

- $\sigma_{\mathbf{s}}(A_{\mathbf{s}}) = \sigma_{\mathbf{t}}(A_{\mathbf{t}}) = \alpha_{\mathbf{s}} = \alpha_{\mathbf{t}} = \ell q(\mathbf{s}) = \ell q(\mathbf{t}) =: \alpha \text{ say.}$
- $h: \alpha \to \alpha$ is a bijection.
- $\sigma_{\mathbf{t}}^{-1} \circ h \circ \sigma_{\mathbf{s}} : A_{\mathbf{s}} \cong A_{\mathbf{t}}$.
- For some or any extension g of $\sigma_{\mathbf{t}}^{-1} \circ h \circ \sigma_{\mathbf{s}}$ to $A_{\mathbf{s}^{\frown}(0)}$, the \mathbb{P} -amalgamation problem $((A_{\mathbf{t}}, g(\Phi_{\mathbf{s}}) \cup \Phi_{\mathbf{t}}), (g(A_{\mathbf{s}^{\frown}(0)}), g(\Phi_{\mathbf{s}^{\frown}(0)}) \cup \Phi_{\mathbf{t}}), (A_{\mathbf{t}^{\frown}(0)}, g(\Phi_{\mathbf{s}}) \cup \Phi_{\mathbf{t}^{\frown}(0)}))$ has a solution.

 $F(\mathbf{s}, \mathbf{t}, h) = 0$ else.

Now choose disjoint stationary subsets $(S_{\alpha} : \alpha < \omega_1)$ of S; then $\Phi(S_{\alpha})$ holds for each $\alpha < \omega_1$. For each $\alpha < \omega_1$ choose $g_{\alpha} : S_{\alpha} \to 2$ such that for every $(\eta, \tau, f) \in (2 \times 2 \times \omega_1)^{\omega_1}$, the set of all $\beta \in S_{\alpha}$ with $g_{\alpha}(\beta) = F(\eta \upharpoonright_{\beta}, \tau \upharpoonright_{\beta}, f \upharpoonright_{\beta})$ is stationary.

For $X \subset \omega_1$ define $\eta_X : \omega_1 \to 2$ by: $\eta_X(\beta) = g_{\alpha}(\beta)$ if $\beta \in S_{\alpha}$ and $\alpha \in X$, and $\eta_X(\beta) = 0$ else.

I claim that for all $X \neq Y$, $M_X \ncong M_Y$, which suffices.

Indeed, suppose $X \neq Y$ and yet $f : \omega_1 \to \omega_1$ is a bijection with $\phi := \sigma_{\tau}^{-1} \circ f \circ \sigma_{\eta} : M_{\eta} \cong M_{\tau}$. We can suppose $\alpha \in X \setminus Y$. Let $\eta = \eta_X$ and let $\tau = \tau_Y$. Let C be the club set of all $\beta < \omega_1$ such that $\sigma_{\eta \restriction_{\beta}}(A_{\eta \restriction_{\beta}}) = \sigma_{\tau \restriction_{\beta}}(A_{\tau \restriction_{\beta}}) = \alpha_{\eta \restriction_{\beta}} = \alpha_{\tau \restriction_{\beta}} = f[\beta] = \beta$.

set of all $\beta < \omega_1$ such that $\sigma_{\eta \upharpoonright_{\beta}}(A_{\eta \upharpoonright_{\beta}}) = \sigma_{\tau \upharpoonright_{\beta}}(A_{\tau \upharpoonright_{\beta}}) = \alpha_{\eta \upharpoonright_{\beta}} = \alpha_{\tau \upharpoonright_{\beta}} = f[\beta] = \beta$. Choose $\beta \in S_{\alpha} \cap C$ such that $F(\eta \upharpoonright_{\beta}, \tau \upharpoonright_{\beta}, f \upharpoonright_{\beta}) = g_{\alpha}(\beta)$. Write $\mathbf{s} = \eta \upharpoonright_{\beta}, \mathbf{t} = \tau \upharpoonright_{\beta}, h = f \upharpoonright_{\beta}$.

Note that the first two items of the definition of F are met, so $F(\mathbf{s}, \mathbf{t}, h) = 1$ iff the third item holds. Also note that $\tau(\beta) = 0$. There are two cases:

Case B0. $F(\mathbf{s}, \mathbf{t}, h) = 0$. Then $\eta(\beta) = 0$. But then clearly the isomorphism $\phi : M_{\eta} \cong M_{\tau}$ witnesses that the \mathbb{P} -amalgamation problem in the third item of the definition of F has a solution, contradicting the case.

Case B1. $F(\mathbf{s},\mathbf{t},h)=1$. Then $\eta(\beta)=1$. Let g be any extension of ϕ to $A_{\mathbf{s}^{\frown}(0)}$; then we can choose a solution $(i_0,i_1,(B,\Psi_0))$ to the \mathbb{P} -amalgamation problem $((A_{\mathbf{t}},g(\Phi_{\mathbf{s}})\cup\Phi_{\mathbf{t}}),(g(A_{\mathbf{s}^{\frown}(0)}),g(\Phi_{\mathbf{s}^{\frown}(0)})\cup\Phi_{\mathbf{t}}),(A_{\mathbf{t}^{\frown}(0)},g(\Phi_{\mathbf{s}})\cup\Phi_{\mathbf{t}^{\frown}(0)}))$, where moreover $i_1:A_{\mathbf{t}^{\frown}(0)}\to B$ is the inclusion. We can use the isomorphism ϕ to get a solution $(j_0,j_1,(C,\Psi_1))$ to the \mathbb{P} -amalgamation problem $((A_{\mathbf{t}},\phi(\Phi_{\mathbf{s}})\cup\Phi_{\mathbf{t}}),(\phi(A_{\mathbf{s}^{\frown}(1)}),\phi(\Phi_{\mathbf{s}^{\frown}(1)})\cup\Phi_{\mathbf{t}}),(A_{\mathbf{t}^{\frown}(0)},\phi(\Phi_{\mathbf{s}})\cup\Phi_{\mathbf{t}^{\frown}(0)})$ where again $j_1:A_{\mathbf{t}^{\frown}(0)}\to C$ is the inclusion. Then by the construction of the system $(A_{\mathbf{s}},\Phi_{\mathbf{s}},\alpha_{\mathbf{s}})$, the \mathbb{P} -amalgamation problem $((A_{\mathbf{t}},\Phi_{\mathbf{t}}),(B,\Psi_0),(C,\Psi_1))$ has a solution. But this yields a solution to the \mathbb{P} -amalgamation problem $((A_{\mathbf{s}},\Phi_{\mathbf{s}}),(A_{\mathbf{s}^{\frown}(0)},\Phi_{\mathbf{s}^{\frown}(0)}),(A_{\mathbf{s}^{\frown}(1)},\Phi_{\mathbf{s}^{\frown}(1)}))$, contradiction.

5.5 Proof of Theorem 2.8

Throughout this section, we suppose Φ^* holds and $Cov(\mathcal{K}) \geq \aleph_2$, and \mathbf{K}_T is not club totally transcendental. We aim to construct 2^{\aleph_1} atomic models of T of size \aleph_1 .

Recall that if we let $\mathbb{P} = \emptyset$ be the empty system of promises, then $\operatorname{Spec}_{\mathbf{K}_T}(\mathbb{P}) = \operatorname{Spec}_{\mathbf{K}_T}(AP)$; so if \mathbf{K}_T fails the club amalgamation property then by Lemma 5.7 we are done. Hence we can suppose that \mathbf{K}_T has the club amalgamation property.

Lemma 5.8. If $\alpha \in \operatorname{Spec}_{\mathbf{K}_T}(AP)$, A is an α -base, $B \supseteq A$ is atomic, and $p(\overline{x}) \in S_{at}(A)$, then $p(\overline{x})$ extends to a type in $S_{at}(B)$.

Proof. Let \overline{a} be a realization of $p(\overline{x})$. Then the amalgamation problem $(A, A\overline{a}, B)$ has a solution, which is equivalent to the claim.

Lemma 5.9. There is a club $\mathbf{C}_1 \subseteq \operatorname{Spec}_{\mathbf{K}_T}(AP)$ and a number n_0 such that for every $\alpha < \beta$ both in \mathbf{C}_1 , if A is the α -base then $S_{at}^{n_0,\beta}(A)$ has size continuum. In particular, $\operatorname{Spec}_{\mathbf{K}_T}(t.t.) \cap \mathbf{C}_1 = \emptyset$.

Proof. We can choose some club $C \subseteq \operatorname{Spec}_{\mathbf{K}_T}(AP)$ by assumption.

Let $\alpha \in C \backslash \operatorname{Spec}_{\mathbf{K}_T}(t.t.)$. Let A be an α -base. Let n_0 be such that $S_{at}^{n_0}(A)$ has size continuum. Let $\beta > \alpha$ with $\beta \in C$ and let $B \supseteq A$ be a β -base. Then by Lemma 5.8, $S_{at}^{n_0}(B)$ has size continuum. It follows that $S_{at}^{n_0}(B)$ has a perfect subset, since otherwise we would have $|S_{at}^{n_0}(B)| \leq \aleph_1 < 2^{\aleph_0}$. Hence there is some $f(\beta)$ such that $S_{at}^{n_0,\beta}(B)$ has size continuum; we can choose $f(\beta) \in C \backslash \beta$.

Let C_1 be the club $(\alpha, f(\alpha), f^2(\alpha), \dots, f^{\gamma}(\alpha), \dots)$ (take unions at limit stages). \square

For the rest of the proof, fix \mathbf{C}_1 , n_0 as above. For each ordinal $\alpha < \omega_1$ let α^+ denote the least ordinal $\beta > \alpha$ with $\beta \in \mathbf{C}_1$. So for any $\alpha \in \mathbf{C}_1$ and for any α -base A, $S_{at}^{n_0,\alpha^+}(A)$ has size continuum. Let K(A) denote the perfect kernel of $S_{at}^{n_0,\alpha^+}(A)$.

The following definition gives a nice description of K(A).

Definition 5.10. Let $\alpha \in \mathbf{C}_1$, and let $\phi(\overline{x}; \overline{y})$ be a partitioned \mathcal{L}_{α^+} formula with $|\overline{x}| = n_0$. Then say that $\phi(\overline{x}; \overline{y})$ is α -unbounded if for some (any) α -base A, there is some $p(\overline{x}) \in K(A)$ and some $\overline{a} \in A$ with $\phi(\overline{x}; \overline{a}) \in p(\overline{x})$. Note that, since K(A) is fixed under A-automorphisms, we have that $p(\overline{x}) \in K(A)$ if and only if for all $\phi(\overline{x}; \overline{a}) \in p(\overline{x})$, $\phi(\overline{x}; \overline{y})$ is α -unbounded.

Now fix for the time being $\alpha \in \mathbf{C}_1$ and an α -base A. We identify closed subsets of $S^n_{at}(A)$ with the corresponding partial n-types over A. So for instance if $C \subseteq S^n_{at}(A)$ is closed then we write $C(\overline{x}) \models \phi(\overline{x})$ to indicate that for every $p(\overline{x}) \in C$, $\phi(\overline{x}) \in p(\overline{x})$. Let $\Phi_{at}(A)$ denote the subsets of $S^n_{at}(A)$ which are in fact closed subsets of $S^{n,\beta}_{at}(A)$ for some $n \in \omega$ and some $\beta \in \mathbf{C}_1$. For example, each $S^{n,\beta}_{at}(A) \in \Phi_{at}(A)$.

We define a (pre)-partial ordering \leq on $\Phi_{at}(A)$, with the idea that $C \leq D$ means that if we realize D over A, then it is hard to realize C over A.

First we define the immediate successors of \leq :

Definition 5.11. Let $C, F \in \Phi_{at}(A)$ be given. Then $C \prec F$ if and only if one of the following holds:

- 1. For some $\beta \geq \alpha$ in C_1 , $C(y, \overline{z})$ is a closed subset of $S_{at}^{1+n,\beta}(A)$, $F(\overline{z}, \overline{w})$ is a closed subset of $S_{at}^{n+m,\beta}(A)$, and there is some \mathcal{L}_{β} -formula $\phi(y, \overline{z}, \overline{w})$ such that $F(\overline{z}, \overline{w})$ is defined by the intersection of the following closed sets:
 - $S_{at}^{n+m,\beta}(A)$;
 - " $\exists y \phi(y, \overline{z}, \overline{w})$;"
 - " $\forall y (\phi(y, \overline{z}, \overline{w}) \to \psi(y, \overline{z}, \overline{d})$ " for each formula $\psi(y, \overline{z}, \overline{d})$ with $C(y, \overline{z}) \models \psi(y, \overline{z}, \overline{d})$.

So, whenever $B \supseteq A$ is β -atomic and whenever $\overline{ab} \in B^{n+m}$, then $tp(\overline{ab}/A) \in F$ iff the following holds: there is some $q(y) \in S^{1,\beta}_{at}(B)$ with $q(y) \models \phi(y,\overline{a},\overline{b})$, and moreover, for any such q(y), if we let $r(y,\overline{z})$ be the set of all $\mathcal{L}(A)$ -formulas $\psi(y,\overline{z})$ such that $q(y) \models \psi(y,\overline{a})$, then $r(y,\overline{z}) \in C$.

- 2. For some $\beta \geq \alpha$ in \mathbf{C}_1 , $C(\overline{y}, \overline{z}) \subseteq S_{at}^{n_0+n,\beta^+}(A)$ is closed, $F(\overline{z}, \overline{w})$ is a closed subset of $S_{at}^{n+m,\beta}(A)$, and there is some β -unbounded, complete formula $\phi(\overline{y}; \overline{z}, \overline{w})$ such that $F(\overline{z}, \overline{w})$ is defined by the intersection of the following closed sets:
 - $S_{at}^{n+m,\beta}(A)$;
 - " $\exists \overline{y} \phi(\overline{y}, \overline{z}, \overline{w});$ "
 - " $\forall \overline{y}(\tau(\overline{y}, \overline{z}, \overline{w}, \overline{d}) \land \phi(\overline{y}, \overline{z}, \overline{w}) \rightarrow \psi(\overline{y}, \overline{z}, \overline{d}))$," for all β -unbounded, complete formulas $\tau(\overline{y}; \overline{z}, \overline{w}, \overline{u})$, and all $\mathcal{L}(A)$ -formulas $\psi(\overline{y}, \overline{z}, \overline{d})$ such that $|\overline{d}| = |\overline{u}|$ and such that $C(\overline{y}, \overline{z}) \models \psi(\overline{y}, \overline{z}, \overline{d})$.

So, whenever $B \supseteq A$ is a β -base and whenever $\overline{ab} \in B^{n+m}$, we have $tp(\overline{ab}/A) \in F$ iff the following holds: there is some $q(\overline{y}) \in K(B)$ with $q(\overline{y}) \models \phi(\overline{y}, \overline{a}, \overline{b})$, and moreover, for any such $q(\overline{y})$, if we let $r(\overline{y}, \overline{z})$ be the set of all $\mathcal{L}(A)$ -formulas $\psi(\overline{y}, \overline{z})$ such that $q(\overline{y}) \models \psi(\overline{y}, \overline{a})$, then $r(\overline{y}, \overline{z}) \in C$.

Now let \leq be the least partial order containing \prec , i.e. $C \leq F$ iff there is a sequence $C = C_0 \prec C_1 \prec \ldots \prec C_{n-1} = F$.

Note that for each C, there are at most \aleph_1 -many F with $C \prec F$, and so there are at most \aleph_1 -many F with $C \leq F$.

Given $p(\overline{x}) \in S_{at}(A)$, let $\Gamma(p(\overline{x})) = \bigcup_{C \geq \{p(\overline{x})\}} C$, so this is the union of \aleph_1 -many closed

subsets of $S_{at}(A)$.

Let $K^*(A) := \{p(\overline{x}) \in K(A) : A \text{ omits } \Gamma(p(\overline{x}))\}$. Finally let $\mathbb{P} = \{\Gamma(p(\overline{x})) : p(\overline{x}) \in K^*(A) \text{ where } A \text{ is an } \alpha\text{-base for some } \alpha \in \mathbf{C}_1\}$.

Then it suffices to establish that \mathbb{P} is a system of promises, with $\operatorname{Spec}_{\mathbf{K}_T}(\mathbb{P}) \cap \mathbf{C}_1 = \emptyset$. Towards this we prove the following three lemmas.

Lemma 5.12. For every $\alpha \in \mathbb{C}_1$ and every α -base A, $|K(A)\backslash K^*(A)| \leq \aleph_1$, in particular $K^*(A)$ is \aleph_1 -comeager in K(A).

Proof. Suppose $p(\overline{x}) \notin K^*(A)$. Then there are closed sets $\{p(\overline{x})\} = C_0 \prec C_1 \prec \ldots \prec C_n$, and some $\overline{a} \in A$ realizing C_n . Then from examining the definition of \prec , we see that we can recover $p(\overline{x})$ from $tp(\overline{a}/A)$ and from the formulas and ordinals witnessing that $C_i \prec C_{i+1}$ for i < n. There are only \aleph_1 -many possibilities for the latter, and so there are only \aleph_1 -many $p(\overline{x})$ not in $K^*(A)$.

Lemma 5.13. Suppose $\alpha \leq \beta$ are both in \mathbb{C}_1 , A is an α -base, $p(\overline{x}) \in K^*(A)$, and $B \supseteq A$ is a β -atomic set which omits $\Gamma(p(\overline{x}))$. Let

 $X=\{q(y)\in S^{1,\beta}_{at}(B): Ba \text{ omits } \Gamma(p(\overline{x})) \text{ for some (any) realization } a \text{ of } q(y)\}.$

Then X is \aleph_1 -comeager in $S_{at}^{1,\beta}(B)$.

Proof. For each $q(y) \in S_{at}^{1,\beta}(B)$ and each $\overline{a} \in B$, let $[q, \overline{a}](y, \overline{z}) \in S_{at}^{1+|a|,\beta}(A)$ be the set of all $\mathcal{L}_{\beta}(A)$ -formulas $\phi(y, \overline{z})$ such that $\phi(y, \overline{a}) \in q(y)$.

Fix $\overline{a} \in B$, say $|\overline{a}| = n$, and fix $C \ge \{p(\overline{x})\}$ with C a closed subset of $S_{at}^{1+n,\beta}(A)$. It suffices to show that $D := \{q(y) \in S_{at}^{1,\beta}(B) : [q,\overline{a}](y,\overline{z}) \in C\}$ is closed nowhere dense. It is clearly closed, since C is.

Suppose it weren't nowhere dense, say $\mathcal{O} = \{q(y) \in S_{at}^{1,\beta}(B) : \phi(y; \overline{a}, \overline{b}) \in q(y)\}$ is such that $\emptyset \neq \mathcal{O} \subseteq D$.

Let $m = |\overline{b}|$ and let \overline{w} be a tuple of variables of length m. Let $F(\overline{z}, \overline{w}) \subseteq S_{at}^{n+m,\beta}(A)$ be the closed set defined as in the first clause of Definition 5.11.

Then $F \succ C$ so $F \subset \Gamma(p(\overline{x}))$, but \overline{ab} realizes F, contradiction.

Lemma 5.14. Suppose $\alpha \leq \beta$ are both in C_1 , A is an α -base, $p(\overline{x}) \in K^*(A)$, $B \supseteq A$ is a β -base, and B omits $\Gamma(p(\overline{x}))$. Let

$$X = \{q(\overline{y}) \in K(B) : B\overline{d} \text{ omits } \Gamma(p(\overline{x})) \text{ for some (any) } \overline{d} \text{ realizing } q(\overline{y})\}.$$

Then X is \aleph_1 -comeager in K(B).

Proof. For each $q(\overline{y}) \in K(B)$ and each $\overline{a} \in B$, let $[q, \overline{a}](\overline{y}, \overline{z}) \in S_{at}^{n_0 + |\overline{a}|, \beta^+}(A)$ be the set of all $\mathcal{L}_{\beta^+}(A)$ -formulas $\phi(\overline{y}, \overline{z})$ such that $\phi(\overline{y}, \overline{a}) \in q(y)$.

Fix $\overline{a} \in B$, say $|\overline{a}| = n$, and fix $C \subseteq S_{at}^{n_0 + n, \beta^+}(A)$ closed, with $C \ge \{p(\overline{x})\}$. It suffices to show that $D := \{q(\overline{y}) \in K(B) : [q, \overline{a}](\overline{y}, \overline{z}) \in C\}$ is closed nowhere dense in K(B). It is clearly closed, since C is.

Suppose it weren't nowhere dense, say $\mathcal{O} = \{q(\overline{y}) \in K(B) : \phi(\overline{y}; \overline{a}, \overline{b}) \in q(\overline{y})\}$ is such that $\emptyset \neq \mathcal{O} \subseteq D$. We can suppose $\phi(\overline{y}, \overline{z}, \overline{w})$ is complete.

Let $m = |\overline{b}|$ and let \overline{w} be a tuple of variables of length m. Let $F(\overline{z}, \overline{w}) \subseteq S_{at}^{n+m,\beta}(A)$ be the closed set defined as in the second clause of Definition 5.11.

Then
$$F \succ C$$
 so $F \subset \Gamma(p(\overline{x}))$, but \overline{ab} realizes F , contradiction.

We conclude the proof of Theorem 2.8 with:

Lemma 5.15. \mathbb{P} is a system of promises, and $\operatorname{Spec}_{\mathbf{K}_T}(\mathbb{P}) \cap \mathbf{C}_1 = \emptyset$.

Proof. Invariance for \mathbb{P} is clear.

Extendibility follows from an iterated application of Lemma 5.13.

Finally, suppose $\alpha \in \mathbf{C}_1$ and (A, Φ) is an (α, \mathbb{P}) -base. Write $\Phi = \{\Gamma(p_n(\overline{x}_n)) : n < \omega\}$, where $p_n(\overline{x}_n) \in K^*(A_n)$ for some $A_n \subseteq A$. Let $X = \{q(\overline{x}) \in K(A) : A\overline{a}$ omits Φ for some (any) realization \overline{a} of $q(\overline{x})$. By applying Lemma 5.14 to each $p_n(\overline{x}_n)$ we get that X is \aleph_1 -comeager in K(A). Hence by Lemma 5.12 we can find $q(\overline{x}) \in X \cap K^*(A)$. Let \overline{a} realize $q(\overline{x})$; then the \mathbb{P} -amalgamation problem $((A, \Phi), (A\overline{a}, \Phi), (A, \Phi \cup \{q(\overline{x})\}))$ has no solution.

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