ZFC proves that the class of ordinals is not weakly compact for definable classes

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Abstract

In ZFC, the class Ord of ordinals is easily seen to satisfy the definable version of strong inaccessibility. Here we explore deeper ZFC-verifiable combinatorial properties of Ord, as indicated in Theorems A & B below. Note that Theorem A shows the unexpected result that Ord is never definably weakly compact in any model of ZFC.

Theorem A. *Let $\mathcal{M}$ be any model of* ZFC.

(1) *The definable tree property fails in $\mathcal{M}$: There is an $\mathcal{M}$-definable* Ord*-tree with no $\mathcal{M}$-definable cofinal branch.*

(2) *The definable partition property fails in $\mathcal{M}$: There is an $\mathcal{M}$-definable 2-coloring $f : [X]^2 \to 2$ for some $\mathcal{M}$-definable proper class X such that no $\mathcal{M}$-definable proper classs is monochromatic for f.*

(3) *The definable compactness property for $\mathcal{L}_{\infty,\omega}$ fails in $\mathcal{M}$: There is a definable theory Γ in the logic $\mathcal{L}_{\infty,\omega}$ (in the sense of $\mathcal{M}$) of size* Ord *such that every set-sized subtheory of Γ is satisfiable in $\mathcal{M}$, but there is no $\mathcal{M}$-definable model of Γ.*

Theorem B. *The definable $\Diamond_{\mathrm{Ord}}$ principle holds in a model $\mathcal{M}$ of* ZFC *iff $\mathcal{M}$ carries an $\mathcal{M}$-definable global well-ordering.*

Theorems A and B above can be recast as theorem schemes in ZFC, or as asserting that a single statement in the language of class theory holds in all 'spartan' models of GB (Gödel-Bernays class theory); where a spartan model of GB is any structure of the form $(\mathcal{M}, D_{\mathcal{M}})$, where $\mathcal{M} \models$ ZF and $D_{\mathcal{M}}$ is the family of $\mathcal{M}$-definable classes. Theorem C gauges the complexity of the collection $\mathrm{GB}_{\mathrm{spa}}$ of (Gödel-numbers of) sentences that hold in all spartan models of GB.

Theorem C. $\mathrm{GB}_{\mathrm{spa}}$ *is Π^1_1-complete.*

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1. Introduction & Preliminaries

In ZFC, the class Ord of ordinals satisfies the definable version of strong inaccessibility since the power set axiom and the axiom of choice together make it evident that Ord is closed under cardinal exponentiation; and the scheme of replacement ensures the definable regularity of Ord in the sense that for each cardinal $\kappa < \mathrm{Ord}$, the range of every definable ordinal-valued map f with domain κ is bounded in Ord. In this paper we investigate more subtle definable combinatorial properties of Ord in the context of ZFC to obtain results, each of which takes the form of a *theorem scheme* within ZFC. In Section 2 we establish a number of results that culminate in Theorem 2.6, which states that the tree property fails for definable classes across all models of ZFC; this result is then used in Section 3 to show the failure of the partition property for definable classes, and the failure of weak compactness of Ord for definable classes in all models of ZFC. *Thus, the results in Sections 2 and 3 together demonstrate the unexpected* ZFC*-provable failure of the definable version of a large cardinal property for* Ord. In Section 4 we establish the equivalence of the combinatorial principle $\Diamond_{\mathrm{Ord}}$ and the existence of a definable global choice function across all models of ZFC.

The results in Sections 2 through 4 can be viewed as stating that certain sentences in the language of class theory hold in all 'spartan' models of GB (Gödel-Bernays class theory), i.e., in all models of GB of the form $(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$, where $\mathcal{M}$ is a model of ZF and $\mathcal{D}_{\mathcal{M}}$ is the collection of $\mathcal{M}$-definable subsets of M. For example Theorem 2.6 is equivalent to the veracity of the statement "if the axiom of choice for sets holds, then there is an Ord-Aronszajn tree" in every spartan model of GB. In Section 5 we show that the *theory* of all spartan models of GB, when viewed as a subset of ω via Gödel-numbering, is Π^1_1-complete; and a fortiori, it is not computably axiomatizable.

We now turn to reviewing pertinent preliminaries concerning models of set theory. Our meta-theory is ZFC.

1.1. Definition. Suppose $\mathcal{M} = (M, \in^{\mathcal{M}})$ and $\mathcal{N} = (N, \in^{\mathcal{N}})$ are models of set theory. Note that we are not assuming that either $\mathcal{M}$ or $\mathcal{N}$ is well-founded.

(a) For $m \in M$, let $m_{\mathcal{M}} := \{x \in M : x \in^{\mathcal{M}} m\}$. If $\mathcal{M} \subseteq \mathcal{N}$ (i.e., $\mathcal{M}$ is a submodel of $\mathcal{N}$) and $m \in M$, then $\mathcal{N}$ *fixes* m if $m_{\mathcal{M}} = m_{\mathcal{N}}$. $\mathcal{N}$ *end extends* $\mathcal{M}$, written $\mathcal{M} \subseteq_e \mathcal{N}$, iff $\mathcal{N}$ fixes every $m \in M$. Equivalently: $\mathcal{M} \subseteq_e \mathcal{N}$ iff $\mathcal{M}$ is a transitive submodel of $\mathcal{N}$ in the sense that if $x \in^{\mathcal{N}} y$ for some $x \in N$ and some $y \in M$, then $x \in^{\mathcal{M}} y$.

(b) Given $n \in \omega$, $\mathcal{N}$ is a *proper* Σ_n*-e.e.e.* of $\mathcal{M}$ ("e.e.e." stands for "elementary end extension"), iff $\mathcal{M} \subsetneq_e \mathcal{N}$, and $\mathcal{M} \prec_{\Sigma_n} \mathcal{N}$ (i.e., Σ_n-statements with parameters from M are absolute in the passage between $\mathcal{N}$ and $\mathcal{M}$). It is well-known that if $\mathcal{M} \prec_{\Sigma_2} \mathcal{N}$ and $\mathcal{M} \models \mathrm{ZF}$, then $\mathcal{N}$ is a *rank extension* of $\mathcal{M}$, i.e., whenever $a \in M$ and $b \in N \backslash M$, then $\mathcal{N} \models \rho(a) \in \rho(b)$, where ρ is the usual ordinal-valued rank function on sets.

(c) Given $\alpha \in \mathrm{Ord}^{\mathcal{M}}$, $\mathcal{M}_\alpha$ denotes the structure $(\mathrm{V}_\alpha, \in)^{\mathcal{M}}$, and $M_\alpha = \mathrm{V}_\alpha^M$.

(d) For $X \subseteq M^n$ (where $n \in \omega$), we say that X is $\mathcal{M}$-*definable* iff X is parametrically definable in $\mathcal{M}$.

(e) $\mathcal{N}$ is a *conservative* extension of $\mathcal{M}$, written $\mathcal{M} \subseteq_{\mathrm{cons}} \mathcal{N}$, iff the intersection of any $\mathcal{N}$-definable subset of N with M is $\mathcal{M}$-definable.

For models of ZF, the set-theoretical sentence $\exists p\, (\mathrm{V} = \mathrm{HOD}(p))$ expresses: "there is some p such that every set is first order definable in some structure of the form $(\mathrm{V}_\alpha, \in, p)$ with $p \in \mathrm{V}_\alpha$". The following theorem is well-known; the equivalence of (a) and (b) will be revisited in Theorem 4.2.

1.2. Theorem. *The following statements are equivalent for* $\mathcal{M} \models \mathrm{ZF}$:

(a) $\mathcal{M} \models \exists p\, (\mathrm{V} = \mathrm{HOD}(p))$.

(b) *For some* $p \in M$ *and some set-theoretic formula* $\varphi(x, y, \overline{p})$ (*where* $\overline{p}$ *is a name for* p) $\mathcal{M}$ *satisfies* "φ *well-orders the universe*".

(c) *For some* $p \in M$ *and some* Σ_2-*formula* $\varphi(x, y, \overline{p})$ $\mathcal{M}$ *satisfies* "φ *well-orders the universe*".

(d) $\mathcal{M} \models \forall x (x \neq \varnothing \rightarrow f(x) \in x)$ *for some* $\mathcal{M}$-*definable* $f : M \rightarrow M$.

Next we use definable classes to lift certain combinatorial properties of cardinals to the class of ordinals.

1.3. Definitions. Suppose $\mathcal{M} \models \mathrm{ZFC}$.

(a) Suppose $\tau = (T, <_T)$ is a tree ordering, where both T and $<_T$ are $\mathcal{M}$-definable. τ is an Ord-*tree in* $\mathcal{M}$ iff $\mathcal{M}$ satisfies "τ is a well-founded tree of height Ord and for all $\alpha \in \mathrm{Ord}$, the collection T_α of elements of T at level α of τ form a set". Such a tree τ is said to be a *definably* Ord-*Aronszajn* tree in $\mathcal{M}$ iff no cofinal branch of τ is $\mathcal{M}$-definable.

(b) *The definable tree property for* Ord *fails in* $\mathcal{M}$ iff there exists a definably Ord-*Aronszajn* tree in $\mathcal{M}$.[1]

[1] This notion should not be confused with the *definable tree property of a cardinal* κ, first introduced and studied by Leshem [L], which stipulates that every κ-tree that is first order definable (parameters allowed) in the structure $(H(\kappa), \in)$ has a cofinal branch B (where $H(\kappa)$ is the collection of sets that are hereditarily of cardinality less than κ). Note that in this definition B is not required to be first order definable in $(H(\kappa), \in)$; so every weakly compact cardinal has the definable tree property.

(c) *The definable proper class partition property fails in* $\mathcal{M}$ iff there is an $\mathcal{M}$-definable proper class X of M with an $\mathcal{M}$-definable 2-coloring $f : [X]^2 \to 2$ such that there is no $\mathcal{M}$-definable monochromatic proper class for f. We also say that $\mathrm{Ord} \to (\mathrm{Ord})^2_2$ *fails in* $\mathcal{M}$ iff there is an $\mathcal{M}$-definable 2-coloring $f : [\mathrm{Ord}]^2 \to 2$ such that there is no $\mathcal{M}$-definable monochromatic proper class for f.

(d) *The definable compactness property for* $\mathcal{L}_{\infty,\omega}$ *fails in* $\mathcal{M}$ iff there is an $\mathcal{M}$-definable theory Γ formulated in the logic $\mathcal{L}_{\infty,\omega}$ (in the sense of $\mathcal{M}$) such that every set-sized subtheory of Γ is satisfiable in $\mathcal{M}$, but there is no $\mathcal{M}$-definable model of T. Here $\mathcal{L}_{\infty,\omega}$ is the extension of first order logic that allows conjunctions and disjunctions applied to *sets* of formulae (of any cardinality) with only a finite number of free variables, as in [B, Ch.III].

(e) An $\mathcal{M}$-definable subset E of $\mathrm{Ord}^{\mathcal{M}}$ is said to be *definably* $\mathcal{M}$*-stationary* iff $E \cap C \neq \varnothing$ for every $\mathcal{M}$-definable subset C of $\mathcal{M}$ such that C is closed and unbounded in $\mathrm{Ord}^{\mathcal{M}}$.

(f) *The definable* $\diamondsuit_{\mathrm{Ord}}$ *holds in* $\mathcal{M}$ iff there is some $\mathcal{M}$-definable $\vec{A} = \left\langle A_\alpha : \alpha \in \mathrm{Ord}^{\mathcal{M}} \right\rangle$ such that $\mathcal{M}$ satisfies "$A_\alpha \subseteq \alpha$ for all $\alpha \in \mathrm{Ord}$", and for all $\mathcal{M}$-definable $A \subseteq \mathrm{Ord}^{\mathcal{M}}$ there is $E \subseteq \mathrm{Ord}^{\mathcal{M}}$ such that E is definably $\mathcal{M}$-stationary and $A_\alpha = A \cap \alpha$ for all $\alpha \in E$. Here $\vec{A}$ is said to be $\mathcal{M}$-definable if there is an $\mathcal{M}$-definable A such that $A_\alpha = \{m : \langle m, \alpha\rangle \in A\}$ for each $\alpha \in \mathrm{Ord}^{\mathcal{M}}$.

2. The failure of the definable tree property for the class of ordinals

The proof of the main result of this section (Theorem 2.6) is based on a number of preliminary model-theoretic results which are of interest in their own right. We should point out that a proof of a special case of Theorem 2.6 was sketched in [En-2, Remark 3.5] for models of set theory with built-in global choice functions, using a more technical argument than the one presented here.

We begin with the following theorem which refines a result of Kaufmann [Ka, Theorem 4.6]. The proof uses an adaptation of Kaufmann's proof based on a strategy introduced in [En-1, Theorem 1.5(a)].

2.1. Theorem. *No model of* ZFC *has a proper conservative* Σ_3*-e.e.e.*

Proof. Suppose to the contrary that $\mathcal{M} \models \mathrm{ZF}$ and $\mathcal{M} \prec_{\Sigma_3,\mathrm{e},\mathrm{cons}} \mathcal{N}$ for some $\mathcal{N}$. Let φ be the statement that expresses the following instance of the reflection theorem:

$$\forall \lambda \in \mathrm{Ord}\ \exists \beta \in \mathrm{Ord}\,(\lambda \in \beta \wedge\ (\mathrm{V}_\beta, \in) \prec_{\Sigma_1} (\mathrm{V}, \in)).$$

Using the fact that the satisfaction predicate for Σ_1-formulae is Σ_1-definable it is easy to see that φ is a Π_3-statement, and thus φ also holds in $\mathcal{N}$ since φ holds in $\mathcal{M}$ by the reflection theorem.[2] So we can fix some $\lambda \in \mathrm{Ord}^{\mathcal{N}} \backslash \mathrm{Ord}^{\mathcal{M}}$ and some $\mathcal{N}$-ordinal $\beta > \lambda$ of $\mathcal{N}$ such that:

$$\mathcal{N}_\beta \prec_{\Sigma_1} \mathcal{N}.$$

Note that this implies that $\mathcal{N}_\beta$ can meaningfully define the satisfaction predicate for every set-structure 'living in' $\mathcal{N}_\beta$ since that $\mathcal{N}_\beta$ is a model of a substantial fragment of ZF, including KP (Kripke-Platek set theory), and already KP is sufficient for this purpose [B, III.2]. Also, since the statement "every set can be well-ordered" is a Π_2-statement which holds in $\mathcal{M}$ by assumption, it also holds in $\mathcal{N}$, and therefore we can fix a binary relation w in $\mathcal{N}$ such that, as viewed in $\mathcal{N}$, w is a well-ordering of V_β. Hence for any $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ with $\alpha < \beta$, *within* $\mathcal{N}$ one can define the submodel $\mathcal{K}_\alpha$ of $\mathcal{N}_\beta$ whose universe K_α is defined via:

$$K_\alpha := \{a \in \mathrm{V}_\beta : a \text{ is first order definable in } (\mathcal{N}_\beta, w, \lambda, m)_{m \in \mathrm{V}_\alpha}\}.$$

Clearly $M_\alpha \cup \{\lambda\} \subsetneq \mathcal{K}_\alpha \prec \mathcal{N}_\beta$, and of course $\mathcal{K}_\alpha$ is a member of $\mathcal{N}$. Next let:

$$\mathcal{K} := \bigcup_{\alpha \in \mathrm{Ord}^{\mathcal{M}}} \mathcal{K}_\alpha.$$

Note that we have:

$$\mathcal{M} \subsetneq_{\mathrm{e}} \mathcal{K} \preceq \mathcal{N}_\beta \prec_1 \mathcal{N}.$$

We now make a crucial case distinction: either (a) $\mathrm{Ord}^{\mathcal{K}} \backslash \mathrm{Ord}^{\mathcal{M}}$ has minimum element, or (b) it does not. The proof will be complete once we verify that both cases lead to a contradiction.

Case (a). Let $\eta = \min(\mathrm{Ord}^{\mathcal{K}} \backslash \mathrm{Ord}^{\mathcal{M}})$. We claim that $\mathcal{M} \prec \mathcal{N}_\eta$. To see this, we use Tarski's test for elementarity: suppose $\mathcal{N}_\eta \models \exists x \varphi(x, \overline{m})$ for some $m \in M$ and some formula $\varphi(x, y)$, and let θ_0 be defined in $\mathcal{N}_\beta$ as the least ordinal θ such that $x \in \mathrm{V}_\theta$ and $\mathrm{V}_\eta \models \exists x\ \varphi(x, \overline{m})$. Then $\theta_0 \in K$ and clearly $\theta_0 < \eta$, which shows that $\theta_0 \in \mathrm{Ord}^{\mathcal{M}}$. Hence $\mathcal{N}_\eta \models \varphi(\overline{m_0}, \overline{m})$ for some $m_0 \in M$, thus completing the proof of $\mathcal{M} \prec \mathcal{N}_\eta$. But if $\mathcal{M} \prec \mathcal{N}_\eta$, then we can choose S in $\mathcal{N}$ such that:

[2]Recall that, provably in ZF, the ordinals β such that $(\mathrm{V}_\beta, \in) \prec_{\Sigma_1} (\mathrm{V}, \in)$ are precisely the fixed points of the $\beth$-function.

$$\mathcal{N} \models S = \{\ulcorner \varphi(\overline{m}) \urcorner \in \mathrm{V}_\eta : \mathcal{N} \models \text{“}(\mathrm{V}_\eta, \in) \models \varphi(\overline{m})\text{”}\}.$$

Based on the assumption that $\mathcal{N}$ is a conservative extension of $\mathcal{M}$, $S \cap M$ should be an $\mathcal{M}$-definable satisfaction predicate for $\mathcal{M}$, which contradicts (a version of) Tarski's undefinability of truth theorem.

Case (b). This is the more difficult case, where $\mathrm{Ord}^{\mathcal{K}} \backslash \mathrm{Ord}^{\mathcal{M}}$ has no least element. Let $\Phi := \bigcup\limits_{\alpha \in \mathrm{Ord}^{\mathcal{M}}} \Phi_\alpha$, where

$$\Phi_\alpha := \{\ulcorner \varphi(c, \overline{m}) \urcorner \in M : \mathcal{N} \models \text{“}(\mathrm{V}_\beta, \in, w, \lambda, m)_{m \in \mathrm{V}_\alpha} \models \varphi(c, \overline{m})\text{”}\}.$$

In the above definition of Φ_α, the constant c is interpreted as λ and $\varphi(c, \overline{m})$ ranges over first order formulae in the sense of $\mathcal{M}$ (or equivalently: in the sense of $\mathcal{N}$) in the language

$$\mathcal{L}_\alpha = \{\in, \lhd, c\} \cup \{\overline{m} : m \in \mathrm{V}_\alpha\},$$

where c is a new constant symbol and $\lhd$ is a binary relation symbol interpreted by w. Thus Φ can be thought of as the type of λ in $\mathcal{N}_\beta$ over M. Since $\mathcal{N}$ is assumed to be a conservative extension of $\mathcal{M}$, Φ is $\mathcal{M}$-definable via some unary formula ϕ. Hence Γ below is also $\mathcal{M}$-definable via some unary formula γ :

$$\overbrace{\left\{\ulcorner t(c, \overline{m}) \urcorner \in M : \phi\left(\ulcorner t(c, \overline{m}) \in \mathrm{Ord} \urcorner\right) \text{ and } \forall \theta \in \mathrm{Ord}(\phi\left(\ulcorner t(c, \overline{m}) > \overline{\theta} \urcorner\right)\right\}}^{\Gamma},$$

where t is a definable term in the language $\mathcal{L}$, i.e., $t(c, \overline{m})$ is an $\mathcal{L}$-definition $\varphi(c, \overline{m}, x)$ of some element x. So, officially speaking, Γ consists of $\ulcorner \varphi(c, \overline{m}, x) \urcorner \in M$ that satisfy the following three conditions:

(1) $\phi\left(\ulcorner \exists! x \varphi(c, \overline{m}, x \urcorner\right)$.
(2) $\phi\left(\ulcorner \forall x\left(\varphi(c, \overline{m}, x) \rightarrow x \in \mathrm{Ord}\right) \urcorner\right)$.
(3) $\forall \theta \in \mathrm{Ord}\ \phi\left(\ulcorner \forall x\left(\varphi(c, \overline{m}, x) \rightarrow x > \overline{\theta}\right) \urcorner\right)$.

Since $\mathrm{Ord}^{\mathcal{K}} \backslash \mathrm{Ord}^{\mathcal{M}}$ has no minimum element (recall: we are analysing case (b)), $\mathcal{M} \models \psi$, where:

$$\psi := \forall t\left(\gamma(t) \rightarrow \exists t'(\gamma(t') \wedge \phi(\ulcorner t' \in t \urcorner)\right).$$

Choose k such that ψ is a Σ_k-statement, and use the reflection theorem in $\mathcal{M}$ to pick $\mu \in \mathrm{Ord}^{\mathcal{M}}$ such that $\mathcal{M}_\mu \prec_{\Sigma_k} \mathcal{M}$. Then ψ holds in $\mathcal{M}_\mu$, so by DC (dependent choice, which holds in $\mathcal{M}$ since AC holds in $\mathcal{M}$), there is some function f_c in $\mathcal{M}$ such that:

$$\mathcal{M} \models \forall n \in \omega \ \phi\left(\ulcorner f_c(n+1) \in f_c(n) \urcorner\right).$$

Let $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ be large enough so that M_α contains all constants $\overline{m}$ that occur in any of the terms in the range of f; let $f_\lambda(n)$ be defined in $\mathcal{N}$ as the result of replacing all occurrences of the constant c with $\overline{\lambda}$ in $f_c(n)$; and let $g(n)$ be defined in $\mathcal{N}$ as the interpretation of $f_\lambda(n)$ in $(\mathrm{V}_\beta, \in, w, \lambda, m)_{m \in \mathrm{V}_\alpha}$. Then $\mathcal{N}$ satisfies:

$$\forall n \in \omega \ \ (g(n) \in g(n+1)),$$

which contradicts the foundation axiom in $\mathcal{N}$. The proof is now complete. □

2.2. Definition.
(a) Given ordinals $\alpha < \beta$, $\mathcal{V}_{\beta,\alpha}$ denotes the structure $(\mathrm{V}_\beta, \in, a)_{a \in \mathrm{V}_\alpha}$, and for a model $\mathcal{M} \models \mathrm{ZF}$,

$$\mathcal{M}_{\beta,\alpha} := (\mathcal{V}_{\beta,\alpha})^{\mathcal{M}}.$$

(b) Given a meta-theoretic natural number n, τ_n denotes the definable tree whose nodes at level α consist of first order theories of the form $\mathrm{Th}(\mathcal{V}_{\beta,\alpha}, s)$, where $s \in \mathrm{V}_\beta \backslash \mathrm{V}_\alpha$, and β is n-correct[3]. The language of $\mathrm{Th}(\mathcal{V}_{\beta,\alpha}, s)$ consists of $\{\in\}$ plus constants $\overline{m}$ for each $m \in \mathrm{V}_\alpha$, and a new constant c whose denotation is s. The ordering of the tree is by set-inclusion.

2.3. Lemma. *For each meta-theoretic natural number n,* ZFC *proves "τ_n is an* Ord-*tree".*

Proof. Thanks to the Montague-Vaught reflection theorem, there are plenty of nodes at any ordinal level α. On the other hand, since each $\mathrm{Th}(\mathcal{V}_{\beta,\alpha}, s)$ can be canonically coded as a subset of V_α, and $|\mathrm{V}_{\omega+\alpha}| = \beth_\alpha$, there are at most $\beth_\alpha$-many nodes at level α □

2.4. **Remark.** One may 'prune' every Ord-tree τ to obtain a definable subtree τ^* which has nodes of arbitrarily high level in Ord by simply throwing away the nodes whose set of successors have bounded height and then using the replacement scheme to verify that the subtree τ^* thus obtained has height Ord. See [K, Lemma 3.11] for a similar construction for κ-trees (where κ is a regular cardinal).

2.5. Lemma. *Suppose $\mathcal{M}$ is a model of* ZFC *that carries an $\mathcal{M}$-definable global well-ordering. Furthermore, suppose that $n \geq 3$ and the tree $\tau_n^{\mathcal{M}}$ has a branch B.* ***Then:***

[3] An ordinal β is n-correct when $(\mathrm{V}_\beta, \in) \prec_{\Sigma_n} (\mathrm{V}, \in)$.

(a) *There is a model* $\mathcal{N}$ *and a proper embedding* $j : \mathcal{M} \to \mathcal{N}$ *such that* $j(\mathcal{M}) \prec_{\mathrm{e},n} \mathcal{N}$.

(b) *Both* $\mathcal{N}$ *and* j *are* $\mathcal{M}$*-definable if* B *is* $\mathcal{M}$*-definable.*

(c) $\mathcal{N}$ *is a conservative extension of* $j(\mathcal{M})$ *if* B *is* $\mathcal{M}$*-definable.*

Proof. We will only prove (a) since the proof of (b) will be clear by an inspection of the proof of (a), and (c) is an immediate consequence of (b). Let B be a branch of $\tau_n^{\mathcal{M}}$. Each node in B is a first order theory in the sense of $\mathcal{M}$ and is of the form $(\mathrm{Th}(\mathrm{V}_{\beta,\alpha}, s))^{\mathcal{M}}$. Note that $(\mathrm{Th}(\mathcal{V}_{\beta,\alpha}, s))^{\mathcal{M}}$ is not the necessarily the same as $\mathrm{Th}(\mathcal{M}_{\beta,\alpha}, s)$, since the latter is the collection of *standard* sentences in $(\mathrm{Th}(\mathcal{V}_{\beta,\alpha}, s))^{\mathcal{M}}$. In particular, if $\mathcal{M}$ not ω-standard, then:

$$\mathrm{Th}(\mathcal{M}_{\beta,\alpha}, s) \subsetneq (\mathrm{Th}(\mathcal{V}_{\beta,\alpha}, s))^{\mathcal{M}}.$$

For each $\alpha \in \mathrm{Ord}^{\mathcal{M}}$, let b_α be the node of B at level α. We may choose some $\beta_\alpha \in \mathrm{Ord}^{\mathcal{M}}$ and some $s_\alpha \in (\mathrm{V}_{\beta_\alpha} \backslash \mathrm{V}_\alpha)^{\mathcal{M}}$ such that:

$$b_\alpha = (\mathrm{Th}(\mathcal{V}_{\beta_\alpha,\alpha}, s_\alpha))^{\mathcal{M}}.$$

The above choices of β_α and s_α are performed at the meta-theoretic level (where ZFC is assumed); however if B is $\mathcal{M}$-definable, then so is the map $\alpha \mapsto b_\alpha$, which in turn shows that the maps $\alpha \mapsto \beta_\alpha$ and $\alpha \mapsto s_\alpha$ can also be arranged to be $\mathcal{M}$-definable since $\mathcal{M}$ is assumed to carry an $\mathcal{M}$-definable global well-ordering (the definability of these two maps plays a key role in verifying that an inspection of the proof of (a) yields a proof of (b)).

We now explain how to use B to construct the desired structure $\mathcal{N}$. In order to do so, we need some definitions:

(i) Let $\mathcal{L}$ be the language consisting of the usual language $\{\in\}$ of set theory, augmented with a binary relation symbol $\lhd$, constants $\overline{m}$ for each $m \in M$, and a new constant c.

(ii) For each $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ let $\mathcal{N}_\alpha$ be the submodel of $\mathcal{M}_{\beta_\alpha}$ whose universe N_α consists of elements of M_{β_α} that are first order definable in the structure $(\mathcal{V}_{\beta_\alpha,\alpha}, s_\alpha)$, *as viewed from* $\mathcal{M}$ (so the available parameters for the definitions come from $M_\alpha \cup \{s_\alpha\}$ and consequently $M_\alpha \cup \{s_\alpha\} \subseteq N_\alpha$). By Theorem 1.2 we may assume that for some formula $W(x, y, \overline{m})$ the sentence "W is a global well-ordering" is equivalent to a Π_3-statement in $\mathcal{M}$. Therefore, since $n \geq 3$, the statement "there is a well-ordering of $\mathrm{V}_{\beta_\alpha}$ that is definable in $(\mathrm{V}_{\beta_\alpha}, \in)$" holds in $\mathcal{M}$, which immediately shows (by Tarski's elementarity test) that the statement expressing $\mathcal{N}_\alpha \prec \mathcal{V}_{\beta_\alpha}$ holds in $\mathcal{M}$.[4] It is important to have in mind that, as viewed from $\mathcal{M}$, each member of $\mathcal{N}_\alpha$ can be written as the denotation $\delta^{\mathcal{N}_\alpha}$ of a definable term $\delta = \delta(\overline{m_\delta}, c)$ for some $m \in M$ in the language $\mathcal{L}$ described above (where c is interpreted by s_α) so δ might be of nonstandard length if $\mathcal{M}$ is not ω-standard (here we are taking advantage of the definability of a sequence-coding function in $\mathcal{M}_{\beta_\alpha}$ to reduce the number of parameters of a definable term that come from M_α to one).

(*iii*) Given ordinals $\alpha_1, \alpha_2 \in \mathrm{Ord}^{\mathcal{M}}$ with $\alpha_1 < \alpha_2$, in $\mathcal{M}$ consider:

$$j_{\alpha_1,\alpha_2} : \mathcal{N}_{\alpha_1} \to \mathcal{N}_{\alpha_2}, \text{ where } j_{\alpha_1,\alpha_2}(\delta^{\mathcal{N}_{\alpha_1}}) := \delta^{\mathcal{N}_{\alpha_2}}.$$

It is not hard to see that j_{α_1,α_2} is an *elementary* embedding as viewed from $\mathcal{M}$. This follows from the following key facts:

- $\left(\mathrm{Th}(\mathcal{V}_{\beta_{\alpha_1},\alpha_1}, s_{\alpha_1})\right)^{\mathcal{M}} = \left(\mathrm{Th}(\mathcal{V}_{\beta_{\alpha_2},\alpha_1}, s_{\alpha_2})\right)^{\mathcal{M}}$, whenever $\alpha_1, \alpha_2 \in \mathrm{Ord}^{\mathcal{M}}$ with $\alpha_1 < \alpha_2$; and

- $\mathcal{M} \models \mathcal{N}_\alpha \prec \mathcal{V}_{\beta_\alpha}$ for each $\alpha \in \mathrm{Ord}^{\mathcal{M}}$.

(*iv*) Hence $\langle j_{\alpha_1,\alpha_2} : \alpha_1 < \alpha_2 \in \mathrm{Ord}^{\mathcal{M}} \rangle$ is a *directed system of elementary embeddings*. The desired $\mathcal{N}$ is the direct limit of this system. Thus, the elements of $\mathcal{N}$ are equivalence classes $[f]$ of "strings" f of the form:

$$f : \{\alpha \in \mathrm{Ord}^{\mathcal{M}} : \alpha \geq \alpha_0\} \to \bigcup_{\alpha \in \mathrm{Ord}^{\mathcal{M}}} N_\alpha,$$

where $\alpha_0 \in \mathrm{Ord}^{\mathcal{M}}$ and there is some $\mathcal{L}$-term δ such that $m_\delta \in M_{\alpha_0}$ and $f(\alpha) = \delta^{\mathcal{N}_\alpha} \in N_\alpha$ (two strings are identified iff they agree on a tail of $\mathrm{Ord}^{\mathcal{M}}$). In particular, for each $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ there is an embedding:

$$j_{\alpha,\infty} : \mathcal{N}_\alpha \to \mathcal{N}, \text{ where } j_{\alpha,\infty}(\delta^{\mathcal{N}_\alpha}) := [h], \text{ and}$$
$$h(\alpha) := \delta^{\mathcal{N}_\alpha} \text{ for all } \alpha \text{ such that } m_\delta \in M_\alpha.$$

A routine variant of Tarski's elementary chains theorem guarantees that $j_{\alpha,\infty}$ is an *elementary embedding* for all $\alpha \in \mathrm{Ord}^{\mathcal{M}}$.

(*v*) For $m \in M$, let $f_m(\alpha) := m = \overline{m}^{\mathcal{N}_\alpha}$ for all $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ such that $m \in M_\alpha$, and consider the embedding

[4]This is the only part of the proof that takes advantage of the assumption that $\mathcal{M}$ carries a definable global well-ordering.

$$j : \mathcal{M} \to \mathcal{N}, \text{ where } j(m) := [f_m(\alpha)].$$

By identifying m with $[f_m]$ we can, without loss of generality, construe $\mathcal{M}$ as a *submodel of* $\mathcal{N}$.

A distinguished element of $\mathcal{N}$ is $[g]$, where $g(\alpha) = s_\alpha$ for $\alpha \in \mathrm{Ord}^{\mathcal{M}}$. $[g] \neq [f_m]$ for all $m \in M$ since $s_\alpha \notin \mathrm{V}_\alpha$ for all α and therefore g and f_m differ on a tail of $\alpha \in \mathrm{Ord}^{\mathcal{M}}$. This shows that $\mathcal{M}$ is a *proper submodel* of $\mathcal{N}$. To see that $\mathcal{N}$ *end* extends $\mathcal{M}$, suppose $m \in M$ and for some $\mathcal{L}$-definable term δ, $\delta^{\mathcal{N}_\alpha} \in \overline{m}$ holds in $\mathcal{N}_\alpha$ for sufficiently large α, i.e., for any α such that $\{m, m_\delta\} \subseteq M_\alpha$. Therefore there is some $m_0 \in \mathrm{V}_\alpha^{\mathcal{M}}$ such that $\delta^{\mathcal{N}_\alpha} = \overline{m_0}$ holds in $\mathcal{N}_\alpha$ for sufficiently large α, and therefore also in $\mathcal{N}$, hence $\mathcal{N}$ end extends $\mathcal{M}$.

Finally, let's verify that $\mathcal{M} \prec_{\Sigma_n} \mathcal{N}$. Suppose $\mathcal{M} \models \varphi(\overline{m})$, where φ is Σ_n and $m \in M$. Then $\varphi(\overline{m})$ holds for all sufficiently large $\mathcal{N}_\alpha$, since by design we have:

$$\mathcal{N}_\alpha \prec \mathcal{M}_{\beta_\alpha} \prec_{\Sigma_n} \mathcal{M}.$$

This shows that $\mathcal{N} \models \varphi(\overline{m})$ since, as observed earlier, each $\mathcal{N}_\alpha$ is elementarily embeddable in $\mathcal{N}$ via $j_{\alpha,\infty}$. □

We are now ready to verify that the tree property for Ord fails in the sense of $\mathcal{M}$ for all $\mathcal{M} \models \mathrm{ZFC}$.

2.6. Theorem. *Every model* $\mathcal{M}$ *of* ZFC *carries an* $\mathcal{M}$*-definable* $\mathrm{Ord}^{\mathcal{M}}$*-tree no cofinal branch of which is* $\mathcal{M}$*-definable.*

Proof. The proof splits into two cases, depending on whether $\mathcal{M}$ satisfies $\exists p\,(\mathrm{V} = \mathrm{HOD}(p))$ or not.[5]

Case 1. Suppose that $\exists p\,(\mathrm{V} = \mathrm{HOD}(p))$ fails in $\mathcal{M}$. Within ZFC we can define the tree τ_{Choice} whose nodes at level α are choice functions f for V_α, i.e., $f : \mathrm{V}_\alpha \to \mathrm{V}_\alpha$, where $f(x) \in x$ for all nonempty $x \in \mathrm{V}_\alpha$, and the tree ordering is set inclusion. Clearly ZFC can verify that τ is an Ord-tree. It is also clear that every $\mathcal{M}$-definable branch of $\tau^{\mathcal{M}}$ (if any) is an $\mathcal{M}$-definable global choice function. By Theorem 1.2 this shows that no branch of τ_{Choice} is $\mathcal{M}$-definable.

[5]Easton proved (in his unpublished dissertation [Ea]) that assuming Con(ZF) there is a model $\mathcal{M}$ of ZFC which carries no $\mathcal{M}$-definable global choice function for the class of pairs in $\mathcal{M}$; and in particular $\exists p\,(\mathrm{V} = \mathrm{HOD}(p))$ fails in $\mathcal{M}$. Easton's theorem was exposited by Felgner [F, p.231]; for a more recent and streamlined account, see Hamkins' MathOverflow answer [H].

Case 2. Now suppose $\exists p\,(\mathrm{V} = \mathrm{HOD}(p))$ holds in $\mathcal{M}$. Then by Theorem 1.2 there is some Σ_2-formula $W(x, y)$ that defines a global well-ordering of $\mathcal{M}$. Note that "W is a global well-ordering" is Π_3-expressible in $\mathcal{M}$. We claim that for any fixed $n \geq 3$, no branch of $\tau_n^{\mathcal{M}}$ is $\mathcal{M}$-definable. If not, then by Lemma 2.4 there is an $\mathcal{M}$-definable structure $\mathcal{N}$, and an $\mathcal{M}$-definable embedding $j : \mathcal{M} \to \mathcal{N}$ such that $\mathcal{N}$ is a proper is a Σ_n-e.e.e. of $j(\mathcal{M})$, which contradicts Theorem 2.1. □

3. Consequences of the failure of the definable tree property for the class of ordinals

In this section we use Theorem 2.6 to establish further results about definable combinatorial properties of proper classes within ZFC. Our first result improves Theorem 2.6 by combining its proof with appropriate combinatorial and coding techniques so as to obtain the description of a single subtree of ${}^{<\mathrm{Ord}}2$ that is Ord-Aronszajn across all models of ZFC; here

$$ {}^{<\mathrm{Ord}}2 = \bigcup_{\alpha \in \mathrm{Ord}} {}^{\alpha}2, $$

where ${}^{\alpha}2$ is the set of binary sequences of length α. The ordering on $2^{<\mathrm{Ord}}$ is 'end extension', denoted $\sqsubseteq$. Given a tree τ we say τ is a *subtree* of $\left({}^{<\mathrm{Ord}}2, \sqsubseteq\right)$ if each node of τ is an element of ${}^{<\mathrm{Ord}}2$, and the nodes of τ are ordered by $\sqsubseteq$

3.1. Theorem. *There is a definable class σ that satisfies the following three properties*:

(a) ZFC $\vdash$ σ *is a subtree of* $\left({}^{<\mathrm{Ord}}2, \sqsubseteq\right)$.

(b) ZFC $\vdash$ σ *is an* Ord-*tree.*

(c) *For all formulae $\beta(x, y)$ of set theory,* ZFC $\vdash$ "$\{x : \beta(x, y)\}$ *is not a branch of σ for any parameter y*".

Proof. The proof has two stages. In the first stage we construct an Ord-tree that satisfies properties (b) and (c); and then in the second stage we construct an appropriate variant of the tree constructed in the first stage which satisfies properties (a), (b) and (c).

Stage 1. Given Ord-trees $\sigma_1 = (S_1, <_1)$ and $\sigma_2 = (S_2, <_2)$, let $\sigma_1 \otimes \sigma_2$ be the tree whose set of nodes is:

$$ S_1 \otimes S_2 := \{(p, q) \in S_1 \times S_2 : h_1(p) = h_2(q)\}, $$

where $h_i(x)$ is the height (level) of x, i.e., the ordinal that measures the order-type of the set of predecessors of x in τ_i. The ordering on $\sigma_1 \otimes \sigma_2$ is given by:

$$(p,q) \lhd (p',q') \text{ iff } p <_1 p' \text{ and } q <_2 q'.$$

Routine considerations show that the following two assertions are verifiable in ZFC:

(i) $\sigma_1 \otimes \sigma_2$ is an Ord-tree.

(ii) Every branch B of $\sigma_1 \otimes \sigma_2$ is of the form:

$$\{(p,q) \in S_1 \otimes S_2 : p \in B_1 \text{ and } q \in B_2\},$$

where B_i is the branch of τ_i obtained by projecting B on its i-th coordinate. In particular, for any model $\mathcal{M} \models$ ZFC we have:

(iii) If $(\sigma_1 \otimes \sigma_2)^{\mathcal{M}}$ has an $\mathcal{M}$-definable branch, so do $\sigma_1^{\mathcal{M}}$ and $\sigma_2^{\mathcal{M}}$.

Let $\sigma_0 := \tau_{\text{Choice}} \otimes \tau_3$; where τ_{Choice} and τ_3 are as in the proof of Theorem 2.6. It is easy to see that σ_0 is an Ord-tree (provably in ZFC). The proof of Theorem 2.6, coupled with (iii) above shows that no branch of $\sigma_0^{\mathcal{M}}$ is $\mathcal{M}$-definable for any $\mathcal{M} \models$ ZFC.

Stage 2. The tools of this stage of the construction are Lemmas 3.1.1 and 3.1.2. Recall that the ordering on both trees τ_{Choice} and τ_3 is set-inclusion $\subseteq$.

Lemma 3.1.1. *Given $\mathcal{M} \models$ ZFC and Ord-trees σ_1 and σ_2 in $\mathcal{M}$ whose ordering (as viewed in $\mathcal{M}$) are set-inclusion, there is an $\mathcal{M}$-definable Ord-tree $\sigma_1 \oplus \sigma_2$ whose ordering is also set-inclusion such that $\sigma_1 \otimes \sigma_2$ is isomorphic to $\sigma_1 \oplus \sigma_2$ via an $\mathcal{M}$-definable isomorphism.*

Proof. Let S_i be the collection of nodes of σ_i, and consider the tree $\sigma_1 \oplus \sigma_2$ whose sets of nodes, $S_1 \oplus S_2$, is defined as:

$$\{(p \times \{0\}) \cup (q \times \{1\}) : (p,q) \in S_1 \otimes S_2\},$$

and whose ordering is set inclusion. It is easy to see the desired isomorphism between $\sigma_1 \otimes \sigma_2$ and $\sigma_1 \oplus \sigma_2$ is described by:

$$(p,q) \mapsto (p \times \{0\}) \cup (q \times \{1\}).$$

□ (Lemma 3.1.1)

Lemma 3.1.2. *Given $\mathcal{M} \models$ ZFC and any* Ord*-tree τ in $\mathcal{M}$ whose ordering (as viewed in $\mathcal{M}$) is set-inclusion, there is an* Ord*-tree $\widetilde{\tau}$ of $\mathcal{M}$ satisfying the following properties*:

(a) $\widetilde{\tau} = (\widetilde{T}, \sqsubseteq)$, *for some* $\widetilde{T} \subseteq {}^{<\mathrm{Ord}}2$.

(b) *If $\widetilde{\tau}$ has an $\mathcal{M}$-definable branch, then τ has an $\mathcal{M}$-definable branch.*

Proof. We will first describe a ZFC-construction that should be understood to be carried out within $\mathcal{M}$. Given a set s, let $\overline{s}$ be the transitive closure of $\{s\}$, and let $\kappa_s := |\overline{s}|$. It is well-known that given a bijection $g : \overline{s} \to |\overline{s}|$, s can be canonically coded by some binary sequence $v_g(s) \in {}^{|\overline{s}|}2$. More specifically, the $\in$ relation on $\overline{s}$ can be readily copied over κ_s with the help of g so as to obtain a binary relation $R_g(s)$ such that $(\overline{s}, \in) \cong (\kappa_s, R_g(s))$. Since, $R_g(s)$ is an extensional well-founded relation, s can thus be recovered from $R_g(s)$ as "the top element of the transitive collapse of $R_g(s)$". On the other hand, $R_g(s)$ can be coded-up as $X_g(s) \subseteq \kappa_s$ with the help of a canonical pairing function $p : \mathrm{Ord}^2 \to \mathrm{Ord}$. Thus, if $v_g(s) : \kappa_s \to \{0, 1\}$ is defined as the characteristic function of $X_g(s)$, then $s = F(v_g(s))$, where $F(x)$ is the parameter-free definable class function given by:

If $x \in {}^{<\mathrm{Ord}}2$, and $\overbrace{\{p^{-1}(t) : x(t) = 1\}}^{x^\circ}$ is well-founded, extensional, and has a top element, then $F(x)$ is the top element of the transitive collapse of x°; otherwise $F(x) = 0$.

Given an Ord-tree $\tau = (T, \subseteq)$, let T_α be the set of elements of T of height $\alpha \in \mathrm{Ord}$, and for $s \in T_\alpha$, and $\beta \leq \alpha$, let s_β be the unique element in T_β that is a subset of s. Let

$$h_g(s) := \bigoplus_{\beta \leq \alpha} v_g(s_\beta),$$

where $g : \overline{s} \to |\overline{s}|$ is a bijection and the operation $\oplus$ is defined as follows: given a transfinite sequence $\langle m_\beta : \beta \leq \alpha \rangle$ of binary sequences, $\bigoplus\limits_{\beta \leq \alpha} m_\beta$ is the *ternary* sequence obtained by concatenating the sequence of sequences $\langle m_\beta * \langle 2 \rangle : \beta \leq \alpha \rangle$, where $m_\beta * \langle 2 \rangle$ is the concatenation of the sequence m_β and the sequence $\langle 2 \rangle$. Thus the 'maximal binary blocks' of $\bigoplus\limits_{\beta \leq \alpha} m_\beta$ are precisely sequences of the form m_β for some $\beta \leq \alpha$. This makes it clear that s can be readily 'read off' $h_g(s)$ as the result of applying F to last binary block of $h_g(s)$.

Let $\widetilde{T}_0 := \{h_g(s) : s \in T$, and g is a bijection between $\overline{s}$ and $|\overline{s}|\}$. We are now ready to define the desired $\widetilde{T}$. Fix a canonical embedding G of ${}^{<\mathrm{Ord}}3$ into ${}^{<\mathrm{Ord}}2$, and let:

$$\widetilde{T} := \{G(v) : v \in \widetilde{T}_0\}.$$

It is easy to see, using the assumption that $(T, \subseteq)$ is an Ord-tree, that $\widetilde{\tau} := (\widetilde{T}, \sqsubseteq)$ is an Ord-tree. Since $\widetilde{T} \subseteq {}^{<\mathrm{Ord}}2$, it remains to show that if $\widetilde{\tau}$ has an $\mathcal{M}$-definable branch, then τ also has an $\mathcal{M}$-definable branch. Suppose $\widetilde{B} = \{\widetilde{b}_\alpha : \alpha \in \mathrm{Ord}^{\mathcal{M}}\}$ is a branch of $\widetilde{\tau}$. Let

$$\widetilde{B}_0 := \{G^{-1}(\widetilde{b}_\alpha) : \alpha \in \mathrm{Ord}^{\mathcal{M}}\}$$

Note that $\widetilde{B}_0$ is a cofinal branch of the tree $\widetilde{\tau}_0$; and the maximal binary blocks of $\widetilde{B}_0$ form a proper class, and are linearly ordered by set-inclusion (in the sense of $\mathcal{M}$) by design. Let B be the collection of elements $b \in T$ that are of the form $F(m)$, where m is the last binary block of $G^{-1}(\widetilde{b}_\alpha)$. Then B is a cofinal branch of τ and is definable from $\widetilde{B}$. $\square$ (Lemma 3.1.2)

Let $\delta := (\tau_{\mathrm{Choice}} \oplus \tau_3)$, and $\tau := \widetilde{\delta}$. Theorem 2.6 together with Lemmas 3.1.1 and 3.1.2 make it clear that in every model $\mathcal{M}$ of ZFC, $\tau^{\mathcal{M}}$ is a definably Ord-Aronszajn subtree of $\left({}^{<\mathrm{Ord}}2\right)^{\mathcal{M}}$; so by the completeness theorem of first order logic, the proof is complete. $\square$ (Theorem 3.1)

Theorem 3.1 has the following immediate consequence for spartan models of GB + AC, where AC is the axiom of choice for sets:

3.2. Corollary. *There is a definable class σ in the language of class theory satisfying the following properties:*

(a) GB + AC $\vdash$ *σ is a subtree of ${}^{<\mathrm{Ord}}2$ and σ is a proper class.*

(b) *The statement "σ is an* Ord-*Aronszajn tree" holds in every spartan model of* GB + AC.

3.3. Remark. It is known [En-3, Corollary 2.2.1] that the set-theoretical consequences of GB + AC + "Ord has the tree property" is precisely ZFC + Φ, where Φ is the scheme whose instances are of the form "there is an n-Mahlo cardinal κ such that κ is n-correct", and n ranges over meta-theoretic natural numbers. Also note that one can derive global choice from local choice in GB + AC + "Ord is weakly compact" (using τ_{Choice} of the proof of Theorem 2.6). Moreover, by an unpublished result of the first-named-author, there are (non ω-) models $(\mathcal{M}, \mathcal{S})$ of GB + AC + "Ord has the tree property" in which the partition property $\mathrm{Ord} \to (\mathrm{Ord})^k_2$ fails for some nonstandard $k \in \omega^{\mathcal{M}}$, which implies that for models of $\mathrm{GB}+\mathrm{AC}$, the condition "$\forall k \in \omega \text{ Ord} \to (\mathrm{Ord})^k_2$" is strictly stronger than "Ord has the tree property".[6] But of course in the Kelley-Morse theory of classes these two statements are equivalent.

3.4. Theorem. *The definable proper class partition property fails in every model of* ZFC. *That is, there is a definable* 2*-coloring of pairs of sets having no definable monochromatic proper class.*

Proof. Let $\tau = (T, \sqsubseteq)$ be as in Theorem 3.1 and $\mathcal{M} \models \mathrm{ZFC}$. We argue in $\mathcal{M}$. For p, q in T, we will say that p is *to the right* of q, written $p \rhd q$, if $p >_T q$, or at the point of first difference, the bit of p is larger than q at that coordinate. Also, as in the proof of Theorem 3.1, we use $h(p)$ for the height of p in τ. Define a coloring $f : [T]^2 \to \{0,1\}$ by:

$$f(\{p,q\}) = \begin{cases} 0, & \text{if } h(p) > h(q), \text{ and } p \rhd q; \\ 1, & \text{otherwise.} \end{cases}$$

Suppose that H is a definable proper subclass of T that is f-monochromatic. Next, color pairs from H with color blue if they are of the same height, and red otherwise. Since the collection of elements of τ of a given height are sets, there cannot be a proper subclass colored blue, and so we can find a subclass of H with all elements on different levels. So without loss of generality, all elements on H are on different levels. If the monochromatic value of pairs from H is 0, then as one goes up the tree, the nodes in H are always to the right. Let B consist of the nodes in τ that are eventually below the nodes of H, that is, $p \in B$ just in case there is some ordinal α such that all nodes in H above α are above p. It is clear that B is downward closed. We claim that B is a branch through τ. B is linearly ordered, since there can be no first point of nonlinearity: if eventually the nodes of H are above $p * 1$, then they cannot be eventually above $p * 0$ (where $*$ is the concatenation operation on sequences). Finally, B is closed under limits, since if p has length δ and $p|\alpha$ is in B for all $\alpha < \delta$, then take the supremum of the levels witnessing that, so you find a single level such that all nodes in H above that level are above every $p|\alpha$, and so they are above p. Thus, B is a branch through τ. But τ has

[6] A similar phenomena occurs in the arithmetic setting in relation to Ramsey's Theorem: even though the predicative extension ACA_0 of PA can prove every instance of Ramsey's Theorem of the form $\omega \to (\omega)^n_2$, where n is any meta-theoretic natural number (by a routine arithmetization of any of the usual proofs of Ramsey's theorem), ACA_0 cannot prove the stronger statement $\forall k \in \omega\ \omega \to (\omega)^k_2$. This natural incompleteness phenomena follows from a subtle recursion-theoretic theorem of Jockusch [Jo], which states that for each natural number $n \geq 2$ there is a recursive partition P_n of $[\omega]^n$ into two parts such that P_n has no infinite Σ^0_n-homogeneous subset. For more detail, see Wang's exposition [W, p.25]; note that Wang refers to ACA_0 as PPA.

no definable branches, and so there cannot be such a monochromatic set H. Finally, if the monochromatic value of H is 1, then as one goes up, the nodes go to the left, and a similar argument works. □

3.5. Corollary. *$\mathrm{Ord} \to (\mathrm{Ord})^2_2$ fails for definable classes in every model of $\mathrm{ZFC} + \exists p\,(\mathrm{V} = \mathrm{HOD}(p))$. Indeed, $\mathrm{Ord} \to (\mathrm{Ord})^2_2$ fails for definable classes in every model of ZFC in which there is a definable well-ordering of ${}^{<\mathrm{Ord}}2$.*[7]

3.6. Remark. We do not know whether $\mathrm{Ord} \to (\mathrm{Ord})^2_2$ fails for definable classes in every model of ZFC. Some of the usual proofs of the infinite Ramsey theorem use König's lemma, which is exactly what is going wrong with our definably Ord-Aronszajn tree; this suggest that perhaps there is a definable coloring of pairs of ordinals for which there is no definable monochromatic proper class of ordinals.

3.7. Theorem. *The definable compactness property fails for $\mathcal{L}_{\infty,\omega}$ in every model $\mathcal{M}$ of ZFC.*

Proof. Fix a definable Ord-Aronszajn tree $\tau = (T, <_T)$ of $\mathcal{M}$, and let $\mathcal{L}$ be the language having a constant $\overline{p}$ for every element $p \in T$ and a binary relation $<$ for the order of τ, together with a new constant c. Let Γ be the theory in $\mathcal{M}$ consisting of the atomic diagram of τ, together with the assertion that $<$ is a tree order and the assertions of the form:

$$\varphi_\alpha := \bigvee_{p \in T_\alpha} (\overline{p} < c).$$

That is, φ_α asserts that the new constant b lies above one of the elements on the α-th level T_α of τ. In ZFC, having 'size Ord' is a stronger property than 'proper class', if global choice fails. Nevertheless, we can organize Γ into an equivalent theory of size Ord as follows. Instead of taking the whole atomic diagram as separate statements, which may not be well-orderable, since we can't seem to well-order the nodes of τ, we instead for each ordinal α let σ_α be the conjunction of the set of atomic assertions that hold in the tree up to level α. Recall that the logic $\mathcal{L}_{\mathrm{Ord},\,\omega}$ allows the formation of conjunctions of any set of assertions, without needing to put them into any order. Hence Γ is defined in $\mathcal{M}$ as $\{\sigma_\alpha \wedge \varphi_\alpha : \alpha \in \mathrm{Ord}\}$ plus the sentence that expresses that $<$ is a tree order.

[7] The existence of a global definable well-ordering of ${}^{<\mathrm{Ord}}2$ is equivalent over ZF to the so-called Leibniz-Mycielski principle (LM), explored in [En-4], which includes a result of Solovay that shows that if ZF is consistent, then there is a model of ZF + LM in which AC fails (such a model, a fortiori, does not carry a parametrically definable global well-ordering). The conjecture that there is a model of ZFC + LM which does not carry a parametrically definable global well-ordering remains open.

Every set-sized subtheory of Γ mentions only bounded many sentences of the form $\sigma_\alpha \wedge \varphi_\alpha$, so we can find a model in $\mathcal{M}$ of the subtheory by interpreting c as any element of the tree τ on a sufficiently high level. But if there is an $\mathcal{M}$-definable model of Γ, then from that model we can extract the predecessors of the interpretation of the element c, and this will give an $\mathcal{M}$-definable branch through τ, contradicting that τ is definably Ord-Aronszajn in $\mathcal{M}$. $\square$

We close this section with a conjecture. In what follows $\mathcal{D}_{\mathcal{M}}$ is the collection of $\mathcal{M}$-definable subsets of M, and "τ is a definably Ord-Suslin tree in $\mathcal{M}$" means that $(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ satisfies "τ is an Ord-Aronszajn tree and every anti-chain of τ has cardinality less than Ord".

3.8. Conjecture. *Suppose $\mathcal{M}$ is a model of* $\mathrm{ZFC} + \mathrm{V} = \mathrm{L}$. *Then there is some $\tau_S \in \mathcal{D}_{\mathcal{M}}$ such that τ_S is a definably* Ord-*Suslin tree in $\mathcal{M}$.*

Let us motivate the above conjecture. By a theorem of Jensen [D, Theorem VII.1.3] , if $\mathrm{V} = \mathrm{L}$ holds, then every cardinal κ that is not weakly compact carries a κ-Suslin tree. The relevant case for us of Jensen's proof is when κ is a strongly inaccessible cardinal. Jensen's proof takes advantage of (1) the existence of a κ-Aronszajn tree, and (2) the combinatorial principle "for some stationary subset set E of κ, $\square_\kappa(E)$ holds". We know, by Theorem 2.6, that the definable version of (1) can be arranged for Ord. On the other hand, by adapting Jensen's proof to the definable context, the analogue of (2) might also be true (using the $\mathrm{V} = \mathrm{L}$ assumption) in $(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$. The result in the next section suggests that perhaps the definable version of (2) holds with the assumption $\mathrm{V} = \mathrm{L}$ weakened to $\exists p\,(\mathrm{V} = \mathrm{HOD}(p))$. This motivates a stronger form of Conjecture 3.8 in which the assumption that $\mathrm{V} = \mathrm{L}$ holds in $\mathcal{M}$ is weakened to the $\mathcal{M}$-definability of a global well-ordering of the universe.

4. The definable version of $\Diamond_{\mathrm{Ord}}$ and global definable well-orderings

In this section we show that the definable version of $\Diamond_{\mathrm{Ord}}$ holds in a model $\mathcal{M}$ of ZFC iff $\mathcal{M}$ carries a definable well-ordering of the universe. In light of Theorem 1.2 it follows as a consequence that the definable $\Diamond_{\mathrm{Ord}}$, although seeming to be fundamentally scheme-theoretic, is actually expressible in the first-order language of set theory as $\exists p\,(\mathrm{V} = \mathrm{HOD}(p))$.

In set theory, the diamond principle asserts the existence of a sequence of objects, of growing size, such that any large object at the end is very often anticipated by these approximations. In the case of diamond on the ordinals, what we will have is a definable sequence of $A_\alpha \subseteq \alpha$, such that for any definable class of ordinals A and any definable class club set C, there are ordinals $\theta \in C$ with $A \cap \theta = A_\theta$. This kind of principle typically allows one to undertake long constructions that will diagonalize against all the large objects, by considering and reacting to their approximations A_α. Since every large object A is often correctly approximated that way, this enables many such constructions to succeed.

4.1. Theorem. *For any model $\mathcal{M}$ of* ZFC, *if there is an $\mathcal{M}$-definable well-ordering of the universe, then the definable $\Diamond_{\mathrm{Ord}}$ holds in $\mathcal{M}$.*

Proof. We argue in $\mathcal{M}$ to establish the theorem as a theorem scheme; namely, we shall provide a specific definition within $\mathcal{M}$ for the sequence $\vec{A} = \langle A_\alpha : \theta < \mathrm{Ord} \rangle$, using the same parameter p as the definition of the global well-order and with a definition of closely related syntactic complexity, and then prove as a scheme, a separate statement for each $\mathcal{M}$-definable class $A \subseteq \mathrm{Ord}$ and class club $C \subseteq \mathrm{Ord}$, that there is some $\theta \in C$ with $A \cap \theta = A_\alpha$. The definitions of the classes A and C may involve parameters and have arbitrary complexity.

Let $\lhd$ be the definable well-ordering of the universe, definable by a specific formula using some parameter p. We define the $\Diamond_{\mathrm{Ord}}$-sequence $\vec{A} = \langle A_\alpha : \theta < \mathrm{Ord} \rangle$ by transfinite recursion. Suppose that $\vec{A} \restriction \theta$ has been defined. We shall let $A_\theta = \varnothing$ unless θ is a $\beth$-fixed point above the rank of p and there is a set $A \subseteq \theta$ and a closed unbounded set $C \subseteq \theta$, with both A and C definable in the structure $(\mathrm{V}_\theta, \in)$ (allowing parameters), such that $A \cap \theta \neq A_\alpha$ for every $\alpha \in C$. In this case, we choose the least such pair (A, C), minimizing first on the maximum of the logical complexities of the definitions of A and of C, and then minimizing on the total length of the defining formulas of A and C, and then minimizing on the Gödel codes of those formulas, and finally on the parameters used in the definitions, using the well-order $\lhd \ \restriction \mathrm{V}_\theta$. For this minimal pair, let $A_\theta = A$. This completes the definition of the sequence $\vec{A} = \langle A_\alpha : \theta < \mathrm{Ord} \rangle$.

Let us remark on a subtle point, since the meta-mathematical issues loom large here. The definition of $\vec{A}$ is internal to the model $\mathcal{M}$, and at stage θ we ask about subsets of θ definable in $(\mathrm{V}_\theta, \in)$, using the truth predicate for this structure. If we were to run this definition inside an ω-nonstandard model $\mathcal{M}$, it could happen that the minimal formula we get is nonstandard, and in this case, the set A would not actually be definable by a standard formula. Also, even when A is definable by a standard formula, it might be paired (with some constants), with a club set C that is defined only by a nonstandard formula (and this is why we minimize on the maximum of the complexities of the definitions of A and C together). So one must give care in the main argument keeping straight the distinction between the meta-theoretic natural numbers and the internal natural numbers of the object theory ZFC.

Let us now prove that the sequence $\vec{A}$ is indeed a $\Diamond_{\mathrm{Ord}}$-sequence for $\mathcal{M}$-definable classes. The argument follows in spirit the classical proof of $\Diamond$ in the constructible universe L, subject to the metamathematical issues we mentioned. If the sequence $\vec{A}$ does not witness the veracity of the definable $\Diamond_{\mathrm{Ord}}$ in $\mathcal{M}$, then there is some $\mathcal{M}$-definable class $A \subseteq \mathrm{Ord}$, defined in $\mathcal{M}$ by a specific formula φ and parameter z, and definable club $C \subseteq \mathrm{Ord}$, defined by some ψ and parameter y, with $A \cap \alpha \neq A_\alpha$ for every $\alpha \in C$. We may assume without loss of generality that these formulas are chosen so as to be minimal in the sense of the construction, so that the maximum of the complexities of φ and ψ are as small as possible, and the lengths of the formulas, and the Gödel codes and finally the parameters z, y are $\lhd$-minimal, respectively, successively. Let m be a sufficiently large natural number, larger than the complexity of the definitions of $\lhd$, A, C, and large enough so that the minimality condition we just discussed is expressible by a Σ_m formula. Let θ be any Σ_m-correct ordinal above the ranks of the parameters used in the definitions. It follows that the restrictions $\lhd \restriction \mathrm{V}_\theta$ and also $A \cap \theta$ and $C \cap \theta$ are definable in $(\mathrm{V}_\theta, \in)$ by the same definitions and parameters as their counterparts in V, that $C \cap \theta$ is club in θ, and $A \cap \theta$ and $C \cap \theta$ form a form a minimal pair using those definitions $A \cap \alpha \neq \alpha$ for any $\alpha \in C \cap \theta$. Thus, by the definition of $\vec{A}$, it follows that $A_\theta = A \cap \theta$. Since $C \cap \theta$ is unbounded in θ and C is closed, it follows that $\theta \in C$, and so $A_\theta = A \cap \theta$ contradicts our assumption about A and C. So there are no such counterexample classes, and thus $\vec{A}$ is a $\Diamond_{\mathrm{Ord}}$-sequence with respect to $\mathcal{M}$-definable classes, as claimed. $\square$

4.2. Theorem. *The following are equivalent for* $\mathcal{M} \models \mathrm{ZFC}$.

(**a**) $\mathcal{M}$ *carries an* $\mathcal{M}$*-definable global well-ordering.*

(**b**) $\exists p\, (\mathrm{V} = \mathrm{HOD}(p))$ *holds in* $\mathcal{M}$.

(**c**) *The definable* $\Diamond_{\mathrm{Ord}}$ *holds in* $\mathcal{M}$.

Proof. We will first give the argument, and then in Remark 4.3 discuss some issues about the formalization, which involves some subtle issues.

(**a**) $\Rightarrow$ (**b**). Suppose that $\lhd$ is a global well-ordering that is definable in $\mathcal{M}$ from a parameter p. In particular in $\mathcal{M}$ every set has a $\lhd$-minimal element. Let us refine this order by defining $x \lhd' y$, just in case $\rho(x) < \rho(y)$ or $\rho(x) = \rho(y)$ and $x \lhd y$ (where ρ is the usual ordinal-valued rank function). The new order is also a well-order, which now respects rank. In particular, the order $\lhd'$ is set-like, and so every object x is the θ-th element with respect to the

$\lhd'$-order, for some ordinal θ. Thus, every object is definable in $\mathcal{M}$ from p and an ordinal, and so $\mathrm{V} = \mathrm{HOD}(p)$ holds in $\mathcal{M}$, as desired.

(**b**) $\Rightarrow$ (**a**). If $\mathcal{M}$ satisfies $\exists p\ \mathrm{V} = \mathrm{HOD}(p)$, then we have the canonical well-order of HOD using parameter p, similar to how one shows that the axiom of choice holds in HOD. Namely, define $x \lhd y$ if and only if $\rho(x) < \rho(y)$, or the ranks are the same, but x is definable from p and ordinal parameters in some V_θ with a smaller θ than y is, or the ranks are the same and the θ is the same, but x is definable in that V_θ by a formula with a smaller Gödel code, or with the same formula but smaller ordinal parameters. It is easy to see that this is an $\mathcal{M}$-definable well-ordering of the universe.

(**a**) $\Rightarrow$ (**c**). This is the content of the Theorem 4.1.

(**c**) $\Rightarrow$ (**a**). If $\vec{A}$ is an $\mathcal{M}$-definable $\Diamond_{\mathrm{Ord}}$-sequence for $\mathcal{M}$-definable classes, then it is easy to see that if A is a set of ordinals in the sense of $\mathcal{M}$, then A must arise as A_θ for unboundedly many $\theta \in \mathrm{Ord}^{\mathcal{M}}$. As recalled in the proof of Lemma 3.1.2, in ZFC every set is coded by a set of ordinals. So let us define that $x \lhd y$, just in case x is coded by a set of ordinals that appears earlier on $\vec{A}$ than any set of ordinals coding y. This is clearly a well-ordering, since the map sending x to the ordinal θ for which codes x is an Ord-ranking of $\lhd$. So there is an $\mathcal{M}$-definable well-ordering of the universe. $\square$

4.3. Remark. An observant reader will notice some meta-mathematical issues concerning Theorem 4.2. The issue is that statements (a) and (b) are known to be expressible by statements in the first-order language of set theory, as single statements, but for statement (c) we have previously expressed it only as a scheme of first-order statements. So how can they be equivalent? The answer is that the full scheme-theoretic content of statement (3) follows already from instances in which the complexity of the definitions of A and C are bounded. Basically, once one gets the global well-order, then one can construct a $\Diamond_{\mathrm{Ord}}$-sequence that works for all definable classes. In this sense, we may regard the diamond principle $\Diamond_{\mathrm{Ord}}$ for definable classes as not really a scheme of statements, but rather equivalent to a single first-order assertion.

Lastly, let us consider the content of Theorem 4.2 in Gödel-Bernays set theory or Kelley-Morse set theory. Of course, we know that there can be models of these theories that do not have $\Diamond_{\mathrm{Ord}}$ in the full second-order sense. For example, it is relatively consistent with ZFC that an inaccessible cardinal κ does not have $\Diamond_\kappa$, and in this case, the structure $(\mathrm{V}_{\kappa+1}, \mathrm{V}_\kappa, \in)$ will satisfy GBC and even KMC, but it will not satisfy $\Diamond_{\mathrm{Ord}}$ with respect to all classes, even though it has a well-ordering of the universe (since there is such a well-ordering in $\mathrm{V}_{\kappa+1}$). But meanwhile, there will be a $\Diamond_{\mathrm{Ord}}$-sequence that works with respect to classes that are definable from that well-ordering and parameters, simply by following the construction given in Theorem 4.2.

4.4. A minor adaptation of the proof of Theorem 4.1 shows that if $\mathcal{M}$ is a model of ZFC that carries an $\mathcal{M}$-definable global well-ordering, then the definable version of $\Diamond_{\mathrm{Ord}}(E)$ holds in $\mathcal{M}$ for any definably $\mathcal{M}$-stationary $E \subseteq \mathrm{Ord}^{\mathcal{M}}$: use the same argument, but only define A_α for $\alpha \in E$; and in the reflection step of the argument use $\theta \in E \cap C$. Theorem 4.2 can be also accordingly strengthened.

5. The theory of spartan models of GB

Recall from Section 1 that $\mathrm{GB_{spa}}$ is the collection of all sentences that hold in all spartan models of GB. As mentioned earlier, each theorem scheme of Sections 2 through 4 can be readily reformulated as demonstrating that a certain sentence belongs to $\mathrm{GB_{spa}}$. Note that the purely set-theoretical consequences of $\mathrm{GB_{spa}}$ coincides with the deductive closure of ZF; this is an immediate consequence of coupling the completeness theorem for first order logic with the fact that $(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ is a model of GB whenever $\mathcal{M}$ is a model of ZF. A natural question is whether $\mathrm{GB_{spa}}$ is computably axiomatizable. The following result provides a strong negative answer to this question.

5.1. Theorem. $\mathrm{GB_{spa}}$ *is* Π^1_1*-complete.*

Proof. We need to use both the meta-theoretic natural numbers, which we will denote by ω, and the object-theoretic natural numbers, which we denote by $\mathbb{N}$. It is not hard to see that $\mathrm{GB_{spa}}$ has a Π^1_1-description. To see this, consider the following predicates, where $r, s \subseteq \omega$:

(1) $\mathrm{Sat_{ZF}}(r)$ expresses "the structure canonically coded by r is a model of ZF".
(2) $s = \mathrm{Def}(r)$ expresses "$\mathrm{Sat_{ZF}}(r)$ and s codes the collection of r-definable subsets of the domain of discourse of the structure (coded by) r".
(3) $\mathrm{Sat}((s,r),\varphi)$ expresses "$s = \mathrm{Def}(r)$, φ is a sentence of $\mathcal{L}_{\mathrm{GB}}$, and the GB-model coded by (r, s) satisfies φ".

Usual arguments show that each of the above three predicates is Δ^1_1 in the Baire space. In light of the fact that Δ^1_1-predicates are closed under Boolean operations, this makes it clear that $\mathrm{GB_{spa}}$ is Π^1_1, since by the Löwenheim-Skolem theorem, we have:

$$\varphi \in \mathrm{GB_{spa}} \text{ iff } \forall r \subseteq \omega\ \forall s \subseteq \omega\ \left((\mathrm{Sat_{ZF}}(r) \wedge s = \mathrm{Def}(r)) \rightarrow \mathrm{Sat}((s,r),\varphi)\right)$$

We next show that $\mathrm{GB}_{\mathrm{spa}}$ is Π^1_1-complete. The revelatory idea here is that within GB one can define – via an existential quantification over classes – a nonempty 'cut' I of ambient natural numbers $\mathbb{N}$ (i.e., a nonempty initial segment I of $\mathbb{N}$ that contains 0 and is closed under successors) such that:

$(*)$ If $(\mathcal{M}, D_{\mathcal{M}})$ is a spartan model of GB, then $\mathrm{I}^{(\mathcal{M}, D_{\mathcal{M}})} \cong \omega$; i.e., $\mathrm{I}^{(\mathcal{M}, D_{\mathcal{M}})}$ has no nonstandard elements.

The cut I has a simple definition within GB. In the definition below F_n is the collection of set theoretical formulae of complexity at most n, where 'complexity' can be taken as the number of occurrences of logical symbols (i.e. the Boolean connectives and the quantifiers)[8]

$$\begin{aligned}\mathrm{I} := \{n \in \mathbb{N} : \text{ there is a proper class } C \text{ such that } C \text{ is the}\\ \text{satisfaction-predicate for } F_n\},\end{aligned}$$

The relevant insight is that in spartan models of GB*, the only members of* I *are the standard natural numbers ω, thanks to Tarski's undefinability of truth theorem,* which explains the veracity of $(*)$.

Using $(*)$, and the fact that every real can be included in the standard system of a model of ZF, we will show that every Π^1_1-subset of ω is many-one reducible to $\mathrm{GB}_{\mathrm{spa}}$. Suppose P is a Π^1_1-subset of ω, and let ω^ω be the Baire space. Then by Kleene normal form for Π^1_1-sets [R], there is some recursive predicate $R(x, y)$ such that:

$$\forall n \left(n \in P \leftrightarrow \forall F \in \omega^\omega \ \exists m \in \omega \ R(F \restriction m, n)\right),$$

where $F \restriction m$ is the canonical code for the finite set of ordered pairs of the form $\langle i, F(i)\rangle$ with $i < m$. Let R be the formula that numeralwise represents R in GB, and given $n \in \omega$, consider the sentence φ_n in the language of GB that expresses:

$$\forall s \left(s \in \mathbb{N}\backslash\mathrm{I} \rightarrow \exists m \in I \ \mathsf{R}^I(\mathrm{F}_s \restriction m, n)\right),$$

where R^{I} is the result of restricting all of the quantifiers of the R to I, and F_s is the function defined in GB with domain $\mathbb{N}$ such that:

$$\mathrm{GB} \vdash \text{“}\mathrm{F}_s(x) \text{ is the } x\text{-th digit of the binary expansion of } s\text{”}.$$

[8]The idea of defining the cut I goes back to Mostowski [Mos], who used it to show that the scheme of induction over $\mathbb{N}$ is not provable in GB.

It is evident that $n \mapsto \ulcorner \varphi_n \urcorner$ is a computable function. We claim:

$(**)$ $\forall n(n \in P \leftrightarrow \varphi_n \in \mathrm{GB}_{\mathrm{spa}})$.

The left-to-right direction of $(**)$ should be clear. The right-to-left direction is also easy to see, using the fact (proved by a simple compactness argument) that for every $F \in \omega^\omega$ there is a non ω-standard model $\mathcal{M} \models \mathrm{ZF}$ and some nonstandard $s \in \mathbb{N}^{\mathcal{M}}$ such that the 'standard part' of the $\mathcal{M}$-finite function coded by s agrees with F, i.e., $\forall m \in \omega\ \mathcal{M} \models (\mathrm{F}_s \upharpoonright m = F \upharpoonright m)$. □

5.2. Remark. The above proof strategy can be used to show that the following theories are also Π^1_1-complete:

(a) The theory $(\mathrm{ACA}_0)_{\mathrm{spa}}$ of all spartan[9] models of ACA_0.

(b) The theory of all models of the form $(\mathcal{M}, \omega)$, where $\mathcal{M}$ is a model of ZF or PA, and $(\mathcal{M}, \omega)$ is the expansion of $\mathcal{M}$ by a new predicate ω consisting of all *standard* natural numbers in $\mathcal{M}$.

(c) The theory of all models the form $(\mathcal{M}, \mathrm{Sat}_{\mathcal{M}})$, where $\mathcal{M}$ is a model of ZF or PA, and $\mathrm{Sat}_{\mathcal{M}}$ is the satisfaction predicate for $\mathcal{M}$.

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