

## ON CUTS IN ULTRAPRODUCTS OF LINEAR ORDERS II

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ABSTRACT. We continue our study of the class  $\mathcal{C}(D)$ , where  $D$  is a uniform ultrafilter on a cardinal  $\kappa$  and  $\mathcal{C}(D)$  is the class of all pairs  $(\theta_1, \theta_2)$ , where  $(\theta_1, \theta_2)$  is the cofinality of a cut in  $J^\kappa/D$  and  $J$  is some  $(\theta_1 + \theta_2)^+$ -saturated dense linear order. We give a combinatorial characterization of the class  $\mathcal{C}(D)$ . We also show that if  $(\theta_1, \theta_2) \in \mathcal{C}(D)$  and  $D$  is  $\aleph_1$ -complete or  $\theta_1 + \theta_2 > 2^\kappa$ , then  $\theta_1 = \theta_2$ .

## 1. INTRODUCTION

Assume  $\kappa$  is an infinite cardinal and  $D$  is an ultrafilter on  $\kappa$ . Recall that  $\mathcal{C}(D)$  is defined to be the class of all pairs  $(\theta_1, \theta_2)$ , where  $(\theta_1, \theta_2)$  is the cofinality of a cut in  $J^\kappa/D$  and  $J$  is some (equivalently any)  $(\theta_1 + \theta_2)^+$ -saturated dense linear order. Also  $\mathcal{C}_{>\lambda}(D)$  is defined to be the class of all pairs  $(\theta_1, \theta_2) \in \mathcal{C}(D)$ , such that  $\theta_1 + \theta_2 > \lambda$ . The classes  $\mathcal{C}_{\geq\lambda}(D)$ ,  $\mathcal{C}_{<\lambda}(D)$  and  $\mathcal{C}_{\leq\lambda}(D)$  are defined similarly.

The works [2], [3] and [4] of Malliaris and Shelah have started the study of this class for the case  $\theta_1 + \theta_2 \leq 2^\kappa$  and [1] started the study of the case  $\theta_1 + \theta_2 > 2^\kappa$ . As it was observed in [1], the study of the class  $\mathcal{C}_{>2^\kappa}(D)$  is very different from the case  $\mathcal{C}_{\leq 2^\kappa}(D)$ , and to prove results about it, usually some extra set theoretic assumptions are needed. In this paper we continue [1] and prove more results related to the class  $\mathcal{C}(D)$ .

In the first part of the paper (Sections 2 and 3) we give a combinatorial characterization of  $\mathcal{C}(D)$ . Using notions defined in section 2, we can state our first main theorem as follows.

**Theorem 1.1.** *Assume  $D$  is an ultrafilter on  $\kappa$  and  $\lambda_1, \lambda_2 > \kappa$  are regular cardinals. The following are equivalent:*

- (a) *There is  $\bar{a} \in \mathcal{S}_c$  which is not  $c$ -solvable, where  $c = \langle \kappa, D, \lambda_1, \lambda_2 \rangle$ .*

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(b)  $(\lambda_1, \lambda_2) \in \mathcal{C}(D)$ .

In the second part of the paper (Sections 4 and 5) we study the existence of non-symmetric pairs (i.e., pairs  $(\lambda_1, \lambda_2)$  with  $\lambda_1 \neq \lambda_2$ ) in  $\mathcal{C}(D)$ . By [5], we can find a regular ultrafilter  $D$  on  $\kappa$  such that

$$\mathcal{C}(D) \supseteq \{(\lambda_1, \lambda_2) : \aleph_0 < \lambda_1 < \lambda_2 \leq 2^\kappa, \lambda_1, \lambda_2 \text{ regular}\}.$$

In particular,  $\mathcal{C}(D)$  contains non-symmetric pairs. On the other hand, results of [1] show that if  $(\lambda_1, \lambda_2) \in \mathcal{C}_{>2^\kappa}(D)$ , then we must have  $\lambda_1^\kappa = \lambda_2^\kappa$ , in particular if SCH, the singular cardinals hypothesis, holds, then  $\lambda_1 = \lambda_2$ , and so  $\mathcal{C}_{>2^\kappa}(D)$  just contains symmetric pairs. We then prove the following theorem (in ZFC):

**Theorem 1.2.** (a) *Assume  $D$  is a uniform  $\aleph_1$ -complete ultrafilter on  $\kappa$  and  $(\lambda_1, \lambda_2) \in \mathcal{C}(D)$ . Then  $\lambda_1 = \lambda_2$ .*

(b) *Assume  $D$  is a uniform ultrafilter on  $\kappa$  and  $(\lambda_1, \lambda_2) \in \mathcal{C}_{>2^\kappa}(D)$ . Then  $\lambda_1 = \lambda_2$ .*

The theorem shows some restrictions on the pairs  $(\lambda_1, \lambda_2)$  that  $\mathcal{C}(D)$  can have, in particular, it shows that in the result of [5] stated above, we can never take the ultrafilter  $D$  to be  $\aleph_1$ -complete and that  $\mathcal{C}_{>2^\kappa}(D)$  can not have non-symmetric pairs.

The paper is organized as follows. In section 2 we give the required definitions, which lead us to the notion of  $c$ -solvability and in section 3 we complete the proof of Theorem 1.1. In section 4 we prove part (a) of Theorem 1.2 and in section 5 we complete the proof of part (b) of Theorem 1.2. We may note that parts one (Sections 2 and 3) and two (Sections 4 and 5) can be read independently of each other.

## 2. ON THE NOTION OF $c$ -SOLVABILITY

In this section we give the required definitions which are used in Theorem 1.1.

**Definition 2.1.** (a) *Let  $\mathcal{C}$  be the class of tuples  $c = \langle \kappa_c, D_c, \lambda_{c,1}, \lambda_{c,2} \rangle$  where*

(a-1)  $\lambda_{c,1}, \lambda_{c,2}$  are regular cardinals  $> \kappa_c$ ,

(a-2)  $D_c$  is a uniform ultrafilter on  $\kappa_c$ .

*Also let  $\lambda_c = 2^{<\lambda_{c,1}} + 2^{<\lambda_{c,2}}$  and  $\lambda_{c,0} = \min\{\lambda_{c,1}, \lambda_{c,2}\}$ .*

(b) For  $c \in \mathcal{C}$  let  $N_c = N_{c,1} + N_{c,2}$  be a linear order of size  $\leq \lambda_c$  in such a way that  $N_{c,1}$  has cofinality  $\lambda_{c,1}$ ,  $N_{c,2}$  has co-initiality  $\lambda_{c,2}$  and both  $N_{c,1}, N_{c,2}$  are  $\lambda_{c,0}$ -saturated dense linear orders <sup>1</sup>.

(c) For  $c \in \mathcal{C}$  let  $\mathcal{S}_c$  be the set of all sequences  $\bar{a} = \langle a_{s,t} : s, t \in N_c \rangle$  such that

(c-1) Each  $a_{s,t}$  is a subset of  $\kappa_c$ ,

(c-2)  $a_{s,s} = \emptyset$ ,

(c-3) For  $s \neq t$ ,  $a_{s,t} = \kappa_c \setminus a_{t,s}$ ,

(c-4)  $s <_{N_c} t \Rightarrow a_{s,t} \in D_c$ ,

(c-5) If  $s_1 <_{N_c} s_2 <_{N_c} s_3$ , then

$$(a_{s_1, s_3} \supseteq a_{s_1, s_2} \cap a_{s_2, s_3}) \ \& \ (a_{s_3, s_1} \supseteq a_{s_3, s_2} \cap a_{s_2, s_1}).$$

(d) For  $c \in \mathcal{C}$  let  $N_c^+ = N_{c,1} + N_0 + N_{c,2}$ , where  $N_0$  is a singleton, say  $N_0 = \{s_*\}$ .

We now define the notion of  $c$ -solvability.

**Definition 2.2.** Let  $c \in \mathcal{C}$ . We say  $\bar{a} \in \mathcal{S}_c$  is  $c$ -solvable, if there exists a sequence  $\bar{b} = \langle b_s : s \in N_c \rangle$ , such that the sequence  $\bar{a}^1 = \bar{a} * \bar{b}$  satisfies clauses (c-1)-(c-5) above, where the sequence  $\bar{a}^1 = \langle a_{s,t}^1 : s, t \in N_c^+ \rangle$  is defined as follows:

- (1) If  $s, t \in N_c$ , then  $a_{s,t}^1 = a_{s,t}$ ,
- (2) For  $s \in N_{c,1}$ ,  $a_{s, s_*}^1 = b_s$  and  $a_{s_*, s}^1 = \kappa_c \setminus b_s$ ,
- (3) For  $s \in N_{c,2}$ ,  $a_{s, s_*}^1 = b_s$  and  $a_{s_*, s}^1 = \kappa_c \setminus b_s$ ,
- (4)  $a_{s_*, s_*}^1 = \emptyset$ .

Then  $\bar{b}$  is called a  $c$ -solution for  $\bar{a}$ .

### 3. A COMBINATORIAL CHARACTERIZATION OF $\mathcal{C}(D)$

In this section we give a proof of Theorem 1.1.

**Lemma 3.1.** Assume  $c \in \mathcal{C}$  and  $\bar{a} \in \mathcal{S}_c$ . Then

(a) There are  $M, \bar{f}$  such that

(a-1)  $M$  is a  $(\lambda_{c,1} + \lambda_{c,2})^+$ -saturated dense linear order,

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<sup>1</sup> $N_c$  is some fixed linear order which we choose in advance. We may assume global choice and let  $N_c$  be the least such order.

$$(a-2) \quad \bar{f} = \langle f_s : s \in N_c \rangle,$$

$$(a-3) \quad \text{Each } f_s \in {}^{\kappa_c}M,$$

$$(a-4) \quad \text{If } s <_{N_c} t, \text{ then } a_{s,t} = \{i < \kappa_c : f_s(i) <_M f_t(i)\},$$

$$(a-4) \quad \langle \text{range}(f_s) : s \in N_c \rangle \text{ is a sequence of pairwise disjoint sets.}$$

(b) If  $M, \bar{f}$  are as in (a), then

$$(b-1) \quad \langle f_s/D_c : s \in N_c \rangle \text{ is an increasing sequence in } {}^{\kappa_c}M/D_c,$$

$$(b-2) \quad \bar{a} \text{ is } c\text{-solvable iff } {}^{\kappa_c}M/D_c \text{ realizes the type}$$

$$q(x) = \{f_s/D_c < x < f_t/D_c : s \in N_{c,1} \text{ and } t \in N_{c,2}\}.$$

*Proof.* (a) Let  $A = \{(i, s) : i < \kappa_c, s \in N_c\}$ , and define the order  $<_A$  on  $A$  by

$$(i_1, s_1) <_A (i_2, s_2) \iff (i_1 < i_2) \text{ or } (i_1 = i_2 \in a_{s_1, s_2}).$$

Also let  $\leq_A$  be defined on  $A$  in the natural way from  $<_A$ , so

$$(i_1, s_1) \leq_A (i_2, s_2) \iff (i_1, s_1) = (i_2, s_2) \text{ or } (i_1, s_1) <_A (i_2, s_2).$$

It is easily seen that  $\leq_A$  is a linear order on  $A$ . Now let  $M$  be a  $(\lambda_{c,1} + \lambda_{c,2})^+$ -saturated dense linear order which contains  $(A, <_A)$  as a sub-order. Also let  $\bar{f} = \langle f_s : s \in N_c \rangle$ , where for  $s \in N_c$   $f_s \in {}^{\kappa_c}M$  is defined by  $f_s(i) = (i, s)$ . It is clear that  $M$ , and  $\bar{f}$  satisfy clauses (a-1)-(a-3). For (a-4), assume  $s <_{N_c} t$  are given. Then

$$a_{s,t} = \{i < \kappa_c : i \in a_{s,t}\} = \{i < \kappa_c : (i, s) <_A (i, t)\} = \{i < \kappa_c : f_s(i) <_M f_t(i)\}.$$

Finally note that for  $s \neq t$  in  $N_c$ ,

$$\text{range}(f_s) \cap \text{range}(f_t) = \{(i, s) : i < \kappa_c\} \cap \{(i, t) : i < \kappa_c\} = \emptyset.$$

So  $M$  and  $\bar{f}$  are as required.

(b) (b-1) follows from (a-4) and the fact that for  $s <_{N_c} t, a_{s,t} \in D_c$ . Let's prove (b-2).

First assume that  $\bar{a}$  is  $c$ -solvable and let  $\bar{b}$  be a solution for  $\bar{a}$ . For each  $i < \kappa_c$  let  $p_i(x)$  be the following type over  $M$ :

$$p_i(x) = \{f_s(i) <_M x : s \in N_{c,1} \text{ and } i \in b_s\} \cup \{x <_M f_t(i) : t \in N_{c,2} \text{ and } i \in \kappa_c \setminus b_t\}.$$

**Claim 3.2.** For each  $i < \kappa_c$ , the type  $p_i(x)$  is finitely satisfiable in  $M$ .

*Proof.* Let  $s_0 <_{N_{c,1}} \cdots <_{N_{c,1}} s_{n-1}$  be in  $N_{c,1}$  and  $t_{m-1} <_{N_{c,2}} < \cdots <_{N_{c,2}} t_0$  be in  $N_{c,2}$ . Also suppose that  $i \in \bigcap_{k < n} b_{s_k} \cap \bigcap_{l < m} (\kappa_c \setminus b_{t_l})$ . Then for  $k < n$  and  $l < m$  we have

$$a_{s_k, t_l} \supseteq a_{s_k, s_*}^1 \cap a_{s_*, t_l}^1 = b_{s_k} \cap (\kappa_c \setminus b_{t_l}),$$

and so  $i \in a_{s_k, t_l}$ , which implies  $f_{s_k}(i) < f_{t_l}(i)$ . Take  $x \in M$  so that

$$\forall k < n, \forall l < m, f_{s_k}(i) < x < f_{t_l}(i),$$

which exists as  $M$  is dense. It follows that  $p_i(x)$  is finitely satisfiable in  $M$ .  $\square$

It follows that there exists  $f \in {}^{\kappa_c}M$  such that for each  $i < \kappa_c$ ,  $f(i)$  realizes the type  $p_i(x)$  over  $M$ . Then  $f/D_c$  realizes  $q(x)$  over  ${}^{\kappa_c}M/D_c$ .

Conversely assume that  $f \in {}^{\kappa_c}M$  is such that  $f/D_c$  realizes the type  $q(x)$  over  ${}^{\kappa_c}M/D_c$ .

**Claim 3.3.** *We can assume that  $\text{range}(f)$  is disjoint from  $A$ .*

*Proof.* As  $\langle \text{range}(f_s) : s \in N_c \rangle$  is a sequence of pairwise disjoint sets and  $\lambda_{c,1}, \lambda_{c,2} > \kappa_c$  are regular, there are  $s_1 \in N_{c,1}$  and  $s_2 \in N_{c,2}$  such that  $s_1 <_{N_c} s <_{N_c} s_2$  implies  $\text{range}(f_{s_1}) \cap \text{range}(f_{s_2}) = \emptyset$ . As  $M$  is a  $(\lambda_{c,1} + \lambda_{c,2})^+$ -saturated dense linear order, there is  $f'$  such that

- $f' \in {}^{\kappa_c}M$ ,
- $\text{range}(f') \cap A = \emptyset$ ,
- If  $s_1 <_{N_c} s <_{N_c} s_2$  and  $i < \kappa_c$ , then  $f_s(i) <_{N_c} f'(i) \Rightarrow f_s(i) <_{N_c} f(i)$  and  $f'(i) <_{N_c} f_s(i) \Rightarrow f(i) <_{N_c} f_s(i)$ .

So we can replace  $f$  by  $f'$  and  $f'$  satisfies the requirements on  $f$ ; i.e.,  $f'/D_c$  realizes  $q(x)$  over  ${}^{\kappa_c}M/D_c$  and further  $\text{range}(f') \cap A = \emptyset$ .  $\square$

Now define  $\bar{b} = \langle b_s : s \in N_c \rangle$  by

$$b_s = \begin{cases} \{i < \kappa_c : f_s(i) <_{N_c} f(i)\} & \text{if } s \in N_{c,1}, \\ \{i < \kappa_c : f(i) <_{N_c} f_s(i)\} & \text{if } s \in N_{c,2}. \end{cases}$$

**Claim 3.4.**  *$\bar{b}$  is a  $c$ -solution for  $\bar{a}$ .*

*Proof.* We show that conditions (c-1)-(c-5) of Definition 2.1 are satisfied by  $\bar{a}^1 = \bar{a} * \bar{b}$  (see Definition 2.2). (c-1) and (c-2) are trivial and (c-3) follows from the fact that  $\forall i < \kappa_c, f_s(i) \neq f(i)$  (as  $\text{range}(f) \cap A = \emptyset$ ).

For (c-4), suppose that  $s <_{N_c^+} t$ . If both  $s, t$  are in  $N_c$ , then we are done. So suppose otherwise. There are two cases to consider.

- If  $s = s_*$ , then  $t \in N_{c,2}$  and as  $f/D_c$  realizes  $q(x)$ , we have  $f/D_c < f_t/D_c$ , which implies  $a_{s_*,t}^1 = b_t = \{i < \kappa_c : f(i) <_{N_c} f_t(i)\} \in D_c$ .
- If  $t = s_*$ , then  $s \in N_{c,1}$  and as  $f/D_c$  realizes  $q(x)$ , we have  $f_s <_{D_c} f$ , which implies  $a_{s,s_*}^1 = b_s = \{i < \kappa_c : f_s(i) <_{N_c} f(i)\} \in D_c$ .

For (c-5), assume  $s_1 <_{N_c^+} s_2 <_{N_c^+} s_3$  are in  $N_c^+$ . If all  $s_1, s_2$  and  $s_3$  are in  $N_c$ , then we are done. So assume otherwise. There are three cases to be considered:

- If  $s_1 = s_*$ , then  $s_2, s_3 \in N_{c,2}$ , and we have

$$\begin{aligned} a_{s_*,s_2}^1 \cap a_{s_2,s_3}^1 &= b_{s_2} \cap a_{s_2,s_3} \\ &= \{i < \kappa_c : (f(i) <_{N_c} f_{s_2}(i)) \wedge (f_{s_2}(i) <_{N_c} f_{s_3}(i))\} \\ &\subseteq \{i < \kappa_c : f(i) <_{N_c} f_{s_3}(i)\} \\ &= b_{s_3} \\ &= a_{s_*,s_3}^1. \end{aligned}$$

Similarly,

$$\begin{aligned} a_{s_3,s_2}^1 \cap a_{s_2,s_*}^1 &= a_{s_3,s_2} \cap (\kappa_c \setminus b_{s_2}) \\ &= \{i < \kappa_c : (f_{s_2}(i) \geq_{N_c} f_{s_3}(i)) \wedge (f(i) >_{N_c} f_{s_2}(i))\} \\ &\subseteq \{i < \kappa_c : f(i) >_{N_c} f_{s_3}(i)\} \\ &= \kappa_c \setminus b_{s_3} \\ &= a_{s_3,s_*}^1. \end{aligned}$$

- If  $s_2 = s_*$ , then  $s_1 \in N_{c,1}$ ,  $s_3 \in N_{c,2}$  and we have

$$\begin{aligned} a_{s_1,s_*}^1 \cap a_{s_*,s_3}^1 &= b_{s_1} \cap b_{s_3} \\ &= \{i < \kappa_c : (f_{s_1}(i) <_{N_c} f(i)) \wedge (f(i) <_{N_c} f_{s_3}(i))\} \\ &\subseteq \{i < \kappa_c : f_{s_1}(i) <_{N_c} f_{s_3}(i)\} \\ &= a_{s_1,s_3} \\ &= a_{s_1,s_3}^1. \end{aligned}$$

Also,

$$\begin{aligned}
a_{s_3, s_*}^1 \cap a_{s_*, s_1}^1 &= (\kappa_c \setminus b_{s_3}) \cap (\kappa_c \setminus b_{s_1}) \\
&= \{i < \kappa_c : (f_{s_3}(i) <_{N_c} f(i)) \wedge (f(i) <_{N_c} f_{s_1}(i))\} \\
&\subseteq \{i < \kappa_c : f_{s_3}(i) \leq_{N_c} f_{s_1}(i)\} \\
&= a_{s_3, s_1} \\
&= a_{s_3, s_1}^1.
\end{aligned}$$

- If  $s_3 = s_*$ , then  $s_1, s_2 \in N_{c,1}$  and we have

$$\begin{aligned}
a_{s_1, s_2}^1 \cap a_{s_2, s_*}^1 &= a_{s_1, s_2} \cap b_{s_2} \\
&= \{i < \kappa_c : (f_{s_1}(i) <_{N_c} f_{s_2}(i)) \wedge (f_{s_2}(i) <_{N_c} f(i))\} \\
&\subseteq \{i < \kappa_c : f_{s_1}(i) <_{N_c} f(i)\} \\
&= b_{s_1}. \\
&= a_{s_1, s_*}^1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
a_{s_*, s_2}^1 \cap a_{s_2, s_1}^1 &= (\kappa_c \setminus b_{s_2}) \cap a_{s_2, s_1} \\
&= \{i < \kappa_c : (f_{s_2}(i) >_{N_c} f(i)) \wedge (f_{s_1}(i) \geq_{N_c} f_{s_2}(i))\} \\
&\subseteq \{i < \kappa_c : f_{s_1}(i) >_{N_c} f(i)\} \\
&= \kappa_c \setminus b_{s_1} \\
&= a_{s_*, s_1}^1.
\end{aligned}$$

Hence,  $\bar{b}$  is a  $c$ -solution for  $\bar{a}$ , as required.  $\square$

The lemma follows.  $\square$

Given  $c \in \mathcal{C}$ , the next lemma gives a characterization, in terms of  $c$ -solvability, of when  $(\lambda_{c,1}, \lambda_{c,2})$  is in  $\mathcal{C}(D_c)$ , which also completes the proof of Theorem 1.1.

**Lemma 3.5.** *Assume  $c \in \mathcal{C}$  and  $M$  is a  $\lambda_c^+$ -saturated dense linear order. The following are equivalent:*

- (a) *There is  $\bar{a} \in \mathcal{S}_c$  which is not  $c$ -solvable.*
- (b)  $(\lambda_{c,1}, \lambda_{c,2}) \in \mathcal{C}(D_c)$ .

*Proof.* First assume there exists  $\bar{a} \in \mathcal{S}_c$  which is not  $c$ -solvable. By Lemma 3.1, there are  $M, \bar{f}$  which satisfy clauses (1-a)-(1-e) of that lemma. But then as  $\bar{a}$  is not  $c$ -solvable, by

Lemma 2.3(b-2), the type

$$q(x) = \{f_s/D_c < x < f_t/D_c : s \in N_{c,1} \text{ and } t \in N_{c,2}\}.$$

is not realized. It follows that  $(\lambda_{c,1}, \lambda_{c,2}) \in \mathcal{C}(D_c)$ .

Conversely assume that  $M$  and  $\bar{f} = \langle f_s : s \in N_c \rangle$  witness  $(\lambda_{c,1}, \lambda_{c,2}) \in \mathcal{C}(D_c)$ . Let  $A = \bigcup \{\text{range}(f_s) : s \in N_c\}$ , and let  $\langle I_d : d \in A \rangle$  be a sequence of pairwise disjoint intervals of  $M$  such that  $d \in I_d$ <sup>2</sup>. For  $s \in N_c$ , let  $f'_s \in {}^{\kappa_c}M$  be such that  $f'_s(i) \in I_{f_s(i)}$  and  $\langle f'_s(i) : s \in N_c, i < \kappa \rangle$  is with no repetitions. Define the sequence  $\bar{a} = \langle a_{s,t} : s, t \in N_c \rangle$ , such that for  $s <_{N_c} t$ ,  $a_{s,t} = \{i < \kappa : f_s(i) < f_t(i)\}$  and  $a_{t,s} = \kappa_c \setminus a_{s,t}$ . Also set  $a_{s,s} = \emptyset$ . It is evident that  $\bar{a} \in \mathcal{S}_c$ .

**Claim 3.6.**  $\bar{a}$  is not  $c$ -solvable.

*Proof.* Assume not. Then by Lemma 3.1(b-2), the type

$$q(x) = \{f_s/D_c < x < f_t/D_c : s \in N_{c,1} \text{ and } t \in N_{c,2}\}$$

is realized in  ${}^{\kappa_c}M/D_c$ , which contradicts the choice of  $M, \bar{f}$ . □

The Lemma follows. □

#### 4. FOR $\aleph_1$ -COMPLETE ULTRAFILTER, $\mathcal{C}(D)$ CONTAINS NO NON-SYMMETRIC PAIRS

In this section we prove part (a) of Theorem 1.2. In fact we will prove something stronger, that is of interest in its own sake.

**Definition 4.1.** Assume  $D$  is an ultrafilter on  $\kappa$ ,  $\langle I_i : i < \kappa \rangle$  is a sequence of linear orders and  $I = \prod_{i < \kappa} I_i/D$ .

(a) a subset  $K$  of  $I$  is called internal if there are subsets  $K_i \subseteq I_i$  such that  $K =$

$$\prod_{i < \kappa} K_i/D.$$

(b) The cut  $(J^1, J^2)$  of  $I$  is called internal, if there are cuts  $(J_i^1, J_i^2)$  of  $I_i$ ,  $i < \kappa$ , such

$$\text{that } J^l = \prod_{i < \kappa} J_i^l/D \text{ (} l = 1, 2\text{)}.$$

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<sup>2</sup>The existence of the sequence  $\langle I_d : d \in A \rangle$  follows from the fact that  $|A| \leq \kappa_c \cdot |N_c| \leq \lambda_c$  and  $M$  is  $\lambda_c^+$ -saturated.

**Remark 4.2.** Assume  $J$  is an initial segment of  $I$  which is internal and suppose that  $(J, I \setminus J)$  is a cut of  $I$ . Then  $(J, I \setminus J)$  is in fact an internal cut of  $I$ . Similarly, if  $J$  is an end segment of  $I$  which is internal and if  $(I \setminus J, J)$  is a cut of  $I$ , then  $(I \setminus J, J)$  is an internal cut of  $I$ .

**Theorem 4.3.** Assume  $D$  is a uniform  $\aleph_1$ -complete ultrafilter on  $\kappa$ ,  $\langle I_i : i < \kappa \rangle$  is a sequence of non-empty linear orders and  $I = \prod_{i < \kappa} I_i / D$ . Also assume  $(J^1, J^2)$  is a cut of  $I$  of cofinality  $(\theta_1, \theta_2)$ , where  $\theta_1 \neq \theta_2$ . Then the cut  $(J^1, J^2)$  is internal.

Before giving the proof of Theorem 4.3, let us show that it implies Theorem 1.2(a).

**Proof of Theorem 1.2(a) from Theorem 4.3.** Suppose  $D$  is an  $\aleph_1$ -complete ultrafilter on  $\kappa$ ,  $J$  is a  $(\lambda_1 + \lambda_2)^+$ -saturated dense linear order and  $(J^1, J^2)$  is a cut of  $J^\kappa / D$  of cofinality  $(\lambda_1, \lambda_2)$ . Towards contradiction assume that  $\lambda_1 \neq \lambda_2$ . It follows from Theorem 4.3 that the cut  $(J^1, J^2)$  is internal, and so that there are cuts  $(J_i^1, J_i^2)$  of  $J$ ,  $i < \kappa$ , such that  $J^l = \prod_{i < \kappa} J_i^l / D$  (for  $l = 1, 2$ ). Let  $(\lambda_i^1, \lambda_i^2) = \text{cf}(J_i^1, J_i^2)$ . It follows that  $\lambda_l = \prod_{i < \kappa} \lambda_i^l / D$ ,  $l = 1, 2$ .

By the choice of  $J$ , for every  $i < \kappa$ , either  $\lambda_i^1 \geq (\lambda_1 + \lambda_2)^+$  or  $\lambda_i^2 \geq (\lambda_1 + \lambda_2)^+$ , hence for some  $l \in \{1, 2\}$ , we have

$$A = \{i < \kappa : \lambda_i^l \geq (\lambda_1 + \lambda_2)^+\} \in D.$$

It follows that  $\lambda_l = \prod_{i < \kappa} \lambda_i^l / D \geq (\lambda_1 + \lambda_2)^+$ , which is a contradiction.  $\square$

We are now ready to complete the proof of Theorem 4.3.

*Proof.* We can assume that  $\theta_1, \theta_2$  are infinite. Let  $<_i^1 = <_{I_i}$  ( $i < \kappa$ ) and  $<_1 = <_I$ . Let  $<_i^2$  be a well-ordering of  $I_i$  with a last element and let  $<_2$  be such that  $(I, <_2) = \prod_{i < \kappa} (I_i, <_i^2) / D$ . Then  $<_2$  is a linear ordering of  $I$  with a last element and since  $D$  is  $\aleph_1$ -complete, it is well-founded, so  $<_2$  is in fact a well-ordering of  $I$  with a last element.

As  $(J^1, <_1)$  has cofinality  $\theta_1$ , we can find  $f_\alpha \in \prod_{i < \kappa} I_i$ , for  $\alpha < \theta_1$ , such that

- (1)  $\forall \alpha < \theta_1, f_\alpha / D \in J^1$ ,
- (2)  $\langle f_\alpha / D : \alpha < \theta_1 \rangle$  is  $<_1$ -increasing,
- (3)  $\langle f_\alpha / D : \alpha < \theta_1 \rangle$  is a  $<_1$ -cofinal subset of  $J^1$ .

Let  $B = \{t \in I : \{s \in J^1 : s <_2 t\} \text{ is } <_1\text{-unbounded in } J^1\}$ . As  $\theta_1$  is infinite, the  $<_2$ -last element of  $I$  belongs to  $B$ , which implies  $B \neq \emptyset$  and hence  $B$  has a  $<_2$ -minimal element; call it  $t_*$ . Let  $g_* \in \prod_{i < \kappa} I_i$  be such that  $t_* = g_*/D$ .

Note that for each  $\alpha < \theta_1$  there are  $s \in J^1$  and  $\beta > \alpha$  such that  $s <_2 g_*/D$  and  $f_\alpha/D <_1 s <_1 f_\beta/D$ , so we can assume that for all  $\alpha < \theta_1$ ,  $f_\alpha/D <_2 g_*/D$ . This implies

$$\bigwedge_{\alpha < \theta_1} [\{i < \kappa : f_\alpha(i) <_i^2 g_*(i)\} \in D].$$

Also note that

$$\{i < \kappa : g_*(i) \text{ is } <_i^1\text{-minimal or } <_i^1\text{-maximal}\} \notin D,$$

so, without loss of generality, it is empty. Hence, without loss of generality

$$\bigwedge_{\alpha < \theta_1} \bigwedge_{i < \kappa} [f_\alpha(i) <_i^2 g_*(i) \text{ and } f_\alpha(i) \text{ is not } <_i^1\text{-minimal}].$$

Let  $f_{\theta_1} = g_*$  and for  $\alpha \leq \theta_1$  set  $K_\alpha = \{s \in I : s <_2 f_\alpha/D\}$ . Thus  $K_\alpha$  is a  $<_2$ -initial segment of  $I$ .

**Claim 4.4.**  $K_\alpha \cap J^1$  is  $<_1$ -bounded in  $J^1$ .

*Proof.* As  $f_\alpha/D <_2 t_*$ , it follows from our choice of  $t_*$  that  $K_\alpha$  is  $<_1$  bounded in  $J^1$ .  $\square$

**Claim 4.5.** If  $\alpha < \theta_1$ , then  $K_\alpha$  is an internal subset of  $I$ .

*Proof.* For each  $i < \kappa$  set

$$K_{\alpha,i} = \{s \in I_i : s <_i^2 f_\alpha(i)\}.$$

Then  $K_\alpha = \prod K_{\alpha,i}/D$  and the result follows.  $\square$

Now consider the following statement:

(\*) There is  $\alpha < \theta_1$  such that  $J^2 \cap K_\alpha$  is  $<_1$ -unbounded from below in  $J^2$ .

We split the proof into two cases.

**Case 1.** (\*) holds: Fix  $\alpha$  witnessing (\*). It follows that  $(J^1 \cap K_\alpha, J^2 \cap K_\alpha)$  is internal in  $K_\alpha$ , so there are end segments  $L_i$  of  $I_i \upharpoonright \{s \in I_i : s <_i^2 f_\alpha(i)\}$ , for  $i < \kappa$ , such that  $J^2 \cap K_\alpha = \prod_{i < \kappa} L_i/D$ , hence by the assumption,  $J^2 = \prod_{i < \kappa} L'_i/D$ , where  $L'_i = \{t \in I_i : \exists s \in L_i, s \leq_i^1 t\}$ , so  $J^2$  is internal. It follows from Remark 4.2 that  $(J^1, J^2)$  is an internal cut of  $I$  and we are done.

**Case 2. (\*) fails:** So for any  $\alpha < \theta_1$ , there is  $s_\alpha \in J^2$  such that

$$\{s \in J^2 : s <_1 s_\alpha\} \cap K_\alpha = \emptyset.$$

As  $\theta_1 \neq \theta_2$  are regular cardinals, there is  $s_* \in J^2$  such that

$$\sup\{\alpha < \theta_1 : s_* \leq_1 s_\alpha\} = \theta_1,$$

hence

$$\{s \in J^2 : s <_1 s_*\} \cap \left( \bigcup_{\alpha < \theta_1} K_\alpha \right) = \emptyset.$$

**Claim 4.6.**  $\bigcup_{\alpha < \theta_1} K_\alpha = K_{\theta_1}$ .

*Proof.* It is clear that  $\bigcup_{\alpha < \theta_1} K_\alpha \subseteq K_{\theta_1}$ . Now suppose  $s \in K_{\theta_1}$ , so  $s <_2 g_*/D$ . If  $s \notin \bigcup_{\alpha < \theta_1} K_\alpha$ , then for any  $\alpha < \theta_1$ ,  $f_\alpha/D <_2 s$ . So by the minimal choice of  $t_*$  and the fact that  $\langle f_\alpha/D : \alpha < \theta_1 \rangle$  is  $<_1$ -cofinal in  $J^1$ , we have  $g_*/D \leq_2 s$  which is a contradiction.  $\square$

So we have  $\{s \in J^2 : s <_1 s_*\} \cap K_{\theta_1} = \emptyset$ . Let  $h_* \in \prod_{i < \kappa} I_i$  be such that  $s_* = h_*/D$ .

**Claim 4.7.** (a)  $K_{\theta_1}$  is internal.

(b)  $J^1 \cap K_{\theta_1}$  is  $<_1$ -unbounded in  $J^1$ .

(c)  $J^1 \cap K_{\theta_1}$  is internal.

*Proof.* (a) can be proved as in Claim 4.5 using  $f_{\theta_1}$  instead of  $f_\alpha$ . (b) is also clear as  $J^1 \cap K_{\theta_1} \supseteq \{f_\alpha/D : \alpha < \theta_1\}$  and  $\langle f_\alpha/D : \alpha < \theta_1 \rangle$  is  $<_1$ -unbounded in  $J^1$ . Let's prove (c). As  $\{s \in J^2 : s <_1 s_*\} \cap K_{\theta_1} = \emptyset$  and  $I = J^1 \cup J^2$ , we can easily see that

$$J^1 \cap K_{\theta_1} = \{s \in K_{\theta_1} : s <_1 s_*\}.$$

For each  $i < \kappa$  set

$$L_i = \{s \in I_i : s <_i^2 f_{\theta_1}(i) \text{ and } s <_i^1 h_*(i)\}.$$

It follows that  $J^1 \cap K_{\theta_1} = \prod_{i < \kappa} L_i/D$ , and so  $J^1 \cap K_{\theta_1}$  is internal.  $\square$

It follows from the above claim that  $J^1 = \prod_{i < \kappa} L'_i/D$ , where for  $i < \kappa$ ,  $L'_i = \{t \in I_i : \exists s \in L_i, t \leq_i^1 s\}$ . Hence  $J^1$  is internal and so by Remark 4.2,  $(J^1, J^2)$  is an internal cut of  $I$  which completes the proof of Case 2. The theorem follows.  $\square$

5.  $\mathcal{C}_{>2^\kappa}(D)$  CONTAINS NO NON-SYMMETRIC PAIRS

In this section we show that if  $D$  is a uniform ultrafilter on  $\kappa$ , then  $\mathcal{C}_{>2^\kappa}(D)$  does not contain any non-symmetric pairs. Again, we prove a stronger result from which the above claim, and hence Theorem 1.2(b) follows.

**Theorem 5.1.** *Assume  $D$  is a uniform ultrafilter on  $\kappa$ ,  $\langle I_i : i < \kappa \rangle$  is a sequence of linear orders and  $I = \prod_{i < \kappa} I_i/D$ . Also assume  $(J^1, J^2)$  is a cut of  $I$  of cofinality  $(\theta_1, \theta_2)$ , where  $\theta_1 \neq \theta_2$  are bigger than  $2^\kappa$ . Then the cut  $(J^1, J^2)$  is internal.*

*Proof.* Let  $<_i^1 = <_{I_i}$  ( $i < \kappa$ ) and  $<_1 = <_I$ . Let  $<_i^2$ , for  $i < \kappa$ , be a well-ordering of  $I_i$  with a last element and let  $<_2$  be such that  $(I, <_2) = \prod_{i < \kappa} (I_i, <_i^2)/D$ ; so  $<_2$  is a linear ordering of  $I$  with a last element.

We say a sequence  $\bar{K} = \langle K_i : i < \kappa \rangle$  catches  $(J^1, J^2)$  if each  $K_i \subseteq I_i$  is non-empty and for every  $s_1 \in J^1$  and  $s_2 \in J^2$  there is  $t \in \prod_{i < \kappa} K_i/D$  such that  $s_1 \leq_1 t \leq_1 s_2$ . Set

$$S = \{\bar{K} : \bar{K} \text{ catches } (J^1, J^2)\},$$

and

$$C = \{\bar{\mu} = \langle \mu_i : i < \kappa \rangle : \text{There exists } \bar{K} \in S \text{ such that } \bigwedge_{i < \kappa} |K_i| = \mu_i\}.$$

We can define an order on  $C$  by

$$\bar{\mu}^1 = \langle \mu_i^1 : i < \kappa \rangle <_D \bar{\mu}^2 = \langle \mu_i^2 : i < \kappa \rangle \iff \{i < \kappa : \mu_i^1 < \mu_i^2\} \in D.$$

Now consider the following statement:

(\*) There is  $\bar{\mu} \in C$  which is  $<_D$ -minimal.

We consider two cases.

**Case 1. (\*) holds:** Fix  $\bar{\mu} = \langle \mu_i : i < \kappa \rangle$  witnessing (\*), and let  $\bar{K} \in S$  be such that for all  $i < \kappa$ ,  $|K_i| = \mu_i$ . Let  $<_i^3$  be a well-ordering of  $K_i$  of order type  $\mu_i$  and let  $<_3$  be such that  $(K, <_3) = \prod_{i < \kappa} (K_i, <_i^3)/D$ , where  $K = \prod_{i < \kappa} K_i/D \subseteq I$ . Let  $\theta_3 = \text{cf}(\prod_{i < \kappa} \mu_i/D)$  and let  $g_\alpha \in \prod_{i < \kappa} K_i$ ,  $\alpha < \theta_3$ , be such that  $\langle g_\alpha/D : \alpha < \theta_3 \rangle$  is  $<_3$ -increasing and cofinal in  $(K, <_3)$ . As  $\theta_1 \neq \theta_2$ , for some  $l \in \{1, 2\}$ ,  $\theta_3 \neq \theta_l$ . Assume without loss of generality that  $\theta_3 \neq \theta_1$ .

For  $\alpha < \theta_3$  and  $i < \kappa$  set

$$K_{\alpha, i} = \{s \in K_i : s <_i^3 g_\alpha(i)\}.$$

and

$$K_\alpha = K \upharpoonright \{s : s <_3 g_\alpha/D\} = \prod_{i < \kappa} K_{\alpha,i}/D.$$

Then the sequence  $\langle K_\alpha : \alpha < \theta_3 \rangle$  is  $\subseteq$ -increasing and  $K = \bigcup_{\alpha < \theta_3} K_\alpha$ . The next claim is evident from our construction.

**Claim 5.2.** *K is internal.*

By our choice of  $\bar{\mu}$ , the sequence  $\langle K_{\alpha,i} : i < \kappa \rangle$  does not catch  $(J^1, J^2)$ , and hence we can find  $s_\alpha \in J^1$  and  $t_\alpha \in J^2$  such that

$$K_\alpha \cap \{s \in I : s_\alpha <_1 s <_1 t_\alpha\} = \emptyset.$$

As  $\theta_3 \neq \theta_1$ , there is  $s_* \in J^1$  such that

$$\sup\{\alpha < \theta_3 : s_\alpha \leq_1 s_*\} = \theta_3.$$

It follows that  $K \cap \{s \in J^1 : s_* \leq_1 s\} = \emptyset$ . As  $K$  catches  $(J^1, J^2)$ , it follows that  $J^2 \cap K$  is  $<_1$ -cofinal in  $J^2$  from below, and since  $K$  is internal, the arguments of section 2 show that  $J^2$  is also internal, and hence by Remark 4.2,  $(J^1, J^2)$  is an internal cut of  $I$ , as required.

**Case 2. (\*) fails:** Clearly  $<_D$  is a linear order on  $C$ , so it has a co-initiality, call it  $\theta_3$ . As (\*) fails,  $<_D$  is not well-founded and so  $\theta_3 \geq \aleph_0$ .

**Claim 5.3.**  $\theta_3 \leq 2^\kappa$ .

*Proof.* Suppose not. Let  $\langle \bar{\mu}_\xi : \xi < (2^\kappa)^+ \rangle$  be a  $<_D$ -decreasing chain of elements of  $C$ . Define a partition  $F : [(2^\kappa)^+]^2 \rightarrow \kappa$  by

$$F(\xi, \zeta) = \min\{i < \kappa : \mu_i^\zeta < \mu_i^\xi\},$$

which is well-defined as  $\{i < \kappa : \mu_i^\zeta < \mu_i^\xi\} \in D$ , in particular it is non-empty. By the Erdős-Rado partition theorem, there are  $X \subseteq (2^\kappa)^+$  of size  $\kappa^+$  and some fixed  $i_* < \kappa$  such that for all  $\xi < \zeta$  in  $X$ ,  $F(\xi, \zeta) = i_*$ . Thus

$$\xi < \zeta \in X \implies \mu_{i_*}^\zeta < \mu_{i_*}^\xi,$$

which is impossible. □

Let  $\langle \bar{\mu}_\xi = \langle \mu_{\xi,i} : i < \kappa \rangle : \xi < \theta_3 \rangle$  be  $<_D$ -decreasing which is unbounded from below in  $(C, <_D)$ . For  $\xi < \theta_3$  choose  $\bar{K}_\xi = \langle K_{\xi,i} : i < \kappa \rangle \in S$  such that for all  $i < \kappa$ ,  $|K_{\xi,i}| = \mu_{\xi,i}$ . Let  $K_\xi = \prod_{i < \kappa} K_{\xi,i}/D \subseteq I$ . We consider two subcases.

**Subcase 2.1. For some  $\xi < \theta_3$ ,  $K_\xi \cap J^1$  is bounded in  $(J^1, <_1)$ :** Fix such a  $\xi < \theta_3$  and let  $s_* \in J^1$  be a bound. Then as  $\bar{K}_\xi$  catches  $(J^1, J^2)$ , it follows that  $K_\xi \cap J^2$  is unbounded in  $J^2$  from below. Since  $K_\xi$  is internal and  $K_\xi \cap J^2$  is unbounded in  $J^2$  from below, so  $J^2$  is internal. It follows that the cut  $(J^1, J^2)$  is internal and we are done.

**Subcase 2.2. For all  $\xi < \theta_3$ ,  $K_\xi \cap J^1$  is unbounded in  $(J^1, <_1)$ :** Since  $\text{cf}(J^1, <_1) = \theta_1$ , there are functions  $f_\alpha \in \prod_{i < \kappa} I_i$ , for  $\alpha < \theta_1$ , such that

- (1)  $\forall \alpha < \theta_1, f_\alpha/D \in J^1$ ,
- (2)  $\langle f_\alpha/D : \alpha < \theta_1 \rangle$  is  $<_1$ -increasing,
- (3)  $\langle f_\alpha/D : \alpha < \theta_1 \rangle$  is a  $<_1$ -cofinal subset of  $J^1$ .

For every  $\alpha < \theta_1$  and  $\xi < \theta_3$  there are  $\beta$  and  $g$  such that

- (4)  $\alpha < \beta < \theta_1$ ,
- (5)  $g \in \prod_{i < \kappa} K_{\xi,i}$ ,
- (6)  $f_\alpha/D <_1 g/D <_1 f_\beta/D$ .

For  $\alpha < \beta < \theta_1$  set

$$\Lambda_{\alpha,\beta} = \{(\xi, i) : f_\alpha(i) \leq_i^1 f_\beta(i) \text{ and there is } s \in K_{\xi,i} \text{ such that } f_\alpha(i) \leq_i^1 s \leq_i^1 f_\beta(i)\}.$$

For  $i < \kappa$  set  $\Lambda_{\alpha,\beta,i} = \{\xi < \theta_3 : (\xi, i) \in \Lambda_{\alpha,\beta}\}$  and  $\Xi_{\alpha,\beta} = \{i < \kappa : \Lambda_{\alpha,\beta,i} \neq \emptyset\}$ . Also let  $F_{\alpha,\beta}^1, F_{\alpha,\beta}^2$  be functions with domain  $\kappa$  such that

- If  $i \in \Xi_{\alpha,\beta}$ , then

$$F_{\alpha,\beta}^1(i) = \min\{\mu_{\xi,i} : (\xi, i) \in \Lambda_{\alpha,\beta}\},$$

and

$$F_{\alpha,\beta}^2(i) = \min\{\xi : \mu_{\xi,i} = F_{\alpha,\beta}^1(i)\}.$$

- If  $i \in \kappa \setminus \Xi_{\alpha,\beta}$ , then  $F_{\alpha,\beta}^1(i) = F_{\alpha,\beta}^2(i) = 0$ .

**Claim 5.4.** For each  $\alpha < \theta_1$  there exist  $A_\alpha \subseteq (\alpha, \theta_1)$ , functions  $F_\alpha^1, F_\alpha^2$  and a set  $\Xi_\alpha$  such that

- (1)  $A_\alpha = \{\beta \in (\alpha, \theta_1) : F_{\alpha,\beta}^1 = F_\alpha^1, F_{\alpha,\beta}^2 = F_\alpha^2 \text{ and } \Xi_{\alpha,\beta} = \Xi_\alpha\}$ .
- (2)  $\sup(A_\alpha) = \theta_1$ .

*Proof.* As  $\theta_3 \leq 2^\kappa$ , we have

$$|\{(F_{\alpha,\beta}^1, F_{\alpha,\beta}^2, \Xi_{\alpha,\beta}) : \alpha < \beta < \theta_1\}| \leq \theta_3^\kappa = 2^\kappa < \theta_1.$$

So there is an unbounded subset  $A_\alpha$  of  $\theta_1$  such that all tuples  $(F_{\alpha,\beta}^1, F_{\alpha,\beta}^2, \Xi_{\alpha,\beta}), \beta \in A_\alpha$ , are the same. The result follows immediately.  $\square$

The next claim can be proved in a similar way.

**Claim 5.5.** *There are  $A \subseteq \theta_1$ , functions  $F_1, F_2$  and a set  $\Xi$  such that*

- (1)  $A = \{\alpha < \theta_1 : F_\alpha^1 = F_1, F_\alpha^2 = F_2 \text{ and } \Xi_\alpha = \Xi\}$ .
- (2)  $\sup(A) = \theta_1$ .

Let  $\bar{K}^* = \langle K_i^* : i < \kappa \rangle$  where  $K_i^* = K_{F_2(i),i}$  and let  $\bar{\mu}^* = \langle \mu_i^* : i < \kappa \rangle$  be defined by  $\mu_i^* = |K_i^*|$ . Note that

$$\mu_i^* = |K_i^*| = |K_{F_2(i),i}| = \mu_{F_2(i),i} = F_1(i).$$

**Claim 5.6.** *For every  $\xi < \theta_3$ ,  $\bar{\mu}^* \leq_D \bar{\mu}_\xi$ .*

*Proof.* Choose  $\alpha \in A$  and  $\beta \in A_\alpha$ . So we have  $F_1 = F_{\alpha,\beta}^1, F_2 = F_{\alpha,\beta}^2$  and  $\Xi = \Xi_{\alpha,\beta}$ . By the construction, there is  $t \in K_\xi$  such that  $f_\alpha/D <_1 t <_1 f_\beta/D$ . Let  $t = g/D$ , where  $g \in \prod_{i < \kappa} K_{\xi,i}$ . Then

$$f_\alpha(i) <_i^1 g(i) <_i^1 f_\beta(i) \Rightarrow (\xi, i) \in \Lambda_{\alpha,\beta} \Rightarrow \mu_i^* = F_1(i) = F_{\alpha,\beta}^1(i) \leq \mu_{\xi,i}.$$

So

$$\{i < \kappa : \mu_i^* \leq \mu_{\xi,i}\} \supseteq \{i < \kappa : f_\alpha(i) <_i^1 g(i) <_i^1 f_\beta(i)\} \in D,$$

and the result follows.  $\square$

**Claim 5.7.**  $\bar{\mu}^* \in C$ .

*Proof.* We show that  $\bar{K}^*$  catches  $(J^1, J^2)$ , so that  $\bar{K}^* \in S$  witnesses  $\bar{\mu}^* \in C$ . So let  $s_1 \in J^1$  and  $s_2 \in J^2$ . Pick  $\alpha \in A$  such that  $s_1 <_1 f_\alpha/D$ . Let  $\beta \in A_\alpha$ . By our construction there is  $g \in \prod_{i < \kappa} K_i^*$  such that  $f_\alpha/D <_1 g/D <_1 f_\beta/D$  and hence

$$s_1 <_1 f_\alpha/D <_1 g/D <_1 f_\beta/D <_1 s_2.$$

The claim follows. □

But Claims 5.6 and 5.7 give us a contradiction to the choice of the sequence  $\langle \bar{\mu}_\xi : \xi < \theta_3 \rangle$ .

This contradiction finishes the proof. □

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