On the structure of groups endowed with a compatible C-relation*

Gabriel Lehéricy
June 29, 2018

Abstract

We use quasi-orders to describe the structure of C-groups. We do this by associating a quasi-order to each compatible C-relation of a group, and then give the structure of such quasi-ordered groups. We also reformulate in terms of quasi-orders some results concerning C-minimal groups given in [6].

Keywords: Ordered group, valuation, C-minimality, quasi-order

MSC: 13A18, 20F60, 06F15, 03C60

Introduction

The notion of C-relation was first introduced by Adeleke and Neumann in [1], where it was used to study certain groups of automorphisms called Jordan groups. In [6], Macpherson and Steinhorn introduced the notion of C-group and C-minimal structure and gave a partial description of C-minimal groups. The notion of C-relation defined in [6] is now called a dense C-relation (see the definition in Section 4 below). A more general notion of C-relation was introduced by Delon in [3] (see Delon's definition of a C-relation in Section 1 below). In Delon's context, o-minimality and strong minimality both become special cases of C-minimality. Until now, all the work concerning C-groups (see for example [6],[7] and [8]) has focused on the study of C-minimal groups. The main motivation behind this paper is to understand the structure of an arbitrary C-group, i.e without any assumption of minimality. We then apply our general theory to the special case of dense C-minimal groups in the last section of the paper.

We already know two examples of C-groups: those whose C-relation comes from an order and those whose C-relation comes from a valuation. The goal of this paper is to

^{*}All results presented here are part of my PhD. In that regard I thank my supervisors Salma Kuhlmann and Françoise Point for the help and support they gave me during its completion. I am particularly grateful to Salma Kuhlmann for taking the time to read and discuss my paper, as her many comments and suggestions greatly improved it. I also thank Françoise Delon for her questions which motivated me to investigate the connections between C-relations and quasi-orders. I would also like to thank the referee for their careful reading of my paper and for their useful comments.

show that these two fundamental examples are the "building blocks" of the class of C-groups, in the sense that any compatible C-relation on a group can be constructed from C-relations induced by valuations and C-relations induced by orders. This is achieved not by working directly with a C-relation but with a quasi-order canonically associated to the C-relation, which we call a C-quasi-order (abbreviated as C-q.o).

Except for Section 2.3, which is not essential to understand the main results of this paper, all results presented here are independent from our work on compatible quasi-orders done in [5]. However, the main ideas behind the method used in the current paper are greatly inspired by what we did in [5], which is why we would like to briefly recall the important results of [5]. We defined a compatible quasi-ordered abelian group (q.o.a.g) as a pair (G, \lesssim) where G is an abelian group and \lesssim a compatible quasi-order, i.e a quasi-order satisfying the following axioms (see Section 1 below for the definition of " \sim "):

$$(Q_1) \ \forall x \ (x \sim 0 \Rightarrow x = 0).$$

$$(Q_2) \ \forall x, y, z \ (x \lesssim y \nsim z \Rightarrow x + z \lesssim y + z).$$

Fixing a compatible q.o.a.g (G, \lesssim) , we distinguished two kinds of elements in G respectively called o-type and v-type elements. The v-type elements are characterized by the fact that they are equivalent to their inverse, whereas o-type elements are not. We showed that the set G^o of o-type elements of G is a subgroup of G and that \lesssim is actually an order on G^o , whereas \lesssim is valuational on the set G^v of v-type elements. We also showed that any compatible quasi-order naturally induces a compatible C-relation. It was however quickly established that some compatible C-relations are not induced by a compatible quasi-order, so that the notion of compatible quasi-order was not appropriate to describe the entire class of compatible C-relations. This is what lead us to develop the notion of C-quasi-order introduced in this paper.

C-quasi-orders are quasi-orders canonically induced by a compatible C-relation on a group. Since there is a bijective correspondence between compatible C-relations and C-quasi-orders, we can study the class of C-groups by studying the class of C-quasi-orders, and this is what we do in this paper. Taking a group G with a C-quasi-order \lesssim , we show that \lesssim is basically a mix of valuational quasi-orders with C-quasi-orders induced by group orderings. The main idea is to distinguish two kinds of elements, respectively called o-type and v-type (analogously to what was done in [5]) and to associate to each $g \in G$ a subset T_g of G called the type-component of G. This set T_g is characterized by two properties: T_g is strictly convex, and if g is v-type (respectively o-type), then the C-quasi-order \lesssim is valuational-like (respectively, order-type-like) on T_g (see Remark 2.10 for the definitions of "valuation-like" and "order-type-like"). Moreover, T_g is maximal with these properties. We can then show that the family of all type-components form a partition of G.

We also draw attention to a counter-intuitive phenomenon, which we call welding, which occurs in certain C-quasi-ordered groups. Welding happens when the group contains an o-type element which is equivalent to a v-type element. This is counter-intuitive, since one would expect the quasi-order to separate elements of different types. If there is

no welding in the group, then the T_g 's are actually convex. However, if there is welding at a point g, then the maximum of T_g is equivalent to the minimum of a T_h , which means that the type-components are only strictly convex. This also means that a C-q.o cannot in general be obtained by lifting C-q.o's of elementary type (i.e C-q.o's induced by a valuation or an ordering). However, we will show (see Theorem 3.39) that any C-q.o can be obtained by first lifting C-q.o's of elementary type and then "welding" (see Proposition 3.38), i.e coarsening the quasi-order in a certain way.

The first section gives preliminaries on C-relations and quasi-orders. In Section 2 we introduce C-quasi-orders. We then give an axiomatization of the class of C-quasi-orders and describe the structure of a C-quasi-order induced by a group ordering. Section 3 is dedicated to the study of an arbitrary C-quasi-ordered group (G, \preceq) . We start by giving five examples of C-q.o's. In Section 3.1 we give some results describing the relation between a C-q.o and the group operation, which will be essential in proving the main results of Section 3.3. Section 3.2 shows that ≾ induces a C-quasi-order on any quotient G/H where H is a strictly convex normal subgroup. In Section 3.3 we define the typecomponent T_q of an element g and describe its properties. We also associate to g two subgroups G^g and G_q of G and show that the C-q.o induced by \lesssim on the quotient G^g/G_q comes from a valuation (respectively, from an ordering) if g is v-type (respectively, if gis o-type). We start Section 3.4 by giving two ways of constructing C-q.o's: lifting and welding. We then give our main result, Theorem 3.39, which states that any C-q.o can be obtained from C-q.o's of elementary type by lifting and welding. Finally, in Section 4, we reinterpret the results on dense C-minimal groups given in [6] in view of our main theorem 3.39. More precisely, we show that the assumption of C-minimality imposes conditions on the type-components.

1 Preliminaries

In this paper, \mathbb{N} denotes the set of natural numbers $\{1, 2, 3, ...\}$ without zero. The set $\mathbb{N} \cup \{0\}$ is denoted by \mathbb{N}_0 . An **ordered group** is a pair (G, \leq) consisting of a group G with a total order \leq satisfying:

$$\forall x, y, z \in G, x \le y \Rightarrow xz \le yz \land zx \le zy. \tag{OG}$$

For any group G and $g, z \in G$, g^z denotes zgz^{-1} . A **valuation** on a group G is a map $v: G \to \Gamma \cup \{\infty\}$ such that:

- (i) Γ is a totally ordered set, and this order is extended to $\Gamma \cup \{\infty\}$ by declaring $\gamma < \infty$ for all $\gamma \in \Gamma$.
- (ii) For any $g \in G$, $v(g) = \infty \Leftrightarrow g = 1$.
- (iii) For any $g, h \in G$, $v(gh^{-1}) \ge \min(v(g), v(h))$.
- (iv) For any $g, h, z \in G$, $v(g) \le v(h) \Leftrightarrow v(g^z) \le v(h^z)$.

If $v: G \to \Gamma \cup \{\infty\}$ is a valuation, then, for any $\gamma \in \Gamma$, G^{γ} and G_{γ} respectively denote $\{g \in G \mid v(g) \geq \gamma\}$ and $\{g \in G \mid v(g) > \gamma\}$.

Remark 1.1

Note that due to the fact that $(g^z)^{z^{-1}} = g$, we can replace " \Leftrightarrow " by " \Rightarrow " in (iv). Also, assuming that (ii) holds, one easily sees that (iii) holds if and only if for any $g, h \in G$, $v(g) = v(g^{-1}) \wedge v(gh) \geq \min(v(g), v(h))$ holds. Moreover, we can easily show that following facts are true for any valued group (G, v):

- (a) For any $g, h \in G$, $v(g) < v(h) \Rightarrow v(g^z) < v(h^z)$ and $v(g) = v(h) \Rightarrow v(g^z) = v(h^z)$ (it follows from (iv)).
- (b) If v(g) < v(h), then v(gh) = v(g) = v(hg).
- (c) For any $\gamma \in \Gamma$, G_{γ} is a normal subgroup of G^{γ} . Note however that it can happen that $v(g) \neq v(g^z)$, and in particular G^{γ} and G_{γ} are not always normal in G. This is showed by Example 3.41.
- (d) Thanks to axiom (iv) of valuations, conjugation by an element $z \in G$ induces an automorphism of Γ defined by $v(g) \mapsto v(g^z)$ (note that this map is onto since $v(g^{z^{-1}})$ is a pre-image of v(g)). If $\gamma = v(g)$, then we denote $v(g^z)$ by γ^z . Conjugation by z also induces a group homomorphism $G^{\gamma} \to G^{\gamma^z}$ and another one from G^{γ}/G_{γ} to $G^{\gamma^z}/G_{\gamma^z}$.

A **C-relation** on a set M (see [3]) is a ternary relation C satisfying the universal closure of the following axioms:

- (C_1) $C(x,y,z) \Rightarrow C(x,z,y)$.
- (C_2) $C(x, y, z) \Rightarrow \neg C(y, x, z)$.
- (C_3) $C(x,y,z) \Rightarrow C(w,y,z) \lor C(x,w,z)$.
- (C_4) $x \neq y \Rightarrow C(x, y, y).$

Note that (C_2) implies $\neg C(x, x, x)$ for all x. If G is a group and C a C-relation on G, then we say that C is **compatible** (with the group operation) if C(x, y, z) implies C(vxu, vyu, vzu) for any $x, y, z, u, v \in G$. A **C-group** is a pair (G, C) consisting of a group G with a compatible C-relation C.

Example 1.2

There are two fundamental examples of C-groups:

- (a) If (G, \leq) is a totally ordered group, then \leq induces a compatible C-relation defined by $C(x, y, z) \Leftrightarrow (y < x \land z < x) \lor (y = z \neq x)$. Such a C-relation is called an **order-type** C-relation.
- (b) If (G, v) is a valued group, then v induces a compatible C-relation by $C(x, y, z) \Leftrightarrow v(yz^{-1}) > v(xz^{-1})$. Such a C-relation is called a **valuational** C-relation.

If (G, C) is a C-group, then we say that C is a C-relation of elementary type if it is either order-type or valuational.

We say that a structure $\mathcal{M}=(M,C,\dots)$ endowed with a C-relation is **C-minimal** if for every $\mathcal{N}=(N,C,\dots)$ such that $\mathcal{N}\equiv\mathcal{M}$ every definable subset of N is quantifier-free definable in the language $\{C\}$. If \mathfrak{T} is a meet-semilattice tree and M a set of maximal branches of \mathfrak{T} , then we can define a C-relation on M as follows: C(x,y,z) holds if and only if the branching point of x and z lies strictly below the branching point of y and z. Conversely, if (M,C) is an arbitrary C-structure, then we can canonically associate a meet-semilattice tree \mathfrak{T} , called the canonical tree of (M,C), so that (M,C) is isomorphic to a set of maximal branches of \mathfrak{T} endowed with the C-relation given above. To study C-minimal structures it might be practical to consider their canonical tree. In [6], the authors described dense C-minimal groups by looking at the action induced by the group on its canonical tree. We will do the same in Section 4.

A quasi-order (q.o) is a binary relation which is reflexive and transitive. If \lesssim is a quasi-order on a set A, then it induces an equivalence relation on A by $a \sim b$ if and only if $a \lesssim b \lesssim a$.

Notation

The symbol \lesssim will always denote a quasi-order, whereas \leq will always denote an order. The symbol \sim will always denote the equivalence relation induced by the quasi-order \lesssim and cl(a) will denote the class of a for this equivalence relation. The notation $a \lesssim b$ means $a \lesssim b \wedge a \nsim b$. If S, T are two subsets of a quasi-ordered set (A, \lesssim) , then the notation $S \lesssim T$ (respectively $S \lesssim T$) means that $s \lesssim t$ (respectively $s \lesssim t$) for any $(s,t) \in S \times T$. If $a \in A$, then we write $S \lesssim a$ instead of $S \lesssim \{a\}$.

The q.o \lesssim induces an order on the quotient A/\sim by $cl(a)\leq cl(b)$ if and only if $a\lesssim b$. We say that a q.o \lesssim is **total** if for every $a,b\in A$, either $a\lesssim b$ or $b\lesssim a$ holds. Note that \lesssim is total if and only if it induces a total order on A/\sim . Unless explicitly stated otherwise, every q.o considered in this paper is total.

A **coarsening** of \lesssim is a q.o \lesssim^* such that $a \lesssim b \Rightarrow a \lesssim^* b$ for any $a, b \in A$. In that case, we also say that \lesssim is a **refinement** of \lesssim^* . The **trivial q.o** on A is the q.o which only has one equivalence class, i.e $a \lesssim b$ for every $a, b \in A$. We usually denote it by $\lesssim_t L$. If $a, c, b \in A$, then we say that c is **between a and b** if $a \lesssim c \lesssim b$ or $b \lesssim c \lesssim a$ holds. If the stronger condition $a \lesssim_t c \lesssim_t b \lor b \lesssim_t c \lesssim_t a$ holds, then we then say that c is **strictly between** a and b. If S is a subset of A, then we define the **maximum** (respectively **minimum**) of S as the set of all elements s of S such that $t \lesssim_t s$ (respectively $s \lesssim_t t$) for every $t \in S$. We denote it by $\max(S)$ (respectively $\min(S)$). Note that the maximum of S is always defined but can be empty. We say that S is:

- An initial segment if $s \in S$ and $a \lesssim s$ implies $a \in S$.
- Convex if $s, t \in S$ and $s \preceq a \preceq t$ implies $a \in S$.
- Strictly convex if $s, t \in S$ and $s \nleq a \nleq t$ implies $a \in S$.

• Left-convex (respectively, right-convex) if $s, t \in S$ and $s \preceq a \not \preceq t$ (respectively $s \not \preceq a \preceq t$) implies $a \in S$.

If S is strictly convex, then we define the **convexity complement** of S as the smallest subset T of $A \setminus S$ such that $S \cup T$ is convex. Note that being left-convex or right-convex implies being strictly convex. We can characterize strict convexity by the following lemma:

Lemma 1.3

For any $S \subseteq A$, S is strictly convex if and only if one of the following conditions holds:

- (i) S is convex. In that case the convexity complement of S is \varnothing .
- (ii) $\min(S) \neq \emptyset$ and $S \cup cl(m)$ is convex for any $m \in \min(S)$. In that case S is right-convex and its convexity complement is $cl(m) \setminus S$.
- (iii) $\max(S) \neq \emptyset$ and $S \cup cl(M)$ is convex for any $M \in \max(S)$. In that case S is left-convex and its convexity complement is $cl(M) \setminus S$.
- (iv) $\min(S)$, $\max(S)$ are both non-empty and $S \cup cl(m) \cup cl(M)$ is convex for any $m \in \min(S)$ and $M \in \max(S)$. In that case the convexity complement of S is $(cl(m) \cup cl(M)) \setminus S$.

Proof. It is easy to check that if one of these conditions holds, then S is strictly convex. Let us prove the converse. Assume that S is not convex. This means that there exists $m, t \in S$ and $a \notin S$ such that $m \lesssim a \lesssim t$. However, since S is strictly convex, we cannot have $m \not \lesssim a \not \lesssim t$. Without loss of generality, we can thus assume that $m \sim a$. Assume that $m \notin \min(S)$ and $m \notin \max(S)$. Then there are $s, M \in S$ with $s \nleq a \sim m \nleq M$. Since S is strictly convex, it follows that $a \in S$, which is a contradiction. Thus, we either have $m \in \min(S)$ or $m \in \max(S)$. If $S \cup cl(m)$ is convex, then we are in case (ii) or (iii). Assume then that it is not convex. Without loss of generality, we may assume $m \in \min(S)$. Take $b \notin S \cup cl(m)$ and $M \in S \cup cl(m)$ with $m \not\preceq b \preceq M$. Since $M \notin cl(m)$, we have $M \in S$. By strict convexity of S, we must have $b \sim M$. If $M \notin \max(S)$, then we would have $m \not \preceq b \not \preceq M'$ for a certain $M' \in S$, which would imply $b \in S$. Therefore, we must have $M \in \max(S)$. Now let us proves that $S \cup cl(m) \cup cl(M)$ is convex, so that we are in case (iv). Let $c \in A$ such that there is $s,t \in S \cup cl(m) \cup cl(M)$ with $s \lesssim c \lesssim t$. Since m, M are respectively minimal and maximal in S, we have $m \lesssim c \lesssim M$. If $c \notin cl(m) \cup cl(M)$, then we even have $m \npreceq c \npreceq M$. By strict convexity of S, this implies $c \in S$. The statements about the convexity complement are clear.

In this paper, a **quasi-ordered group** is just a group endowed with a quasi-order without any further assumption. An element g of a quasi-ordered group (G, \preceq) is called **v-type** if $g \sim g^{-1}$ and **o-type** if $g = 1 \vee g \nsim g^{-1}$. Moreover, g is called **o⁺-type** if $g^{-1} \not\preceq g$ and **o⁻-type** if $g \not\preceq g^{-1}$. Note that 1 is the only element which is both v-type and o-type. If (G, v) is a valued group, then v induces a quasi-order on G via $g \preceq h \Leftrightarrow v(g) \geq v(h)$. If (G, \preceq_G) and (H, \preceq_H) are two quasi-ordered groups and

 $\phi: G \to H$ a group homomorphism, then we say that ϕ is **quasi-order-preserving** if for any $g, h \in G$, $g \lesssim h$ if and only if $\phi(g) \lesssim \phi(h)$. It will be convenient to consider quotients, which is why we need the following lemma from [5]:

Lemma 1.4

Let (G, \lesssim) be a quasi-ordered group and H a normal subgroup of G such that the following condition is satisfied:

$$\forall g_1, g_2 \in G((g_1g_2^{-1} \notin H \land g_1 \precsim g_2) \Rightarrow (\forall h_1, h_2 \in H, g_1h_1 \precsim g_2h_2 \land h_1g_1 \precsim h_2g_2)).$$

Then \lesssim induces a q.o on the quotient group G/H defined by:

$$gH \lesssim hH \Leftrightarrow gh^{-1} \in H \lor (gh^{-1} \notin H \land g \lesssim h).$$

Lemma 1.4 was only proved for abelian groups in [5], but we can easily see that the proof is exactly the same in the general case. The opposite process of quotienting a q.o is lifting, which we will also need. Let G be an abelian group and $v: G \to \Gamma \cup \{\infty\}$ a valuation. Assume that for each $\gamma \in \Gamma$, the quotient G^{γ}/G_{γ} is endowed with a q.o \lesssim_{γ} . We define the **lifting** of $(\lesssim_{\gamma})_{\gamma \in \Gamma}$ to G as the quasi-order defined on G by the following formula:

$$g \lesssim h \Leftrightarrow v(g) > v(h) \lor (v(g) = v(h) = \gamma \land gG_{\gamma} \lesssim_{\gamma} hG_{\gamma}).$$

Let us check that \precsim is indeed a q.o. Reflexivity is clear. Assume $f \precsim g \precsim h$. If v(f) > v(g) or v(g) > v(h), then clearly v(f) > v(h), hence $f \precsim h$. Thus, we can assume $v(f) = v(h) = v(g) = \gamma$. But then $f \precsim h$ follows from the transitivity of \precsim_{γ} . Assume now that $g \precsim h$ does not hold. In particular, we must have $v(g) \le v(h)$. If v(g) < v(h), then $h \precsim g$. If $v(h) = v(g) = \gamma$, then we cannot have $gG_{\gamma} \precsim_{\gamma} hG_{\gamma}$, but since \precsim_{γ} is total it follows that $hG_{\gamma} \precsim_{\gamma} gG_{\gamma}$, hence $h \precsim g$. This shows that \precsim is total.

2 C-quasi-orders

2.1 Definition and axiomatization

As mentioned in the introduction, we want to associate a quasi-order to every compatible C-relation. This idea originates from the following general fact:

Lemma 2.1

Let A be a set (not necessarily a group), C a C-relation on A and take $z \in A$. Then z induces a quasi-order on A by $a \lesssim b \Leftrightarrow \neg C(a, b, z)$.

Proof. Note that $\neg C(z, z, z)$ follows from (C_2) , so we have $z \preceq z$. Let $a \in A$ with $a \neq z$. By (C_4) , we have C(z, a, a). By (C_2) , this implies $\neg C(a, z, a)$, which by (C_1) implies $\neg C(a, a, z)$. This proves that \preceq is reflexive. Transitivity is the contra-position of axiom (C_3) . Totality is given by axiom (C_2) .

In the context of groups, the natural candidate for the parameter z is z=1, hence the following definition:

Definition 2.2

Let G be a group. For any compatible C-relation C on G, we define the **q.o** induced by C as the q.o given by the formula $x \lesssim y \Leftrightarrow \neg C(x,y,1)$. A **C-quasi-order** (C-q.o) on G is the q.o induced by a compatible C-relation on G. A **C-quasi-ordered group** (C-q.o.g) is a pair (G, \lesssim) consisting of a group G endowed with a C-q.o. \lesssim .

Remark 2.3

If \lesssim is the q.o induced by C, then we have $C(x, y, 1) \Leftrightarrow y \lesssim x$.

If \lesssim is a C-q.o induced by the C-relation C, then we say that \lesssim is **order-type** (respectively **valuational**/ **of elementary type**) if C is order-type (respectively valuational/ of elementary type). These definitions make sense thanks to the following proposition:

Proposition 2.4

Let \lesssim be a C-q.o. Then there is only one compatible C-relation inducing it, namely the one given by the formula $C(x,y,z) \Leftrightarrow yz^{-1} \lesssim xz^{-1}$.

Proof. Let
$$C$$
 be a compatible C-relation inducing \lesssim . C is compatible so we have $C(x,y,z)\Leftrightarrow C(xz^{-1},yz^{-1},1)\Leftrightarrow yz^{-1} \lesssim xz^{-1}$.

We now want to axiomatize the class of C-q.o's. Proposition 2.4 states that \lesssim is a C-q.o if and only if the formula $yz^{-1} \lesssim xz^{-1}$ defines a compatible C-relation. We thus want to answer the question: When does this formula define a compatible C-relation?

Lemma 2.5

Let \lesssim be a quasi-order on a group G and define a ternary relation C(x, y, z) by the formula $yz^{-1} \lesssim xz^{-1}$. Then the relation C satisfies (C_2) and (C_3) .

Proof.
$$C$$
 clearly satisfies (C_2) . Assume $C(x,y,z)$ and $\neg C(w,y,z)$ hold. This means $yz^{-1} \not \precsim xz^{-1}$ and $\neg (yz^{-1} \not \precsim wz^{-1})$. Since \precsim is total, this implies $wz^{-1} \precsim yz^{-1} \not \precsim xz^{-1}$, hence $wz^{-1} \not \precsim xz^{-1}$ i.e $C(x,w,z)$. This proves (C_3) .

This gives us an axiomatization of C-q.o's:

Proposition 2.6 (Axiomatization of C-q.o's)

Let G be a group and \lesssim a q.o on G. Then \lesssim is a C-q.o if and only if the following three axioms are satisfied:

$$(CQ_1) \ \forall x \in G \setminus \{1\}, \ 1 \preceq x.$$

$$(CQ_2) \ \forall x, y(x \lesssim y \Leftrightarrow xy^{-1} \lesssim y^{-1}).$$

$$(CQ_3) \ \forall x, y, z \in G, \ x \lesssim y \Leftrightarrow x^z \lesssim y^z.$$

Note that " \Leftrightarrow " can be replaced by " \Rightarrow " in (CQ_2) and (CQ_3) since $(xy^{-1})(y^{-1})^{-1} = x$, $(y^{-1})^{-1} = y$ and $(x^z)^{z^{-1}} = x$.

Proof. Define $C(x,y,z):=yz^{-1} \not \subset xz^{-1}$. By Proposition 2.4, \preceq is a C-q.o if and only if C is a compatible C-relation. Assume C is a compatible C-relation. By (C_4) , we have C(x,1,1) for any $x \neq 1$, which means $1 \not \subset x$. Take $x,y,z \in G$ with $x \preceq y$, which means $\neg C(x,y,1)$. By (C_1) , we then have $\neg C(x,1,y)$. By compatibility, this implies $\neg C(xy^{-1},y^{-1},1)$ i.e $xy^{-1} \preceq y^{-1}$, hence (CQ_2) . By compatibility we also have $\neg C(x^z,y^z,1)$, hence (CQ_3) . Conversely, assume $(CQ_1),(CQ_2),(CQ_3)$ hold. By Lemma 2.5, we already know that C satisfies (C_2) and (C_3) . We first prove that C is compatible. Take $x,y,z,u,v\in G$ with C(x,y,z). We thus have $yz^{-1} \preceq xz^{-1}$. By (CQ_3) , this implies $uyz^{-1}u^{-1} \preceq uxz^{-1}u^{-1}$ i.e $(uyv)(uzv)^{-1} \preceq (uxv)(uzv)^{-1}$, so C(uxv,uyv,uzv). This proves compatibility. Let $x\neq y$ in G. (CQ_1) implies $1 \preceq xy^{-1}$ which means C(x,y,y), so C satisfies (C_4) . Now assume $\neg C(x,y,z)$, i.e $xz^{-1} \preceq yz^{-1}$. By applying (CQ_2) to this inequality, we get $xy^{-1} \preceq zy^{-1}$, hence $\neg C(x,z,y)$, which proves that C satisfies (C_1) .

Remark 2.7

By combining (CQ_3) and (CQ_2) we obtain an improved version of (CQ_2) : $x \preceq y \Rightarrow xy^{-1} \preceq y^{-1} \wedge y^{-1}x \preceq y^{-1}$. We will also often use the contra-position of (CQ_2) : (CQ_2') $y \preceq x \Rightarrow y^{-1} \preceq xy^{-1}$.

2.2 C-q.o's of elementary type

Before investigating the structure of an arbitrary C-q.o.g, we want to understand the structure of C-q.o's of elementary type. Assume first that \lesssim is a valuational C-q.o on G. We then have $g \lesssim h \Leftrightarrow v(g) \geq v(h)$ for any $g,h \in G$. In other words, \lesssim is the C-q.o. induced by v. The order-type case is a bit more complicated. Note first that if we start with an ordered group (G, \leq) , if C is the C-relation induced by \leq and if \lesssim is the corresponding C-q.o, then there is no reason for \leq and \lesssim to be the same. In fact, an order-type C-q.o can never be an order. Let us have a closer look at \lesssim . It is easy to see from the definition of C and \lesssim that $x \lesssim y$ is equivalent to the formula $(x = y \lor y \neq 1) \land (x \leq y \lor x \leq 1)$. From this formula we immediately see that the following holds:

- (i) If $\{x, y\} < 1$, then $x \sim y$.
- (ii) If x < 1 < y, then $x \preceq y$.
- (iii) If 1 < x and 1 < y, then $x \lesssim y \Leftrightarrow x \leq y$.

In other words, \lesssim is given by: $1 \lesssim (G^-, \lesssim_t) \lesssim (G^+, \leq)$, where \lesssim_t is the trivial q.o on G^- . This structure completely characterizes order-type C-q.o's:

Proposition 2.8

Let (G, \preceq) be a C-q.o.g. The C-q.o \preceq is order-type if and only if there exists a subset G^+ of G such that the following holds:

- (i) $G = G^+ \sqcup G^- \sqcup \{1\}$ (disjoint unions), where $G^- := \{g^{-1} \mid g \in G^+\}$.
- (ii) $1 \preceq G^- \preceq G^+$.
- (iii) \lesssim is trivial on G^- and coincides with an order \leq on G^+ .

Proof. We already showed that order-type q.o's satisfy this condition with $G^+ = \{g \mid 1 < g\}$. Let us prove the converse. We denote by C the C-relation corresponding to \lesssim . Define an order on G^- as follows: $h \leq g \Leftrightarrow h^{-1} \geq g^{-1}$. Now extend \leq on all of G by $G^- < 1 < G^+$. Note that $x \in G^-$ if and only if $x \lesssim x^{-1}$. By (CQ_3) , it then follows that G^- and G^+ are stable under conjugation. We first want to show that (G, \leq) is an ordered group. This will be a consequence of the following claim:

Claim: For any $x, y \in G$, $xy^{-1} < 1 \Leftrightarrow x < y$.

<u>Proof:</u> Note that it is sufficient to prove \Rightarrow . Indeed, assume \Rightarrow has been proved, and assume $\neg(xy^{-1} < 1)$. This implies $yx^{-1} \le 1$, which by \Rightarrow implies $y < x \lor y = x$, so $\neg(x < y)$.

Assume then that $xy^{-1} < 1$. The case $x = 1 \lor y = 1$ is clear, so assume $y \neq 1 \land x \neq 1$. Since $x \neq y$, it is sufficient to prove $x \leq y$. If $y \not\lesssim x$, then by (CQ_2') we have $y^{-1} \not\lesssim xy^{-1}$. Since \lesssim is trivial on G^- , this implies $xy^{-1} \in G^+$, which contradicts $xy^{-1} < 1$. Thus, we have $x \lesssim y$. We consider two cases:

Case 1: $y \in G^+$. It follows immediately from $x \lesssim y$ that $x \leq y$.

Case 2: Assume $y \in G^-$. It follows from $1 \neq x \lesssim y$ that $x \in G^-$. Note that we have $yx^{-1} \in G^+$. By conjugation, this implies $x^{-1}y \in G^+$, hence $y \not \lesssim x^{-1}y$. By (CQ_2') , this implies $y^{-1} \lesssim x^{-1}$, hence $x \leq y$. This proves the claim.

Now let us show that (G, \leq) is an ordered group. Assume x < y and take $z \in G$. By the claim, we have $xy^{-1} < 1$, hence $xz(yz)^{-1} < 1$, hence xz < yz. By conjugation, we also have $y^{-1}x < 1$, so $(zy)^{-1}(zx) < 1$. By the claim, this means zx < zy. This proves that (G, \leq) is an ordered group.

Denote by C' the C-relation induced by \leq . We show that C'=C. Assume C(x,y,z) holds. The case $x \neq y = z$ is obvious, so assume $y \neq z$. We have $yz^{-1} \not \supset xz^{-1}$ and $zy^{-1} \not \supset xy^{-1}$. We either have $zy^{-1} \in G^+$ or $yz^{-1} \in G^+$. Without loss of generality, we can assume that the former holds (the other case is done similarly). We then have $zy^{-1}, xy^{-1} \in G^+$ with $zy^{-1} \not \supset xy^{-1}$, which means $1 < zy^{-1} < xy^{-1}$. It follows that y < z < x, hence C'(x,y,z). Conversely, assume C'(x,y,z) holds. Since C' is compatible, this implies $C'(xz^{-1}, yz^{-1}, 1)$, which means $1 < xz^{-1}$ and $yz^{-1} < xz^{-1}$. We thus have $xz^{-1} \in G^+$ and $yz^{-1} < xz^{-1}$, which means $yz^{-1} \not \supset xz^{-1}$, hence C(x,y,z)

All of this shows us how to construct \lesssim from \leq and vice-versa. More precisely, we see that \leq and \lesssim define the same sets:

Proposition 2.9

Let (G, \leq) be an ordered group and \lesssim the corresponding C-q.o. The relation \lesssim is quantifier-free definable in the language $\{1, ., ^{-1}, \leq\}$ and \leq is quantifier-free definable in $\{1, ., ^{-1}, \lesssim\}$.

Proof. As already mentioned, $x \lesssim y$ is equivalent to $(x = y \lor y \neq 1) \land (x \leq y \lor x \leq 1)$. Conversely, $x \leq y$ is equivalent to the formula:

 $(x, y \in G^+ \land x \preceq y) \lor (x, y \in G^- \land y^{-1} \preceq x^{-1}) \lor (x \in G^- \land y \in G^+ \cup \{1\}) \lor (x = 1 \land y \in G^+),$ and G^+ and G^- are respectively defined by the formulas $x^{-1} \preceq x$ and $x \preceq x^{-1}$.

Remark 2.10

We just saw what C-q.o groups of elementary type look like. In Section 3, our work will consist in showing that any C-q.o group is in some sense a "mix" of the elementary ones. This means that we will identify parts of the group where the q.o is "order-type-like" and parts where it is "valuational-like". Intuitively, we want to say that a q.o is "like" an elementary-type q.o on a subset T of G if it shares the important properties of this elementary q.o. We will say that the q.o \preceq is **valuational-like on** T if $gh \preceq \max(\{g,h\})$ for any $g,h \in T$. We will say that \preceq is **order-type-like on** T if T can be partitioned into two subsets, T^- and T^+ , such that the following holds: $T^- = \{g^{-1} \mid g \in T^+\}$, $T^- \preceq T^+$ and \preceq is trivial on T^- (i.e $g \sim h$ for all $g,h \in T^-$). We say that \preceq is **elementary-type-like on** T if it is either valuational-like or order-type-like on T.

2.3 Connection with compatible q.o's

We now want to establish the connection between the notion of C-q.o developed in this paper and the work done in [5] which we mentioned in the introduction. As we showed in [5], we can associate a compatible C-relation to any compatible quasi-order defined on an abelian group. However, this does not mean that compatible q.o's are C-q.o's. In fact, we have the following:

Proposition 2.11

Let (G, \preceq) be a compatible quasi-ordered abelian group. Then \preceq is a C-q.o if and only if every element of G is v-type.

Proof. By Proposition 2.13 of [5], we know that the set G^o of all o-type elements of G is a subgroup of G and that (G^o, \preceq) is an ordered abelian group. If G^o is non-trivial, then G contains negative elements, which contradicts axiom (CQ_1) , so \preceq cannot be a C-q.o. Thus, G^o must be trivial, which means that every element of G is v-type.

Now let (G, \preceq) be a compatible quasi-ordered abelian group. Proposition 2.11 states that, if the subgroup G^o of o-type elements is non-trivial, then \preceq is not a C-q.o. However, we can transform \preceq into a corresponding C-q.o \preceq^* . We know that \preceq coincides with an order \leq on G^o and is valuational on G^v . Now define \preceq^* as follows: on G^o , \preceq^* is the order-type C-q.o corresponding to \leq . On G^v , \preceq^* coincides with \preceq . Finally, declare $G^o \preceq^* G^v$. Then \preceq^* is a C-q.o. Now denote by C^* the C-relation corresponding to the C-q.o \preceq^* and denote by C the C-relation induced by the compatible q.o \preceq as defined in Proposition 4.1 of [5]. We recall that in [5], we defined the C-relation induced by \preceq as a sort of "mix" between the definition of a C-relation induced by an order and the C-relation induced by a valuation. More precisely, Proposition 4.1 of [5] defines C as follows: the relation C(x, y, z) holds if and only if the following formula is true:

 $(x \neq y = z) \lor (xz^{-1} \in G^v \land (yz^{-1} \precsim xz^{-1})) \lor (yz^{-1}, xz^{-1} \in G^o \land (1 \precsim xy^{-1} \land 1 \precsim xz^{-1})).$ By distinguishing the cases $xz^{-1} \in G^v$ and $xz^{-1} \notin G^v$, one can show that C(x,y,z) holds if and only if $yz^{-1} \not \precsim^* xz^{-1}$. It then follows that $C = C^*$.

3 Structure of C-q.o.g's

In this section we describe the structure of an arbitrary C-q.o.g (G, \lesssim) . We start by giving four different examples of C-q.o's. All of them are obtained by lifting (with the notion of lifting defined after Lemma 1.4). It is possible to directly check that each of them satisfy the axioms of C-q.o's, but this will actually be a consequence of Propositions 3.36, 3.37 and 3.38. Examples (a),(c) and (d) are obtained by direct application of 3.36, and example (e) is proved from example (d) with Proposition 3.37. Finally, to prove example (b), apply Proposition 3.38 on the C-q.o group (G, \lesssim) from example (a) with g := (-1,0).

Examples 3.1

Set $G := \mathbb{Z}^2$. We let \lesssim_o denote the C-q.o induced by the usual order of \mathbb{Z} (which is characterized in Proposition 2.8) and \lesssim_v the C-q.o induced by the trivial valuation on

$$\mathbb{Z}$$
. Define the valuation v_G on G by $v_G(a,b) = \begin{cases} 1 & \text{if } a \neq 0. \\ 2 & \text{if } a = 0 \neq b. \\ \infty & \text{if } a = b = 0. \end{cases}$

We have $G^1/G_1 \cong G^2/G_2 \cong \mathbb{Z}$. We define three different C-q.o's on G:

- (a) Choose $\lesssim_1:=\lesssim_o$ and $\lesssim_2:=\lesssim_v$. The lifting of (\lesssim_1,\lesssim_2) to G is the C-q.o given by : $(0,0) \lesssim (\{0\} \times (\mathbb{Z} \setminus \{0\}), \lesssim_t) \lesssim (-\mathbb{N} \times \mathbb{Z}, \lesssim_t) \lesssim (\mathbb{N} \times \mathbb{Z}, \lesssim),$ where \lesssim_t always denotes the trivial q.o and \lesssim is defined on $\mathbb{N} \times \mathbb{Z}$ as follows: $(a,b) \lesssim (c,d) \Leftrightarrow a \leq c$. In this example, \lesssim is valuational on $\{0\} \times \mathbb{Z}$ and order-type-like on $(\mathbb{Z} \setminus \{0\} \times \mathbb{Z})$. The set of v-type elements is $\{0\} \times \mathbb{Z}$, the set of o^- -type elements is $-\mathbb{N} \times \mathbb{Z}$ and the set of o^+ -type elements is $\mathbb{N} \times \mathbb{Z}$.
- (b) Coarsen the C-q.o of the previous example by declaring that $(\{0\} \times (\mathbb{Z} \setminus \{0\}), \preceq_t) \sim (-\mathbb{N} \times \mathbb{Z}, \preceq_t)$. This new C-q.o is now given by: $(0,0) \preceq ((-\mathbb{N}_0 \times \mathbb{Z}) \setminus \{(0,0)\}, \preceq_t) \preceq (\mathbb{N} \times \mathbb{Z}, \preceq)$. All elements of G in this example have the same type as in (a).
- (c) Define $\preceq_1 = \preceq_2 = \preceq_o$. The lifting of (\preceq_1, \preceq_2) to G is the C-q.o:

$$(0,0) \not \preceq (\{0\} \times -\mathbb{N}, \preceq_t) \not \preceq (\{0\} \times \mathbb{N}, \leq) \not \preceq (-\mathbb{N} \times \mathbb{Z}, \preceq_t) \not \preceq (\mathbb{N} \times \mathbb{Z}, \preceq),$$

where \leq is the natural order of \mathbb{Z} and \lesssim is defined on $\mathbb{N} \times \mathbb{Z}$ as follows: $(a,b) \lesssim (c,d) \Leftrightarrow a \leq c$. Here \lesssim is order-type-like on $\{0\} \times \mathbb{Z}$ and on $(\mathbb{Z} \setminus \{0\} \times \mathbb{Z})$. The set of o^- -type elements is $\{0\} \times -\mathbb{N} \cup -\mathbb{N} \times \mathbb{Z}$, the set of o^+ -type elements is $\{0\} \times \mathbb{N} \cup \mathbb{N} \times \mathbb{Z}$, and (0,0) is the only v-type element.

(d) Let \lesssim be the C-q.o of example (a) on G.

Set $H:=\coprod_{\mathbb{Z}}G=\{(g_n)_{n\in\mathbb{Z}}\in G^{\mathbb{Z}}\mid \text{ the support of }(g_n)_{n\in\mathbb{Z}} \text{ is finite}\}\ (H \text{ is thus the Hahn sum of }\mathbb{Z}\text{-many copies of }G).$ We denote the elements of H as formal sums $h=\sum_{n\in\mathbb{Z}}g_n\tau_n$. H can be endowed with a valuation $w_H:H\to\mathbb{Z}\cup\{\infty\}$, where $w_H(h)$ is defined as the minimum of the support of h. In this context, we have $H^{\gamma}=\{\sum_{n\in\mathbb{Z}}g_n\tau_n\mid \forall n<\gamma,g_n=(0,0)\},\ H_{\gamma}=\{\sum_{n\in\mathbb{Z}}g_n\tau_n\mid \forall n\leq\gamma,g_n=(0,0)\}$ and $H^{\gamma}/H_{\gamma}\cong G$ for every $\gamma\in\mathbb{Z}$. We endow H with the lifting \precsim_H of $(\precsim)_{\gamma\in\mathbb{Z}}$. Here the C-q.o alternates infinitely many times between order-type-like parts and valuational-like parts. More precisely, for any $h=\sum_{n\in\mathbb{Z}}g_n\tau_n\in H$ with $\gamma:=w_H(h)$, then h is v-type if and only if $g_{\gamma}\in\{0\}\times\mathbb{Z}$, h is σ^- -type if and only if $g_{\gamma}\in-\mathbb{N}\times\mathbb{Z}$ and h is σ^+ -type if and only if $g_{\gamma}\in\mathbb{N}\times\mathbb{Z}$. For any $\gamma\in\mathbb{Z}$, \precsim_H is valuational-like on $\{h=\sum_{n\in\mathbb{Z}}g_n\tau_n\in H\mid w_H(h)=\gamma,g_{\gamma}\in\{0\}\times\mathbb{Z}\}$ and is order-type-like on $\{h=\sum_{n\in\mathbb{Z}}g_n\tau_n\in H\mid w_H(h)=\gamma,g_{\gamma}\in(\mathbb{Z}\setminus\{0\})\times\mathbb{Z}\}$.

We can also give a non-abelian example:

(e) Let (H, \preceq_H) be as in the previous example. For any $k \in \mathbb{Z}$, let α_k be the k-th shift on H (i.e $\alpha_k(\sum_{n \in \mathbb{Z}} g_n \tau_n) = \sum_{n \in \mathbb{Z}} g_{n-k} \tau_n$). This is a group automorphism of H. Set $F := \mathbb{Z} \ltimes_{\alpha} H$ (\ltimes_{α} denotes the semi-direct product) and define \preceq_F by: $(k, h_1) \preceq_F (l, h_2) \Leftrightarrow (k \preceq_v l) \land (l \neq 0 \lor (l = 0 \land h_1 \preceq_H h_2))$. Here the elements of H have the same type as in (d). Elements of the form (l, h) with $l \neq 0$ are v-type.

We see on each of these examples that G can be partitioned into strictly convex subsets on each of which \leq is elementary-type-like. We want to show that this is true for an arbitrary C-q.o.g. As the terminology and Examples 3.1 suggest, it will turn out that \leq is valuational-like on the set of v-type elements and order-type-like around o-type elements. Note that Example (b) seems counter-intuitive. Indeed, we would expect the C-q.o to separate o-type elements from v-type elements, but we see that $(0,1) \sim (-1,1)$. This means that the C-q.o does not distinguish between the v-type element (0,1) and the o-type element (-1,1). This phenomenon is what we call "welding". We say that G is **welded** at h, or that h is a **welding point** of G if there exists an element g such that g and h are of different type and $g \sim h$. We will see that the existence of welding in certain groups makes things technically slightly more difficult but does not fundamentally change the structure of a C-q.o.g.

The following propositions show the relevance of distinguishing o-type elements from v-type elements and justify our terminology:

Proposition 3.2

The C-q.o \lesssim is valuational if and only if every element of G is v-type.

Proof. If \lesssim is valuational, then every element must obviously be v-type. Conversely, assume that every element is equivalent to its inverse. We only have to check that the ultrametric inequality is satisfied. Let $g, h \in G$. If $h \lesssim g$, then by (CQ_2) we have

 $gh^{-1} \sim hg^{-1} \lesssim g^{-1} \sim g$. If $g \lesssim h$, then we have $gh^{-1} \lesssim h^{-1} \sim h$. In any case, we have $gh^{-1} \lesssim \max(g,h)$.

Proposition 3.3

The C-q.o \lesssim is order-type if and only if every element of G is o-type and G contains exactly one equivalence class of o^- -type elements.

Proof. Both directions are proved with 2.8. If \preceq is order-type, then we see from 2.8 that every element is o-type and that all the o^- -type elements are contained in one class. For the converse, set $G^+ := \{o^+$ -type elements $\}$ and $G^- := \{o^-$ -type elements $\}$. By assumption, \preceq is trivial on G^- . We obviously have $G = \{1\} \sqcup G^+ \sqcup G^-$. Let $g \in G^+$. By definition of o^+ -type, we have $g^{-1} \preceq g$. By assumption, the elements of G^- are all equivalent to one another, hence $G^- \preceq g$. This shows $1 \preceq G^- \preceq G^+$. We just have to check that \preceq is an order on G^+ . Let $g, h \in G^+$ with $g \sim h$. By $(CQ_2), g \preceq h \preceq g$ implies $gh^{-1} \preceq h^{-1}$ and $hg^{-1} \preceq g^{-1}$, so we have $\{gh^{-1}, hg^{-1}\} \preceq G^-$. This is only possible if $gh^{-1} = 1$ i.e g = h.

Remark 3.4

As example 3.1(c) above shows, the fact that every element is o-type is not sufficient to insure that \leq is order-type.

3.1 Some relations between \lesssim and the group operation

Here we investigate the relation between multiplication and \lesssim . More precisely, we want to understand how the equivalence class of the product of two elements relates to the equivalence class of each factor. These results will play a fundamental role in the proofs of Section 3.3. We fix a C-q.o.g (G, \lesssim) . We first note that in many cases the order of the factors will not matter:

Lemma 3.5

For any $g, h \in G$, $hg \sim g \Leftrightarrow gh \sim g$.

Proof. It is a direct consequence of (CQ_3) : take the inequalities $hg \lesssim g \lesssim hg$ and conjugate by g.

Lemma 3.6

Let $g, h \in G$. The following holds:

- (i) If $h \preceq g^{-1}$, then $g \sim hg \sim gh$.
- (ii) Assume that $h \not \lesssim \{g^{-1}, g\}$. Then $h^{-1} \not \lesssim \{g, g^{-1}\}$ and we have $gh \sim g \sim gh^{-1}$ and $g^{-1} \sim hg^{-1} \sim h^{-1}g^{-1}$.
- (iii) If $\{h, h^{-1}\} \lesssim g^{-1} \lesssim g$, then $g \sim gh \sim gh^{-1} \sim hg \sim h^{-1}g$ and $g^{-1} \sim g^{-1}h^{-1} \sim g^{-1}h \sim h^{-1}g^{-1} \sim hg^{-1}$.

Proof. (i) By (CQ_2) , $h \lesssim g^{-1} \Rightarrow hg \lesssim g$. By (CQ_2') , $h \gtrsim g^{-1} \Rightarrow h^{-1} \gtrsim g^{-1}h^{-1}$. By (CQ_2) , $h^{-1} \lesssim g^{-1}h^{-1} \Rightarrow g \lesssim hg$, hence $g \sim hg$.

- (ii) By (i), $g \sim gh$ and $g^{-1} \sim g^{-1}h$. By (CQ_2') , $h \not \gtrsim gh \Rightarrow h^{-1} \not \gtrsim g$ and $h \not \gtrsim g^{-1}h \Rightarrow h^{-1} \not \gtrsim g^{-1}$. In particular, h^{-1} satisfies $h^{-1} \not \gtrsim \{g, g^{-1}\}$, so we get $g \sim gh^{-1}$ and $g^{-1} \sim g^{-1}h^{-1}$, hence the claim.
- (iii) By (i), $\{h, h^{-1}\} \lesssim g$ implies $g^{-1} \sim g^{-1}h^{-1} \sim g^{-1}h$. By (CQ_2) , $h \lesssim g^{-1} \Rightarrow hg \lesssim g$ and $h^{-1} \lesssim g^{-1}h^{-1} \Rightarrow g \lesssim hg$, hence $g \sim hg$. Analogously, $g \sim gh^{-1}$. The rest follows from Lemma 3.5.

We can summarize these results in the following proposition:

Proposition 3.7

Assume
$$g$$
 is v-type. If $h \not \lesssim g$, then $h^{-1} \not \lesssim g$ and we have $hg \sim h^{-1}g \sim gh^{-1} \sim gh \sim g \sim g^{-1} \sim g^{-1}h \sim g^{-1}h^{-1} \sim h^{-1}g^{-1} \sim hg^{-1}$. Assume g is o^+ -type. If $\{h, h^{-1}\} \not \lesssim g^{-1}$, then we have $g^{-1}h^{-1} \sim g^{-1}h \sim g^{-1} \not \lesssim g \sim gh \sim gh^{-1} \sim hg \sim h^{-1}g$.

We now want to find an analog of axiom (Q_2) of compatible q.o's (see [5]).

Lemma 3.8

If $f \preceq g$ and $g^{-1} \preceq h^{-1}g^{-1}$, then $fh \preceq gh$ and $hf \preceq hg$.

Proof. By (CQ_2) , $f \lesssim g$ implies $fg^{-1} \lesssim g^{-1}$. By assumption, this implies $fg^{-1} \lesssim h^{-1}g^{-1}$. By (CQ_2) again, this implies $fh \lesssim gh$. (CQ_3) then implies $hf \lesssim hg$.

Proposition 3.9

Let $f,g \in G$ such that $f \lesssim g$ and assume that either $g \nsim h^{-1}$ or $\{h,h^{-1}\} \lesssim g \lesssim g^{-1}$ holds. Then we have $fh \lesssim gh$ and $hf \lesssim hg$.

Proof. If $h^{-1} \not \supset g$, then by 3.6 we have $g^{-1} \sim h^{-1}g^{-1}$. If $g \not \supset h^{-1}$, then (CQ_2') implies $g^{-1} \not \supset h^{-1}g^{-1}$. In both cases, we have $g^{-1} \not \subset g^{-1}h^{-1}$, so we can apply the previous lemma. For the second claim, we use 3.7 to get $g^{-1} \sim h^{-1}g^{-1}$.

Remark 3.10

We just showed that C-q.o.g's satisfy the formula: $\forall g, h, f \in G, f \lesssim g \nsim h^{-1} \Rightarrow fh \lesssim gh$. This formula is very similar to axiom (Q_2) of compatible q.o's and seems to be more practical to deal with than axiom (CQ_2) of C-q.o's. However, we don't know if we can actually replace (CQ_2) by this formula in our axiomatization of C-q.o's.

3.2 Quotients

In the theory of ordered abelian groups there is a classical notion of the order induced on a quotient G/H where H is a normal convex subgroup of G. In [5], we showed that the same holds for compatible quasi-ordered abelian groups. Here we show a similar

result for C-q.o.g's. However, because of the occasional occurrence of welding, it won't be sufficient for us to only consider convex subgroups, so we will show that a C-q.o \lesssim on G canonically induces a C-q.o on the quotient group G/H if H is a normal strictly convex subgroup of G. This will be useful to describe the structure of the C-q.o on G. Note first that thanks to axiom (CQ_1) any convex subgroup of G is an initial segment. This also means that any non-convex strictly convex subgroup of G is in case (iii) of Lemma 1.3.

Proposition 3.11

Let H be a strictly convex normal subgroup of G. Then \lesssim induces a C-q.o on G/H by the formula: $gH \lesssim hH \Leftrightarrow (g \in H) \lor (h \notin H \land g \lesssim h)$.

The proof of Proposition 3.11 is done in three parts. We first show the case where H is convex:

Proposition 3.12

Let (G, \preceq) be a C-q.o.g and H a convex normal subgroup of G. Then \preceq induces a C-q.o on G/H given by the formula: $gH \preceq hH \Leftrightarrow (g \in H) \lor (h \notin H \land g \preceq h)$.

Proof. We apply Lemma 1.4. Let $g_1, g_2 \in G$ with $g_1 \lesssim g_2$ and $g_1g_2^{-1} \notin H$ and let $h_1, h_2 \in H$. We want to show that $g_1h_1 \lesssim g_2h_2 \wedge h_1g_1 \lesssim h_2g_2$ holds. If $g_1 \in H$, then $g_2 \notin H$ and we have $h_1g_1, g_1h_1 \in H$ and $h_2g_2, g_2h_2 \notin H$. By convexity of H, this implies $g_1h_1 \lesssim g_2h_2 \wedge h_1g_1 \lesssim h_2g_2$. Now assume $g_1 \notin H$. By convexity of H, this implies $g_2 \notin H$. By convexity of H, we have $\{h_1, h_2\} \lesssim \{g_1, g_2, g_1^{-1}, g_2^{-1}\}$. By Lemma 3.6(iii), this implies $h_1g_1 \sim g_1h_1 \sim g_1 \lesssim g_2 \sim g_2h_2 \sim h_2g_2$. This proves that the assumption of Lemma 1.4 is verified, so \lesssim induces a q.o on G/H by the formula $gH \lesssim hH \Leftrightarrow gh^{-1} \in H \vee (gh^{-1} \notin H \wedge g \lesssim h)$. We now want to show that this is equivalent to $(g \in H) \vee (h \notin H \land g \lesssim h)$. Assume $gH \lesssim hH$ and $g \notin H$. If $h \not \gtrsim g$, then by Lemma 3.6 we have $g^{-1} \sim hg^{-1}$. This implies $hg^{-1} \notin H$ and $h \nleq g$, which contradicts the assumption. Thus, $g \lesssim h$. Since $g \notin H$, this implies $h \notin H$, so $h \notin H \land g \lesssim h$ holds. Conversely, assume $(g \in H) \vee (h \notin H \land g \lesssim h)$. If $g \notin H$, then $g \lesssim h$, which implies $gH \lesssim hH$. If $g \in H$, then either $h \in H$, in which case $gh^{-1} \in H$, or $h \notin H$, which implies $gh^{-1} \notin H \land g \lesssim h$ by convexity of H. In both cases, we have $gH \lesssim hH$. Now we can show that the induced q.o is a C-q.o. For (CQ_1) : If $g \notin H$ and $h \in H$, then by convexity of H we have $h \nleq g$, so $1 = hH \nleq gH$. Now let us prove $(CQ_2) \land (CQ_3)$. Assume $gH \lesssim hH$. If $gh^{-1} \in H$, then by (CQ_1) we have $gh^{-1}H \lesssim h^{-1}H$. Since H is normal, we also have $(gh^{-1})^z \in H$, hence $g^zH \lesssim h^zH$. If $gh^{-1} \notin H$, then $g \lesssim h$, which implies $gh^{-1} \lesssim h^{-1}$ and $g^z \lesssim h^z$. This implies $gh^{-1}H \lesssim h^{-1}H$ and $g^zH \lesssim h^zH$

If H is only strictly convex, then the assumption of Lemma 1.4 is in general not verified, which is why we need the following lemma:

Lemma 3.13

Let (G, \lesssim_1) be a C-q.o.g and let H be a strictly convex normal subgroup of (G, \lesssim_1) with convexity complement $F \neq \varnothing$. We are then in case (iii) of Lemma 1.3, so we have $H \lesssim F$. Let \lesssim_2 be the refinement of \lesssim_1 defined by declaring that $H \lesssim_2 F$. Then \lesssim_2 is a C-q.o and H is \lesssim_2 -convex.

Proof. The fact that H is \lesssim_2 -convex is clear, as is the fact that $1 \not\lesssim_2 x$ for every $x \in G$. Since $F \neq \emptyset$, $\max(H)$ is non-empty. Note that the notation $\max(H)$ is unambiguous, since the max of H in (G, \lesssim_1) is the same as in (G, \lesssim_2) . Now assume $x \lesssim_2 y$. Since \lesssim_1 is a coarsening of \lesssim_2 , we have $x \lesssim_1 y$. This implies $xy^{-1} \lesssim_1 y^{-1}$ and $x^z \lesssim_1 y^z$. The only way that we could have $y^{-1} \lesssim_2 xy^{-1}$ is if $y^{-1} \in \max(H)$ and $xy^{-1} \in F$. However, if $y^{-1} \in H$, then $y \in H$. Since we have $x \lesssim_2 y$, $y \in H$ also implies $x \in H$. This means $xy^{-1} \in H$, so $xy^{-1} \notin F$. It follows that $xy^{-1} \lesssim_2 y^{-1}$. By the same reasoning (using the fact that H is normal), we get $x^z \lesssim_2 y^z$.

We can now show Proposition 3.11:

proof of 3.11. Set $\lesssim_1:=\lesssim$ and consider the q.o \lesssim_2 as in Lemma 3.13. Since H is \lesssim_2 -convex, we know that the formula $gH \lesssim hH \Leftrightarrow (g \in H) \lor (h \notin H \land g \lesssim_2 h)$ gives a well-defined C-q.o. It is easy to see that $(g \in H) \lor (h \notin H \land g \lesssim_2 h)$ is equivalent to $(g \in H) \lor (h \notin H \land g \lesssim h)$, since for any $h \notin H$ and any $g \in G$, $g \lesssim h \Leftrightarrow g \lesssim_2 h$.

3.3 Type-components

In this section, we introduce the "type-components" T_g mentioned in the introduction. For $g \neq 1$, we want to find a set T_g which is the biggest strictly convex subset of G containing g on which \preceq is elementary-type-like. For an o⁺-type $g \in G$, one can see that the set of $h \in G$, which are o⁺-type and such that every element strictly between g and h are also o⁺-type is the greatest strictly convex subset of o⁺-type elements containing g. We can define in the same way such a "strictly convex closure" for an o-type element or a v-type element. Now, since by definition T_g contains g and g^{-1} , in the o-type cases T_q cannot be this closure. We will show that T_q is the union of the strictly convex closures of g and g^{-1} . In the v-type case the strictly convex closures of g and g^{-1} are equal. We also introduce the set G_g which should be thought of as the set of elements of G which are "below" T_g . We then introduce the set G^g which should be thought of as the set of elements which are not bigger than T_g . We will show that G_g and G^g are subgroups. For proving the properties of T_g, G^g, G_g , and the welding properties, it is more convenient to define T_g by means of formulas with inequalities instead of strict inequalities. This motivates the following definitions. For an element $1 \neq g \in G$, we define the type-component T_q of g as follows:

- If g is v-type, then T_g is the set of v-type elements $h \neq 1$ such that there is no o^+ -type element between h and g.
- If g is o^+ -type, then T_g^+ is the set of o^+ -type elements h such that every element between g and h is o^+ -type. We then set $T_g^- := (T_g^+)^{-1}$ and $T_g := T_g^+ \bigcup T_g^-$.
- If g is o^- -type, then $T_g := T_{g^{-1}}$.

We define two sets G^g and G_g as follows:

• If g is v-type, then define $G_q := \{h \mid h \preceq T_q\}.$

- If g is o^+ -type, then define $G_q := \{h \mid \{h, h^{-1}\} \lesssim g^{-1}\}.$
- If g is o^- -type, then define $G_g := G_{g^{-1}}$.

In all cases we set $G^g := G_g \cup T_g$. For g = 1, we set $T_g = G^g = G_g = \{1\}$. We will show later that G^g and G_g are actually subgroups of G (see Propositions 3.17 and 3.29). Note that for any $g \in G$, $1 \in G_g$, so G_g and G^g are non-empty.

Example 3.14

Let us have a look again at the groups given in Examples 3.1. Set g = (0,1) and h = (1,0). In examples (a), (b) and (c) we have $T_g = (\{0\} \times \mathbb{Z}) \setminus \{(0,0)\}$, $T_h^+ = \mathbb{N} \times \mathbb{Z}$ and $T_h = (\mathbb{Z} \setminus \{0\}) \times \mathbb{Z}$. We also have $G_g = \{0\}$, $G^g = G_h = \{0\} \times \mathbb{Z}$, $G^h = G$. In other words, we have $G^g = G^2$, $G_g = G_2$, $G^h = G^1$, $G_h = G_1$. It is also easy to see that the q.o induced on the quotients G^g/G_g and G^h/G_h are exactly the q.o's \lesssim_1 and \lesssim_2 which we lifted to construct the q.o on G. Note that the only difference between cases (a) and (b) is that T_g, T_h, G_h, G^g are convex in case (a) but are only strictly convex in case (b) due to welding. Note also that in example (b), each element of the form (x, y) with x < 0 is an o⁻-type welding point with $(x, y) \sim (0, z)$ for every $z \neq 0$. In particular, there is an o⁻-type element (for example (-1,0)) which is contained between g and g^{-1} , even though $g^{-1} \in T_g$. This explains why we restrict to o⁺-type elements in the definition of T_g when g is v-type.

In the next two sections, we describe some properties of the sets T_g , G^g and G_g for $g \neq 1$. As announced in the introduction, we are going to show that T_g is a maximal subset of G with the properties that T_g is strictly convex and that \lesssim is elementary-type-like (of the same type as g) on T_g (see Propositions 3.15 and 3.28). We will also show that G_g and G^g are subgroups of G and that G_g is normal in G^g . We first show these properties for the case where g is o-type and then do the same for the case where g is v-type.

3.3.1 T_q in the o-type case

We now want to describe T_g, G_g, G^g in the case where $g \neq 1$ is o-type. By definition of T_g , we can assume without loss of generality that g is o^+ -type. The following proposition states the main properties of T_g :

Proposition 3.15 (Characteristics of T_q)

The set T_g has the following properties:

- (a) T_g is right-convex with convexity complement $F_g := cl(g^{-1}) \setminus T_g$. Moreover, we have $cl(g^{-1}) = F_g \cup T_g^-$ and F_g can only contain v-type elements.
- (b) T_g is convex if and only if g^{-1} is not a welding point of G.
- (c) T_g is the biggest strictly convex subset of G containing g with the following properties:

- (i) Every element of T_g is o-type.
- (ii) T_g contains exactly one class of o^- -type elements, and this class is smaller than every o^+ -type element.
- (d) for any $f_1, f_2, h \in T_g$ which are o^+ -type, we have $f_1 \lesssim f_2 \Rightarrow f_1 h \lesssim f_2 h \wedge h f_1 \lesssim h f_2$.

Remark 3.16 1. Proposition 3.15(c) basically says that T_g is the biggest strictly convex subset of G containing g on which \lesssim is order-type-like.

- 2. It follows from Proposition 3.15(a) and from Lemma 1.3(ii) that $\min(T_g) = cl(g^{-1}) \cap T_g$.
- 3. If g^{-1} is not a welding point, then we can replace "strictly convex" by "convex" in Proposition 3.15(c).
- 4. Example 3.1(b) shows that T_g is not always convex.
- 5. It is interesting to note that property (d) in 3.15 is the property satisfied by ordered groups (see axiom (OG) in the introduction).

We now state the main properties of G_g and G^g :

Proposition 3.17 (Quotient for o-type elements)

Both G^g and G_g are subgroups of G. Moreover, G^g is convex and G_g is the smallest normal strictly convex subgroup of G^g such that the q.o induced by \lesssim on G^g/G_g is order-type.

Remark 3.18

If g^{-1} is not a welding point, then G_g is actually convex. However, Example 3.1(b) shows that G_g is not convex in general. We see that the existence of welding makes the structure of G less smooth, since it prevents the type-components from being convex.

Our goal is now to prove Propositions 3.15 and 3.17. We start by characterizing the elements of T_g^+ in the next two lemmas:

Lemma 3.19

Assume $g^{-1} \preceq h \preceq g$. Then $h^{-1} \sim g^{-1}$, and in particular h is o^+ -type.

Proof. By (CQ_2) , $h \lesssim g$ implies $hg^{-1} \lesssim g^{-1}$, hence $hg^{-1} \lesssim h$. By (CQ_2) and (CQ_3) , this implies $g^{-1} \lesssim h^{-1}$. Now assume that $g^{-1} \lesssim h^{-1}$ holds. By Lemma 3.6, we then have $h \sim hg^{-1} \lesssim g^{-1}$, which is a contradiction. Therefore, $g^{-1} \sim h^{-1}$.

Lemma 3.20

For any $h \in G$, $h \in T_g^+$ if and only if h is o^+ -type and $h^{-1} \sim g^{-1}$. In particular, $g \in T_g^+$ and $T_g^- \subseteq cl(g^{-1})$.

Proof. Assume $h \in T_g^+$. If $h^{-1} \not \gtrsim g^{-1}$, then $h^{-1} \not \gtrsim g^{-1} \not \gtrsim g$. By Lemma 3.6, this implies $h \not \lesssim g^{-1} \not \lesssim g$, so there is an o^- -type element between h and g, which is a contradiction. If $g^{-1} \not \lesssim h^{-1}$, then by the same reasoning we get $g \not \lesssim h^{-1} \not \lesssim h$, which is also a contradiction. This proves that $h^{-1} \sim g^{-1}$. Conversely, assume that h is o^+ -type and $h^{-1} \sim g^{-1}$. We want to show that every f between h and g is o^+ -type. Since f is between h and g and since $h^{-1} \sim g^{-1}$, we either have $h^{-1} \not \lesssim f \not \lesssim h$ or $g^{-1} \not \lesssim g$. By Lemma 3.19, this implies that f is o^+ -type.

As a direct consequence of these two lemmas, we have that the q.o is order-type-like on T_q :

Proposition 3.21

 T_g contains exactly one class of o^- -type elements, which is T_g^- . Moreover, $T_g^- \not \lesssim T_g^+$ and there is no h such that $T_g^- \not \lesssim h \not \lesssim T_g^+$

Proof. The fact that there is exactly one class of o^- type elements is a consequence of Lemma 3.20. If h satisfies $T_g^- \not \gtrsim h \lesssim T_g^+$, then by Lemma 3.19 $h \in T_g^+$, so we don't have $h \not \lesssim T_g^+$.

We can now show Proposition 3.15:

proof of 3.15. We first prove (a). It is clear from its definition that T_g^+ is convex. We also know that $\min(T_g) = T_g^- \subseteq cl(g^{-1})$ and that there is no element strictly between T_g^- and T_g^+ . It follows that $T_g \cup cl(g^{-1})$ is convex, which in particular means that T_g is right-convex and that $F_g := cl(g^{-1})\backslash T_g$ is the convexity complement of T_g (see Lemma 1.3). Since $T_g^- \subseteq cl(g^{-1})$, it follows from the definition of F_g that $cl(g^{-1}) = F_g \cup T_g^-$. Now let $h \in F_g$. Then $h \sim g^{-1}$. If h were o^+ -type, then we would have $h^{-1} \not \preceq g^{-1} \not \preceq h$. By Lemma 3.19, this would imply that g^{-1} is o^+ -type, which is a contradiction. Thus, h cannot be o^+ -type. If h were o^- -type, then by Lemma 3.20 we would have $h \in T_g^-$, which is excluded, so h cannot be o^- -type. Thus, h must be v-type. It then follows that $F_g = \varnothing$ if and only if g^{-1} is not a welding point, hence (b). Now let us prove (c). It only remains to prove that there is no strictly convex set bigger than T_g satisfying (i) and (ii). Towards a contradiction, let $S \not\supseteq T_g$ be such a set and take $h \in S \backslash T_g$. Assume first that $T_g^+ \not\preceq h$. Then h must be o^+ -type and $h^{-1} \preceq T_g^-$. Let $g \preceq f \preceq h$. We have $h^{-1} \not\preceq f \preceq h$. By Lemma 3.19, this implies that f is o^+ -type. Thus, every element between g and h is o^+ -type, so $h \in T_g^+$, which is a contradiction. Assume that $h \preceq T_g^-$. Then h must be o^- -type and $T_g^+ \not\preceq h^{-1}$. We then have $h \not\preceq g \preceq h^{-1}$. By 3.19, this implies $g^{-1} \sim h$, which means $h^{-1} \in T_g^+$: contradiction. (d) is a direct consequence of Proposition 3.9, since $h^{-1} \approx f_2$.

We mentioned in remark 3.16 that the q.o \lesssim on T_g is order-type-like. In fact, the only difference between the structure of T_g and the group in Proposition 2.8 is that \lesssim is not an order on T_g^+ (see for example T_h^+ in Example 3.14). However, we have the following, which will be useful in the proof of Proposition 3.17:

Lemma 3.22

Let $f, h \in T_g^+$ and $f \sim h$. Then $fh^{-1} \in G_g$.

Proof. By simply applying (CQ_2) to the inequalities $f \lesssim h \lesssim f$ we obtain $fh^{-1} \lesssim h^{-1} \sim g^{-1}$ and $hf^{-1} \lesssim f^{-1} \sim g^{-1}$, hence $fh^{-1} \in G_q$.

Intuitively, we see from Lemma 3.22 that the q.o induced by \lesssim on the quotient G^g/G_g will satisfy the condition of Proposition 2.8, where the sets G^- and G^+ of Proposition 2.8 will respectively correspond to $T_g^-/G_g := \{hG_g \mid h \in T_g^-\}$ and $T^+/G_g := \{hG_g \mid h \in T_g^+\}$. The next two propositions will help us prove Proposition 3.17:

Proposition 3.23

We have $G_g = \{h \not \gtrsim g^{-1}\} \cup F_g$, where F_g is as in Proposition 3.15. In particular, G_g is left-convex, and it is convex if and only if $F_g = \emptyset$. This in turn holds if and only if T_g is convex. Moreover, if G_g is not convex, then its convexity complement is T_g^- and we have $F_g = \max(G_g)$.

Proof. Let $h \in G_g$. Then in particular $\{h,h^{-1}\} \lesssim g^{-1}$. If $h \sim g^{-1}$, then we have $h \in cl(g^{-1}) = T_g^- \cup F_g$. Since $h^{-1} \lesssim g^{-1}$, we cannot have $h \in T_g^-$ (otherwise we would have $h^{-1} \in T_g^+$), so $h \in F_g$. Conversely, assume $h \not \subset g^{-1}$. By Lemma 3.6(ii), this implies $h^{-1} \not \subset g^{-1}$, hence $h \in G_g$. Assume $h \in F_g$. Then $h \sim g^{-1}$ and h is v-type, so $h^{-1} \sim h \lesssim g^{-1}$, hence $h \in G_g$. If $F_g = \varnothing$, then $G_g = \{h \not \subset g^{-1}\}$ is clearly convex. Now assume that $F_g \neq \varnothing$. By definition of F_g , we have $f \sim g^{-1}$ for every $f \in F_g$, hence $h \preceq f$ for every $f \in F_g$ and $h \in G_g$. Since $F_g \subseteq G_g$, it follows that $F_g = \max(G_g)$. Now take any $f \in F_g$. We then have $f \in G_g$, $f \sim g^{-1}$, but $g^{-1} \notin G_g$, so G_g is not convex. Moreover, we have $T_g^- = cl(f) \backslash G_g$. By Lemma 1.3(iii), this implies that T_g^- is the convexity complement of G_g .

Proposition 3.24

 G^g is an initial segment of G.

Proof. We already showed that $G_g \cup T_g^-$ is an initial segment. Since T_g^+ is convex and since there is no element strictly contained between T_g^- and T_g^+ , it follows that $G^g = G_g \cup T_g^- \cup T_g^+$ is an initial segment.

We can now show Proposition 3.17:

proof of 3.17. Let $h_1, h_2 \in G_g$. We have $h_1 \lesssim g^{-1}$ and $\{h_2, h_2^{-1}\} \lesssim g^{-1} \not \gtrsim g$, so we can apply Propositions 3.9 and 3.7 and get $h_1h_2^{-1} \lesssim g^{-1}h_2^{-1} \sim g^{-1}$. By a similar argument, we also have $h_2h_1^{-1} \lesssim g^{-1}$, hence $h_1h_2^{-1} \in G_g$. This proves that G_g is a subgroup of G. Now let us show that G^g is a subgroup of G. Note that by Propositions 3.21, 3.23 and 3.24 we have $G_g \lesssim T_g^- \lesssim T_g^+$. Since G^g is moreover an initial segment of G^g , it follows that an element $h \in G$ is in G^g if and only if there exists $f \in T_g^+$ with $h \lesssim f$. Let $h_1, h_2 \in G^g$. There exists $f \in T_g^+$ with $\{h_1, h_2\} \lesssim f$. Assume $h_2 \sim f$. By convexity of T_g^+ , this implies that $h_2 \in T_g^+$. We then have $h_1 \lesssim h_2$. By (CQ_2) , this implies

 $h_1h_2^{-1} \lesssim h_2^{-1} \lesssim h_2 \in T_g^+$, hence $h_1h_2^{-1} \in G^g$. Assume $h_2 \lesssim f$. By Proposition 3.9, this implies $h_1h_2^{-1} \lesssim fh_2^{-1}$. If $fh_2^{-1} \lesssim f^{-1}$, then $h_1h_2^{-1} \lesssim f \in T_g^+$, hence $h_1h_2^{-1} \in G^g$. Assume then that $f^{-1} \lesssim fh_2^{-1}$. By Lemma 3.6 (i), $h_2 \lesssim f$ implies $f^{-1} \sim h_2f^{-1}$, hence $h_2f^{-1} \lesssim fh_2^{-1}$ which means that fh_2^{-1} is o^+ -type. Since $f \in T_g^+$, we have $f^{-1} \sim g^{-1}$ by Lemma 3.20. We thus have $g^{-1} \sim h_2f^{-1}$ and fh_2^{-1} is o^+ -type. By Lemma 3.20, we then have $fh_2^{-1} \in T_g^+$. Since $h_1h_2^{-1} \lesssim fh_2^{-1}$, it follows that $h_1h_2^{-1} \in G^g$. This proves that G^g is a subgroup of G. Now let us show that G_g is normal in G^g . Let $f_g \in G^g$. By $f_g \in G^g$ is a group, we have $f_g \in G^g$ and since there is no $f_g \in G^g$. Since $f_g \in G^g$ is a group, we have $f_g \in G^g$ and since there is no $f_g \in G^g$ in $f_g \in G^g$ in $f_g \in G^g$.

Now let us prove that the q.o induced on G^g/G_g is order-type. Set $G^+ := T_g^+/G_g$ and $G^- = T_g^-/G_g$. Clearly, $G^- = (G^+)^{-1}$. Remember that, if $f \notin G_g$, then $h \preceq f \Leftrightarrow hG_g \preceq fG_g$. Since $T_g^- \preceq T_g^+$, we have $G^- \preceq G^+$. Let $fG_g, hG_g \in G^-$. Since \preceq is trivial on T_g^- , we have $h \sim f$, which implies $hG_g \sim fG_g$. Moreover, Lemma 3.22 implies that \preceq is an order on G^+ . By proposition 2.8, \preceq is order-type on G^g/G_g . Now assume that $H \subsetneq G_g$ is another strictly convex normal subgroup of G^g . Take $h \in G_g \backslash H$. If G_g is convex, then we have $h \not \preceq T_g^-$, so $H \not \preceq hH \not \preceq G^-$. By Proposition 2.8, it follows that \preceq on G^g/H cannot be order-type. If G_g is not convex, then we can choose $h \in F_g$ so h is v-type and so is hH, so \preceq on G^g/H is also not order-type.

3.3.2 T_q in the v-type case

Assume now that $g \neq 1$ is v-type.

Lemma 3.25

 $g \in T_g$.

Proof. All we have to check is that there is no o^+ -type element equivalent to g. This is given by Lemma 3.19

Lemma 3.26

Let h be o-type. Then either $h \not \subset T_g$ or $T_g \not \subset h$.

Proof. By Proposition 3.15, T_h is right-convex and contains h. Moreover, T_h only contains o-type elements and T_g only contains v-type elements, hence the claim. \square

Proposition 3.27

Define $F_g := \{h \quad o^-\text{-type} \mid h \sim \max(T_g)\}$ if $\max T_g \neq \emptyset$ and $F_g := \emptyset$ otherwise. Then T_g is left-convex with convexity complement F_g . In particular, T_g is convex if it has no maximum.

Proof. Assume T_g is not convex. Then there exists $h_1, h_2 \in T_g$ and $f \notin T_g$ such that $h_1 \lesssim f \lesssim h_2$. If f were v-type, then since $f \notin T_g$ there would an o^+ -type element between g and f. This would imply that there is an o^+ -type element either between g and h_1 or between g and h_2 , which is a contradiction. For the same reason f cannot be

 o^+ -type. Thus, f is o^- -type. It follows from the previous lemma that $T_g \lesssim f$, so $h_2 \lesssim f$, hence $h_2 \sim f$. It follows that $h_2 \in \max(T_g)$. Now let us show that $T_g \cup cl(h_2)$ is convex. Let $f_1, f_2 \in T_g \cup cl(h_2)$ and $f_1 \lesssim f \lesssim f_2$. With the same reasoning as above, f cannot be o^+ -type so it must either be v-type or o^- -type. If it is v-type, then $f \in T_g$. If it is o^- -type, then $f \sim h_2$.

We can now state a v-type analogue of Proposition 3.15:

Proposition 3.28

The set T_g is the biggest strictly convex subset of $G\setminus\{1\}$ containing g such that every element of T_g is v-type. If G has no welding at g, then T_g is even convex.

Proof. Let $S \supseteq T_g$ be strictly convex and let $h \in S \backslash T_g$ with $h \ne 1$ be v-type. Since $h \notin T_g$, then by definition of T_g there must be an o^+ -type element f between g and h. By Lemma 3.19, we have $f \nsim h$ and $f \nsim g$, so f is strictly between g and h, hence $f \in S$. Thus, S must contain o-type elements.

We now want to establish the v-type analogue of Proposition 3.17.

Proposition 3.29

Both G^g and G_g are subgroups of G, G^g is strictly convex with convexity complement F_g and G_g is convex. Moreover, G_g is normal in G^g .

Proof. G_g is clearly an initial segment by definition, so it is convex. Moreover, we know that T_g is left-convex and that there is no element strictly contained between G_g and T_g , so it follows immediately that $G^g = G_g \cup T_g$ is left-convex. We also know that F_g is the convexity complement of G^g .

Let us show that G_g is a group. Let $f_1, f_2 \in G_g$ and $h \in T_g$, so in particular h is v-type. Assume $h \preceq f_1 f_2^{-1}$. We then have $f_1 \nsim f_1 f_2^{-1}$. Applying Proposition 3.9, we get $f_1^{-1}h \preceq f_2^{-1}$. However, by Proposition 3.7, we have $f_2^{-1} \preceq h$ and $f_1^{-1}h \sim h$, so this is a contradiction. Thus, we must have $f_1 f_2^{-1} \preceq h$. Since h is arbitrary in T_g , this means $f_1 f_2^{-1} \in G_g$. Now let us show that G^g is a group. Let $f_1, f_2 \in G^g$. This implies that there is $h \in T_g$ with $\{f_1, f_2\} \preceq h$. If $h \sim f_2$, then $f_2 \in T_g$, so f_2 is v-type and we have $f_1 \preceq f_2$. By (CQ_2) , it follows that $f_1 f_2^{-1} \preceq f_2^{-1} \sim h \in T_g$. If $h \nsim f_2$, then $f_1 \preceq h$ implies $f_1 f_2^{-1} \preceq h f_2^{-1}$ by Proposition 3.9. Since h is v-type, $f_2 \preceq h$ implies $h \sim h f_2^{-1}$ by Proposition 3.7, hence $f_1 f_2^{-1} \preceq h$. In any case we have $f_1 f_2^{-1} \preceq h$, which means $f_1 f_2^{-1} \in G^g \cup F_g$. We can show with the same reasoning that $f_2 f_1^{-1} \preceq h$. This implies that $f_1 f_2^{-1} \not\in F_g$. Indeed, if $f_1 f_2^{-1}$ were in F_g , then it would be o^- -type, so we would have $h \sim f_1 f_2^{-1} \preceq f_2 f_1^{-1}$.

Take $h \in G_g$ and $z \in T_g$. If $h \ne 1$, then there exists an o^+ -type element f between

Take $h \in G_g$ and $z \in T_g$. If $h \neq 1$, then there exists an o^+ -type element f between h and g. We then have $h^z \preceq f^z \preceq g^z \in G^g$, so there is an o^+ -type element between h^z and g^z , hence $h^z \in G_g$.

Proposition 3.30

The group G_g is the smallest normal convex subgroup of G^g such that the quotient G^g/G_g is valuational.

Proof. Remember that for any $f, h \notin G_g$, $f \lesssim h$ if and only if $fG_g \lesssim hG_g$. Since every element of G_g is v-type, it follows that every element of G_g is also v-type, so the q.o is valuational. If H is strictly contained in G_g , then G_g is also v-type element h, and then hH is o-type.

- **Remark 3.31** 1. As happens in the o-type case, welding is the only thing preventing T_g and G^g from being convex. If G has no welding point, then we can replace "strictly convex" by "convex" in Propositions 3.28 and 3.30.
 - 2. In the o-type case as well as in the v-type case, it can happen that G_g and G^g are not normal in G (see Example 3.41 below).

3.3.3 Type-valuation

We can now show that the T_q 's form a partition of G:

Proposition 3.32

The following holds for any $g, h \in G$: $g \in T_h \Leftrightarrow h \in T_q \Leftrightarrow T_q = T_h \Leftrightarrow T_q \cap T_h \neq \emptyset \Leftrightarrow G_q = G_h \Leftrightarrow G^g = G^h$.

Proof. Assume $g \in T_h$. If h,g are v-type, then we use Proposition 3.28. We know that T_g is the biggest strictly convex subset of G containing g whose every element is v-type. Since T_h is strictly convex and only contains v-type elements and $g \in T_h$, it follows that $T_h \subseteq T_g$. This implies $h \in T_g$. By a similar argument, it also follows that $T_g \subseteq T_h$, hence $T_g = T_h$. The case where they are o-type is similar by using Proposition 3.15. This proves the first two equivalences. The third one follows immediately: if $T_g \cap T_h \neq \emptyset$, then there is $f \in G$ with $f \in T_g \cap T_h$, which implies $T_g = T_f = T_h$. Assume $T_g = T_h$. In the v-type case we obviously have $G_h = G_g$ by definition of G_g . If they are o^+ -type, then $g^{-1} \sim h^{-1}$. But then, for any $f \in G$, $\{f, f^{-1}\} \lesssim g^{-1}$ is equivalent to $\{f, f^{-1}\} \lesssim h^{-1}$, hence $G_g = G_h$. Assume $G_g = G_h$. Without loss of generality $g \lesssim h$. Since $G_g = G_h$, we have $g \notin G_h$. Note that there is no element strictly contained between G_h and G_h (otherwise, there would be an element G_h with G_h is strictly convex). Thus, we have G_h is we have G_h is strictly convex. Thus, we have G_h is generally, assume G_h is generally.

We have thus reached the goal we announced in the introduction: we showed that G is partitioned into a family of sets on each of which the C-q.o is elementary-type-like. Our next objective is to reformulate this statement by showing that \lesssim can be obtained by lifting elementary C-q.o's. To do this we need to define a valuation on G whose fibers are the type-components. We first notice that \lesssim naturally induces an order on the set of type-components:

Proposition 3.33

Define \leq on the set of all type-components by $T_g \leq T_h \Leftrightarrow T_g = T_h \vee T_g \lesssim T_h$. This is an order on the set of all type-components of G.

Proof. The fact that \leq is total follows from the fact that the type-components are strictly convex and pairwise disjoint. The relation \leq is clearly reflexive and transitive, let us prove that is is antisymmetric. If $T_g \lesssim T_h \lesssim T_g$, then all elements of $T_g \cup T_h$ are equivalent to one another. It follows that h, g must both be v-type. Since $g \sim h$, this implies $T_g = T_h$.

Remark 3.34

If S is a subset of G which contains elements $s,t \in S$ such that $s \not \lesssim t$, then $S \not \lesssim S$ does not hold (remember that $S \not \lesssim T$ means that $s \not \lesssim t$ for any pair $(s,t) \in S \times T$). Hence the condition $T_g = T_h$ does not imply $T_g \not \lesssim T_h$. Therefore, the condition " $T_g = T_h$ " in the definition of \leq is essential for reflexivity.

Proposition 3.35

Set $\Gamma := \{T_g \mid g \in G\}$ and let \leq^* be the reverse order of the one given in Proposition 3.33. We define a valuation on G called the **type-valuation associated to** \lesssim by

$$v: G \to (\Gamma, \leq^*)$$

 $g \mapsto T_q.$

Proof. Clearly, T_1 is a maximum of (Γ, \leq^*) and $v(g) = v(g^{-1})$ for any $g \in G$. Let $g, h \in G$ with $v(g) \leq^* v(h)$. By definition of \leq^* , it follows that $h \in G^g$. Since G^g is a group, we then have $gh \in G^g$. This implies $T_{gh} = T_g$ or $T_{gh} \lesssim T_g$, which means $v(g) \leq^* v(gh)$, hence $\min(v(g), v(h)) \leq^* v(gh)$. Now let $z \in G$. If $T_h \lesssim T_g$, then in particular $h \lesssim g$, so $h^z \lesssim g^z$. This implies $v(g^z) \leq^* v(h^z)$. Now assume $T_g = T_h$. If g, h are both v-type, then so are g^z and h^z (this follows from (CQ_3)). Since $h \in T_g$, there is no o^+ -type element between g and h. Therefore, by (CQ_3) , there cannot be an o^+ -type elements between g^z and h^z . This proves $T_{g^z} = T_{h^z}$. The same kind of argument show $T_g^+ = T_h^+$ in the case where g, h are both o^+ -type. If one of them is o^- -type, then take their inverse and we are back to the o^+ -type case.

3.4 Structure theorems

We now want to summarize the results of Section 3.3 into a structure theorem of C-q.o.g's. We start by giving two ways of constructing C-q.o's: lifting from quotients and "welding". This will justify the fact that the q.o's given in Example 3.1 are indeed C-q.o's. We then show that any C-q.o can be obtained by lifting C-q.o's of elementary type and then welding if necessary.

Proposition 3.36 (construction by lifting)

Let G be a group, $v: G \to \Gamma \cup \{\infty\}$ a valuation. Assume that for each γ , the quotient G^{γ}/G_{γ} is endowed with a C-q.o \lesssim_{γ} . Assume moreover that for any $z \in G$ and any $\gamma \in \Gamma$, the isomorphism $G^{\gamma}/G_{\gamma} \to G^{\gamma^z}/G_{\gamma^z}$ induced by conjugation by z is quasi-order-preserving. Then the lifting of $(\lesssim_{\gamma})_{\gamma \in \Gamma}$ to G is also a C-q.o.

Proof. Denote by \lesssim the lifting. (CQ_1) is clearly satisfied. Let $x \lesssim y$. If v(x) > v(y), then $v(xy^{-1}) = v(y^{-1}) =: \gamma$ and $xy^{-1}G_{\gamma} = y^{-1}G_{\gamma}$. This implies $xy^{-1}G_{\gamma} \lesssim_{\gamma} y^{-1}G_{\gamma}$,

hence $xy^{-1} \lesssim y^{-1}$. We also have $v(x^z) > v(y^z)$, hence $x^z \lesssim y^z$. Assume $v(x) = v(y) = \gamma$ and $xG_{\gamma} \lesssim_{\gamma} yG_{\gamma}$. This implies $v(x^z) = v(y^z) = \gamma^z$. By assumption, $x^zG_{\gamma^z} \lesssim_{\gamma} y^zG_{\gamma^z}$, hence $x^z \lesssim y^z$. Moreover, we have $v(xy^{-1}) \geq \min(v(x), v(y)) = \gamma$. If $v(xy^{-1}) > \gamma$, then $xy^{-1} \lesssim y^{-1}$, so assume $v(xy^{-1}) = \gamma$. Since \lesssim_{γ} is a C-q.o, we have $xy^{-1}G_{\gamma} \lesssim_{\gamma} y^{-1}G_{\gamma}$, hence $xy^{-1} \lesssim y^{-1}$.

As a special case of lifting we can define a C-q.o on semi-direct products, which is how we obtained Example 3.1(e):

Proposition 3.37

Let $(G, \lesssim_G), (H, \lesssim_H)$ be two C-q.o.g and let $\alpha : G \to Aut(H)$ such that for any $g \in G$, $\alpha(g)$ preserves \lesssim_H . Define a q.o \lesssim on $G \ltimes_{\alpha} H$ by $(g_1, h_1) \lesssim (g_2, h_2) \Leftrightarrow (g_1 \gtrsim_G g_2) \land (g_2 \neq 1 \lor (g_2 = 1 \land h_1 \lesssim_H h_2))$. Then \lesssim is a C-q.o.

Proof. Set $F := G \ltimes_{\alpha} H$, $\Gamma := \{1,2\}$ and define $v : F \to \Gamma \cup \{\infty\}$ as follows:

$$v(g,h) := \begin{cases} 1 \text{ if } g \neq 1. \\ 2 \text{ if } g = 1 \neq h. \\ \infty \text{ if } g = h = 1. \end{cases}$$

This defines a valuation on F. We have $F_2 \cong \{1\}$, $F^2 = F_1 = \{1\} \times H$ and $F^1 = F$. Now take $h_1, h_2 \in H \cong F^2/F_2$ with $h_1 \preceq_H h_2$ and $z = (g, h) \in F$. Since H is normal in F, we have $F^2/F_2 = F^{2^z}/F_{2^z}$. We have $h_i^z = (\alpha(g)(h_i))^h$ for i = 1, 2. By assumption, $\alpha(g)$ preserves \preceq_H , hence $\alpha(g)(h_1) \preceq_H \alpha(g)(h_2)$. By (CQ_3) , it then follows that $h_1^z \preceq_H h_2^z$. This proves that the isomorphism $F^2/F_2 \to F^{2^z}/F_{2^z}$ induced by z preserves \preceq_H . Now note that $G \cong F^1/F_1$, so F^1/F_1 is endowed with a C-q.o. \preceq_G defined by $(g_1, 1).F_1 \preceq_G (g_2, 1).F_1 \Leftrightarrow g_1 \preceq_G g_2$. Take $(g_1, 1).F_1$ and $(g_2, 1).F_1$ in F^1/F_1 with $(g_1, 1).F_1 \preceq_G (g_2, 1).F_1$. By definition, $((g_i, 1).F_1)^z = (g_i, 1)^z.F_1 = (g_i^g, h\alpha(g_i^g)(h^{-1})).F_1 = (g_i^g, 1).F_1$. Because $(g_1, 1).F_1 \preceq_G (g_2, 1).F_1$, it follows from (CQ_3) on G that $(g_1^g, 1) \preceq_G (g_2^g, 1)$, hence $((g_1, 1).F_1)^z \preceq_G ((g_2, 1).F_1)^z$. This proves that the isomorphism $F^1/F_1 \to F^{1z}/F_{1z}$ induced by z preserves \preceq_G . Thus, the hypothesis of Proposition 3.36 are satisfied, so the lifting of (\preceq_H, \preceq_G) to F is a C-q.o.

We now introduce another way of obtaining C-q.o's, which we call welding. Let g be an o⁻-type element, and assume that the maximum M_g of G_g is non-empty. We noted in Proposition 3.23 that, if $F_g \neq \varnothing$, then $M_g = F_g$, and so, by Proposition 3.15(a), we have $M_g \subseteq cl(g)$. If $F_g = \varnothing$, then by Proposition 3.23 we have $G_g = \{h \in G \mid h \not \preceq g\}$. In any case, there is no element strictly contained between M_g and cl(g). This means that we can coarsen \preceq by joining the sets cl(g) and M_g . In other words, we define a coarsening \preceq_2 of \preceq by declaring that $h \sim_2 f$ for any $f, h \in M_g \cup cl(g)$ and $h \preceq_2 f \Leftrightarrow h \preceq f$ whenever $h \notin M_g \cup cl(g)$ or $f \notin M_g \cup cl(g)$. Note that, in example 3.1(b), if we set g := (-1,0), then we have $G_g = \{0\} \times \mathbb{Z}$ and $M_g = \{0\} \times (\mathbb{Z} \setminus \{0\}) \subseteq cl(g)$. Therefore, it can happen that $M_g \subseteq cl(g)$, in which case nothing changes. But if T_g is convex, then by 3.23 we have $M_g \cap cl(g) = \varnothing$, and then \preceq_2 is different from \preceq . If we apply this coarsening

operation simultaneously at each g^z for $z \in G$, then we will obtain a new C-q.o, as the next proposition shows:

Proposition 3.38 (Construction by welding)

Let (G, \preceq) be a C-q.o.g and $g \in G$ an o^- -type element such that $M_g := \max(G_g)$ is non-empty. Then for any $z \in G$, $M_{g^z} := \max(G_{g^z})$ is also non-empty, so we can define a coarsening \preceq_2 of \preceq by declaring $M_{g^z} \sim_2 g^z$ for every $z \in G$. Moreover, this coarsening is a C-q.o.

Proof. Note that by (CQ_3) , we have $g \not\gtrsim g^{-1} \Rightarrow g^z \not\gtrsim (g^{-1})^z = (g^z)^{-1}$, so g^z is o⁻-type. The fact that M_{g^z} is non-empty is also a direct consequence of (CQ_3) . It also follows from (CQ_3) that $F_g \neq \varnothing \Leftrightarrow F_{g^z} \neq \varnothing$. Note also that if $F_g \neq \varnothing$, then by Proposition 3.23 we have $M_g = F_g$, so we already have $M_g \sim g$. By (CQ_3) , this implies $M_{g^z} \sim g^z$ for all $z \in G$. It then follows that $\preceq = \preceq_2$, so there is nothing to prove. Therefore, we can assume without loss of generality that $F_{g^z} = \varnothing$ for all $z \in G$.

Set $\preceq_1:=\preceq$. We want to show that \preceq_2 is a C-q.o. Let $x,y,z\in G$ with $x\preceq_2 y$. If $x\preceq_1 y$, then we have $xy^{-1}\preceq_1 y^{-1}$ and $x^z\preceq_1 y^z$. Since \preceq_2 is a coarsening of \preceq_1 , this implies $xy^{-1}\preceq_2 y^{-1}$ and $x^z\preceq_2 y^z$. Now assume $y \not \preceq_1 x$. This can only happen if there is $w\in G$ with $y\in M_{g^w}$ and $x\sim_1 g^w$. Since we assumed that $F_{g^w}=\varnothing$, it follows that x is o⁻-type. By maximality of y, we have $y^{-1}\preceq_1 y$. We thus have $\{y,y^{-1}\}\preceq_1 x \not \preceq_1 x^{-1}$. By Lemma 3.6(iii), this implies $xy^{-1}\sim_1 x$. By $(CQ'_2), y^{-1}\not \preceq_1 y$ would imply $y\not \preceq_1 y^2$, which would contradict the maximality of y. It follows that y is v-type. We thus have $xy^{-1}\sim_1 g^w$ and $y^{-1}\in M_{g^w}$. By definition of z, this implies $zy^{-1}\sim_2 z^{-1}$. Moreover, we have $z \in M_{g^{wz}}$ and $z \sim_1 g^{wz}$, which also implies $z \sim_2 z^z$.

We see that, if we lift a family of C-q.o's of elementary types as in Proposition 3.36 and then apply welding, then the q.o which we obtain is again a C-q.o. Our main theorem states that any C-q.o is obtained through this process:

Theorem 3.39 (Structure theorem of a C-q.o.g)

Let (G, \preceq) be a C-q.o.g. There exists a valuation v on G with value set $\Gamma \cup \{\infty\}$, called the type-valuation associated to \preceq , such that the following holds:

- (i) For any $\gamma \in \Gamma$, G^{γ} and G_{γ} are \preceq -strictly-convex subgroups of G.
- (ii) The q.o \lesssim_{γ} induced by \lesssim on $H_{\gamma} := G^{\gamma}/G_{\gamma}$ is of elementary type.
- (iii) If $\gamma \leq \delta$, if \lesssim_{γ} , \lesssim_{δ} are both valuational, then there exists α between γ and δ such that \lesssim_{α} is order-type.

Moreover, the q.o \lesssim can be obtained by lifting the family $(\lesssim_{\gamma})_{\gamma\in\Gamma}$ to G and then welding if necessary.

Proof. We already defined the type-valuation v in Proposition 3.35. Note that for any $g \in G$, we have $G^{v(g)} = G^g$ and $G_{v(g)} = G_g$. (i) and (ii) follow from Propositions 3.17, 3.29 and 3.30, (iii) follows from 3.28. Denote by \preceq^* the lifting of $(\preceq_{\gamma})_{\gamma \in \Gamma}$ to G. Note that an element $g \in G$ is v-type (respectively, o⁻-type) with respect to \preceq if and

only if it is v-type (respectively, o[−]-type) with respect to \precsim^* (this follows easily from Propositions 3.17 and 3.30 and from the definition of the the lifting). We first show that \lesssim is a coarsening of \lesssim^* . Let $g, h \in G$ with $g \lesssim^* h$. By definition of \lesssim^* , we either have v(h) > v(g) or $v(g) = v(h) \land gG_g \lesssim_{v(g)} hG_g$. In the first case we have by definition of v: $g \lesssim h$. In the second case, since $G_h = G_g$, we have $h \notin G_g$. Thus, by definition of the q.o induced on the quotient, we must have $g \lesssim h$. This proves that \lesssim is a coarsening of \lesssim^* . Now let $g, h \in G$ be such that $g \lesssim h$ but $h \lesssim^* g$. We will show that h is v-type, g is o⁻-type and $h \in \max(G_q, \preceq^*)$. It will then follow that \preceq is obtained from \preceq^* by welding g and $\max(G_g, \lesssim^*)$. By definition of \lesssim^* , $h \not\lesssim^* g$ means either v(h) > v(g) or v(g) = v(h)and $hG_g \not \lesssim_{v(g)} gG_g$. But the latter case would imply $h \not \lesssim g$, so we must have v(h) > v(g)i.e $h \in G_g$. This implies $h \lesssim g$, so $g \sim h$. If h were o-type, then by Proposition 3.17 G^h would be convex with respect to \lesssim . The inequality $g \lesssim h$ would then imply $g \in G^h$, which contradicts v(g) < v(h). Therefore, h must be v-type. Assume for a contradiction that g is v-type. Since v(h) > v(g), we have $g \notin T_h$. By definition of T_h , it follows that there is an o⁺-type element f between g and h. But since $g \sim h$, it follows that $f \sim h$. This contradicts Lemma 3.19. Therefore, g is o-type. Since $h \sim g$, h is in the convexity complement of T_q . By Proposition 3.23, we thus have $h \in \max(G_q, \lesssim)$. Now let $f \in G_g$ with $h \lesssim^* f$. Since \lesssim is a coarsening of \lesssim^* , we then have $h \lesssim f$, hence $h \sim f$ by maximality of h. Now $h \lesssim^* f$ implies $v(f) \leq v(h)$ and $f \in G_g$ implies v(f) > v(g). Since $h \in \max(G_q, \lesssim)$, there is no element strictly contained between T_h and T_q , so we must have v(h) = v(f). Since h is v-type, T_h is left-convex, so $h \sim f$ implies $f \notin G_h$. By definition of $\lesssim_{v(h)}$ (see Proposition 3.12), it then follows that $hG_h \sim_{v(h)} fG_h$, hence $h \sim^* f$. This shows that h is maximal in (G_g, \lesssim^*) . Thus, the only point on which \lesssim and \preceq^* disagree are welding points, so \preceq is obtained from \preceq^* by welding.

Remark 3.40

If there is no welding point, then \lesssim actually coincides with the lifting of $(\lesssim_{\gamma})_{\gamma\in\Gamma}$.

Example 3.41

We take notations from Examples 3.1. We are going to give an explicit definition to the type-valuation associated to the C-q.o's \lesssim_H and \lesssim_F of examples (d) and (e). We already defined a valuation $v_G: G \to \{1,2\} \cup \{\infty\}$ on G and a valuation $w_H: H \to \mathbb{Z}$ on G and a valuation w_H

$$v_F(k,h) = \begin{cases} a \text{ if } k \neq 0. \\ v_H(h) \text{ if } k = 0. \end{cases}$$

If we assimilate elements of $\Gamma \cup \{a\}$ with their v_F -fiber, then v_F is the type-valuation associated to \lesssim_F . Now take $z := (-1, \sum_{n \in \mathbb{Z}} (0,0)\tau_n) \in F$ and $f := (0, \sum_{n \in \mathbb{Z}} g_n\tau_n) \in F$, where $g_0 = (1,0)$ and $g_n = (0,0)$ for $n \neq 0$. We have $v_F(f) = (0,1)$ but $v_F(z+g-z) = (-1,1) < v_F(f)$. In particular, $F^{(0,1)} = F_{(-1,2)}$ is not normal in F. This shows that the

groups G^{γ} and G_{γ} of theorem 3.39 are not always normal in G.

We can also reformulate Theorem 3.39 in terms of C-relations:

Theorem 3.42

Let (G,C) be a C-group. There exists a valuation $v:G\to \Gamma\cup\{\infty\}$ such that the following holds:

- 1. For any $\gamma \in \Gamma$, C induces a C-relation C_{γ} on the quotient G^{γ}/G_{γ} defined by the formula $C_{\gamma}(fG_{\gamma}, gG_{\gamma}, hG_{\gamma}) \Leftrightarrow fh^{-1} \notin G_{\gamma} \wedge (gh^{-1} \in G_{\gamma} \vee C(f, g, h))$.
- 2. For each $\gamma \in \Gamma$, C_{γ} is of elementary type.
- 3. If $\gamma \leq \delta$, if C_{γ} , C_{δ} are both valuational, then there exists α between γ and δ such that C_{α} is order-type.

4 C-minimal groups

We now want to interpret the results on C-minimal groups given in [6] in view of our structure theorem 3.39. Note that the C-relations considered in [6] are dense, i.e they satisfy the extra axioms: $x \neq y \Rightarrow \exists z, (z \neq y \land C(x,y,z))$ and $\exists x \exists y, y \neq x$. The authors of [3] and [2] described how to obtain the canonical tree associated to a given C-structure (see Proposition 1.5 in [3] and Theorem 12.4 in [2]). If (M,C) is a C-structure, then we can define a partial quasi-order \preceq on the set M^2 by $(x,y) \preceq (u,v) \Leftrightarrow \neg C(u,x,y) \land \neg C(v,x,y)$. We then define the canonical tree (\mathfrak{T},\leq) of (M,C) as the quotient $\mathfrak{T}:=M^2/\sim$ endowed with the partial order \leq induced by \preceq . To simplify notations, we will refer to elements of \mathfrak{T} by one of their representatives in M^2 . Note that (x,y)=(y,x) for any x,y.

If (G, C) is a C-group with canonical tree \mathfrak{T} , then we see that G induces a right action on \mathfrak{T} by (x,y).g := (xg,yg). Note that the partial order on \mathfrak{T} is compatible with this action in the sense that $(x,y) \leq (u,v) \Rightarrow (x,y).g \lesssim (u,v).g$ (this follows directly from the fact that C is compatible). The authors of [6] described dense C-minimal groups by looking at the orbits of this action. They distinguished three cases:

- 1. All orbits are antichains.
- 2. One orbit is a non-trivial chain.
- 3. No orbit is a non-trivial chain and there exists one non-trivial orbit which is not an antichain.

Now let \lesssim be the C-q.o associated to C. We want to interpret this trichotomy in terms of \lesssim . More precisely, we want to see how the type of elements x and y influences the orbit of (x,y). Note that the partial order \leq of \mathfrak{T} is given by $(x,y) \leq (u,v) \Leftrightarrow \{uy^{-1},vy^{-1}\} \lesssim xy^{-1}$. We first want to describe the structure of the tree \mathfrak{T} in the order-type case:

Lemma 4.1

Assume (G, \preceq) is an order-type C-q.o.g and set $\mathcal{C} := \{(x,y) \in \mathfrak{T} \mid x \neq y\}$. Then \mathcal{C} is a non-trivial chain and an orbit under the action of G.

Proof. Denote by \leq the underlying order on G. Let $(x,y), (u,v) \in \mathcal{C}$. Note that since (x,y)=(y,x), we can assume that x < y and u < v. We have $(x,y) \leq (u,v) \Leftrightarrow \{uy^{-1},vy^{-1}\} \lesssim xy^{-1}$. We saw in the proof of Proposition 2.8 that x < y is equivalent to $xy^{-1} \in G^-$. Since \lesssim is trivial on G^- , $\{uy^{-1},vy^{-1}\} \lesssim xy^{-1}$ is equivalent to $uy^{-1},vy^{-1} \in G^- \cup \{1\}$. This in turn is equivalent to $u \leq y \land v \leq y$. Since u < v, this is equivalent to $v \leq y$. Thus, we have $(x,y) \leq (u,v) \Leftrightarrow v \leq y$ and it follows that $\mathcal C$ is a chain. Note that it also shows:

 $(*) (u < v \land x < y \land y = v) \Rightarrow (x, y) = (u, v).$

Now we want to show that (x, y) and (u, v) are in the same orbit. Set $g := y^{-1}v$. Note that by definition of order-type C-relations in Example 1.2(a), < is compatible with the group operation, so we have xg < yg. Moreover, we have u < v and yg = v. By (*), this implies that we have (x, y).g = (u, v).

Lemma 4.2

Let (G, \preceq) be a C-q.o.g (not necessarily minimal) and $g \in G$. Let (\mathfrak{T}, \leq) be the canonical tree associated to G^g and (\mathfrak{T}', \leq') the canonical tree associated to G^g/G_g . If $x, y, u, v \in T_g$ are such that $xy^{-1}, uv^{-1} \in T_g$, then $(x, y) \leq (u, v)$ if and only if $(xG_g, yG_g) \leq' (uG_g, vG_g)$.

Proof. By definition of the q.o on G^g/G_g and since $xy^{-1} \notin G_g$, we have $\{uy^{-1}, vy^{-1}\} \lesssim xy^{-1}$ if and only if $\{uy^{-1}G_g, vy^{-1}G_g\} \lesssim xy^{-1}G_g$.

Lemma 4.3

Let (G, \preceq) be a C-q.o.g. Let $x \in G$ and $y \in G^x$. The following holds:

- (i) If xy^{-1} is v-type, then the orbit of (x,y) under the action of G is an antichain.
- (ii) If $xy^{-1} \notin T_x$, then the orbit of (x,y) under the action of G^x is not a chain.
- (iii) The orbit of (x, y) under the action of G^x is a non-trivial chain if and only if x is o-type and $xy^{-1} \in T_x$.
- Proof. (i) Assume that xy^{-1} is v-type and let $g \in G$. We want to show that (x,y) and (xg,yg) are either incomparable or equal. Assume $(xg,yg) \leq (x,y)$. This means $\{xg^{-1}y^{-1}, yg^{-1}y^{-1}\} \lesssim xy^{-1}$. Since xy^{-1} is v-type, $yg^{-1}y^{-1} \lesssim xy^{-1}$ implies $ygy^{-1} \lesssim xy^{-1}$ (indeed, if $yg^{-1}y^{-1}$ is v-type, then $ygy^{-1} \sim yg^{-1}y^{-1}$. If $yg^{-1}y^{-1}$ is otype, then we have $yg^{-1}y^{-1} \in G_{xy^{-1}}$. Since $G_{xy^{-1}}$ is a group, this implies $ygy^{-1} \in G_{xy^{-1}}$, hence $ygy^{-1} \lesssim xy^{-1}$. Moreover, if we conjugate the inequality $yg^{-1}y^{-1} \lesssim xy^{-1}$ by xy^{-1} , then we obtain $xgx^{-1} \lesssim xy^{-1} \sim yx^{-1}$. By (CQ_2) , $xgx^{-1} \lesssim yx^{-1}$ implies $xgy^{-1} \lesssim xy^{-1}$. Thus, we have $\{xgy^{-1}, ygy^{-1}\} \lesssim xy^{-1}$, which means $(x,y) \leq (xg,yg)$, so (x,y) and (xg,yg) are equal. Now if we assume that $(x,y) \leq (x,y)$. g instead of $(x,y).g \leq (x,y)$ at the beginning, then by compatibility of the action we have $(x,y).g^{-1} \leq (x,y)$, which brings us back to the previous case.

- (ii) Assume $xy^{-1} \notin T_x$. Since $x \notin G_x$ and $x, y^{-1} \in G^x$, it follows that $y \notin G_x$, hence $y \in T_x$. We thus have $xy^{-1} \not \subset \{x, y^{-1}\}$. Taking $g := y^{-1}$, we cannot have $ygy^{-1} \not \subset xy^{-1}$ and we also cannot have $xg^{-1}y^{-1} \not \subset xy^{-1}$. Therefore, neither $(x,y) \leq (xg,yg)$ nor $(xg,yg) \leq (x,y)$ is true.
- (iii) If the orbit of (x,y) under G^x is a chain, then by (ii) we must have $xy^{-1} \in T_x$. By (i), x cannot be v-type. Conversely, assume x is o-type with $xy^{-1} \in T_x$. Since G^x/G_x is order-type, and since $xG_g \neq yG_g$, it follows from Lemma 4.1 that the orbit of (xG_g, yG_g) under the action of G^g/G_g is a non-trivial chain. It then follows from lemma 4.2 that the orbit of (x,y) under G^x is also a non-trivial chain.

Proposition 4.4

Let (G, \preceq) be a C-q.o.g. The following holds:

- (i) All orbits are antichains if and only if every element if v-type.
- (ii) There exists an orbit which is a chain if and only if there exists $g \in G$ o-type such that T_g is maximal in the set of type-components of G (for the order given in Proposition 3.33).

Proof. If every orbit is an antichain, then by Lemma 4.3(iii) every element of G must be v-type (otherwise we can always choose $x, y \in G$ o-type with $xy^{-1} \in T_x$, for example choose any o^+ -type element x and $y := x^2$). The converse follows from 4.3(i). Now assume that $x \in G$ is an o-type element such that T_g is maximal in the set of type-components of G. Take $y \in T_x$ with $xy^{-1} \in T_x$. It follows from Lemma 4.3(iii) that the orbit of (x, y) under G is a chain. Conversely, assume there is an orbit of an element (x, y) which is a chain. Since (x, y) = (y, x), we can assume without loss of generality that $y \preceq x$, hence $y \in G^x$. By Lemma 4.3(iii), this implies in particular that x is o-type with $xy^{-1} \in T_x$. Assume that there is some $g \notin G^x$. We then have $xgy^{-1}, ygy^{-1}, xg^{-1}y^{-1}, yg^{-1}y^{-1} \notin G^x$, so neither $(x, y) \le (xg, yg)$ nor $(xg, yg) \le (x, y)$ can be true.

We can now reformulate Theorems 4.4, 4.8 and 4.9 of [6] into the following result:

Theorem 4.5

Let (G, \preceq) be a C-minimal C-q.o.g and assume that C is a dense C-relation. Then exactly one of the following holds:

- (i) \leq comes from a valuation $v: G \to \Gamma \cup \{\infty\}$. In that case, we have the following:
 - (1) For any $\gamma \in \Gamma$, G_{γ} and G^{γ} are normal in G.
 - (2) The quotient G^{γ}/G_{γ} is abelian for all but finitely many $\gamma \in \Gamma$.
 - (3) If G^{γ}/G_{γ} is infinite, then it is elementary abelian or divisible abelian. If it is divisible, then G^{γ} is also abelian.
 - (4) There is a definable abelian subgroup H of G such that G/H has finite exponent.

- (ii) There exists an o-type element $g \in G$ such that T_g is maximal in the set of type-components of G. In that case G is abelian and divisible, G_g is C-minimal and G^g/G_g is o-minimal.
- (iii) G contains o-type elements, but T_g is never maximal for any g o-type. In that case there exists $g \in G$ such that the following holds:
 - (1) The final segment $\{h \in G \mid g \lesssim h\}$ only contains v-type elements.
 - (2) There is a definable subgroup H of G such that G/H has finite exponent.

Proof. Cases (i) and (ii) are direct reformulations of theorems 4.4 and 4.8 from [6] using our Proposition 4.4. For (iii), we know from Proposition 4.4 and from Theorem 4.9 of [6] that there exists $w := (g,1) \in \mathfrak{T}, g \in G$, such that for any $t \leq w$, the orbit of t under G is an antichain. Since $g.1^{-1} \in T_g$, and since the orbit of w under G is an antichain, Lemma 4.3(iii) implies that g is v-type. Now let $u \in G$ with $g \lesssim u$. We have $\{1,g\} \lesssim u$. By definition of \leq , this implies $(u,1) \leq (g,1) = w$, so the orbit of (u,1) under G is an antichain. Since moreover $u \in T_u$, it follows from Lemma 4.3(iii) that u is v-type. \square

- Remark 4.6 1. Theorem 4.5 shows in particular that, if G is C-minimal, then the set of type-components has a maximum. Thus, the "ordered" parts cannot alternate indefinitely with the "valued" parts. Eventually, the group has to either stay valuational-like or stay order-type-like.
 - 2. Theorem 4.5 leaves open the question of welding in the case of C-minimality. More precisely, we don't know if it is possible to have welding in case (ii).
 - 3. We needed the assumption of density in Theorem 4.5 in order to apply the results of [6]. In a coming paper, we will go further and explore the structure of general C-minimal groups, i.e without the density assumption.

References

- [1] S.A. Adeleke and P.M. Neumann *Primitive permutation groups with primitive Jordan* sets, Journal of the London Mathematical Society 53(2), April 1996.
- [2] S.A. Adeleke and P.M. Neumann Relations related to betweenness: their structure and their automorphisms, Memoirs of the American Mathematical Society 623, January 1998.
- [3] Françoise Delon: C-minimal structures without the density assumption, In Raf Cluckers, Johannes Nicaise et Julien Sebag, éditeurs: Motivic Integration and its Interactions with Model Theory and Non-Archimedean Geometry. Cambridge University Press, Berlin, 2011.
- [4] Syed M.Fakhruddin, *Quasi-ordered fields*, Journal of Pure and Applied Algebra 45, 207-210, 1987.

- [5] Gabriel Lehéricy, A structure theorem for abelian quasi-ordered groups, preprint, arXiv number 1606.07710v4, 2016, submitted.
- [6] Dugald Macpherson and Charles Steinhorn, On variants of o-minimality, Annals of Pure and Applied Logic 79, 165-209, 1996.
- [7] Patrick Simonetta, Abelian C-minimal groups, Annals of pure and applied logic, 110 (1-3):1-22, 2001.
- [8] Patrick Simonetta, On non-abelian C-minimal groups, Annals of pure and applied logic, 122, 263 287, 2003.

FACHBEREICH MATHEMATIK UND STATISTIK, UNIVERSITÄT KONSTANZ, 78457, GERMANY.

UNIVERSITÉ PARIS DIDEROT, IMJ-PRG, 75013 PARIS, FRANCE.

Email address: gabriel.lehericy@uni-konstanz.de