

Proof-theoretic strengths of weak theories for positive inductive definitions

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Abstract

In this paper the lightface Π_1^1 -Comprehension axiom is shown to be proof-theoretically strong even over RCA_0^* , and we calibrate the proof-theoretic ordinals of weak fragments of the theory ID_1 of positive inductive definitions over natural numbers. Conjunctions of negative and positive formulas in the transfinite induction axiom of ID_1 are shown to be weak, and disjunctions are strong. Thus we draw a boundary line between predicatively reducible and impredicative fragments of ID_1 .

1 The lightface Π_1^1 -Comprehension axiom over RCA_0^*

This research is motivated to answer the questions raised by J. Van der Meeren, M. Rathjen and A. Weiermann [10, 11]: Let $|T|$ denote the proof-theoretic ordinal of a subsystem T of second order arithmetic.

Conjecture([10, 11])

1. $|\text{RCA}_0^* + (\Pi_1^1(\Pi_3^0)\text{-CA})^-| = \varphi\omega 0$.
2. $|\text{RCA}_0^* + (\Pi_1^1\text{-CA})^-| = \varphi\omega 0$.
3. $|\text{RCA}_0 + (\Pi_1^1\text{-CA})^-| = \vartheta(\Omega^\omega)$.

where RCA_0^* defined [9] is obtained from RCA_0 by adding a function symbol for the exponential function 2^x together with an axiom for the function 2^x , and restricting the induction axiom schema to bounded formulas in the expanded language. In RCA_0 the induction axiom schema is available for Σ_1^0 -formulas. $(\Pi_1^1(\Pi_3^0)\text{-CA})^-$ denotes the axiom schema of lightface, i.e., set parameter-free Π_1^1 -Comprehension Axiom with Π_3^0 -matrix λ :

$$\exists Y \forall n [n \in Y \leftrightarrow \forall X \lambda(X, n)] \quad (1)$$

$(\Pi_1^1\text{-CA})^-$ is the axiom schema of set parameter-free Π_1^1 -Comprehension Axiom with arbitrary arithmetical formulas λ . φ in the ordinal $\varphi\omega 0$ denotes the binary

Veblen function, and ϑ in $\vartheta(\Omega^\omega)$ is a collapsing function introduced in [8]. The ordinal $\Gamma_0 = \vartheta(\Omega^2)$ is known to be the limit of predicativity.

When Σ_1^0 -formulas are available in the induction axiom schema, the proof-theoretic ordinal is shown to be the small Veblen ordinal $\vartheta(\Omega^\omega)$.

Theorem 1.1 ([11]) $|\text{RCA}_0 + (\Pi_1^1(\Pi_3^0)\text{-CA})^-| = \vartheta(\Omega^\omega)$.

According to [10] A. Weiermann showed that the wellfoundedness of ordinals up to each ordinal $< \vartheta(\Omega^\omega)$ is provable in $\text{RCA}_0 + (\Pi_1^1(\Pi_3^0)\text{-CA})^-$, and M. Rathjen showed that $\text{RCA}_0 + (\Pi_1^1(\Pi_3^0)\text{-CA})^-$ is reducible to $\Pi_2^1\text{-BI}_0$, whose proof-theoretic ordinal, $|\Pi_2^1\text{-BI}_0|$, is $\vartheta(\Omega^\omega)$, cf. [8].

In trying to settle the Conjecture affirmatively, we have first investigated weak fragments of the theory ID_1 of positive inductive definitions over natural numbers, and found a line between predicatively reducible and impredicative fragments of ID_1 , cf. Theorem 1.8 below. One fragment is proof-theoretically strong in the sense that the fragment proves the wellfoundedness up to each ordinal $< \vartheta(\Omega^\omega)$, cf. Lemma 2.3. The proof can be transformed to one in $\text{RCA}_0^* + (\Pi_1^1(\Pi_3^0)\text{-CA})^-$, and thereby we obtain $|\text{RCA}_0^* + (\Pi_1^1(\Pi_3^0)\text{-CA})^-| \geq \vartheta(\Omega^\omega)$. By combining Theorem 1.1 we arrive at a negative answer to the Conjecture 1, $|\text{RCA}_0^* + (\Pi_1^1(\Pi_3^0)\text{-CA})^-| = \vartheta(\Omega^\omega)$. Actually in Proposition 1.2.1 we will see that RCA_0^* is equal to RCA_0 with the help of the lightface Π_1^1 -Comprehension Axiom. Thus the whole of the Conjecture is refuted.

The theory ID_1 for non-iterated positive inductive definitions over natural numbers is an extension of the first-order arithmetic PA in a language $\mathcal{L}(\text{ID})$, which is obtained from an arithmetic language by adding unary predicate constant R_φ for each X -positive formula $\varphi(X, x)$. Axioms are

$$\theta(0) \wedge \forall x(\theta(x) \rightarrow \theta(x+1)) \rightarrow \forall y \theta(y) \quad (2)$$

for each $\mathcal{L}(\text{ID})$ -formula θ .

$$\forall x[\varphi(R_\varphi, x) \rightarrow R_\varphi(x)] \quad (3)$$

$$\forall u[R_\varphi(u) \rightarrow \forall x(\varphi(\sigma, x) \rightarrow \sigma(x)) \rightarrow \sigma(u)] \quad (4)$$

for each $\mathcal{L}(\text{ID})$ -formula σ .

Note that ID_1 proves that R_φ is a fixed point of positive φ ,

$$\forall x[R_\varphi(x) \rightarrow \varphi(R_\varphi, x)] \quad (5)$$

since $\varphi(\varphi(R_\varphi)) \subset \varphi(R_\varphi)$ by (3), and then apply (4) to the formula $\sigma(x) \equiv \varphi(R_\varphi, x)$.

$\Pi_k^0(\Omega)\text{-ID}(\Pi_1^0)$ is a fragment of ID_1 in which X -positive formulas $\varphi(X, x) \in \Pi_1^0$, the formulas θ in the complete induction schema (2) as well as the formulas σ in the axiom (4) are restricted to Π_k^0 -formulas $\theta, \sigma \in \Pi_k^0(\Omega)$ in the language $\mathcal{L}(\text{ID})$ with atomic formulas $R_\varphi(t)$.

Let EA^2 be the elementary recursive arithmetic in the second order logic, i.e., no comprehension axiom such as Δ_0^0 -Comprehension is assumed. IND denotes

the Π_1^1 -sentence $\forall X \forall a [X(0) \wedge \forall y (X(y) \rightarrow X(y+1)) \rightarrow X(a)]$. For a first-order positive formula $\varphi(X, y)$, let

$$I_\varphi = \bigcap \{X : \varphi(X) \subset X\}.$$

The following Proposition 1.2 is utilized to show Theorem 1.3, in which the base theory \mathbf{EA}^2 can be the stronger \mathbf{RCA}_0^* .

Proposition 1.2 *Let $k \geq 1$.*

1. $\mathbf{EA}^2 + \text{IND} + (\Pi_1^1(\Sigma_{k+1}^0)\text{-CA})^- \vdash \Sigma_k^0\text{-IND}$.
2. *Let $\varphi(X, y) \in \Pi_1^0$ be an X -positive formula. If $\Pi_k^0(\Omega)\text{-ID}(\Pi_1^0) \vdash A(R_\varphi)$, then $\mathbf{EA}^2 + \text{IND} + (\Pi_1^1(\Sigma_{k+1}^0)\text{-CA})^- \vdash A(I_\varphi)$.*
3. *For any X -positive formula $\varphi(X, y)$, if $\text{ID}_1 \vdash A(R_\varphi)$, then $\mathbf{EA}^2 + \text{IND} + (\Pi_1^1\text{-CA})^- \vdash A(I_\varphi)$.*

Proof. Let $k \geq 1$.

1.2.1. For a Σ_k^0 -formula $\varphi(a, X, z)$ let

$$N(a, z) := \forall X [\varphi(0, X, z) \wedge \forall y (\varphi(y, X, z) \rightarrow \varphi(y+1, X, z)) \rightarrow \varphi(a, X, z)]$$

$N(a, z)$ is a $\Pi_1^1(\Sigma_{k+1}^0)$ -formula without set parameter, and exists as a set by the axiom $(\Pi_1^1(\Sigma_{k+1}^0)\text{-CA})^-$. It is obvious that $N(a, z)$ is inductive with respect to a , i.e., $N(0, z)$ and $\forall a [N(a, z) \rightarrow N(a+1, z)]$. Therefore by IND we obtain $\forall a N(a, z)$, i.e., $\forall X [\varphi(0, X, z) \wedge \forall y (\varphi(y, X, z) \rightarrow \varphi(y+1, X, z)) \rightarrow \forall a \varphi(a, X, z)]$.

1.2.2 and 1.2.3. We show Proposition 1.2.2. Proposition 1.2.3 is similarly seen. Argue in $\mathbf{EA}^2 + \text{IND} + (\Pi_1^1(\Sigma_{k+1}^0)\text{-CA})^-$, and let φ, ψ be positive Π_1^0 -formulas. First note that I_φ exists as a set by $(\Pi_1^1(\Sigma_2^0)\text{-CA})^-$. By Proposition 1.2.1 we have $\Sigma_k^0\text{-IND}$, and hence $A(I_\varphi, 0) \wedge \forall x (A(I_\varphi, x) \rightarrow A(I_\varphi, x+1)) \rightarrow \forall z A(I_\varphi, z)$ for any Σ_k^0 - or Π_k^0 -formula A . $\varphi(I_\varphi) \subset I_\varphi$ is seen logically.

Let $A(I_\psi, y)$ be a Π_k^0 -formula. We show

$$\varphi(A(I_\psi)) \subset A(I_\psi) \rightarrow I_\varphi \subset A(I_\psi) \quad (6)$$

Let

$$B(y) := \forall X [\varphi(A(X)) \subset A(X) \rightarrow A(X, y)].$$

B exists as a set by $(\Pi_1^1(\Sigma_{k+1}^0)\text{-CA})^-$. We claim that

$$\varphi(B) \subset B \quad (7)$$

Assume $\varphi(B, y)$ and $\varphi(A(X)) \subset A(X)$. We need to show $A(X, y)$. We first show $B \subset A(X)$. Suppose $B(z)$. Then by the assumption $\varphi(A(X)) \subset A(X)$ we have $A(X, z)$. Hence $B \subset A(X)$, and $\varphi(B, y) \rightarrow \varphi(A(X), y)$ by the positivity of $\varphi(X)$. The assumption $\varphi(B, y)$ yields $\varphi(A(X), y)$, and we conclude $A(X, y)$ by $\varphi(A(X)) \subset A(X)$.

Since B is a set, we obtain $I_\varphi \subset B$ by (7). On the other hand we have $B(y) \rightarrow \varphi(A(I_\psi)) \subset A(I_\psi) \rightarrow A(I_\psi, y)$ since I_ψ is a set. Therefore $\varphi(A(I_\psi)) \subset A(I_\psi) \rightarrow B \subset A(I_\psi)$. This together with $I_\varphi \subset B$ yields (6). \square

Let $\omega_0(\alpha) = \alpha$, and $\omega_{n+1}(\alpha) = \omega^{\omega_n(\alpha)}$ for $n \geq 0$. Thus $\Omega^\omega = \omega_1(\Omega \cdot \omega)$.

Theorem 1.3 *Let $k \geq 1$.*

1. $\text{RCA}_0^* + (\Pi_1^1(\Pi_{k+2}^0)\text{-CA})^- = \text{RCA}_0 + (\Pi_1^1(\Pi_{k+2}^0)\text{-CA})^-$.
2. $\Pi_k^0(\Omega)\text{-ID}(\Pi_1^0)$ *is interpreted canonically in* $\text{RCA}_0 + (\Pi_1^1(\Pi_{k+2}^0)\text{-CA})^-$.

$$\begin{aligned} |\Pi_k^0(\Omega)\text{-ID}(\Pi_1^0)| &= |\text{RCA}_0^* + (\Pi_1^1(\Pi_{k+2}^0)\text{-CA})^-| \\ &= |\text{RCA}_0 + (\Pi_1^1(\Pi_{k+2}^0)\text{-CA})^-| = |\Pi_{k+1}^1\text{-BI}_0| = \vartheta(\omega_k(\Omega \cdot \omega)) \end{aligned}$$

3. ID_1 *is interpreted canonically in* $\text{RCA}_0 + (\Pi_1^1\text{-CA})^-$.

$$|\text{ID}_1| = |\text{RCA}_0^* + (\Pi_1^1\text{-CA})^-| = |\text{RCA}_0 + (\Pi_1^1\text{-CA})^-| = \vartheta(\varepsilon_{\Omega+1}).$$

Proof. Theorem 1.3.1 follows from Proposition 1.2.1.

Let us consider Theorem 1.3.2. From Proposition 1.2.2 we see that $\Pi_k^0(\Omega)\text{-ID}(\Pi_1^0)$ is interpreted canonically in $\text{RCA}_0 + (\Pi_1^1(\Pi_{k+2}^0)\text{-CA})^-$. As in [11] using [6] we see that $\Pi_{k+1}^1\text{-BI}_0$ comprises $\text{RCA}_0 + (\Pi_1^1(\Pi_{k+2}^0)\text{-CA})^-$. In [8] it is shown that $|\Pi_{k+1}^1\text{-BI}_0| = \vartheta(\omega_k(\Omega \cdot \omega))$. The fact $\vartheta(\omega_k(\Omega \cdot \omega)) \leq |\Pi_k^0(\Omega)\text{-ID}(\Pi_1^0)|$ is shown in Proposition 2.2 below. \square

Therefore even the weakest fragment $\text{RCA}_0^* + (\Pi_1^1(\Pi_3^0)\text{-CA})^-$ is not predicatively reducible. In [10] it is reported that $|\text{RCA}_0^* + (\Pi_1^1(\Pi_2^0)\text{-CA})^-| = |\text{IS}_2|$ and $|\text{RCA}_0 + (\Pi_1^1(\Pi_2^0)\text{-CA})^-| = |\text{IS}_3|$.

This indicates that fragments of the light-face $\Pi_1^1\text{-CA}$ could be graded according to another hierarchy of formulas for matrices λ in (1) rather than the usual arithmetic hierarchy. Let $\Pi_0^0(\mathbf{P}^X) = \Sigma_0^0(\mathbf{P}^X)$ denote the class of first-order formulas $\lambda(X)$ obtained from arithmetic atomic formulas and X -positive formulas by means of Boolean connectives and bounded quantifications. Classes $\Pi_k^0(\mathbf{P}^X)$ and $\Sigma_k^0(\mathbf{P}^X)$ of first-order formulas are defined from the class by prefixing alternating (unbounded) quantifiers. It is open for us, but seems to me plausible that $\text{RCA}_0^* + (\Pi_2^0(\mathbf{P}^X)\text{-CA})^-$ is, or even $\text{RCA}_0 + (\Pi_2^0(\mathbf{P}^X)\text{-CA})^-$ is predicatively reducible.

1.1 Weak fragments

Let us introduce weak fragments of the theory ID_1 .

Let \mathcal{L} be a language for arithmetic having function constants¹ for each elementary recursive functions. Relation symbols in \mathcal{L} are $=, <$. Δ_0^- denotes the set of bounded formulas in \mathcal{L} , and Π_0^{1-} the set of formulas in \mathcal{L} , called *arithmetical* formulas. The elementary recursive arithmetic EA is the theory in \mathcal{L}

¹The proof-theoretic strength does not increase with more constants, e.g., with function constants for primitive recursive functions.

whose axioms are defining axioms for function constants, axioms for $=, <$ and Δ_0^- -IND: (2) is restricted to $\theta \in \Delta_0^-$.

For a second-order arithmetic T , its *proof-theoretic ordinal* $|T|$ is defined to be the supremum of the order types $|\prec|$ of elementary recursive and transitive relations \prec for which $T \vdash \forall y (\forall x \prec y X(x) \rightarrow X(y)) \rightarrow \forall y X(y)$. When T is a theory for positive inductive definitions, $|T|$ is defined to be the supremum of the order types $|\prec|$ of elementary recursive and transitive relations \prec for which $T \vdash \forall x (x \in W_\prec)$ for the accessible (well founded) part W_\prec of the relation \prec .

For a class Φ of X -positive formulas $\varphi(X, x)$, let $\mathcal{L}(\Phi) = \mathcal{L} \cup \{R_\varphi : \varphi \in \Phi\}$ denote the language obtained from the language \mathcal{L} by adding unary predicate constants R_φ for each $\varphi \in \Phi$. The unary predicate constant R_φ is intended to denote the least fixed point of the monotone operator defined from φ : $\mathbb{N} \supset \mathcal{X} \mapsto \{n \in \mathbb{N} : \mathbb{N} \models \varphi[\mathcal{X}, n]\}$.

For classes Θ, Γ of formulas in $\mathcal{L}(\Phi)$, let (Θ, Γ) -ID(Φ) denote the fragment of ID_1 defined as follows. (Θ, Γ) -ID(Φ) is an extension of EA . In (Θ, Γ) -ID(Φ), the positive formula φ is in Φ , the formulas θ in complete induction schema (2) are in Θ , and the formulas σ in the axiom (4) are in Γ .

When $\Theta = \Gamma$, let us write Γ -ID(Φ) for (Γ, Γ) -ID(Φ), and when Φ is the class of all positive formulas, let us write Γ -ID for Γ -ID(Φ).

Acc denotes the class of X -positive formulas φ

$$\varphi(X, x) \equiv [\forall y (\theta_0(x, y) \rightarrow t_0(x, y) \in X)] \quad (8)$$

with an arithmetic bounded formula $\theta_0(x, y)$ and a term $t_0(x, y)$. In $\theta_0(x, y)$ and $t_0(x, y)$ first-order parameters other than x, y may occur. For an elementary recursive relation \prec , $\forall y (y \prec x \rightarrow y \in X)$ is a typical example of an Acc -operator. Acc denotes the class of formulas $\sigma(x)$ which are obtained from an Acc -operator by substituting any predicate constant R for X

$$\sigma(x) \equiv [\forall y (\theta_0(x, y) \rightarrow t_0(x, y) \in R)] \quad (9)$$

where $\theta_0(x, y)$ is an arithmetic bounded formula and $t_0(x, y)$ a term possibly with first-order parameters other than x, y .

Definition 1.4 A formula is said to be *positive* [*negative*] if each predicate constant R_φ for least fixed point occurs only positively [negatively] in it, resp. Pos [Neg] denotes the class of all positive formulas [the class of all negative formulas], resp.

Also let $\text{P} \cup \text{N} := \text{Pos} \cup \text{Neg}$, $\text{P} \wedge \text{N} := \{C \wedge D : C \in \text{Pos}, D \in \text{Neg}\}$ and $\text{N} \vee \text{P} := \{D \vee C : D \in \text{Neg}, C \in \text{Pos}\}$.

Definition 1.5 For $k \geq 0$, classes $\Pi_k^0(\text{P})$ and $\Sigma_k^0(\text{P})$ of formulas in the language $\mathcal{L}(\text{ID})$ are defined recursively.

1. $\Pi_0^0(\text{P}) = \Sigma_0^0(\text{P})$ denotes the class of bounded formulas in positive formulas. Each formula in $\Pi_0^0(\text{P})$ is obtained from positive formulas by means of propositional connectives \neg, \vee, \wedge and bounded quantifiers $\exists x < t, \forall x < t$.

2. $\Pi_k^0(P) \cup \Sigma_k^0(P) \subset \Pi_{k+1}^0(P) \cap \Sigma_{k+1}^0(P)$.
3. Each class $\Pi_k^0(P)$ and $\Sigma_k^0(P)$ is closed under positive boolean combinations \vee, \wedge and bounded quantifications.
4. If $A \in \Pi_k^0(P)$ [$A \in \Sigma_k^0(P)$], then $\neg A \in \Sigma_k^0(P)$ [$\neg A \in \Pi_k^0(P)$], resp.
5. If $A \in \Pi_k^0(P)$ [$A \in \Sigma_k^0(P)$], then $\forall x A \in \Pi_k^0(P)$ [$\exists x A \in \Sigma_k^0(P)$], resp.

Let $\Pi_\infty^0(P) = \bigcup_{k < \omega} \Pi_k^0(P)$.

Classes $\Pi_k^0(\Omega)$ and $\Sigma_k^0(\Omega)$ are defined similarly by letting $\Pi_0^0(\Omega) = \Sigma_0^0(\Omega)$ denote the class of bounded formulas in atomic formulas $t \in R_\varphi, t = s, t < s$. The predicates R_φ may occur positively and/or negatively in $\Pi_0^0(P)$ -formulas and in $\Pi_0^0(\Omega)$ -formulas.

P-ID denotes the theory, in which X -positive formulas $\varphi(X, x)$ are arbitrary, the formulas θ in the complete induction schema (2) as well as the formulas σ in the axiom (4) are restricted to positive formulas $\theta, \sigma \in \text{Pos}$. $(\Pi_\infty^0(P), P)$ -ID is an extension of P-ID in which the formulas θ in (2) are arbitrary, but $\sigma \in \text{Pos}$.

The following theorem is shown by D. Probst [7], and independently by B. Afshari and M. Rathjen [1]. P-ID $[(\Pi_\infty^0(P), P)$ -ID] is denoted as $\text{ID}_1^* \upharpoonright [\text{ID}_1^*]$ in [7], and as $\text{ID}_1^* [\text{ID}_1^* + \text{IND}_{\mathbb{N}}]$ in [1], resp.

Theorem 1.6 (Probst [7], Afshari and Rathjen [1])

1. $|\text{P-ID}| = \varphi\omega 0 = \vartheta(\Omega \cdot \omega)$.
2. $|(\Pi_\infty^0(P), P)\text{-ID}| = \varphi\varepsilon_0 0 = \vartheta(\Omega \cdot \varepsilon_0)$.

They show that P-ID $[(\Pi_\infty^0(P), P)$ -ID] is interpreted in $\Sigma_1^1\text{-DC}_0$ [in $\Sigma_1^1\text{-DC}$], resp. It seems that their proofs do not work when σ in the axiom (4) is a negative formula.

On the other side G. Jäger and T. Strahm [5] show directly the following. Let $\text{ID}_1^\#$ be a subtheory of $\widehat{\text{ID}}_1$ for fixed points with the axioms (3) and (5). In $\text{ID}_1^\#$ the complete induction schema (2) is restricted to positive formulas $\theta \in \text{POS}$.

Theorem 1.7 (Jäger and Strahm [5]) $|\text{ID}_1^\#| = \varphi\omega 0$.

In this paper we show the following theorem 1.8. $(\text{Acc}, N \vee P)\text{-ID}(\text{Acc})$ is the theory, in which X -positive formulas $\varphi(X, x)$ are restricted to **Acc**-operators (8), the formulas θ in the complete induction schema (2) are restricted to $\theta \in \text{Acc}$ (9), and the formulas σ in the axiom (4) are restricted to a disjunction of negative formula and a positive formula $\sigma \in N \vee P$.

$(\Pi_k^0(P), P \cup N)\text{-ID}$ denotes the theory, in which $\varphi(X, x)$ is an arbitrary X -positive formula, $\theta \in \Pi_k^0(P)$ in (2), and $\sigma \in P \cup N$ in (4).

$(\Pi_k^0(P), P \wedge N)\text{-ID}(\text{Acc})$ is the theory, in which $\varphi(X, x)$ are restricted to **Acc**-operators (8), $\theta \in \Pi_k^0(P)$ in (2), and σ in (4) are restricted to a conjunction of positive formula and a negative formula $\sigma \in P \wedge N$. P-ID $\subset (\Pi_0^0(P), P \cup N)\text{-ID}$ is obvious.

Let $\omega_0 = 1$ and $\omega_{n+1} = \omega^{\omega_n}$.

Theorem 1.8 1.

$$\begin{aligned} |(Acc, N \vee P)\text{-ID}(Acc)| &= |(Acc, \Pi_0^0(P))\text{-ID}(Acc)| = |\Pi_1^0(P)\text{-ID}(Acc)| \\ &= \vartheta(\Omega^\omega) \end{aligned}$$

2.

$$\begin{aligned} |Acc\text{-ID}(Acc)| &= |P\text{-ID}| = |(\Pi_0^0(P), P \cup N)\text{-ID}| \\ &= |(\Pi_0^0(P), P \wedge N)\text{-ID}(Acc)| = \vartheta(\Omega \cdot \omega) = \varphi\omega 0 \end{aligned}$$

3. For each $k > 0$

$$\begin{aligned} |(\Pi_k^0(P), Acc)\text{-ID}(Acc)| &= |(\Pi_k^0(P), P \cup N)\text{-ID}| \\ &= |(\Pi_k^0(P), P \wedge N)\text{-ID}(Acc)| = \vartheta(\Omega \cdot \omega_{1+k}) = \varphi\omega_{1+k} 0 \end{aligned}$$

In particular $|(\Pi_\infty^0(P), Acc)\text{-ID}(Acc)| = |(\Pi_\infty^0(P), P)\text{-ID}| = |(\Pi_\infty^0(P), P \cup N)\text{-ID}| = |(\Pi_\infty^0(P), P \wedge N)\text{-ID}(Acc)| = \vartheta(\Omega \cdot \varepsilon_0) = \varphi\varepsilon_0 0$.

Among other things this means that negative formulas σ in the axiom (4) does not raise the proof-theoretic ordinals. Theorem 1.8.2 strengthens Theorems 1.6 in [1, 7], and Theorem 1.7 in [5]. Our proof of the upper bound is directly done by cut-eliminations in infinitary derivations.

Let us mention the contents of the paper. In Section 2 the easy halves in Theorem 1.8 are shown by giving some wellfoundedness proofs. In Section 3 theories to be considered are reformulated in one-sided sequent calculi. In Section 4 finitary proofs in sequent calculi are first embedded to infinitary derivations to eliminate cut inferences partially. This first step is needed to unfold complex induction formulas. Second finitary proofs and infinitary derivations are embedded into a system with the operator controlled derivations due to W. Buchholz [3]. In the latter derivations, cut formulas are restricted to boolean combinations of positive formulas. The upper bounds of the proof-theoretic ordinals are obtained through collapsing and bounding lemmas. Finally we conclude the other halves in Theorem 1.8.

2 Wellfoundedness proofs

In this section the easy halves in Theorem 1.8 are shown by giving some wellfoundedness proofs. First let us recall the notation system $OT'(\vartheta)$ in [10]. $OT'(\vartheta)$ denotes a notation system of ordinals based on symbols $\{0, \Omega, +, \vartheta\}$.

1. $0 \in OT'(\vartheta)$. 0 is the least element in $OT'(\vartheta)$, and $K(0) = \emptyset$.
2. If $\{\beta_k, \alpha_k : k < n\} \subset OT'(\vartheta)$ with $n > 0$, $\alpha_{n-1} > \dots > \alpha_0$, $0 \neq \beta_k < \Omega$ and $\alpha_0 > 0 \vee n > 1$, then $\Omega^{\alpha_{n-1}}\beta_{n-1} + \dots + \Omega^{\alpha_0}\beta_0 \in OT'(\vartheta)$.
 $K(\Omega^{\alpha_{n-1}}\beta_{n-1} + \dots + \Omega^{\alpha_0}\beta_0) = \bigcup \{K(\alpha_k) \cup \{\beta_k\} : k < n\}$.

3. If $\beta \in OT'(\vartheta)$, then $\vartheta(\beta) \in OT'(\vartheta) \cap \Omega$. $K(\vartheta(\beta)) = \{\vartheta(\beta)\}$.
4. $\vartheta(\alpha) < \vartheta(\beta) \Leftrightarrow [\alpha < \beta \wedge \forall \gamma \in K(\alpha)(\gamma < \vartheta(\beta))] \vee [\exists \delta \in K(\beta)(\vartheta(\alpha) \leq \delta)]$.
5. Each ordinal $\vartheta(\alpha)$ is defined to be additively closed. This means that $\beta, \gamma < \vartheta(\alpha) \Rightarrow \beta + \gamma < \vartheta(\alpha)$.

Note that the system $OT'(\vartheta)$ is ω -exponential-free except $\vartheta(\alpha) = \omega^{\alpha_0}$ for some α_0 . An inspection of the proof in [11] shows that $Acc\text{-}ID(Acc)$ suffices to prove the wellfoundedness of ordinals up to each ordinal $< \vartheta(\Omega \cdot \omega)$.

Let $<$ be the elementary recursive relation obtained from the relation $<$ on $OT'(\vartheta)$ through a suitable encoding. For the formula $\forall y(y < x \rightarrow y \in X)$ in Acc , let W denote the accessible part of $<$, and $Prog(X) :\Leftrightarrow \forall \alpha[\forall \beta < \alpha(\beta \in X) \rightarrow \alpha \in X]$. Then the axiom (3) states $Prog(W)$, and the axiom (4) runs $\forall x[x \in W \rightarrow Prog(\sigma) \rightarrow \sigma(x)]$ for $\sigma \in Acc$.

The following lemma shows the easy half in Theorem 1.8.2.

Lemma 2.1 $Acc\text{-}ID(Acc) \vdash \forall \beta < \vartheta(\Omega \cdot k)(\beta \in W)$ for each $k < \omega$.

Proof. We see that the following are provable in $Acc\text{-}ID(Acc)$. Note that $A, B, C \in Acc$ for the formulas A, B, C below.

- (a) $x \in W \rightarrow \forall y(y < x \rightarrow y \in W)$ by the axiom (4) for the formula $A(x) \Leftrightarrow \forall y(y < x \rightarrow y \in W)$.
- (b) $y \in W \rightarrow x \in W \rightarrow x + y \in W$ by the axiom (4) for the formula $B(y) \Leftrightarrow (x + y \in W)$.
- (c) Assume $K(a) \subset W$, $\forall \beta <_\Omega \Omega \cdot a(\vartheta(\beta) \in W)$ and $\zeta \in W \cap \Omega$, where $\alpha <_\Omega \beta :\Leftrightarrow (K(\alpha) \cup K(\beta) \subset W \wedge \alpha < \beta)$. Then $Prog(C)$ for $C(\xi) :\Leftrightarrow (\xi < \zeta \rightarrow \vartheta(\Omega \cdot a + \xi) \in W)$.

Suppose $\xi < \zeta$ and $\forall \eta < \xi C(\eta)$. Then $\xi \in W$ by $\zeta \in W$, and $K(\xi) \subset W$. We show $\forall \alpha < \vartheta(\Omega \cdot a + \xi)(\alpha \in W)$ by Acc -induction on the length of α . By (b) we can assume that $\alpha = \vartheta(\beta)$. If $\alpha \leq \xi_0$ for a $\xi_0 \in K(a) \cup K(\xi) \subset W$, then $\alpha \in W$. Otherwise $K(\beta) \subset W$ by the induction hypothesis, and $\beta < \Omega \cdot a + \xi$. We can assume $\beta = \Omega \cdot a + \eta$ for an $\eta < \xi$ by the assumption $\forall \beta <_\Omega \Omega \cdot a(\vartheta(\beta) \in W)$. We obtain $\alpha \in W$ by $C(\eta)$.

- (d) $K(a) \subset W \rightarrow \forall \beta <_\Omega \Omega \cdot a(\vartheta(\beta) \in W) \rightarrow \forall \beta <_\Omega \Omega \cdot (a + 1)(\vartheta(\beta) \in W)$.

Assume $K(a) \subset W$, $\forall \beta <_\Omega \Omega \cdot a(\vartheta(\beta) \in W)$ and $\beta <_\Omega \Omega \cdot (a + 1)$. We need to show $\vartheta(\beta) \in W$. We can assume $\beta = \Omega \cdot a + \zeta$ for a $\zeta < \Omega$. By $K(\beta) \subset W$ we have $\zeta \in W$. Then by (c) we have $Prog(C)$, which yields $\forall \xi \in W \cap \zeta(\vartheta(\Omega \cdot a + \xi) \in W)$ by the axiom (4) for the Acc -formula C . From this we see that $\forall \alpha < \vartheta(\Omega \cdot a + \zeta)(\alpha \in W)$ by Acc -induction on the length of α , and hence $\vartheta(\Omega \cdot a + \zeta) \in W$ as desired.

By (d) we obtain $\forall \beta <_\Omega \Omega \cdot k C(\beta)$, i.e., $\forall \beta <_\Omega \Omega \cdot k(\vartheta(\beta) \in W)$ for each $k < \omega = \vartheta(1)$ with $K(k) = \{1\} = \{\vartheta(0)\} \subset W$. Using this and (b), we see that

$\forall \beta < \vartheta(\Omega \cdot k)(\beta \in W)$ by *Acc*-induction on the length of β . This shows Lemma 2.1. \square

Although the fact $\vartheta(\omega_k(\Omega \cdot \omega)) \leq |\Pi_k^0(\Omega)\text{-ID}(\Pi_1^0)|$ is assumed to be a folklore, let us give a proof of it for completeness, cf. Theorem 1.3.2. Let $\Omega_0(\alpha) = \alpha$, and $\Omega_{n+1}(\alpha) = \Omega^{\Omega_n(\alpha)}$. Then $\omega_k(\Omega \cdot \omega) = \Omega_k(\omega)$ for $k \geq 1$.

Proposition 2.2 *Let $k \geq 1$. $\Pi_k^0(\Omega)\text{-ID}(\Pi_1^0) \vdash \vartheta(\Omega_k(\ell)) \in W$ for each $\ell < \omega$.*

Proof. Let $\beta \in M : \Leftrightarrow (K(\beta) \subset W)$. For a formula $C(\beta)$, let $(M \rightarrow C)(\beta) : \Leftrightarrow (\beta \in M \rightarrow C(\beta))$, and $J[C](\zeta) : \Leftrightarrow \forall \alpha (M \cap \alpha \subset C \rightarrow M \cap (\alpha + \Omega^\zeta) \subset C)$. Then we claim that $\Pi_k^0(\Omega)\text{-ID}(\Pi_1^0)$ proves $\text{Prog}(M \rightarrow C) \rightarrow \text{Prog}(M \rightarrow J[C])$ for $\Pi_k^0(\Omega)$ -formula C . Argue in $\Pi_k^0(\Omega)\text{-ID}(\Pi_1^0)$, and assume $\text{Prog}(M \rightarrow C)$, $\forall \xi \in M \cap \zeta J[C](\xi)$, $M \cap \alpha \subset C$, $\zeta \in M$, and $\beta \in M \cap (\alpha + \Omega^\zeta)$. We need to show $C(\beta)$. We can assume $\beta \geq \alpha$ by $M \cap \alpha \subset C$. When $\zeta = 0$, we have $M \ni \beta = \alpha$. Then $\text{Prog}(M \rightarrow C)$ together with $M \cap \alpha \subset C$ yields $C(\alpha)$. Next consider the case when ζ is a limit number. Then $\beta < \alpha + \Omega^\xi$ for a $\xi \in M \cap \zeta$. $\forall \xi \in M \cap \zeta J[C](\xi)$ yields $C(\beta)$.

Finally let $\zeta = \xi + 1$. Then $\xi \in M$, and we see from $\beta \in M$ that there exists a $\gamma_1 \in W \cap \Omega$ such that $\beta < \alpha + \Omega^\xi \gamma_1$. We claim that $\text{Prog}(\sigma_1)$ for $\sigma_1(\gamma) : \Leftrightarrow (M \cap (\alpha + \Omega^\xi \gamma) \subset C)$. Assuming $\forall \gamma < \gamma_0 \sigma_1(\gamma)$, we need to show $M \cap (\alpha + \Omega^\xi \gamma_0) \subset C$. The case $\gamma_0 = 0$ follows from $M \cap \alpha \subset C$, and the case when γ_0 is a limit number is readily seen. Let $\gamma_0 = \gamma + 1$. From $J[C](\xi)$ and $\sigma_1(\gamma)$, i.e., $M \cap (\alpha + \Omega^\xi \gamma) \subset C$, we see that $M \cap (\alpha + \Omega^\xi \gamma + \Omega^\xi) \subset C$. Thus $\text{Prog}(\sigma_1)$ is shown. Since σ_1 is a $\Pi_k^0(\Omega)$ -formula for $k \geq 1$, we obtain $W \subset \sigma_1$, i.e., $\forall \gamma \in W (M \cap (\alpha + \Omega^\xi \gamma) \subset C)$. We conclude $C(\beta)$ from $\beta \in M \cap (\alpha + \Omega^\xi \gamma_1)$ and $\gamma_1 \in W$.

Thus we have shown the claim $\text{Prog}(M \rightarrow C) \rightarrow \text{Prog}(M \rightarrow J[C])$ for $\Pi_k^0(\Omega)$ -formula C . Now let $C_0(\beta) : \Leftrightarrow (\vartheta(\beta) \in W)$, and $C_{i+1} := J[C_i]$. Then C_i is a $\Pi_{i+1}^0(\Omega)$ -formula. It is clear that $\text{Prog}(M \rightarrow C_0)$, i.e., $\forall \alpha (M \cap \alpha \subset C_0 \rightarrow \alpha \in M \rightarrow \vartheta(\alpha) \in W)$, cf. the proof of Lemma 2.1. By metainduction on $i \leq k$ we obtain $\text{Prog}(M \rightarrow C_i)$. In particular for each $\ell < \omega$, $C_k(\ell)$ follows from $\text{Prog}(M \rightarrow C_k)$. By metainduction on $i \leq k$ we see from this that $M \cap \Omega_i(\ell) \subset C_{k-i}$, and hence $C_{k-i}(\Omega_i(\ell))$. Therefore $C_0(\Omega_k(\ell))$, i.e., $\vartheta(\Omega_k(\ell)) \in W$ for each $\ell < \omega$. \square

The next lemma shows the easy half in Theorem 1.8.1, and the power of disjunctions of negative and positive formulas, i.e., implications of positive formulas in the axiom (4). Note that our proof of the lemma is formalizable in $\text{RCA}_0^* + (\Pi_1^1(\Pi_3^0)\text{-CA})^-$.

Lemma 2.3 *$(\text{Acc}, N \vee P)\text{-ID}(\text{Acc}) \vdash \vartheta(\Omega^\ell) \in W$ for each $\ell < \omega$.*

Proof. Argue in $(\text{Acc}, N \vee P)\text{-ID}(\text{Acc})$. We claim that $\text{Prog}(\omega \rightarrow J[C_0])$ for $C_0(\beta) \Leftrightarrow (\vartheta(\beta) \in W)$ and $J[C_0](\ell) \Leftrightarrow \forall \alpha (M \cap \alpha \subset C_0 \rightarrow M \cap (\alpha + \Omega^\ell) \subset C_0)$ with $\beta \in M \Leftrightarrow (K(\beta) \subset W)$. $J[C_0](0)$ is seen from $\text{Prog}(M \subset C_0)$. Assuming $J[C_0](\ell)$, and $M \cap \alpha \subset C_0$, we need to show $M \cap (\alpha + \Omega^{\ell+1}) \subset C_0$.

Let $<_{lx}$ denote the lexicographic ordering on $OT'(\vartheta) \times OT'(\vartheta)$, in which the first components are ordered in the ordering $<$ on $OT'(\vartheta)$ and the second

components are ordered in the ω -ordering $<^{\mathbb{N}}$ on $OT'(\vartheta) \subset \mathbb{N}$:

$$(\xi, \gamma) <_{lx} (\zeta, \beta) :\Leftrightarrow (\xi < \zeta) \vee (\xi = \zeta \wedge \gamma <^{\mathbb{N}} \beta)$$

Let W_{lx} denote the accessible part of $<_{lx}$, which is the least fixed point of the operator $\forall(\xi, \gamma) <_{lx} (\zeta, \beta) X(\xi, \gamma)$. Let $Prog_{lx}(X) :\Leftrightarrow \forall(\zeta, \beta)[\forall(\xi, \gamma) <_{lx} (\zeta, \beta) X(\xi, \gamma) \rightarrow X(\zeta, \beta)]$.

$$\zeta \in W \rightarrow \forall\beta[(\zeta, \beta) \in W_{lx}] \quad (10)$$

This follows from $Prog(D)$ for $D(\zeta) \Leftrightarrow \forall\beta[(\zeta, \beta) \in W_{lx}]$ with the positive formula $D \in \text{Pos} \subset \mathbb{N} \vee \mathbb{P}$. $Prog(D)$ is seen from *Acc*-induction on β .

Now let $\sigma_0(\zeta, \beta) :\Leftrightarrow (\beta \in M \wedge \beta < \alpha + \Omega^\ell \zeta \rightarrow \vartheta(\beta) \in W)$. We claim that σ_0 is progressive with respect to the lexicographic ordering $<_{lx}$, $Prog_{lx}(\sigma_0)$. Suppose $\forall(\xi, \gamma) <_{lx} (\zeta, \beta) \sigma_0(\xi, \gamma)$, $\beta \in M$ and $\beta < \alpha + \Omega^\ell \zeta$. We need to show $\vartheta(\beta) \in W$. We can assume that $\beta = \alpha + \Omega^\ell \xi + \delta$ with $\xi < \zeta$ and $\delta < \Omega^\ell$. We claim that $M \cap \alpha_0 \subset C_0$ for $\alpha_0 = \alpha + \Omega^\ell \xi$. Let $\gamma \in M \cap \alpha_0$. We have $(\xi, \gamma) <_{lx} (\zeta, \beta)$, $\gamma \in M$ and $\gamma < \alpha + \Omega^\ell \xi$. $\sigma_0(\xi, \gamma)$ yields $\vartheta(\gamma) \in W$, i.e., $\gamma \in C_0$. $J[C_0](\ell)$ yields $\beta \in M \cap (\alpha_0 + \Omega^\ell) \subset C_0$ from $M \cap \alpha_0 \subset C_0$. Thus $\vartheta(\beta) \in W$.

From $Prog_{lx}(\sigma_0)$ we obtain $\forall(\zeta, \beta) \in W_{lx} \sigma_0(\zeta, \beta)$ for $\sigma_0 \in \mathbb{N} \vee \mathbb{P}$. By (10) we conclude $\forall\zeta \in W \forall\beta \sigma_0(\zeta, \beta)$, and hence $M \cap (\alpha + \Omega^{\ell+1}) \subset C_0$.

We have shown $Prog(\omega \rightarrow J[C_0])$. By meta induction on ℓ we obtain $J[C_0](\ell)$, and $\vartheta(\Omega^\ell) \in W$. \square

Lemma 2.3 shows that

$$\vartheta(\Omega^\omega) \leq |(Acc, \mathbb{N} \vee \mathbb{P})\text{-ID}(Acc)| \leq |(Acc, \Pi_0^0(\mathbb{P}))\text{-ID}(Acc)| \leq |\Pi_1^0(\mathbb{P})\text{-ID}(Acc)|.$$

The non-trivial halves of Theorem 1.8 follow from the following theorem. For a positive operator $\varphi(X, x)$ and a number n in the least fixed point I_φ of the monotonic operator $\omega \supset \mathcal{X} \mapsto \{n : \mathbb{N} \models \varphi[\mathcal{X}, n]\}$, $|n|_\varphi := \min\{\alpha : n \in I_\varphi^{\alpha+1}\}$ denotes the inductive norm of n . $Th(\mathbb{N})$ denotes the set of true arithmetic sentences.

Theorem 2.4 1. For each $k \geq 0$ and positive operator $\varphi(X, x)$,

$$Th(\mathbb{N}) + (\Pi_k^0(\mathbb{P}), \mathbb{P} \cup \mathbb{N})\text{-ID} \vdash R_\varphi(n) \Rightarrow |n|_\varphi < \vartheta(\Omega \cdot \omega_{1+k}).$$

2. For each $k \geq 0$ and *Acc*-operator $\varphi(X, x)$,

$$Th(\mathbb{N}) + (\Pi_k^0(\mathbb{P}), \mathbb{P} \wedge \mathbb{N})\text{-ID}(Acc) \vdash R_\varphi(n) \Rightarrow |n|_\varphi < \vartheta(\Omega \cdot \omega_{1+k}).$$

3. For each *Acc*-operator $\varphi(X, x)$,

$$Th(\mathbb{N}) + \Pi_1^0(\mathbb{P})\text{-ID}(Acc) \vdash R_\varphi(n) \Rightarrow |n|_\varphi < \vartheta(\Omega^\omega).$$

Our proof of Theorem 2.4 is based on an analysis through the operator controlled derivations due to W. Buchholz [3]. An ordinal notation system with the ψ -function also due to W. Buchholz [2] (but without the exponential function below Ω) is convenient for our proof.

Definition 2.5 Let Ω be the least uncountable ordinal ω_1 , and $\varepsilon_{\Omega+1}$ the next epsilon number above Ω . Define simultaneously on ordinals $\alpha < \varepsilon_{\Omega+1}$, operators \mathcal{H}_α on the power set of $\varepsilon_{\Omega+1}$, and ordinals ψ_α as follows. Let $X \subset \varepsilon_{\Omega+1}$.

1. $\{0, \Omega\} \cup X \subset \mathcal{H}_\alpha(X)$.
2. If $\Omega < \beta \in \mathcal{H}_\alpha(X)$, then $\omega^\beta \in \mathcal{H}_\alpha(X)$.
3. $\{\beta, \gamma\} \subset \mathcal{H}_\alpha(X) \Rightarrow \beta + \gamma \in \mathcal{H}_\alpha(X)$.
4. $\beta \in \mathcal{H}_\alpha(X) \cap \alpha \Rightarrow \psi\beta \in \mathcal{H}_\alpha(X)$.

Let

$$\psi\alpha := \min\{\beta \leq \Omega : \mathcal{H}_\alpha(\beta) \cap \Omega \subset \beta\}.$$

It is well known that $\mathcal{H}_{\varepsilon_{\Omega+1}}(0)$ is a computable notation system, and $\psi\alpha$ is in normal form if $G\alpha < \alpha$ for $\alpha \in \mathcal{H}_{\varepsilon_{\Omega+1}}(0)$, where $G0 = G\Omega = \emptyset$, $G(\psi\alpha) = \{\alpha\} \cup G\alpha$, $G\omega^\alpha = G\alpha$ and $G(\beta + \gamma) = G\beta \cup G\gamma$. Also it is shown the following in [4].

Proposition 2.6 $\vartheta(\Omega \cdot \omega_{1+k}) = \psi(\Omega^{\omega_{1+k}})$, $\vartheta(\Omega \cdot \varepsilon_0) = \psi(\Omega^{\varepsilon_0})$ and $\vartheta(\Omega^\omega) = \psi(\Omega^{\Omega^\omega}) = \psi(\omega^{\Omega^\omega})$.

Let W denote the accessible part of $<$ on $\mathcal{H}_{\varepsilon_{\Omega+1}}(0)$. The easy half in Theorem 1.8.3 follows from the following lemma.

Lemma 2.7 For each $\alpha < \psi(\Omega^{\omega_{1+k}})$, $(\Pi_k^0(\mathbf{P}), \text{Acc})\text{-ID}(\text{Acc}) \vdash \alpha \in W$.

Proof. It is clear that $\text{Acc-ID}(\text{Acc}) \subset (\Pi_0^0(\mathbf{P}), \text{Acc})\text{-ID}(\text{Acc})$, and we have (a) and (b) in the proof of Lemma 2.1 in hand. The following (e) and (f) are provable in $\text{Acc-ID}(\text{Acc})$.

- (e) $G\beta < \beta \rightarrow [\forall \gamma < \beta (\mathbb{P}(\gamma) \subset W \rightarrow w(\gamma)) \leftrightarrow w(\beta)]$, where $w(\gamma) := (G\gamma < \gamma \rightarrow \psi(\gamma) \in W)$ and $\mathbb{P}(\gamma)$ denotes the set of ordinal terms $\psi\alpha$ occurring in γ .

Assume $G\beta < \beta$ and $\forall \gamma < \beta (\mathbb{P}(\gamma) \subset W \rightarrow w(\gamma))$. By *Acc*-induction on the length of α we see that $\forall \alpha < \psi\beta (\alpha \in W)$. For $\alpha = \psi\gamma$ with $G\gamma < \gamma$, $\mathbb{P}(\gamma) \subset W$ follows from the induction hypothesis and $\mathbb{P}(\gamma) < \psi\gamma$.

- (f) $\text{Prog}(E)$ for $E(a) := (\forall \beta [\forall \gamma < \beta w(\gamma) \rightarrow \forall \gamma < \beta + \Omega^a w(\gamma)])$.

It suffices to show $E(a+1)$ assuming $E(a)$, which follows from $\text{Prog}(D)$, and the axiom (4) for the *Acc*-formula $D(\zeta) := (\zeta < \Omega \rightarrow w(\beta + \Omega^a \zeta))$.

From (f) we see that $\text{Acc-ID}(\text{Acc}) \vdash \forall \beta < \Omega^n w(\beta)$, i.e., $\text{Acc-ID}(\text{Acc}) \vdash \forall \alpha < \psi(\Omega^n)(\alpha \in W)$ for each n .

In what follows argue in $(\Pi_k^0(\mathbf{P}), \text{Acc})\text{-ID}(\text{Acc})$.

For a formula A , let $j[A](\alpha) := \forall \beta [\forall \gamma < \beta A(\gamma) \rightarrow \forall \gamma < \beta + \omega^\alpha A(\gamma)]$. Then $\text{Prog}(A) \rightarrow \text{Prog}(j[A])$ for $A \in \Pi_k^0(\mathbf{P})$.

Let $E_1 = E$ for the formula E in (f), and $E_{n+1} = j[E_n]$.

Then $E_n \in \Pi_n^0(\mathbf{P})$ and $\text{Prog}(E_{k+1})$. This yields $E_{k+1}(n)$ for each n . Hence $E_{k+1-m}(\omega_m(n))$ for each n and $m \leq k$, where $\omega_0(n) = n$ and $\omega_{m+1}(n) = \omega^{\omega_m(n)}$, i.e., $\omega_m = \omega_m(1)$. In particular $E_1(\omega_k(n))$ for each n . Therefore $w(\Omega^{\omega_k(n)})$ for each n . We conclude $\forall \alpha < \psi(\Omega^{\omega_k(n)})(\alpha \in W)$ in $(\Pi_k^0(\mathbf{P}), \text{Acc})\text{-ID}(\text{Acc})$. \square

3 Sequent calculi for weak fragments

To establish upper bounds in Theorem 1.8, let us reformulate $Th(\mathbb{N})+(\Pi_k^0(P), \text{PU N})\text{-ID}$, $Th(\mathbb{N})+(\Pi_k^0(P), P \wedge \text{N})\text{-ID(Acc)}$ and $Th(\mathbb{N})+\Pi_1^0(P)\text{-ID(Acc)}$ in one-sided sequent calculi. We assume that for each predicate symbol R , its complement or negation \bar{R} is in the language. For example, we have negations $\neq, \not<$ of the predicate constants $=, <$. Logical connectives are $\vee, \wedge, \exists, \forall$. Negations $\neg A$ of formulas A are defined recursively by de Morgan's law and elimination of double negations. $A \rightarrow B$ denotes $\neg A \vee B$ for formulas A, B . $\neg A$ is also denoted by \bar{A} .

The followings are *initial sequents*.

1. (logical initial sequent)

$$\bar{L}, L, \Gamma \text{ where } L \text{ is a literal.}$$

2. (equality initial sequent)

$$t \neq s, \bar{L}(t), L(s), \Gamma \text{ for literals } L(x).$$

3. (arithmetical initial sequent)

$$A, \Gamma$$

where A is one of formulas $t = t$, a defining axiom for an elementary recursive function, or a true arithmetical sentence in \mathcal{L} .

Inference rules are (cut) , (\exists) , (\forall) , $(b\exists)$, $(b\forall)$, (\vee) , (\wedge) , (R) , (\bar{R}) , and (ind) .

$$\frac{\Gamma, \bar{C} \quad C, \Delta}{\Gamma, \Delta} (cut)$$

where C is the *cut formula* of the (cut) .

$$\frac{A(t), \Gamma}{\Gamma} (\exists) \frac{A(a), \Gamma}{\Gamma} (\forall)$$

where $(\exists x A(x)) \in \Gamma$ in (\exists) , and a is an eigenvariable and $(\forall x A(x)) \in \Gamma$.

$$\frac{A(t), \Gamma \quad t < s, \Gamma}{\Gamma} (b\exists) \frac{a \not< s, A(a), \Gamma}{\Gamma} (b\forall)$$

where $(\exists x < s A(x)) \in \Gamma$ in $(b\exists)$, and a is an eigenvariable and $(\forall x < s A(x)) \in \Gamma$.

$$\frac{A_i, \Gamma}{\Gamma} (\vee) \frac{A_0, \Gamma \quad A_1, \Gamma}{\Gamma} (\wedge)$$

for an $i = 0, 1$ with $(A_0 \vee A_1) \in \Gamma$ in (\vee) , and $(A_0 \wedge A_1) \in \Gamma$.

For each theory the inference rule for the predicates R_φ is the following:

$$\frac{\varphi(R_\varphi, t), \Gamma}{\Gamma} (R)$$

with $(R_\varphi(t)) \in \Gamma$.

1. For the theory $Th(\mathbb{N}) + (\Pi_k^0(\mathbf{P}), \mathbf{P} \cup \mathbf{N})\text{-ID}$, the following (\bar{R}) is the inference rule for \bar{R}_φ :

$$\frac{\bar{\varphi}(\sigma, a), \sigma(a), \Gamma \quad \bar{\sigma}(t), \Gamma}{\Gamma} (\bar{R})$$

with $(\bar{R}_\varphi(t)) \in \Gamma$ and an eigenvariable a , where $\varphi(X, x)$ is an X -positive formula, and $\sigma \in \mathbf{P} \cup \mathbf{N}$.

2. For the theory $Th(\mathbb{N}) + (\Pi_k^0(\mathbf{P}), \mathbf{P} \wedge \mathbf{N})\text{-ID}(\mathbf{Acc})$, let $\sigma \equiv (\bar{D} \wedge C)$ for positive formulas D, C , and $\varphi(X, x)$ an \mathbf{Acc} -operator in (8). Then the following (\bar{R}) is the inference rule for \bar{R}_φ :

$$\frac{\neg\varphi(\bar{D}, a) \vee \neg\varphi(C, a), \sigma(a), \Gamma \quad \bar{\sigma}(t), \Gamma}{\Gamma} (\bar{R})$$

with $(\bar{R}_\varphi(t)) \in \Gamma$ and an eigenvariable a . Note that $\varphi(\bar{D}, a) \wedge \varphi(C, a)$ is logically equivalent to $\varphi(\sigma, a)$.

3. For the theory $Th(\mathbb{N}) + \Pi_1^0(\mathbf{P})\text{-ID}(\mathbf{Acc})$, let $\sigma(u) \equiv (\forall z \sigma_0(z, u))$ for $\sigma_0 \in \Pi_0^0(\mathbf{P})$, and $\varphi(X, x)$ an \mathbf{Acc} -operator $\forall y \{\theta_0(x, y) \rightarrow t_0(x, y) \in X\}$ with an arithmetic bounded formula $\theta_0(x, y)$ and a term $t_0(x, y)$. Let

$$\varphi_\sigma(x) := [\forall w \{\theta_0(x, p_0(w)) \rightarrow \sigma_0(p_1(w), t_1)\}]$$

for $t_1 \equiv (t_0(x, p_0(w)))$ and inverses p_0, p_1 of a surjective pairing function. Note that $\varphi_\sigma(x) \leftrightarrow \varphi(\sigma, x)$ over \mathbf{EA} . Then the following (\bar{R}) is the inference rule for \bar{R}_φ with $(\bar{R}_\varphi(t)) \in \Gamma$ and an eigenvariable a :

$$\frac{\neg\varphi_\sigma(a), \sigma(a), \Gamma \quad \bar{\sigma}(t), \Gamma}{\Gamma} (\bar{R})$$

$$\frac{\Delta, \theta(0) \quad \Delta, \bar{\theta}(a), \theta(a+1) \quad \bar{\theta}(t), \Delta}{\Delta} (ind)$$

where a is the eigenvariable.

1. The *induction formula* $\theta \in \Pi_k^0(\mathbf{P})$ for $Th(\mathbb{N}) + (\Pi_k^0(\mathbf{P}), \mathbf{P} \cup \mathbf{N})\text{-ID}$ and for $Th(\mathbb{N}) + (\Pi_k^0(\mathbf{P}), \mathbf{P} \wedge \mathbf{N})\text{-ID}(\mathbf{Acc})$.
2. $\theta \in \Pi_1^0(\mathbf{P})$ for $Th(\mathbb{N}) + \Pi_1^0(\mathbf{P})\text{-ID}(\mathbf{Acc})$.

Note that we can assume that when $k = 0$, $\theta \in \Pi_0^0(\mathbf{P})$ is either a formula $\exists y < t \forall z < s \bigwedge_i (C_i \rightarrow D_i)$ for some positive formulas C_i, D_i , or its complement $\forall y < t \exists z < s \bigvee_i (C_i \wedge \bar{D}_i)$. When $k > 0$, we can assume that $\theta \in \Pi_k^0(\mathbf{P})$ is of the form $\forall x_k \exists x_{k-1} \dots Q x_1 \theta_0$, where $Q = \forall$ if k is odd, and $Q = \exists$ else, and $\theta_0 \in \Pi_0^0(\mathbf{P})$ is one of formulas $\exists y < t \forall z < s \bigwedge_i (C_i \rightarrow D_i)$ and $\forall y < t \exists z < s \bigvee_i (C_i \wedge \bar{D}_i)$.

A *proof* is defined from these initial sequents and inference rules.

4 Infinitary derivations

In what follows we assume that each formula has no free variable, and a closed term t is identified with the numeral n of the value of t . Furthermore assume that there occurs no bounded quantifiers in any formula. Each bounded quantifier $\exists x < n B(x), \forall x < n B(x)$ is replaced by $\bigvee_{i < n} B(i), \bigwedge_{i < n} B(i)$, resp. In other words, $\bigvee_{i < n} B(i), \bigwedge_{i < n} B(i)$ are formulas for formulas $\{B_i\}_{i < n}$.

4.1 ω -rule

Finitary proof in the sequent calculus for $Th(\mathbb{N}) + (\Pi_k^0(P), P \cup N)$ -ID or for $Th(\mathbb{N}) + (\Pi_k^0(P), P \wedge N)$ -ID(Acc) with $k > 0$ is embedded in infinitary derivations with the ω -rule:

$$\frac{\{\Gamma, A(n) : n \in \mathbb{N}\}}{\Gamma}$$

with $(\forall x A) \in \Gamma$.

Let $I_\varphi^{<\Omega} := R_\varphi$ and $\bar{I}_\varphi^{<\Omega} := \bar{R}_\varphi$. A formula is said to be *positive* [*negative*] if the predicates $\bar{I}_\varphi^{<\Omega}$ [the predicates $I_\varphi^{<\Omega}$] do not occur in it.

Definition 1.5 is modified as follows.

- Definition 4.1** 1. $\Pi_0^0(P) = \Sigma_0^0(P)$ denotes a class of formulas of the form $\bigvee_i \bigwedge_j (C_{ij} \rightarrow D_{ij})$ for some positive formulas C_{ij}, D_{ij} , or its complement $\bigwedge_i \bigvee_j (C_{ij} \wedge \bar{D}_{ij})$.
2. If $A \in \Sigma_k^0(P)$, then $(\forall x A) \in \Pi_{k+1}^0(P)$. If $A \in \Pi_k^0(P)$, then $(\exists x A) \in \Sigma_{k+1}^0(P)$.

Definition 4.2 The *degree* $\text{dg}(A) < \omega$ of the formula $A \in \bigcup_{k < \omega} (\Sigma_k^0(P) \cup \Pi_k^0(P))$ is defined as follows.

1. $\text{dg}(A) = 0$ if no predicate $I_\varphi^{<\Omega}, \bar{I}_\varphi^{<\Omega}$ occurs in A .
2. $\text{dg}(A) = 1 + \min\{k : A \in \Sigma_k^0(P) \cup \Pi_k^0(P)\}$ if one of the predicates $I_\varphi^{<\Omega}, \bar{I}_\varphi^{<\Omega}$ occurs in A .

Definition 4.3 For finite sets Γ of formulas, ordinals $a < \varepsilon_0$ and $d < \omega$,

$$\vdash_d^a \Gamma$$

designates that there exists an infinitary derivation with its ordinal depth $\leq a$ and its cut degree $< d$, where an infinitary derivation is a well founded tree of sequents locally correct with inference rules in the sequent calculus for $Th(\mathbb{N}) + (\Pi_k^0(P), P \cup N)$ -ID or for $Th(\mathbb{N}) + (\Pi_k^0(P), P \wedge N)$ -ID(Acc) except the inference rule (\forall) is replaced by the ω -rule. By an infinitary derivation with cut degree $< d$, we mean a derivation in which $\text{dg}(A) < d$ for every cut formula A .

Let $\Gamma[\vec{a}]$ be a sequent in the language of $\mathcal{L}(\text{ID})$, where $\vec{a} = (a_1, \dots, a_p)$ is a list of free variables occurring in the sequent $\Gamma[\vec{a}]$. For lists $\vec{n} = (n_1, \dots, n_p) \subset \mathbb{N}$ of natural numbers, $\Gamma^*[\vec{n}] = \{A^*[\vec{n}] : A \in \Gamma\}$ and $A^*[\vec{n}]$ denotes the result of replacing every occurrence of the variable a_i in the list \vec{a} by the natural number n_i , and every occurrence of bounded quantifies $\exists x < n B(x)$, $\forall x < n B(x)$ by $\bigvee_{i < n} B^*(i)$, $\bigwedge_{i < n} B^*(i)$, resp.

Lemma 4.4 (Pre-embedding)

1. If $Th(\mathbb{N}) + (\Pi_k^0(\text{P}), \text{P} \cup \text{N})\text{-ID} \vdash \Gamma[\vec{a}]$ for $k > 0$, then there exists $a < \omega_{1+k}$ such that $\vdash_2^a \Gamma^*[\vec{n}]$ for any \vec{n} .
2. If $Th(\mathbb{N}) + (\Pi_k^0(\text{P}), \text{P} \wedge \text{N})\text{-ID}(\text{Acc}) \vdash \Gamma[\vec{a}]$ for $k > 0$, then there exists $a < \omega_{1+k}$ such that $\vdash_2^a \Gamma^*[\vec{n}]$ for any \vec{n} .

Proof. Consider $Th(\mathbb{N}) + (\Pi_k^0(\text{P}), \text{P} \cup \text{N})\text{-ID}$. Let P be a proof of the sequent $\Gamma[\vec{a}]$. By eliminating (*cut*)'s partially we may assume that any cut formula in P is either an arithmetical formula or an atomic formulas $R_\varphi(t)$. We see easily that there exists $c < \omega^2$ such that $\vdash_{2+k}^c \Gamma^*[\vec{n}]$ for any \vec{n} since $\text{dg}(\theta) \leq 1 + k$ for the induction formula $\theta \in \Pi_k^0(\text{P})$. By cut-elimination we obtain $\vdash_2^a \Gamma^*[\vec{n}]$ for $a = 2_k(c) < \omega_{1+k}$ with $2_0(c) = c$ and $2_{m+1}(c) = 2^{2^m(c)}$.

The lemma for $Th(\mathbb{N}) + (\Pi_k^0(\text{P}), \text{P} \wedge \text{N})\text{-ID}(\text{Acc})$ is similarly seen. \square

4.2 Operator controlled derivations

In this subsection let us introduce operator controlled derivations, and prove the remaining halves in Theorem 1.8.

The language $\mathcal{L}^\infty(\text{ID})$ for the next infinitary calculus is obtained from the language $\mathcal{L}(\text{ID})$ by deleting free variables, and adding unary predicate symbols $I_\varphi^{<\alpha}$, $\bar{I}_\varphi^{<\alpha}$ for each positive operator φ and each $\alpha < \vartheta(\Omega^\omega) = \psi(\omega^{\Omega^\omega})$. Recall that $I_\varphi^{<\Omega} \equiv R_\varphi$.

A formula in the language $\mathcal{L}^\infty(\text{ID})$ is said to be *positive* [*negative*] if the predicates $\bar{I}_\varphi^{<\Omega}$ [the predicates $I_\varphi^{<\Omega}$] do not occur in it. In these formulas predicates $I_\varphi^{<\alpha}$, $\bar{I}_\varphi^{<\alpha}$ may occur. Definition 4.1 of classes $\Pi_k^0(\text{P})$, $\Sigma_k^0(\text{P})$ of formulas and Definition 4.2 of the degree $\text{dg}(A)$ of formulas A are modified according to this enlargement of positive/negative formulas. Specifically predicates $I_\varphi^{<\alpha}$, $\bar{I}_\varphi^{<\alpha}$ may occur in formulas A with $\text{dg}(A) = 0$.

A closed term t is identified with the numeral n of the value of t . Γ, Δ, \dots denote finite sets of formulas, *sequents*.

$I_\varphi^{<\alpha}$ is intended to denote the union $\bigcup_{\beta < \alpha} I_\varphi^\beta$ of the β -th stage $I_\varphi^\beta = \{n \in \mathbb{N} : \varphi(I_\varphi^{<\beta}, n)\}$ of the least fixed point $I_\varphi^{<\Omega}$, and $\bar{I}_\varphi^{<\alpha}$ its complement. Thus for any ordinal α and any natural number n

$$I_\varphi^{<\alpha}(n) \leftrightarrow \exists \beta < \alpha \varphi(I_\varphi^{<\beta}, n).$$

For a sequent Γ , $k(\Gamma)$ denotes the set of ordinals $\alpha < \Omega$ such that one of predicates $I_\varphi^{<\alpha}$, $\bar{I}_\varphi^{<\alpha}$ occurs in a formula in the set Γ . $\mathcal{H}[\Theta](X) := \mathcal{H}(\Theta \cup X)$ for sets Θ, X of ordinals and operators $\mathcal{H} : X \mapsto \mathcal{H}(X)$ on the sets X of ordinals.

Definition 4.5 *Inductive definition* of $\mathcal{H} \vdash_d^a \Gamma$.

Let Γ be a sequent, $a < \Omega \cdot \varepsilon_0$ and $d \leq 3$. $\mathcal{H} \vdash_d^a \Gamma$ holds if

$$\{a\} \cup k(\Gamma) \subset \mathcal{H} \quad (11)$$

and one of the followings holds:

(initial) There exists a true arithmetic formula $A \in \mathcal{L}$ in Γ .

(\bigvee) There exist a formula $(\bigvee_{i < n} A_i) \in \Gamma$ with $n > 0$, $a_0 < a$ and $i < n$ such that $\mathcal{H} \vdash_d^{a_0} \Gamma, A_i$.

(\bigwedge) There exist a formula $(\bigwedge_{i < n} A_i) \in \Gamma$ and an $a_0 < a$ such that $\forall i < n \{ \mathcal{H} \vdash_d^{a_0} \Gamma, A_i \}$.

(\exists) There exist a formula $(\exists x A(x)) \in \Gamma$, $n \in \omega$ and $a_0 < a$ such that $\mathcal{H} \vdash_d^{a_0} \Gamma, A(n)$.

(\forall^ω) There exist a formula $(\forall x A(x)) \in \Gamma$ and a sequence of ordinals $\{a_n\}_{n \in \mathbb{N}}$ such that $\forall n (a_n < a)$ and $\forall n \{ \mathcal{H} \vdash_d^{a_n} \Gamma, A(n) \}$.

($I^<$) There exist $\alpha \leq \Omega$, $(I_\varphi^{<\alpha}(n)) \in \Gamma$, $\beta < \alpha$ and $a_0 < a$,

$$\text{if } X \text{ occurs in } \varphi(X, n), \text{ then } \beta < a \quad (12)$$

and $\mathcal{H}' \vdash_d^{a_0} \Gamma, \varphi(I_\varphi^{<\beta}, n)$, where $\mathcal{H}' = \mathcal{H}[\{\beta\}]$ if X occurs in $\varphi(X, n)$, and $\mathcal{H}' = \mathcal{H}$ else.

($\bar{I}^<$) There exist $\alpha \leq \Omega$, $(\bar{I}_\varphi^{<\alpha}(n)) \in \Gamma$ and a sequence of ordinals $\{a_\beta\}_{\beta < \alpha}$ such that $\forall \beta < \alpha (a_\beta < a)$ and $\forall \beta < \alpha \{ \mathcal{H}[\{\beta\}] \vdash_d^{a_\beta} \Gamma, \neg \varphi(I_\varphi^{<\beta}, n) \}$.

(Cl) There exist a formula $(n \in I_\varphi^{<\Omega}) \in \Gamma$ and $a_0 < a$ such that $\mathcal{H} \vdash_d^{a_0} \Gamma, \varphi(I_\varphi^{<\Omega}, n)$.

(cut) There exist a formula C and $a_0 < a$ such that $\text{dg}(C) < d$, $\mathcal{H} \vdash_d^{a_0} \Gamma, \neg C$ and $\mathcal{H} \vdash_d^{a_0} C, \Gamma$.

Lemma 4.6 (Embedding 1)

If $Th(\mathbb{N}) + (\Pi_k^0(\mathbb{P}), \mathbb{P} \cup \mathbb{N})\text{-ID} \vdash \Gamma[\vec{a}]$, there exist an $a < \Omega \cdot \omega_{1+k}$ such that for any $\vec{n} \subset \mathbb{N}$ and any operator $\mathcal{H} = \mathcal{H}_\gamma$ defined in Definition 2.5 with $\gamma \geq 2$, $\mathcal{H} \vdash_2^a \Gamma^*[\vec{n}]$.

Proof. Note that $1 = \psi 0, \omega = \psi 1 \in \mathcal{H}_2$.

First consider the case $k = 0$. Pick a finitary proof of the sequent $\Gamma[\vec{a}]$ in $Th(\mathbb{N}) + (\Pi_0^0(\mathbb{P}), \mathbb{P} \cup \mathbb{N})\text{-ID}$. We show the lemma by induction on the depth of the finitary proof.

Logical initial sequents $\Gamma, \bar{R}_\varphi(t), R_\varphi(t)$ turns to $\mathcal{H} \vdash_0^\Omega \Gamma^*, \bar{I}_\varphi^{<\Omega}(n), I_\varphi^{<\Omega}(n)$, which in turn follows from $\mathcal{H}[\{\alpha\}] \vdash_0^{f(\alpha)} \bar{I}_\varphi^{<\alpha}(n), I_\varphi^{<\alpha}(n)$ for any $\alpha < \Omega$ and any $n \in \mathbb{N}$ with $f(\alpha) = k\alpha$ for a $k < \omega$.

$$\frac{\Delta, \theta(0) \quad \Delta, \bar{\theta}(a), \theta(a+1) \quad \bar{\theta}(t), \Delta}{\Delta} \text{ (ind)}$$

We can assume that the bounded formula $\theta(a)$ is of the form $\exists x < t\forall y < s \bigwedge_{k < m} (\bar{C}_k \wedge D_k)$ for positive formulas C_k, D_k . Then $\theta^*(i) \equiv \bigvee_{j < p} \bigwedge_{k < q} (\bar{C}_{ijk} \wedge D_{ijk})$ with $\text{dg}(\theta^*(i)) = 1$. The inference (*ind*) turns to a series of (*cut*)'s of cut formulas $\theta^*(i)$ for an $a < \Omega \cdot \omega$. From $\mathcal{H} \vdash_2^a \Delta^*, \theta^*(0)$ and $\mathcal{H} \vdash_2^a \Delta^*, \bar{\theta}^*(0), \theta^*(1)$, infer $\mathcal{H} \vdash_2^{a+1} \Delta^*, \theta^*(1)$, and so on. From $\mathcal{H} \vdash_2^{a+n} \Delta^*, \theta^*(n)$ and $\mathcal{H} \vdash_2^a \bar{\theta}^*(n), \Delta^*$, infer $\mathcal{H} \vdash_2^{a+\omega} \Delta^*$.

Next consider

$$\frac{\varphi(R_\varphi, t), \Gamma}{\Gamma} (R)$$

From $\mathcal{H} \vdash_2^a \varphi(I_\varphi^{<\Omega}, n), \Gamma$, infer $\mathcal{H} \vdash_2^{a+1} \Gamma$ by (*Cl*).

Finally consider

$$\frac{\bar{\varphi}(\sigma, a), \sigma(a), \Gamma \quad \bar{\sigma}(t), \Gamma}{\Gamma} (\bar{R})$$

where $\bar{R}_\varphi(t) \in \Gamma$ and $\sigma \in \mathbf{P} \cup \mathbf{N}$.

By the induction hypothesis we have a $c < \Omega \cdot \omega$ such that $\mathcal{H} \vdash_2^c \bar{\varphi}(\sigma^*, n), \sigma^*(n), \Gamma^*$ and $\mathcal{H} \vdash_2^c \bar{\sigma}^*(n), \Gamma^*$. We show by induction on $\alpha < \Omega$ that for $f(\alpha) = c + \omega\alpha + 1$ and any $n \in \mathbb{N}$

$$\mathcal{H}[\{\alpha\}] \vdash_2^{f(\alpha)} \Gamma^*, \bar{I}_\varphi^{<\alpha}(n), \sigma^*(n) \quad (13)$$

By the induction hypothesis we have $\mathcal{H}[\{\beta\}] \vdash_2^{f(\beta)} \Gamma^*, \bar{I}_\varphi^{<\beta}(n), \sigma^*(n)$ for any $\beta < \alpha$ and $n \in \mathbb{N}$. From this we see that $\mathcal{H}[\{\beta\}] \vdash_2^{f(\beta)+m} \Gamma^*, \bar{\varphi}(I^{<\beta}, n), \varphi(\sigma^*, n)$ for some $m < \omega$. ($\bar{I}^{<}$) yields $\mathcal{H}[\{\alpha\}] \vdash_2^{c+\omega\alpha} \Gamma^*, \bar{I}_\varphi^{<\alpha}(n), \varphi(\sigma^*, n)$. A (*cut*) with $\mathcal{H} \vdash_2^c \bar{\varphi}(\sigma^*, n), \sigma^*(n), \Gamma^*$ yields $\mathcal{H}[\{\alpha\}] \vdash_2^{f(\alpha)} \Gamma^*, \bar{I}_\varphi^{<\alpha}(n), \sigma^*(n)$. Here note that $\varphi(\sigma^*, n) \in \mathbf{P} \cup \mathbf{N}$ and

$$\varphi(\sigma^*, n) \in \Pi_0^0(\mathbf{P}) \quad (14)$$

with $\text{dg}(\varphi(\sigma^*, n)) \leq 1$.

From (13) and ($\bar{I}^{<}$) we have $\mathcal{H} \vdash_2^{c+\Omega} \Gamma^*, \bar{I}_\varphi^{<\Omega}(n), \sigma^*(n)$. Finally a (*cut*) with $\text{dg}(\sigma^*(n)) \leq 1$ yields $\mathcal{H} \vdash_2^{c+\Omega+1} \Gamma^*$ for $(\bar{I}_\varphi^{<\Omega}(n)) \in \Gamma^*$ and $c + \Omega + 1 < \Omega \cdot \omega$.

Next consider the case $k > 0$. By Lemma 4.4 there exists $a < \omega_{1+k}$ such that $\vdash_2^a \Gamma^*[\vec{n}]$ for any \vec{n} . We show by induction on a that

$$\vdash_2^a \Gamma \Rightarrow \exists c \leq \Omega \cdot (1 + a) (\mathcal{H}[\mathbf{k}(\Gamma)] \vdash_2^c \Gamma)$$

Consider an ω -rule. Let $(\forall x A(x)) \in \Gamma$ and $\vdash_2^{a_n} \Gamma, A(n)$ with $a_n < a$ for any n . By the induction hypothesis we have $\mathcal{H}[\mathbf{k}(\Gamma)] \vdash_2^{c_n} \Gamma, A(n)$ for $c_n \leq \Omega \cdot (1 + a_n)$. Then by (\forall^ω) we obtain $\mathcal{H}[\mathbf{k}(\Gamma)] \vdash_2^c \Gamma$ for $c = \sup_n \{\Omega \cdot (1 + a_n) + 1\} \leq \Omega \cdot (1 + a)$. \square

Lemma 4.7 (Embedding 2)

If $Th(\mathbb{N}) + (\Pi_k^0(\mathbf{P}), \mathbf{P} \wedge \mathbf{N})\text{-ID}(\text{Acc}) \vdash \Gamma[\vec{a}]$, there exist an $a < \Omega \cdot \omega_{1+k}$ such that for any $\vec{n} \subset \mathbb{N}$ and any operator $\mathcal{H} = \mathcal{H}_\gamma$ defined in Definition 2.5 with $\gamma \geq 2$, $\mathcal{H} \vdash_2^a \Gamma^*[\vec{n}]$.

Proof. This is seen as in Lemma 4.6. Consider

$$\frac{\neg\varphi(\bar{D}, a) \vee \neg\varphi(C, a), \sigma(a), \Gamma \quad \bar{\sigma}(t), \Gamma}{\Gamma} (\bar{R})$$

where $\bar{R}_\varphi(t) \in \Gamma$ and $\sigma \equiv (\bar{D} \wedge C)$ with positive formulas D, C . As in (13) we see for a $c < \Omega \cdot \omega_{1+k}$ and $f(\alpha) = c + \omega\alpha + 1$ that $\mathcal{H}[\{\alpha\}] \vdash_2^{f(\alpha)} \Gamma^*, \bar{I}_\varphi^{<\alpha}(n), \sigma^*(n)$ for any $n \in \mathbb{N}$. Note that the cut formulas $\varphi(\bar{D}^*, n) \wedge \varphi(C^*, n)$ and $\sigma^*(n)$ arise, cf. (14). We have $\varphi(\bar{D}^*, n) \wedge \varphi(C^*, n), \sigma^*(n) \in \Pi_0^0(\mathbf{P})$ and $\text{dg}(\varphi(\bar{D}^*, n) \wedge \varphi(C^*, n), \text{dg}(\sigma^*(n))) \leq 1$. \square

Lemma 4.8 (Embedding 3)

If $\text{Th}(\mathbb{N}) + \Pi_1^0(\mathbf{P})\text{-ID}(\text{Acc}) \vdash \Gamma[\bar{a}]$, there exist an $a < \Omega \cdot \omega$ such that for any $\vec{n} \subset \mathbb{N}$ and any operator $\mathcal{H} = \mathcal{H}_\gamma$ defined in Definition 2.5 with $\gamma \geq 2$, $\mathcal{H} \vdash_3^a \Gamma^[\vec{n}]$.*

Proof. This is seen as in Lemma 4.6 for $k = 0$. Note that the cut formula $\theta^*(i) \in \Pi_1^0(\mathbf{P})$ arises from (ind) with $\text{dg}(\theta^*(i)) \leq 2$. Consider

$$\frac{\bar{\varphi}(\sigma, a), \sigma(a), \Gamma \quad \bar{\sigma}(t), \Gamma}{\Gamma} (\bar{R})$$

where $\bar{R}_\varphi(t) \in \Gamma$ and $\sigma, \varphi_\sigma(a) \in \Pi_1^0(\mathbf{P})$. As in (13) we see for a $c < \Omega \cdot \omega$ and $f(\alpha) = c + \omega\alpha + 1$ that $\mathcal{H}[\{\alpha\}] \vdash_3^{f(\alpha)} \Gamma^*, \bar{I}_\varphi^{<\alpha}(n), \sigma^*(n)$ for any $n \in \mathbb{N}$. For the cut formulas $\varphi_\sigma^*(n)$ and $\sigma^*(n)$, we have $\varphi_\sigma^*(n), \sigma^*(n) \in \Pi_1^0(\mathbf{P})$ and $\text{dg}(\varphi_\sigma^*(n), \text{dg}(\sigma^*(n))) \leq 2$. \square

Lemma 4.9 *For any operator $\mathcal{H} = \mathcal{H}_\gamma$ defined in Definition 2.5 with $\gamma \geq 2$, if $\mathcal{H} \vdash_3^a \Gamma$, then $\mathcal{H} \vdash_2^{\omega^a} \Gamma$.*

In the following lemmas $\Gamma^{(b)} = \{A^{(b)} : A \in \Gamma\}$, while $A^{(b)}$ is obtained from A by replacing some *positive* occurrences of $I_\varphi^{<\Omega}$ by $I_\varphi^{<b}$.

Lemma 4.10 (Bounding)

Let $\mathcal{H} \vdash_1^a \Gamma$ for $a < \Omega$ and $\Gamma \subset \text{Pos}$. Then $\mathcal{H} \vdash_1^a \Gamma^{(b)}$ for $a \leq b \in \mathcal{H} \cap \Omega$.

Proof. This is seen by induction on $a < \Omega$.

Suppose $\mathcal{H} \vdash_1^a \Gamma$ follows from $(I^{<})$ so that $(I_\varphi^{<\Omega}(n)) \in \Gamma$ and $\mathcal{H}[\{\gamma\}] \vdash_1^{a_\gamma} \Gamma, \varphi(I_\varphi^{<\gamma}, n)$ with $\gamma < \Omega$, $a_\gamma < a$ and $\gamma < a$ if X occurs in $\varphi(X, n)$, (12). Then by the induction hypothesis we have $\mathcal{H}[\{\gamma\}] \vdash_1^{a_\gamma} \Gamma^{(b)}, \varphi(I_\varphi^{<\gamma}, n)$. By $(I^{<})$ we obtain $\mathcal{H} \vdash_1^a \Gamma^{(b)}$ for $\gamma < a \leq b$.

Suppose $\mathcal{H} \vdash_1^a \Gamma$ follows from (Cl) with $(I_\varphi^{<\Omega}(n)) \in \Gamma$ and $\mathcal{H} \vdash_1^{a_0} \Gamma, \varphi(I_\varphi^{<\Omega}, n)$ for an $a_0 < a$. By the induction hypothesis we have $\mathcal{H} \vdash_1^{a_0} \Gamma^{(b)}, \varphi(I_\varphi^{<a_0}, n)$ for $a_0 < a \leq b$ and $a_0 \in \mathcal{H}$. An $(I^{<})$ yields $\vdash_1^a \Gamma^{(b)}$. \square

Lemma 4.11 (Collapsing)

Let $\gamma \in \mathcal{H}_\gamma$ and $\Gamma \subset \text{Pos}$. Assume $\mathcal{H}_\gamma \vdash_2^a \Gamma$. Then $\mathcal{H}_{\hat{a}+1} \vdash_1^{\psi \hat{a}} \Gamma$ for $\hat{a} = \gamma + \omega^{\Omega+a}$.

Proof. We show the lemma by induction on a .

First let us verify the condition (11) in $\mathcal{H}_{\hat{a}+1} \vdash_1^{\psi\hat{a}} \Gamma$. From $\gamma < \hat{a} + 1$ we see $k(\Gamma) \subset \mathcal{H}_\gamma \subset \mathcal{H}_{\hat{a}+1}$. Also by $\{\gamma, a\} \subset \mathcal{H}_\gamma$ we have $\hat{a} = \gamma + \omega^{\Omega+a} \in \mathcal{H}_\gamma \subset \mathcal{H}_{\hat{a}}$ and $\psi\hat{a} \in \mathcal{H}_{\hat{a}+1}$. From $\hat{a} \in \mathcal{H}_{\hat{a}}$ we see that if $a_0 < a$ and $\mathcal{H}_\gamma \vdash_2^{a_0} \Gamma_0$, then $\psi\hat{a}_0 < \psi\hat{a}$.

Case 1 ($\bar{I}^<$): For $a_\beta < a$, $\mathcal{H}_\gamma \vdash_2^a \Gamma, \bar{I}_\varphi^{<\alpha}(n)$ follows from $\{\mathcal{H}_\gamma[\{\beta\}] \vdash_2^{a_\beta} \Gamma, \bar{I}_\varphi^{<\alpha}(n), \neg\varphi(I_\varphi^{<\beta}, n) : \beta < \alpha\}$. From $(\bar{I}_\varphi^{<\alpha}(n)) \in \text{Pos}$ we see $\alpha < \Omega$. We claim

$$\forall \beta < \alpha (\beta \in \mathcal{H}_\gamma) \quad (15)$$

Let $\beta < \alpha$. We have $\Omega > \alpha \in k(\bar{I}_\varphi^{<\alpha}(n)) \subset \mathcal{H}_\gamma$, which yields $\beta < \alpha \in \mathcal{H}_\gamma(0) \cap \Omega = \psi\gamma$, and $\beta \in \mathcal{H}_\gamma$. (15) yields $\mathcal{H}_\gamma[\{\beta\}] = \mathcal{H}_\gamma$. By the induction hypothesis we obtain for any $\beta < \alpha$, $\mathcal{H}_{\widehat{a_\beta}+1} \vdash_1^{\psi\widehat{a_\beta}} \Gamma, \bar{I}_\varphi^{<\alpha}(n), \neg\varphi(I_\varphi^{<\beta}, n)$, where $\widehat{a_\beta} = \gamma + \omega^{\Omega+a_\beta}$ and $\psi\widehat{a_\beta} < \psi\hat{a}$. We conclude $\mathcal{H}_{\hat{a}+1} \vdash_1^{\psi\hat{a}} \Gamma, \bar{I}_\varphi^{<\alpha}(n)$ by ($\bar{I}^<$).

Case 2 ($I^<$): For $\beta < \min\{\alpha, a\}$ and $\alpha \leq \Omega$, $\mathcal{H}_\gamma \vdash_2^a \Gamma, I_\varphi^{<\alpha}(n)$ follows from $\mathcal{H}_\gamma \vdash_2^{a_0} \Gamma, I_\varphi^{<\alpha}(n), \varphi(I_\varphi^{<\beta}, n)$. If X occurs in $\varphi(X, n)$, then by (11) we have $\Omega > \beta \in k(\varphi(I_\varphi^{<\beta}, n)) \subset \mathcal{H}_\gamma$, and $\beta < \psi\gamma \leq \psi\hat{a}$. For $\widehat{a_0} = \gamma + \omega^{\Omega+a_0}$ we obtain $\mathcal{H}_{\widehat{a_0}+1} \vdash_1^{\psi\widehat{a_0}} \Gamma, I_\varphi^{<\alpha}(n), \varphi(I_\varphi^{<\beta}, n)$ by the induction hypothesis. ($I^<$) with $\beta < \psi\hat{a}$ for (12) yields $\mathcal{H}_{\hat{a}+1} \vdash_1^{\psi\hat{a}} \Gamma, I_\varphi^{<\alpha}(n)$.

Case 3. $\mathcal{H}_\gamma \vdash_2^a \Gamma$ follows by a (*cut*) from

$$\mathcal{H}_\gamma \vdash_2^{a_0} \Gamma, \bigwedge_i (C_i \vee \bar{D}_i) \quad (16)$$

and

$$\mathcal{H}_\gamma \vdash_2^{a_0} \bigvee_i (\bar{C}_i \wedge D_i), \Gamma \quad (17)$$

for $a_0 < a$ and positive formulas C_i, D_i . For the sake of simplicity let us assume $i = 0, 1$. In the Appendix A the general case is treated.

Let $b_m = \gamma + \omega^{\Omega+a_0} \cdot m$ and $\beta_m = \psi(b_m)$ for $m = 1, 2, \dots, 5$. We have $\beta_m < \beta_{m+1}$. By inversion on (17) we have

$$\mathcal{H}_\gamma \vdash_2^{a_0} D_0, D_1, \Gamma$$

By the induction hypothesis we obtain

$$\mathcal{H}_{b_1+1} \vdash_1^{\beta_1} D_0, D_1, \Gamma$$

From $\beta_1 \in \mathcal{H}_{b_1+1}$ and Bounding lemma 4.10 we obtain

$$\mathcal{H}_{b_1+1} \vdash_1^{\beta_1} D_0^{(\beta_1)}, D_1^{(\beta_1)}, \Gamma \quad (18)$$

For each $i = 0, 1$ we have by inversion on (16)

$$\mathcal{H}_{b_1+1} \vdash_2^{a_0} \Gamma, C_i, \bar{D}_i^{(\beta_1)}$$

From $b_1 + 1 \in \mathcal{H}_{b_1+1}$ and the induction hypothesis we obtain

$$\mathcal{H}_{b_2+1} \vdash_1^{\beta_2} \Gamma, C_i, \bar{D}_i^{(\beta_1)}$$

Once again by Bounding lemma 4.10 we obtain

$$\mathcal{H}_{b_2+1} \vdash_1^{\beta_2} \Gamma, C_i^{(\beta_2)}, \bar{D}_i^{(\beta_1)} \quad (19)$$

Again by inversion on (17) we obtain

$$\mathcal{H}_{b_2+1} \vdash_2^{a_0} \bar{C}_i^{(\beta_2)}, D_{1-i}, \Gamma$$

and the induction hypothesis yields

$$\mathcal{H}_{b_3+1} \vdash_1^{\beta_3} \bar{C}_i^{(\beta_2)}, D_{1-i}^{(\beta_3)}, \Gamma \quad (20)$$

From (16) we see that

$$\mathcal{H}_{b_4+1} \vdash_1^{\beta_4} \Gamma, C_{1-i}^{(\beta_4)}, \bar{D}_{1-i}^{(\beta_3)} \quad (21)$$

and from (17) we see that

$$\mathcal{H}_{b_5+1} \vdash_1^{\beta_5} \Gamma, \bar{C}_i^{(\beta_2)}, \bar{C}_{1-i}^{(\beta_4)} \quad (22)$$

A (*cut*) with (21) and (22) yields

$$\mathcal{H}_{b_5+1} \vdash_1^{\beta_5+1} \Gamma, \bar{C}_i^{(\beta_2)}, \bar{D}_{1-i}^{(\beta_3)} \quad (23)$$

Another (*cut*) with (23) and (20) yields

$$\mathcal{H}_{b_5+1} \vdash_1^{\beta_5+2} \Gamma, \bar{C}_i^{(\beta_2)} \quad (24)$$

One more (*cut*) with (24) and (19) yields for each $i = 0, 1$

$$\mathcal{H}_{b_5+1} \vdash_1^{\beta_5+3} \Gamma, \bar{D}_i^{(\beta_1)} \quad (25)$$

Finally several (*cut*)'s with (25) and (18) yields

$$\mathcal{H}_{b_5+1} \vdash_1^{\beta_5+5} \Gamma$$

Here we have $b_5 = \gamma + \omega^{\Omega+a_0} \cdot 5 < \gamma + \omega^{\Omega+a} = \hat{a}$ and $\beta_5 = \psi(b_5) < \psi(\gamma + \omega^{\Omega+a}) = \psi\hat{a}$.

All other cases are seen easily from the induction hypothesis. \square

(**Proof** of Theorem 2.4). First consider Theorems 2.4.1 and 2.4.2. Let $Th(\mathbb{N}) + (\Pi_k^0(\mathbb{P}), \text{P}\cup\text{N})\text{-ID} \vdash R_\varphi(n)$ for a positive operator $\varphi(X, x)$, or $Th(\mathbb{N}) + (\Pi_k^0(\mathbb{P}), \text{P}\wedge\text{N})\text{-ID}(\text{Acc}) \vdash R_\varphi(n)$ for an **Acc**-operator $\varphi(X, x)$. By Embedding Lemmas 4.6 and 4.7, we have $\mathcal{H}_2 \vdash_2^a I_\varphi^{<\Omega}(n)$ for $0 < a < \Omega \cdot \omega_{1+k}$. Then by Collapsing Lemma 4.11 we obtain $\mathcal{H}_{\omega^{\Omega+a}+1} \vdash_1^{\psi(\omega^{\Omega+a})} I_\varphi^{<\Omega}(n)$, which in turn yields

$\mathcal{H}_{\omega^{\Omega+a}+1} \vdash_1^{\psi(\omega^{\Omega+a})} I_{\varphi}^{<\psi(\omega^{\Omega+a})}(n)$ by Bounding Lemma 4.10. We conclude $|n|_{\varphi} < \psi(\omega^{\Omega+a}) < \psi(\omega^{\Omega \cdot \omega_{1+k}}) = \psi(\Omega^{\omega_{1+k}}) = \vartheta(\Omega \cdot \omega_{1+k})$.

Second consider Theorem 2.4.3. Let $Th(\mathbb{N}) + \Pi_1^0(\text{P})\text{-ID}(\text{Acc}) \vdash R_{\varphi}(n)$ for an Acc-operator $\varphi(X, x)$. By Embedding Lemma 4.8 we have $\mathcal{H}_2 \vdash_3^a I_{\varphi}^{<\Omega}(n)$ for $0 < a < \Omega \cdot \omega$. Lemma 4.9 yields $\mathcal{H}_2 \vdash_2^{\omega^a} I_{\varphi}^{<\Omega}(n)$. Collapsing Lemma 4.11 together with Bounding Lemma 4.10 yields $\mathcal{H}_{\omega^{\Omega+\omega^a}+1} \vdash_1^{\psi(\omega^{\Omega+\omega^a})} I_{\varphi}^{<\psi(\omega^{\Omega+\omega^a})}(n)$. We conclude $|n|_{\varphi} < \psi(\omega^{\Omega+\omega^a}) < \psi(\omega^{\omega^{\Omega \cdot \omega}}) = \psi(\Omega^{\omega^{\Omega \cdot \omega}}) = \vartheta(\Omega^{\omega})$.

A Resolution

In **Case 3** of the proof of Collapsing Lemma 4.11, the following two are given.

$$\mathcal{H}_{\gamma} \vdash_2^{a_0} \Gamma, \bigwedge_i (C_i \vee \bar{D}_i) \quad (16)$$

and

$$\mathcal{H}_{\gamma} \vdash_2^{a_0} \bigvee_i (\bar{C}_i \wedge D_i), \Gamma \quad (17)$$

Let n be the number of conjunctions/disjunctions. From (16) and (17) we obtain the following by inversion. For each $i < n$

$$\mathcal{H}_{\gamma} \vdash_2^{a_0} \Gamma, C_i, \bar{D}_i \quad (26)$$

and for each partition I, J of the index set $\{0, \dots, n-1\}$ ($I \cup J = \{0, \dots, n-1\}$, $I \cap J = \emptyset$)

$$\mathcal{H}_{\gamma} \vdash_2^{a_0} \{\bar{C}_i\}_{i \in I}, \{D_j\}_{j \in J}, \Gamma \quad (27)$$

Let $b_m = \gamma + \omega^{\Omega+a_0} \cdot m$ and $\beta_m = \psi(b_m)$ for positive integers m . From (26) and (27) together with inversion, Bounding Lemma 4.10 and the induction hypothesis, we see the following.

$$\forall m, k \left[m < k \Rightarrow \mathcal{H}_{b_k+1} \vdash_1^{\beta_k} \Gamma, C_i^{(\beta_k)}, \bar{D}_i^{(\beta_m)} \right] \quad (28)$$

and

$$\forall \vec{m}, \vec{k} \left[\max \vec{m} < \min \vec{k} \Rightarrow \mathcal{H}_{b_k+1} \vdash_1^{\beta_k} \{\bar{C}_i^{(\beta_{m_i})}\}_{i \in I}, \{D_j^{(\beta_{k_j})}\}_{j \in J}, \Gamma \right] \quad (29)$$

where (I, J) is a partition, $\vec{m} = (m_i)_{i \in I}$ and $\vec{k} = (k_j)_{j \in J}$ are sequences of positive integers. $k = \max\{\max \vec{k}, 1 + \max \vec{m}\}$, where $\max \vec{k} := 0$ and $\min \vec{k} := \omega$ when \vec{k} is the empty sequence.

We show that there exists a k such that $\mathcal{H}_{b_k+1} \vdash_1^{\beta_k} \Gamma$. Now the above case $n = 2$ indicates us that our task is to find a (ground) resolution refutation and a ‘correct’ assignment of integers to *occurrences* of ‘literals’ $C_i, \bar{C}_i, D_i, \bar{D}_i$ in the refutation.

Since the set Γ of positive formulas plays no rôle, let us suppress it. Second we omit the operator controlled part $\mathcal{H}_\gamma \vdash_d^a$ since we can recover it. Third let us denote $C^{(\beta_m)}$ by $C^{(m)}$. Thus (26) and (27) are written as follows.

$$C_i, \bar{D}_i \quad (26)$$

$$\{\bar{C}_i\}_{i \in I}, \{D_j\}_{j \in J} \quad (27)$$

By the completeness of the ground resolution, there exists a resolution refutation from ‘clauses’ (26), (27) using only the ground resolution rule:

$$\frac{\Gamma, \bar{E} \quad E, \Delta}{\Gamma, \Delta}$$

We need to find an assignment of positive integers to occurrences of literals for which the following hold:

1. If an integer m is assigned to the left cut formula, i.e., the occurrence \bar{E} in the ground resolution rule, then the right cut formula E receives the same number m .
$$\frac{\Gamma, \bar{E}^{(m)} \quad E^{(m)}, \Delta}{\Gamma, \Delta}$$
2. If the two occurrences of the same literal are linked in the ground resolution rule, then these two occurrences receive the same number. This means if a literal F occurs in the left upper sequent Γ and receives m , then the occurrence of F in the lower sequent Γ, Δ receives m . The same for occurrences in Δ and in Γ, Δ .
$$\frac{F^{(m)}, \Gamma, \bar{E} \quad E, \Delta, F^{(m)}}{F^{(m)}, \Gamma, \Delta}$$
3. The assignment to leaf clauses in (26) and (27) has to enjoy the conditions in (28) and (29). This means if we attach numbers k, m to a leaf clause C_i, \bar{D}_i , then $k > m$ has to hold:

$$C_i^{(k)}, \bar{D}_i^{(m)} \Rightarrow k > m \quad (28)$$

Also if we attach numbers \vec{m}, \vec{k} to a leaf clause $\{\bar{C}_i\}_{i \in I}, \{D_j\}_{j \in J}$, then $\max \vec{m} < \min \vec{k}$ has to hold:

$$\{\bar{C}_i^{(m_i)}\}_{i \in I}, \{D_j^{(k_j)}\}_{j \in J} \Rightarrow \max \vec{m} < \min \vec{k} \quad (29)$$

When a ground resolution derivation together with an assignment of positive integers to occurrences of literals enjoys the above three conditions, the derivation together with attached integers is said to be *decorated* derivation.

Proposition A.1 *For each $j < n$, there exist a k_j and a decorated derivation $\pi_j(n)$ of $\bar{D}_j^{(k_j)}$ from clauses (26) and (27).*

Assuming Proposition A.1, the following is a desired decorated refutation.

$$\frac{\begin{array}{c} \vdots \pi_0(n) \\ \bar{D}_0^{(k_0)} \end{array} \quad \dots \quad \begin{array}{c} \vdots \pi_{n-1}(n) \\ \bar{D}_{n-1}^{(k_{n-1})} \end{array} \quad D_0^{(k_0)}, \dots, D_{n-1}^{(k_{n-1})}}{\square}$$

where $D_0^{(k_0)}, \dots, D_{n-1}^{(k_{n-1})}$ is one of clauses in (27), i.e., $I = \emptyset$.

We show Proposition A.1 by induction on $n \geq 1$. The case $n = 1$ is clear: for any $1 \leq k < m$ the following is a decorated derivation of $\bar{D}_0^{(k)}$.

$$\pi_0(1) = \frac{C_0^{(m)}, \bar{D}_0^{(k)} \quad \bar{C}_0^{(m)}}{\bar{D}_0^{(k)}}$$

Assume that Proposition A.1 holds for $n \geq 1$, and let $\pi_j(n)$ be decorated derivation of $\bar{D}_j^{(k_j)}$ for each $j < n$.

Proposition A.2 1. *For each $p < n$ and $m \geq 1$, there exist a $k'_p \geq k_p$ and a decorated derivation $\pi_p(n) * \bar{C}_n^{(m)}$ of $\bar{D}_p^{(k'_p)}, \bar{C}_n^{(m)}$.*
 2. *For each $q < n$, there exist a k and a decorated derivation $\pi_q(n) * D_n^{(m)}$ of $\bar{D}_q^{(k_q)}, D_n^{(k)}$.*

Assuming Proposition A.2 the following with $m > k$ is a desired decorated derivation $\pi_p(n+1)$ of $\bar{D}_p^{(k_p)}$ for $p < n$.

$$\frac{\begin{array}{c} \vdots \pi_0(n) * D_n^{(k)} \\ \bar{D}_0^{(k_0)}, D_n^{(k)} \end{array} \quad \dots \quad \begin{array}{c} \vdots \pi_{n-1}(n) * D_n^{(k)} \\ \bar{D}_{n-1}^{(k_{n-1})}, D_n^{(k)} \end{array} \quad D_0^{(k_0)}, \dots, D_{n-1}^{(k_{n-1})}, D_n^{(k)} \quad \vdots \pi_p(n) * \bar{C}_n^{(m)}}{\frac{C_n^{(m)}, \bar{D}_n^{(k)} \quad D_n^{(m)} \quad \bar{D}_p^{(k'_p)}, \bar{C}_n^{(m)}}{\bar{D}_p^{(k'_p)}}}$$

A decorated derivation $\pi_n(n+1)$ of $\bar{D}_n^{(k'_n)}$ is obtained from $\pi_0(n+1)$ of $\bar{D}_0^{(k'_0)}$ by interchanging indices 0 and n in literals C_0, D_0, C_n, D_n .

Proof of Proposition A.2.

First we show Proposition A.2.1. Let $p < n$ and $m \geq 1$. In the decorated derivation $\pi_p(n)$ of $\bar{D}_p^{(k_p)}$, append the negative decorated literal $\bar{C}_n^{(m)}$ to each leaf clause (27), and then append $\bar{C}_n^{(m)}$ to each clause occurring below a leaf clause (27). The result may violate the condition (29) when $\max \vec{m} < \min \vec{k} \leq m$ in a leaf $\{\bar{C}_i^{(m_i)}\}_{i \in I}, \{D_j^{(k_j)}\}_{j \in J}$. To avoid this, given an m , raise all of the assigned integers to literals in $\pi_p(n)$ by $1 + m$. Each $E_i^{(r)}$ is replaced by $E_i^{(1+m+r)}$. This results in a decorated derivation of $\bar{D}_p^{(k'_p)}$ with $k'_p = 1 + m + k_p$.

Next consider Proposition A.2.2. Let $q < n$. In the decorated derivation $\pi_q(n)$ of $\bar{D}_q^{(k_q)}$, append the positive *undecorated* literal D_n to each leaf clause (27), and then append D_n to each clause occurring below a leaf clause (27). We need to find an integer k such that if we assign the number k to all occurrences of D_n , then the result is a decorated derivation $\pi_q(n) * D_n^{(k)}$ of $\bar{D}_q^{(k_q)}, D_n^{(k)}$. For each leaf clause $\{\bar{C}_i^{(m_i)}\}_{i \in I}, \{D_j^{(k_j)}\}_{j \in J}$ in $\pi_q(n)$, $\max \vec{m} < k$ is required for the condition (29). It suffices for k to be larger than every assigned number m_i to the negative literal \bar{C}_i . Thus $k = 1 + \max(\max \vec{m} : \vec{m})$ suffices, where \vec{m} ranges over every decorated leaf clause $\{\bar{C}_i^{(m_i)}\}_{i \in I}, \{D_j^{(k_j)}\}_{j \in J}$ in $\pi_q(n)$. \square

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